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INTEGRABILITY OF A CLASS OF N–DIMENSIONAL LOTKA–VOLTERRA AND KOLMOGOROV SYSTEMS

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Abstract. We study the integrability of an N–dimensional differential Kolmogorov systems of the form

$$\dot{x}_j = x_j\left(a_j + \sum_{k=1}^{N} a_{jk}x_k\right) + x_j\Psi(x_1,\ldots,x_N), \quad j = 1,\ldots,N,$$

where a_j, and a_{jk} are constants for $j,k = 1,\ldots,N$ and $\Psi(x_1,\ldots, x_N)$ is a homogenous polynomial of degree $n > 2$, with either one additional invariant hyperplane, or with one exponential factor. We also study the integrability of the N–dimensional classical Lotka-Volterra systems (when $\Psi(x_1,\ldots,x_N) = 0$). In particular we consider the integrability of the asymmetric May–Leonard systems.

1. Introduction and main results

For the N–dimensional nonlinear differential systems the existence of $K < N-1$ independent first integrals means that systems is partially integrable. The existence of $N-1$ independent first integrals means that the system is completely integrable, i.e. the intersection of the $N-1$ hypersurfaces obtained fixing the $N-1$ first integrals provide the trajectories of the differential system.

Polynomial differential systems of the form

$$\dot{x}_j = x_j f_j(x_1,\ldots,x_N), \quad \text{for} \quad j = 1,\ldots,N$$

is called the N*-dimensional Kolmogorov differential equations*, where $f_j = f_j(x_1,\ldots, x_N)$ is a given function for $j = 1,\ldots,N$.

We develop a method based in the study of the rank of convenient matrices in order to determine the integrability of a class of N–dimensional Kolmogorov systems of the form

$$\dot{x}_j = x_j\left(a_j + \sum_{k=1}^{N} a_{jk}x_k\right) + x_j\Psi(x_1,\ldots,x_N), \quad j = 1,\ldots,N, \tag{1}$$

where $\Psi(x_1,\ldots,x_N)$ is a homogenous polynomial of degree n, with either one additional invariant hyperplane, or with one exponential factor.

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When $\Psi(x_1, x_2, x_3) \equiv 0$ systems (1) becomes the classical Lotka–Volterra differential systems that describe the evolution of N conflicting species in population biology, and appear in many different topics like neural networks, laser physics, plasma physics, etc. (see for instance [3, 6, 11, 13]).

For simplicity we shall assume that all the functions which appear below are of class C^∞ although most of the results remain valid under weaker hypotheses.

1.1. **Integrability of N- dimensional Lotka-Volterra systems with an additional invariant plane.** We begin by studying the integrability of the N–dimensional classical Lotka–Volterra systems, i.e. systems (1) with $\Psi(x_1, \ldots, x_N) = 0$.

We shall study the classical N- dimensional Lotka -Volterra systems.

$$\dot{x}_j = x_j \left(a_{j\,N+1} + \sum_{k=1}^{N} a_{jk} x_k \right) := X_j, \quad j = 1, \ldots, N, \tag{2}$$

Since populations can not be negative, the vector $\mathbf{x} = (x_1, \ldots, x_N)$ is taken to be non-negative, i.e., $\mathbf{x}$ is an element of

$$\mathbb{R}^N_+ = \left\{ \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^N : x_j \geq 0, \qquad \text{for} \quad j = 1, \ldots, N \right\}.$$

Hence any population might be zero. However if $x_j = 0$ holds at certain point in time, it holds for all time.

The Lotka -Volterra systems were introduced independently by Lotka and Volterra in the 1920's to model the interaction among species, see [11, 13].

Differential system (2) has N invariant hyperplanes $g_j = 0$ with cofactors K_j for $j = 1, 2, \ldots, N$ namely

$$g_j = x_j, \qquad K_j = a_{j\,N+1} + \sum_{k=1}^{N} a_{jk} x_k, \quad \text{for} \quad j = 1, \ldots, N.$$

Clearly that

$$\operatorname{div}\mathcal{X} = \sum_{j=1}^{N} \left(a_j + 2a_{jj} x_j + \sum_{k \neq j}^{N} a_{jk} x_k \right) \tag{3}$$

where $\mathcal{X} = \displaystyle\sum_{j=1}^{N} X_j \frac{\partial}{\partial x_j}$.

The aim of this subsection is to study the problem on the integrability of system (2) by considering that $(x_1, \ldots, x_N) \in \mathbb{R}^N$, with the additional invariant hyperplane $g_{N+1} = 0$ and with the cofactor K_{N+1} such that

$$g_{N+1} = \nu_{N+1} + \sum_{j=1}^{N} \nu_j x_j, \quad K_{N+1} = a_{N+1\,N+1} + \sum_{k=1}^{N} a_{N+1\,k} x_k. \tag{4}$$

First we introduce some definitions and notions which we will use in this paper.

Let U be an open subset of $\mathbb{R}^N$. We say that a non-locally constant function $H : U \longrightarrow \mathbb{R}$ is a *first integral* of the differential system

$$\dot{x}_j = X_j(x_1, x_2, \ldots, x_N), \tag{5}$$

for $j = 1, \ldots, N$, or its associated vector field

$$\mathcal{X} = X_1\frac{\partial}{\partial x_1} + X_2\frac{\partial}{\partial x_2} + \ldots + X_N\frac{\partial}{\partial x_N},$$

if $H = H(x_1(t), \ldots, x_N(t))$ is constant for all values of t for which the solution $(x_1(t), \ldots, x_N(t))$ is defined and contained in U. Clearly a C^1 function H is a first integral of system (5) if and only if

$$\dot{H} = \mathcal{X}(H) = \frac{\partial H}{\partial x_1}X_1 + \frac{\partial H}{\partial x_2}X_2 + \ldots + \frac{\partial H}{\partial x_N}X_N \equiv 0 \quad \text{in } U.$$

If $H_r : U_r \longrightarrow \mathbb{R}$ for $r = 1, \ldots, K$ are K first integrals of system (5), we say that they are *independent* in $\tilde{U}_K := U_1 \cap U_2 \ldots \cap U_K$ if their gradients are independent in all the points of $\tilde{U}$ except perhaps in a zero Lebesgue measure set.

First we characterize the integrability of the N–dimensional Lotka-Volterra systems with $N+1$ invariant hyperplanes.

Theorem 1. *The Lotka-Volterra systems* (2) *with the additional invariant hyper plane $g_{N+1} = 0$ with cofactor K_{N+1} given in* (4) *has the Darboux first integral*

$$H = \log\left(|g_{N+1}|^{\mu_{N+1}}\prod_{j=1}^{N}|x_j|^{\mu_j}\right), \tag{6}$$

if and only if the rank of the $(N+1)\times(N+1)$ *matrix*

$$B_1 = \begin{pmatrix} a_{11} & \ldots & a_{N\,1} & a_{N+1\,1} \\ a_{12} & \ldots & a_{N\,2} & a_{N+1\,2} \\ \vdots & \ldots & \vdots & \\ a_{1\,N+1} & \ldots & a_{N-1,N} & a_{N+1\,N+1} \end{pmatrix} \tag{7}$$

has

$$\mathrm{rankB_1} \,\leq\, \mathrm{N}. \tag{8}$$

We say that system (5) is *completely integrable* in an open set $\tilde{U}_{N-1}$ if it has $N-1$ independent first integrals. In this case the orbits of system (5) are contained in the curves

$$\Big\{H_1 = h_1\Big\} \cap \Big\{H_2 = h_2\Big\} \ldots \cap \Big\{H_{N-1} = h_{N-1}\Big\}$$

where h_1, $h_2 \ldots, h_{N-1}$ vary in $\mathbb{R}$.

Let $J = J(x_1, \ldots, x_N)$ be a non–negative function non–identically zero on an open subset U of $\mathbb{R}^N$. Then J is a *Jacobi multiplier* of the differential system (5) if

$$\int_{\Omega} J(x_1, \ldots, x_N)dx_1 \ldots dx_N = \int_{\varphi_t(\Omega)} J(x_1, \ldots, x_N)dx_1 \ldots dx_N,$$

for all open subset Ω of U, φ_t is the flow defined by the differential system (5), and $\varphi_t(\Omega)$ is the image of Ω under the flow φ_t.

Theorem 2. *The Lotka-Volterra systems with $N+1$ invariant hyperplanes has $N-1$ independent Darboux first integrals $H_1,\, H_2, \ldots, H_{N-1}$ if and only if* $\operatorname{rank} B_1 = 2$. *Moreover the independent first integrals are*

$$\begin{aligned} H_j = &\quad \log\left(|x_1|^{\kappa_{1j}}|x_2|^{\kappa_{2j}}|x_j|\right) \quad \textit{for} \qquad j=3,\ldots,N, \\ H_{N+1} = &\quad \log\left(|x_1|^{\kappa_{1\,N+1}}|x_2|^{\kappa_{2\,N+1}}|g_{N+1}|\right), \end{aligned} \tag{9}$$

where κ_{nj} is a convenient constant, for $n = 1, 2$ and $j = 3, \ldots, N$.

Moreover completely integrable N–dimensional Lotka-Volterra systems (2) can be written as

$$\dot{x}_j = \left(|g_{N+1}|\prod_{j=1}^{N}|x_j|\right)\{H_1,\, H_2, \ldots, H_{N-1}, x_j\}, \quad \textit{for} \quad j=1,\ldots,N,$$

where $J = \left(|g_{N+1}|\prod_{j=1}^{N}|x_j|\right)^{-1}$ *is the Jacobi multiplier.*

Theorem 3. *Let $J = |g_{N+1}|^{\sigma_{N+1}}\prod_{j=1}^{N}|x_j|^{\sigma_j}$ be a non-negative C^1 function non-identically zero defined on an open subset dense in $\mathbb{R}^N$. Assume that the $g_{N+1}=0$ is an invariant hyperplane with cofactor K_{N+1}. Then J is a Jacobi multiplier of the Lotka-Volterra systems* (2) *if and only if*

$$\operatorname{rank}(\mathrm{B}_1) = \operatorname{rank}(\mathrm{B}_2) = \operatorname{rank}(\mathrm{B}_3), \tag{10}$$

where

$$B_2 = \begin{pmatrix} a_{11} & \ldots & a_{N+1\,1} & -2a_{11} - \sum_{j\neq 1}^{N} a_{1j} \\ a_{21} & \ldots & a_{N+1\,2} & -2a_{22} - \sum_{j\neq 2}^{N} a_{2j} \\ \vdots & \ldots & \vdots & \\ a_{1\,N+1} & \ldots & a_{N+1,N+1} & -\sum_{j=1}^{N} a_{j\,N+1} \end{pmatrix}.$$

$$B_3 = \begin{pmatrix} a_{11} & \ldots & a_{N,1} & a_{N+1\,1} & -a_{11} + a_{N+1\,1} \\ a_{21} & \ldots & a_{N\,2} & a_{N+1\,2} & -a_{22} + a_{N+1\,2} \\ \vdots & \ldots & \vdots & & \\ a_{1\,N+1} & \ldots & a_{N,N+1} & a_{N+1,N+1} & a_{N+1,N+1} \end{pmatrix}.$$

Moreover if

$$\operatorname{rank}(\mathrm{B}_1) = \operatorname{rank}(\mathrm{B}_2) = \operatorname{rank}(\mathrm{B}_3) \leq \mathrm{N}, \tag{11}$$

then differential system (2) *is completely integrable.*

Corollary 4. *The N–dimensional Lotka-Volterra systems* (2) *with the invariant hyperplane* $g_{N+1} = \nu_{N+1} + \sum_{j=1}^{N} \nu_j x_j = 0$, *with the cofactor* $K_{N+1} = \sum_{k=1}^{N} a_{j\,j} x_j$ *has the Jacobi multiplier*

$$J = \frac{e^H}{|g_{N+1}| \prod_{j=1}^{N} |x_j|}.$$

and the first integral $H = \log \prod_{j=1}^{N} \left(|x_j||g_{N+1}|\right)^{\mu_j}$.

Theorem 5. *The Lotka-Volterra system* (2) *has the Jacobi multiplier*

$$J = \frac{1}{|g_{N+1}| \prod_{j=1}^{N} |x_j|},$$

and $N-2$ Darboux first integral

$$\begin{aligned} H_j = &\quad \log\left(|x_1|^{\kappa_{1j}}|x_2|^{\kappa_{2j}}|x_3|^{\kappa_{3j}}|x_j|\right) \quad \textit{for} \quad j = 4, \ldots, N, \\ H_{N+1} = &\quad \log\left(|x_1|^{\kappa_{1\,N+1}}|x_2|^{\kappa_{2\,N+1}}|x_3|^{\kappa_{3\,N+1}}|g_{N+1}|\right), \end{aligned} \tag{12}$$

if and only if

$$\mathrm{rank}(\mathrm{B}_1) = \mathrm{rank}(\mathrm{B}_2) = \mathrm{rank}(\mathrm{B}_3) = 3. \tag{13}$$

Moreover this differential system is completely integrable and the complementary first integral H_{N-1} can be determined from the equations

$$\{H_1,\, H_2, \ldots, H_{N-1}, x_j\} = J\, x_j \left(a_{j\,N+1} + \sum_{k=1}^{N} a_{jk} x_k\right) \tag{14}$$

where $j = 1, \ldots, N$.

The proofs of these results are given in section 3.

1.2. **Integrability of N- dimensional Lotka-Volterra systems with one exponential factor.** We shall study the Lotka-Volterra system (2) with an exponential factor.

Let g and h be relative prime polynomials in the variables $x_1, x_2, \ldots, x_N$. Then the function $E = e^{h/g}$ is called an *exponential factor* of the polynomial vector field $\mathcal{X} = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \ldots + X_N \frac{\partial}{\partial x_N}$, of degree m if $\mathcal{X}(E) = LE$, where L is a polynomial of degree at most $m-1$, L is called the cofactor of the exponential factor E. Clearly the exponential factors do not define invariant hypersurfaces of the flow of $\mathcal{X}$.

In fact the exponential factor $E = e^{h/g}$ appears when the hypersurface $g = 0$ is invariant. There are exponential factor of the form $E = e^g$, these appears when the infinity has multiplicity larger then one. For more details see [9, 10].

Analogously we can to study the problem on the existence of first integrals and Jacobi multipliers for N- dimensional Lotka -Volterra with the invariant hyperplanes $g_j = x_j = 0$, for $j = 1, \ldots, N$ and with the exponential factor $E = e^{h/g}$ with cofactor $L_{N+1} = b_{N+1\,N+1} + \sum_{j=1}^{N} b_{N+1\,j} x_j$. Since the results of this subsection can be proved exactly as the ones of the previous section using the cofactors, we do not provide their proofs.

Theorem 6. *Assume that the Lotka–Volterra system* (2) *has the exponential factor* $E = e^{h/g}$. *Let* T_1 *be the* $(N+1) \times (N+1)$ *matrix*

$$T_1 = \begin{pmatrix} a_{11} & \ldots & a_{N\,1} & b_{N+1\,1} \\ a_{12} & \ldots & a_{N\,2} & b_{N+1\,2} \\ \vdots & \ldots & \vdots & \\ a_{1\,N+1} & \ldots & a_{N-1,N} & b_{N+1\,N+1} \end{pmatrix}.$$

Then the Lotka-Volterra system (2) *has the first integral*

$$H = \log\left(e^{\left(\mu_{N+1}\dfrac{h}{g}\right)} \prod_{j1}^{N} |x_j|^{\mu_j} \right),$$

if and only if $\mathrm{rank}(\mathrm{T}_1) \leq \mathrm{N}$. *Moreover if* $\mathrm{rank}(\mathrm{T}_1) = 2$, *then the Lotka-Volterra system* (2) *is completely integrable.*

Example 7. *Three dimensional Lotka-Volterra system*

$$\begin{aligned} \dot{x} = &\quad x\left(\alpha_4 + \alpha_2 y + \beta_3 \frac{\alpha_4\lambda_2 - \alpha_2\lambda_4)}{\lambda_2\beta_4} z\right), \\ \dot{y} = &\quad y\left(\beta_4 + \beta_1 x + \beta_3 z\right), \\ \dot{z} = &\quad z\left(\lambda_4 + \beta_1 \frac{\alpha_2\lambda_4 - \alpha_4\lambda_2}{\beta_4\alpha_2} x + \lambda_2 y\right). \end{aligned} \tag{15}$$

is completely integrable. Indeed in this case we have that

$$T_1 = \begin{pmatrix} 0 & \beta_1 & \beta_1 \dfrac{\alpha_2\lambda_4 - \alpha_4\lambda_2}{\beta_4\alpha_2} & \\ \alpha_2 & 0 & \lambda_2 & \lambda_2\beta_1 \\ \beta_3 \dfrac{\alpha_4\lambda_2 - \alpha_2\lambda_4)}{\lambda_2\beta_4} & 0 & \alpha_2\beta_3 & \\ \alpha_4 & \beta_4 & \lambda_4 & 0 \end{pmatrix}.$$

It is easy to show that $\mathrm{rank}(\mathrm{T}_1) = 2$. *Then in view of Theorem 6 we get that differential system* (15) *is completely integrable. The independent first integrals are*

$$\begin{aligned} H_1 &= \log\left(|x|^{\beta_4}|y^{-\alpha_4}|e^{\lambda_2\beta_1 x - \lambda_2\alpha_2 y + \alpha_2\beta_3 z)/\lambda_2}\right), \\ H_2 &= \log\left(|x|^{-\lambda_4}|z|^{\beta_4} e^{(\beta_1\lambda_2 x - \alpha_2\lambda_2 h y + \alpha_2\beta_3 z)/\alpha_2}\right). \end{aligned}$$

Differential system (15) *can be written as follows*

$$\dot{x} = \frac{1}{J}\{H_1, H_2, x\}, \quad \dot{y} = \frac{1}{J}\{H_1, H_2, y\}, \quad \dot{z} = \frac{1}{J}\{H_1, H_2, z\},$$

where $J = \dfrac{1}{\varrho|xyz|}$, *and* ϱ *is a non-zero constant.*

Theorem 8. *Assume that the Lotka–Volterra system* (2) *has the exponential factor* $E = e^{h/g}$. *Let* J *be a non-negative* C^1 *function non-identically zero on an open subset dense in* $\mathbb{R}^N$. *Then*

$$J = e^{\left(\sigma_{N+1}\dfrac{h}{g}\right)} \prod_{j=1}^{N} |x_j|^{\sigma_j}$$

is a Jacobi multiplier of the Lotka-Volterra systems (2) *if and only if*

$$\mathrm{rank}(\mathrm{T}_1) = \mathrm{rank}(\mathrm{T}_2) = \mathrm{rank}(\mathrm{T}_3),$$

where

$$T_2 = \begin{pmatrix} a_{11} & \dots & a_{N,1} & b_{N+1\,1} & -2a_{11} - \displaystyle\sum_{j\neq 1}^{N} a_{j1} \\ a_{21} & \dots & a_{N\,2} & b_{N+1\,2} & -2a_{22} - \displaystyle\sum_{j\neq 2}^{N} a_{j2} \\ \vdots & \dots & \vdots & & \\ a_{1\,N+1} & \dots & a_{N,N+1} & b_{N+1,N+1} & -\displaystyle\sum_{j=1}^{N} a_{N+1\,j} \end{pmatrix},$$

and

$$T_3 = \begin{pmatrix} a_{11} & \dots & a_{N,1} & b_{N+1\,1} & -a_{11} + b_{N+1\,1} \\ a_{21} & \dots & a_{N\,2} & b_{N+1\,2} & -a_{22} + b_{N+1\,2} \\ \vdots & \dots & \vdots & & \\ a_{1\,N+1} & \dots & a_{N,N+1} & b_{N+1,N+1} & b_{N+1,N+1} \end{pmatrix}$$

Moreover if $\mathrm{rank}(\mathrm{T}_1) = \mathrm{rank}(\mathrm{T}_2) = 3$, *then the Lotka-Volterra system* (2) *is completely integrable.*

As we observe the proofs of these results can be obtained from the proof of the results given in the previous section, by putting $e^{\left(\sigma_{N+1}\dfrac{g}{h}\right)}$ instead of g_{N+1}.

1.3. **Integrability of a class of N- dimensional Kolmogorov systems with one additional invariant hyperplane.** Now we shall study the class of Kolmogorov differential systems

$$\dot{x}_j = x_j \left(a_{j\,N+1} + \sum_{k=1}^{N} a_{jk} x_k \right) + x_j \Psi(x_1, \dots, x_N) := X_j, \quad j = 1, \dots, N, \tag{16}$$

where $\Psi(x_1, \dots, x_N)$ is a homogenous polynomial of degree $n > 2$, and system (16) has $N+1$ invariant hyperplanes.

Differential system (16) has N invariant hyperplanes $g_j = 0$ with cofactor K_j for $j = 1, \dots, N$, namely

$$g_j = x_j, \qquad K_j = a_{j\,N+1} + \sum_{k=1}^{N} a_{jk} x_k + \Psi,$$

where $j = 1, \ldots, N$.

Clearly in view of Euler Theorem for homogenous polynomials we get that

$$\frac{\partial(x_1\Psi)}{\partial x_1} + \frac{\partial(x_2\Psi)}{\partial x_2} + \ldots + \frac{\partial(x_N\Psi)}{\partial x_N} = (N+n)\Psi.$$

Then

$$\mathrm{div}\mathcal{X} = \sum_{\mathrm{j=1}}^{\mathrm{N}} \left(\mathrm{a_j + 2a_{jj}x_j} + \sum_{\mathrm{k \neq j}}^{\mathrm{N}} \mathrm{a_{jk}x_k} \right) + \mathrm{(N+n)}\Psi. \tag{17}$$

Theorem 9. *The differential system* (16) *with the invariant hyperplanes* $x_j = 0$ *for* $j = 1, \ldots, N$, *and with the additional invariant hyperplane* $g_{N+1} = 0$ *with cofactor* K_{N+1} *given by*

$$g_{N+1} = \nu_{N+1} + \sum_{k=1}^{N} \nu_j x_j, \qquad K_{N+1} = a_{N+1\,N+1} + \sum_{k=1}^{N} a_{N+1\,k} x_k + \Psi,$$

has the Darboux first integral

$$H = \log \left(\prod_{j=1}^{N} \left(\frac{|x_j|}{|g_{N+1}|} \right)^{\mu_j} \right) \tag{18}$$

if and only if the matrix $(N+1) \times N$

if and only if the $R(N+1) \times N$ *matrix*

$$W_1 = \begin{pmatrix} a_{11} - a_{N+1\,1} & \ldots & a_{N\,1} - a_{N+1\,1} \\ a_{12} - a_{N+1\,2} & \ldots & a_{N\,2} - a_{N+1\,2} \\ \vdots & \ldots & \vdots \\ a_{1\,N+1} - a_{N+1\,N+1} & \ldots & a_{N-1,N} - a_{N+1\,N+1} \end{pmatrix}$$

is such that

$$\mathrm{rank\,W_1} \le \mathrm{N}. \tag{19}$$

Theorem 10. *Let*

$$J = \frac{\displaystyle\prod_{j=1}^{N} \left(\frac{|x_j|}{|g_{N+1}|} \right)^{\mu_j}}{|g_{N+1}|^n \displaystyle\prod_{j=1}^{N} |x_j|}$$

be a non-negative C^1 *function non-identically zero on an open subset dense in* $\mathbb{R}^N$. *Then* J *is a Jacobi multiplier of differential system* (16) *if and only if*

$$\mathrm{rank(W_1) = rank(W_2)},$$

where

$$W_2 = \begin{pmatrix} 1 & \ldots & 1 & 1 & -n+1 \\ a_{11} & \ldots & a_{N\,1} & a_{N+1\,1} & -a_{11} + a_{N+1\,1} \\ a_{12} & \ldots & a_{N\,2} & a_{N+1\,2} & -a_{22} + a_{N+1\,2} \\ \vdots & \ldots & \vdots & & \\ a_{1\,N} & \ldots & a_{N\,N} & a_{N+1\,N} & -a_{N\,N} + a_{N+1\,N} \\ a_{1\,N+1} & \ldots & a_{N\,N+1} & a_{N+1\,N+1} & a_{N+1\,N+1} \end{pmatrix}.$$

Theorem 11. *Differential system* (16) *with the additional invariant hyperplane* $g_{N+1} = \nu_{N+1} + \sum_{j=1}^{N} \nu_j x_j = 0$ *with cofactor*

$$K_{N+1} = \sum_{k=1}^{N} a_{j\,j} x_j + \Psi, \tag{20}$$

has the Darboux first integral

$$H = \log\left(\prod_{j=1}^{N}\left(\frac{|x_j|}{|g_{N+1}|}\right)^{\mu_j}\right)$$

and the Jacobi multiplier

$$J = \frac{e^H}{|g_{N+1}|^n \prod_{j=1}^{N} |x_j|} \tag{21}$$

if and only if

$$\operatorname{rank} \mathrm{W}_1 = \operatorname{rank} \mathrm{W}_2 \leq \mathrm{N}.$$

Corollary 12. *Differential system* (16) *has a Jacobi multiplier and* $N-2$ *Darboux first integrals if and only if*

$$\operatorname{rank} \mathrm{W}_1 = \operatorname{rank} \mathrm{W}_2 = 3.$$

Moreover this differential system is completely integrable with the $N-2$ *first integrals*

$$\begin{aligned} H_j = &\quad \log\left(|x_1|^{\kappa_{1j}}|x_2|^{\kappa_{2j}}|x_3|^{\kappa_{3j}}|x_j|\right) \quad \textit{for} \quad j = 4, \ldots, N, \\ H_{N+1} = &\quad \log\left(|x_1|^{\kappa_{1\,N+1}}|x_2|^{\kappa_{2\,N+1}}|x_3|^{\kappa_{3\,N+1}}|g_{N+1}|\right), \end{aligned}$$

and with the Jacobi multiplier $J = \dfrac{1}{|g_{N+1}|^n \prod_{j=1}^{N} |x_j|}$.

The proof of these results are given in section 4

2. Preliminary results

The following result of Whittaker [15] plays a main role for detecting a Jacobi multiplier.

Theorem 13. *Let* J *be a non–negative* C^1 *function non–identically zero defined on an open subset of* $\mathbb{R}^N$. *Then* J *is a Jacobi multiplier of the differential system* (5) *if and only if the divergence of the vector field* $J\mathcal{X}$ *is zero, i.e.*

$$\operatorname{div}(J\mathcal{X}) := \frac{\partial(JX_1)}{\partial x_1} + \ldots + \frac{\partial(JX_N)}{\partial x_N} = 0.$$

The following result goes back to Jacobi, for a proof see Theorem 2.7 of [5].

Theorem 14. *Consider the differential system* (5) *and assume that it has a Jacobi multiplier J and $N-2$ independent first integrals $H_1, H_2, \ldots, H_{N-2}$. Then the system admits an additional first integral functionally independent of the previous ones given by*

$$H_{N-1} = \int \frac{\tilde{J}}{\tilde{\Delta}} \left(\tilde{X}_2 dx_1 - \tilde{X}_1 dx_2 \right),$$

where $\tilde{}$ *denotes quantities expressed in the variables* $(x_1, x_2, h_1, \ldots, h_{N-2})$ *with* $H_j = h_j$ *for* $j = 1, \ldots, N-2$ *and*

$$\Delta = \begin{vmatrix} \dfrac{\partial H_1}{\partial x_3} & \dfrac{\partial H_1}{\partial x_4} & \cdots & \dfrac{\partial H_1}{\partial x_N} \\ \dfrac{\partial H_2}{\partial x_3} & \dfrac{\partial H_2}{\partial x_4} & \cdots & \dfrac{\partial H_2}{\partial x_N} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial H_{N-2}}{\partial x_3} & \dfrac{\partial H_{N-2}}{\partial x_4} & \cdots & \dfrac{\partial H_{N-2}}{\partial x_N} \end{vmatrix}$$

Then system (5) *is completely integrable.*

Theorem 15. *Suppose that a polynomial vector field $\mathcal{X}$ defined in $\mathbb{R}^N$ admits M invariant algebraic hypersurfaces $g_j = 0$ with cofactor K_j for $j = 1, \ldots, M$, i.e. the following equations hold*

$$\mathcal{X} g_j = X_1 \frac{\partial g_j}{\partial x_1} + X_2 \frac{\partial g_j}{\partial x_2} + \ldots + X_N \frac{\partial g_j}{\partial x_N} = K_j \, g_j,$$

for $j = 1, \ldots, M$. If there exist $\mu_j \in \mathbb{R}$ not all zero such that

$$\sum_{j=1}^{M} \mu_j K_j + \operatorname{div} \mathcal{X} = 0,$$

if and only if the function $\displaystyle\prod_{j=1}^{M} |g_j|^{\mu_j}$ *is a Jacobi multiplier.*

For a proof of Theorem 15 see [4].

The following result is proved in [**?**].

Theorem 16. *A differential system* (5) *is completely integrable with C^2 first integrals H_j for $j = 1, \ldots, N-1$ if and only if it can be written as*

$$\dot{x}_j = \frac{1}{J} \Big\{ H_1, H_2, \ldots, H_{N-1}, x_j \Big\} = X_j, \quad \textit{for} \quad j = 1, 2, 3,$$

where $J = J(x_1, x_2, \ldots, x_N)$ is a Jacobi multiplier and $\Big\{ H_1, H_2, \ldots, H_{N-1}, * \Big\}$ is the Nambu bracket (see for instance [8, 12]*), i.e.*

$$\Big\{H_1, H_2, \ldots, H_{N-1}, *\Big\} := \begin{vmatrix} \dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \cdots & \dfrac{\partial H_1}{\partial x_N} \\ \dfrac{\partial H_2}{\partial x_1} & \dfrac{\partial H_2}{\partial x_2} & \cdots & \dfrac{\partial H_2}{\partial x_N} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial H_{N-1}}{\partial x_1} & \dfrac{\partial H_{N-1}}{\partial x_2} & \cdots & \dfrac{\partial H_{N-1}}{\partial x_N} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \cdots & \dfrac{\partial}{\partial x_N} \end{vmatrix}.$$

Theorem 16 extends results of [14] and [1]. From Theorem 16 it is immediate to prove the following result (see [7]).

Corollary 17. *Assume that differential systems* (5) *has* $N-2$ *independent first integrals* $H_1, H_2, \ldots, H_{N-2}$ *and a Jacobi multiplier* J, *then another independent first integral* H_{N-1} *can be obtained as a solution of the partial differential equation*

$$\{H_1, H_2, \ldots, H_{N-2}, H_{N-1}, x_j\Big\} = JX_j, \qquad \textit{for} \quad j = 1, \ldots, N.$$

Theorem 18. *Suppose that an* N*–dimensional polynomial vector field* $\mathcal{X}$ *admits* p *invariant algebraic hypersurface* $g_j = 0$ *with cofactors* K_j *for* $j = 1, \ldots, p$ *and* q *exponential factors* $E_j = e^{h_j/g_j}$ *with cofactors* L_j *for* $j = 1, \ldots, q$. *Then there exist complex numbers* μ_j *for* $j = 1, \ldots, p+q$ *not all zero such that*

$$\sum_{j=1}^{p} \mu_j K_j + \sum_{j=1}^{q} \mu_{p+j} L_j = 0,$$

if and only function $H = \log \left(\prod_{j=1}^{p} |g_j|^{\mu_j} \prod_{j=1}^{q} e^{\left(\mu_{j+p} \dfrac{g_j}{h_j}\right)} \right)$, *is a first integral of the vector field* $\mathcal{X}$.

For the proof of Theorem 18 see [4].

3. Proof of Theorems 1, 2 3, 5 and Corollary 4

Proof of Theorem 1. First we suppose that H given in (31) is a first integral of system (2), then by Theorem 18 we have $\displaystyle\sum_{j=1}^{N+1} \mu_j K_j = 0$, with not all the μ_j are zero. Hence

$$(22) \qquad \begin{array}{rl} \mu_1 a_{1\,1} + \mu_2 a_{2\,1} + \ldots + \mu_{N+1} a_{N+1\,1} = & 0, \\ \mu_1 a_{1\,2} + \mu_2 a_{2\,2} + \ldots + \mu_{N+1} a_{N+1\,2} = & 0, \\ \vdots \qquad \vdots \qquad \ldots \qquad & \vdots \\ \mu_1 a_{1\,N+1} + \mu_2 a_{2\,N+1} + \ldots + \mu_{N+1} a_{N+1\,N+1} = & 0. \end{array},$$

Consequently $B_1\mu = 0$ where $\mu = (\mu_1, \mu_2, \ldots, \mu_{N+1})^T$, where B_1 is the matrix given in (7). Since μ is a non-zero vector we obtain condition (8).

Now we suppose that (8) holds and we shall prove that there exists the Darboux first integral (31). Indeed, if (??) holds then (for example)

$$\begin{aligned}
a_{N+1\,1} = & \quad \gamma_1 a_{11} + \gamma_2 a_{21} + \ldots + \gamma_N a_{N\,1},\\
a_{N+1\,2} = & \quad \gamma_1 a_{12} + \gamma_2 a_{22} + \ldots + \gamma_N a_{N\,2},\\
\vdots \quad & \quad \vdots \qquad \ldots \qquad \vdots,\\
a_{N+1\,N+1} = & \quad \gamma_1 a_{1\,N+1} + \gamma_2 a_{2\,N+1} + \ldots + \gamma_N a_{N\,N+1},
\end{aligned}$$

where $\gamma_1, \gamma_2, \ldots, \gamma_N$ are convenient constants. Hence we get that the cofactor of the invariant plane $g_{N+1} = 0$ becomes

$$K_{N+1} = a_{N+1\,N+1} + \sum_{k=1}^{N} a_{N+1\,k} x_k = \sum_{j=1}^{N} \gamma_j K_j.$$

Thus from the equation

$$\frac{\partial g_{N+1}}{\partial x_1} X_1 + \ldots + \frac{\partial g_{N+1}}{\partial x_N} X_N = K_{N+1} g_{N+1} = \left(\frac{\gamma_1}{x_1} X_1 + \ldots + \frac{\gamma_N}{x_N} X_N\right) g_{N+1},$$

we deduce that

$$\left(\frac{\partial g_{N+1}}{\partial x} - \frac{g_{N+1}\gamma_1}{x_1}\right) X_1 + \ldots + \left(\frac{\partial g_{N+1}}{\partial x_N} - \frac{g_{N+1}\gamma_N}{x_N}\right) X_N = 0,$$

Hence

$$\left(\frac{\partial g_{N+1}}{\partial x_1} - g_{N+1} \frac{\partial \log \prod_{j=1}^{N} |x_j|^{\gamma_j}}{\partial x_1}\right) X_1 + \ldots + \left(\frac{\partial g_{N+1}}{\partial x_N} - g_{N+1} \frac{\partial \log \prod_{j=1}^{N} |x_j|^{\gamma_j}}{\partial x_N}\right) X_N = 0,$$

which is equivalent $\dfrac{\partial \tilde H}{\partial x_1} X_1 + \ldots + \dfrac{\partial \tilde H}{\partial x_N} X_N = \dot{\tilde H} = 0$, where $\tilde H = \log \left(\dfrac{|g_{N+1}|}{\prod_{j=1}^{N} |x_j|^{\gamma_j}}\right)$.

Thus $\tilde H$ is a first integral. Hence taking $\gamma_j = -\dfrac{\mu_j}{\mu_{N+1}}$ after some computations we get $H = \left(\tilde H\Big|_{\gamma_j = -\frac{\mu_j}{\mu_{N+1}}}\right)^{\mu_{N+1}}$. In short the theorem is proved. □

Proof of Theorem 2. By assumption the Lotka –Volterra system has $N+1$ invariant hyperplanes. Then by Theorem 1 the system (2) has a Darboux first integral of the form (31) because $\mathrm{rank}\,\mathrm{B}_1 = 2 \leq \mathrm{N}$. Thus (22) holds.

Since the $\mathrm{rank}\mathrm{B}_1 = 2$, by the Rouche– Frobenius Theorem the set of solutions of system (22) form a subspace of dimension $N - 1$. Then without loss of generality we can assume that

$$\mathbf{a}_n = -\kappa_{1n}\mathbf{a}_1 - \kappa_{2n}\mathbf{a}_2, \quad \text{for} \quad n = 3, \dots, N+1, \tag{23}$$

where $\mathbf{a}_n = (a_{1n}, a_{2n}, \dots, a_{N+1,n})$ for $n = 1, \dots, N+1$, and κ_{1n} and κ_{2n} are convenient constants.

Substituting (23) in system (22) we get

$$\mathbf{a}_1(\mu_1 - (\kappa_{13}\mu_3 + \ldots + \kappa_{1N+1}\mu_{N+1})) + \mathbf{a}_2(\mu_2 - (\kappa_{23}\mu_3 + \ldots + \kappa_{2N+1}\mu_{N+1})) = \mathbf{0},$$

where μ_j for $j = 3, \dots, N+1$ are arbitrary constants. Hence since $\mathbf{a}_1$ and $\mathbf{a}_2$ are independent vectors we obtain

$$\mu_1 = \kappa_{13}\mu_3 + \ldots + \kappa_{1N+1}\mu_{N+1}, \qquad \mu_2 = \kappa_{23}\mu_3 + \ldots + \kappa_{2N+1}\mu_{N+1}.$$

Consequently the Darboux first integral H becomes

$$\begin{aligned} H = \quad & \log\left(|x_1|^{\mu_1}|x_2|^{\mu_2}\ldots|x_N|^{\mu_N}|g_{N+1}|^{\mu_{N+1}}\right) \\ = \quad & \log\left(|x_1|^{\kappa_{11}\mu_3+\ldots+\kappa_{1N+1}\mu_{N+1}}|x_2|^{\kappa_{23}\mu_3+\ldots+\kappa_{2N+1}\mu_{N+1}}|x_3|^{\mu_3}|x_N|^{\mu_N}|g_{N+1}|^{\mu_{N+1}}\right) \\ = \quad & \log\left(|x_1|^{\kappa_{13}}|x_2|^{\kappa_{23}}|x_3|\right)^{\mu_3} + \ldots + \log\left(|x_1|^{\kappa_{1N}}|x_2|^{\kappa_{2N}}|x_N|\right)^{\mu_N} \\ & + \log\left(|x_1|^{\kappa_{1\,N+1}}|x_2|^{\kappa_{2\,N+1}}|g_{N+1}|\right)^{\mu_{N+1}} := \sum_{j=3}^{N+1}\mu_j H_j. \end{aligned}$$

In view of arbitrariness of $\mu_3, \dots, \mu_{N+1}$ we have that $H_j^{\mu_j}$ for $j = 3, \dots, N+1$ are independent first integrals (9). Thus, under the assumption rankB $= 2$ the N-dimensional Lotka-Volterra systems is completely Darboux integrable.

The reciprocity it is easy to obtain. Indeed, from Theorem 18 and 1 if $H_k = \log\left(|g_{N+1}|^{\mu^k_{N+1}}\prod_{j=1}^{N}|x_j|^{\mu^k_j}\right)$ for $k = 1, \dots, N-1$ are independent first integrals, then the following relations hold

$$\mu^{(k)}_{N+1}K_{N+1} + \mu^{(k)}_N K_N + \sum_{j=1}^{N-1}\mu^{(k)}_j K_j = 0, \quad \text{for} \quad k = 1, \dots, N-1.$$

We consider that the matrix $\Gamma = (\left.\mu^{(k)}_j\right|_{j,k=1,\dots,N-1})$ is a non-degenerated matrix. Hence after some computations we prove that

$$K_\alpha = \varrho_N K_N + \varrho_{N+1}K_{N+1}, \qquad \text{for} \quad \alpha = 1, \dots, N-1.$$

Consequently if we denote by $\mathbf{a}_n = (a_{1n}, a_{2n}, \dots, a_{N+1,n})$ for $n = 1, \dots, N+1$, we get

$$\mathbf{a}_n = \kappa_N\mathbf{a}_N + \kappa_{N+1}\mathbf{a}_{N+1}, \qquad \text{for} \quad n = 1, \dots, N-1.$$

for convenient constants κ_N and κ_{N+1}. Hence we easily obtain that the matrix B_1 is such that rank$\text{B}_1 = 2$. In short the corollary is proved. $\square$

Example 19. *Three dimensional Lotka-Volterra systems with the invariant plane not passing through the origin* $g_4 = \nu_1 x + \nu_2 y + \nu_3 z + \nu_4 = 0$ *with cofactor* $K_4 =$

$\alpha_1 x + \beta_2 y + \lambda_3 z$, *is*

$$
\begin{aligned}
\dot{x} = &\quad x\left(\frac{\alpha_1\nu_4}{\nu_1} + \alpha_1 x + \left(\beta_2 - \frac{\nu_2(\beta_1-\alpha_1)}{\nu_1}\right) y + \alpha_3\, z\right),\\
\dot{y} = &\quad y\left(\frac{\beta_2\nu_4}{\nu_2} + \beta_1 x + \beta_2 y + \left(\lambda_3 - \frac{\nu_3(\lambda_2-\beta_2)}{\nu_2}\right) z\right),\\
\dot{x} = &\quad z\left(\frac{\lambda_3\nu_4}{\nu_3} + \left(\alpha_1 - \frac{\nu_1(\alpha_3-\lambda_3)}{\nu_3}\right) x + \lambda_2 y + \lambda_3 z\right)
\end{aligned}
\tag{24}
$$

The matrix B_1 under the conditions σ_1, becomes

$$
B_1 = \begin{pmatrix}
\alpha_1 & \beta_1 & \alpha_1 - \dfrac{\nu_1(\alpha_3-\lambda_3)}{\nu_3} & \alpha_1\\
\beta_2 - \dfrac{\nu_2(\beta_1-\alpha_1)}{\nu_1} & \beta_2 & \lambda_2 & \beta_2\\
\alpha_3 & \lambda_3 - \dfrac{\nu_3(\lambda_2-\beta_2)}{\nu_2} & \lambda_3 & \lambda_3\\
\dfrac{\alpha_1\nu_4}{\nu_1} & \dfrac{\beta_2\nu_4}{\nu_2} & \dfrac{\lambda_3\nu_4}{\nu_3} & 0
\end{pmatrix}.
$$

By considering that

$$
\det(B_1) = -\frac{\nu_4}{\nu_1\nu_2\nu_3}\Big(\nu_1\beta_2(\lambda_3-\alpha_3) + \nu_2\lambda_3(\alpha_1-\beta_1) + \nu_3\alpha_1(\beta_2-\lambda_2)\Big)^2.
$$

we get that if $\det B_1 = 0$, *then it is easy to prove that* $\operatorname{rank}(\mathrm{B}_1) = 2$, *hence in view of Theorem 2 differential system* (24) *has two independent Darboux first integrals H_1 and H_2. In particular if*

$$
\nu_1 = \nu_2 = \nu_3 = -\nu_4 = -1, \qquad \alpha_1 = \frac{\alpha_3\beta_2 + \lambda_3(\beta_1-\beta_2)}{\lambda_3-\lambda_2+\beta_2},
\tag{25}
$$

we get that the first integrals are

$$
\begin{aligned}
H_1 = &\quad \log\left(|x|^{-\lambda_3(\lambda_3-\lambda_2+\beta_2}|z|^{\alpha_3\beta_2+\lambda_3(\beta_1-\beta_2)}|1-x-y-z|^{(\alpha_3+\beta_2-\beta_1)\lambda_3-\alpha_3\lambda_2}\right),\\
H_2 = &\quad \log\left(|x|^{-\beta_2(\lambda_3-\lambda_2+\beta_2)}|y|^{\alpha_3\beta_2+\lambda_3(\beta_1-\beta_2)}|1-x-y-z|^{(\beta_2-\beta_1)(\lambda_3-\lambda_2+\beta_2)}\right),
\end{aligned}
$$

After some computations we obtain that differential system (24) *under the conditions* (25) *can be written as*

$$
\dot{x} = \frac{1}{J}\{H_1, H_2, x\}, \quad \dot{y} = \frac{1}{J}\{H_1, H_2, y\}, \quad \dot{z} = \frac{1}{J}\{H_1, H_2, z\},
$$

where $J = \dfrac{1}{\varrho|xyz(1-x-y-z)|}$, *is the Jacobi multiplier, and ϱ is nonzero constant.*

Proof of Theorem 3. From Theorem 13 we get that system (2) has a Jacobi multiplier if and only if

$$
\sum_{j=1}^{N} X_j \frac{\partial \log J}{\partial x_j} + \operatorname{div}\mathcal{X} = 0.
$$

Consequently, if $J=|g_{N+1}|^{\sigma_{N+1}}\prod_{j=1}^{N}|x_j|^{\sigma_j}$ then we get that $\sum_{j=1}^{N+1}\sigma_jK_j+\operatorname{div}\mathcal{X}=0$. In view of (3) we get that this equation holds if and only if

$$
\begin{aligned}
\sigma_1a_{1\,1}+\sigma_2a_{2\,1}+\ldots+\sigma_{N+1}a_{N+1\,1} &= -2a_{11}-\sum_{j\neq 1}^{N}a_{1j},\\
\sigma_1a_{1\,2}+\sigma_2a_{2\,2}+\ldots+\sigma_{N+1}a_{N+1\,2} &= -2a_{22}-\sum_{j\neq 2}^{N}a_{2j},\\
\vdots\qquad\quad \vdots\qquad \ldots\qquad\quad & \quad\vdots\\
\sigma_1a_{1\,N+1}+\sigma_2a_{2\,N+1}+\ldots+\sigma_{N+1}a_{N+1\,N+1} &= -\sum_{j=1}^{N}a_j.
\end{aligned}
\tag{26}
$$

After the change $\sigma_j=-1+\mu_j$ for $j=1,\ldots,N+1$ we obtain that

$$
\begin{aligned}
\mu_1a_{1\,1}+\mu_2a_{2\,1}+\ldots+\mu_{N+1}a_{N+1\,1} &= -a_{11}+a_{N+1\,1},\\
\mu_1a_{1\,2}+\mu_2a_{2\,2}+\ldots+\mu_{N+1}a_{N+1\,2} &= -a_{22}+a_{N+1\,2},\\
\vdots\qquad\quad \vdots\qquad \ldots\qquad\quad & \quad\vdots\qquad\qquad ,\\
\mu_1a_{1\,N}+\mu_2a_{2\,N}+\ldots+\mu_{N+1}a_{N\,N+1} &= -a_{NN}+a_{N+1\,N},\\
\mu_1a_{1\,N+1}+\mu_2a_{2\,N+1}+\ldots+\mu_{N+1}a_{N+1\,N+1} &= a_{N+1\,N+1}.
\end{aligned}
\tag{27}
$$

Consequently in view of the Rouché–Frobenius Theorem we get that system (27) has solution if and only if (10) holds. On the other hand if (11) holds then in view of Theorem 1 we obtain that there exists the first integral (31), consequently from Theorem 14 we get that the Lotka-Volterra system is completely integrable. In short the theorem is proved. $\square$

Proof of Corollary 4. The proof of this corollary is obtained from the proof of Theorem 3. Indeed, from (26) we get that if $a_{N+1\,j}=a_{j\,j}$, and $a_{N+1\,N+1}=0$ then

$$
\begin{aligned}
J &= \left.\left(|g_{N+1}|^{\sigma_{N+1}}\prod_{j=1}^{N}|x_j|^{\sigma_j}\right)\right|_{\sigma_1=-1+\mu_1,\ldots,\sigma_{N+1}=-1+\mu_{N+1}}\\
&= \frac{|g_{N+1}|^{\mu_{N+1}}\prod_{j=1}^{N}|x_j|^{\mu_j}}{|g_{N+1}|\prod_{j=1}^{N}|x_j|}:=\frac{e^H}{|g_{N+1}|\prod_{j=1}^{N}|x_j|},
\end{aligned}
$$

where $\mu_1, \ldots, \mu_{N+1}$ are solutions of (27), i.e.,

$$\begin{array}{rcl} \mu_1 a_{1\,1} + \mu_2 a_{2\,1} + \ldots + \mu_{N+1} a_{N+1\,1} &=& 0, \\ \mu_1 a_{1\,2} + \mu_2 a_{2\,2} + \ldots + \mu_{N+1} a_{N+1\,2} &=& 0, \\ \vdots \qquad \vdots \qquad \ldots && \vdots \\ \mu_1 a_{1\,N} + \mu_2 a_{2\,N} + \ldots + \mu_{N+1} a_{N\,N+1} &=& 0, \\ \mu_1 a_{1\,N+1} + \mu_2 a_{2\,N+1} + \ldots + \mu_{N+1} a_{N+1\,N+1} &=& 0. \end{array},$$

Hence in view of Theorem 1 we obtain that $H = \log\left(|g_{N+1}|^{\mu_{N+1}} \prod_{j=1}^{N} |x_j|^{\mu_j} \right)$ is the Darboux first integral see [**?**]. In short the corollary is proved. $\square$

Proof of Theorem 5. If $\operatorname{rank}(B_1) = 3 \leq N$, hence

$$\mathbf{a}_n = -\kappa_{1n}\mathbf{a}_1 - \kappa_{2n}\mathbf{a}_2 - \kappa_{3n}\mathbf{a}_3, \quad \text{for} \quad n = 4, \ldots, N+1. \tag{28}$$

where $\mathbf{a}_j = (a_{1j}, \ldots, a_{N+1j})$ for $j = 1, \ldots, N+1$, and κ_{1n}, κ_{2n} and κ_{3n} are constants.

From Theorem 1 it follows that there exists the first integral (31). Consequently inserting (28) into (22) we have that

$$\mathbf{a}_1(\mu_1 - \sum_{j=4}^{N+1} \kappa_{1j}\mu_j) + \mathbf{a}_2(\mu_2 - \sum_{j=4}^{N+1} \kappa_{2j}\mu_j) + \mathbf{a}_3(\mu_3 - \sum_{j=4}^{N+1} \kappa_{3j}\mu_j) = \mathbf{0}.$$

On the other hand from (13) we obtain that $\operatorname{rank}(\mathrm{B}_1) = \operatorname{rank}(\mathrm{B}_2) = \operatorname{rank}(\mathrm{B}_3) = 3$, thus in view of Theorem 3 we obtain that there exists the Jacobi multiplier

$$J = |g_{N+1}|^{\sigma_{N+1}} \prod_{j=1}^{N} |x_j|^{\sigma_j} \Bigg|_{\sigma_j = -1+\mu_j}.$$

Then from (27) we have

$$0 = \mathbf{a}_1(\mu_1 - \sum_{j=4}^{N+1} \kappa_{1j}\mu_j) + \mathbf{a}_2(\mu_2 - \sum_{j=4}^{N+1} \kappa_{2j}\mu_j) + \mathbf{a}_3(\mu_3 - \sum_{j=4}^{N+1} \kappa_{3j}\mu_j) = -\mathbf{a}_{N+1} + \mathbf{b},$$

where $\mathbf{b} = (a_{11}, a_{22}, \ldots, a_{NN}, 0)$. Hence by considering that $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ are independent vectors we get that

$$\mu_1 = \sum_{j=4}^{N+1} \kappa_{1j}\mu_j, \quad \mu_2 = \sum_{j=4}^{N+1} \kappa_{2j}\mu_j, \quad \mu_3 = \sum_{j=4}^{N+1} \kappa_{3j}\mu_j,$$

and $\mathbf{a}_{N+1} = \mathbf{b}$, thus $a_{N+1\,j} = a_{jj}$, for $j = 1, \ldots, N$, $a_{N+1\,N+1} = 0$, where $\mu_4, \ldots, \mu_{N+1}$ are arbitrary constants. Consequently the Darboux first integral H

becomes

$$\begin{aligned}
H = &\ \log\left(|x_1|^{\mu_1}|x_2|^{\mu_2}\dots|x_N|^{\mu_N}|g_{N+1}|^{\mu_{N+1}}\right)\\
= &\ \log\left(|x_1|^{\kappa_{11}\mu_3+\ldots+\kappa_{1\,N+1}\mu_{N+1}}|x_2|^{\kappa_{23}\mu_3+\ldots+\kappa_{2N+1}\mu_{N+1}}|x_3|^{\kappa_{33}\mu_3+\ldots+\kappa_{3\,N+1}\mu_{N+1}}|x_4|^{\mu_4}\right)\\
&+\ldots+\log\left(|x_N|^{\mu_N}|g_{N+1}|^{\mu_{N+1}}\right)\\
= &\ \log\left(|x_1|^{\kappa_{13}}|x_2|^{\kappa_{23}}|x_3|^{\kappa_{33}}|x_4|\right)^{\mu_4}+\log\left(|x_1|^{\kappa_{15}}|x_2|^{\kappa_{25}}|x_3|^{\kappa_{35}}|x_5|\right)^{\mu_5}\\
&+\ldots+\log\left(|x_1|^{\kappa_{1N}}|x_2|^{\kappa_{2N}}|x_3|^{\kappa_{3N}}|x_N|\right)^{\mu_N}\\
&+\log\left(|x_1|^{\kappa_{1\,N+1}}|x_2|^{\kappa_{2\,N+1}}|x_3|^{\kappa_{3\,N+1}}|g_{N+1}|\right)^{\mu_{N+1}}:=\sum_{j=4}^{N+1}\mu_j H_j,
\end{aligned}$$

which in view of arbitrariness $\mu_4,\dots,\mu_{N+1}$ we get the existence of first integral (12). The proof of (14) follows from Corollary 17. The reciprocity it is easy to obtain. Thus the theorem is proved. $\square$

4. Proofs of Theorems 9, 10 11 and Corollary 12

Proof of Theorem 9. Let $H=\log\left(\prod_{j=1}^{N}\left(\frac{|x_j|}{|g_{N+1}|}\right)^{\mu_j}\right)$ be a first integral of (16). Then by Theorem 18 we have

$$\sum_{j=1}^{N}\mu_j K_j-\sum_{j=1}^{N}\mu_j K_{N+1}=\sum_{j=1}^{N}\mu_j\left(K_j-K_{N+1}\right)\Longleftrightarrow$$
$$\sum_{j=1}^{N}\mu_j\left(a_{j\,N+1}-a_{N+1\,N+1}+\sum_{k=1}^{N}\left(a_{jk}-a_{N+1\,k}\right)x_k\right)=0.$$

Hence

$$\sum_{j=1}^{N}\mu_j=0,\ \sum_{j=1}^{N}\mu_j\left(a_{j\,N+1}-a_{N+1\,N+1}\right)=0,\ \sum_{j=1}^{N}\mu_j\left(a_{j\,k}-a_{N+1\,k}\right)=0,$$

for $k=1,\dots,N$, which is equivalent to $\sum_{j=1}^{N}\mu_j=0$ and $\sum_{j=1}^{N}\mu_j\left(a_{j\,k}-a_{N+1\,k}\right)=0$, for $k=1,\dots,N+1$, or equivalently

$$\begin{aligned}
(a_{11}-a_{N+1\;1})\mu_1+\ldots+(a_{N\,1}-a_{N+1\;1})\mu_N= &\quad 0,\\
(a_{12}-a_{N+1\;2})\mu_1+\ldots+(a_{N\,2}-a_{N+1\;2})\mu_N= &\quad 0,\\
\vdots \qquad\qquad \vdots \qquad\qquad &\quad \vdots,\\
(a_{1\;N+1}-a_{N+1\;N+1})\mu_1+\ldots+(a_{N-1\,N}-a_{N+1\;N+1})\mu_N= &\quad 0,
\end{aligned}$$

This system can be written in matrix form $W_1\mu=\mathbf{0}$, with $\mu=(\mu_1,\dots,\mu_N)^T\neq\mathbf{0}$ and $\mathbf{0}=(0,\dots,0)\in\mathbb{R}^{N+1}$. So since W_1 is a $(N+1)\times N$ matrix. From the Rouche–Frobenius Theorem we get that this linear system has a nontrivial solution if and only if $\mathrm{rank}\,(\mathrm{W}_1)\leq\mathrm{N}$.

Now we suppose that (19) holds and we shall prove that then there exits the Darboux first integral (18). Indeed, if (19) holds then

$$\mathbf{a}_1 - \mathbf{a}_{N+1} = \sum_{j=2}^{N} \lambda_j \left(\mathbf{a}_j - \mathbf{a}_{N+1}\right),$$

where $\mathbf{a}_n = (a_{1\,n}, \ldots, a_{N+1\,n})$. Hence $\mathbf{a}_{N+1} = \dfrac{-\mathbf{a}_1 + \sum_{j=2}^{N} \lambda_j \mathbf{a}_j}{\left(\sum_{j=}^{N} \lambda_j - 1\right)}$. Consequently we get that $K_{N+1} = \dfrac{-K_1 + \sum_{j=2}^{N} \lambda_j K_j}{\left(\sum_{j=}^{N} \lambda_j - 1\right)}$, i.e. $\dfrac{\dot{g}_{N+1}}{g_{N+1}} \left(\sum_{j=}^{N} \lambda_j - 1\right) = -\dfrac{\dot{x}_1}{x_1} + \sum_{j=1}^{N} \lambda_j \dfrac{\dot{x}_j}{x_j}$,

After some computations we obtain that $\dfrac{d}{dt}\left(|g_{N+1}|^{\sum_{j=1}^{N} \lambda_j - 1} |x_1| \prod_{j=2}^{N} |x_j|^{-\lambda_j} \right) =$ 0. Thus the function $\tilde{H} = \dfrac{|x_1|}{|g_{N+1}|} \prod_{j=2}^{N} \left(\dfrac{|x_j|}{|g_{N+1}|}\right)^{-\lambda_j}$ is a first integral. By choosing λ_j for $j = 2, \ldots, N$, properly we get the first integral (18). $\square$

Remark 20. *If we suppose that* $H = \log\left(|g_{N+1}|^{\mu_{N+1}} \prod_{j=1}^{N} |x_j|^{\mu_j} \right)$ *is a first integral of* (16) *then*

$$\sum_{j=1}^{N+1} \mu_j K_j = \sum_{j=1}^{N+1} \mu_j \left(a_{j\,N+1} + \sum_{j=1}^{N} a_{jk} x_k + \Psi \right) = 0.$$

Hence

$$\begin{aligned} \sum_{j=1}^{N+1} \mu_j = &\quad 0, \\ \sum_{j=1}^{N+1} \mu_j a_{jk} = &\quad 0, \quad \textit{for} \quad k = 1, \ldots, N+1. \end{aligned} \tag{29}$$

Clearly this system can be written in matrix form

$$W_2 \mu = \mathbf{0},$$

where $\mu = (\mu_1, \dots, \mu_{N+1})^T$, *and* $\boldsymbol{0} = (0, \dots, 0) \in \mathbb{R}^{N+2}$, *and*

$$W_2 = \begin{pmatrix} 1 & \dots & 1 & 1 \\ a_{11} & \dots & a_{N\,1} & a_{N+1\,1} \\ a_{12} & \dots & a_{N\,2} & a_{N+1\,2} \\ \vdots & \dots & \vdots & \\ a_{1\,N+1} & \dots & a_{N-1,N} & a_{N+1\,N+1} \end{pmatrix}.$$

From the last equations we obtain the system $B_1\mu = \boldsymbol{0}$, *where* B_1 *is the matrix* (7) *and* $\mu = (\mu_1, \dots, \mu_{N+1})^T$, *is a non-zero vector,* $\boldsymbol{0} = (0, \dots, 0) \in \mathbb{R}^{N+1}$. *Hence in view of Theorem 1 we get that there exist the fist integral given by formula* (31) *if and only if* $\mathrm{rankB_1} \leq \mathrm{N}.$. *Consequently*

$$\mathrm{rankB_1} = \mathrm{rankW_2} \leq \mathrm{N}. \tag{30}$$

On the other hand from the firs equation of (29) *we get that* $\mu_{N+1} = -\sum_{j=1}^{N} \mu_j$. *Inserting this relation into the first integral we obtain*

$$H = \log\left(|g_{N+1}|^{\mu_{N+1}} \prod_{j=1}^{N} |x_j|^{\mu_j}\right)\Bigg|_{\mu_{N+1}=-\sum\limits_{j=1}^{N}\mu_j} = \log\left(\prod_{j=1}^{N}\left(\frac{|x_j|}{|g_{N+1}|}\right)^{\mu_j}\right).$$

Proof of Theorem 10. From Jacobi's Theorem we get that (16) has a Jacobi multiplier if and only if

$$\sum_{j=1}^{N} X_j \frac{\partial \log J}{\partial x_j} + \mathrm{div}\mathcal{X} = 0.$$

Consequently, by considering that $J = |g_{N+1}|^{\sigma_{N+1}} \prod_{j=1}^{N} |x_j|^{\sigma_j}$, where $\sigma_j = \mu_j - 1$ for $j = 1, \dots, N$ and $\sigma_{N+1} = -n - \sum_{j=1}^{N} \mu_j$, from Theorem 15 we obtain that

$$\sum_{j=1}^{N+1} \sigma_j K_j + \mathrm{div}\mathcal{X} = 0. \tag{31}$$

In view of (17) and by considering the coefficients of the polynomial (31) we get that (31) holds if and only if

$$\begin{aligned}
&\sum_{j=1}^{N+1} \sigma_j = -n - N,\\
&\sigma_1 a_{1\,1} + \sigma_2 a_{2\,1} + \ldots + \sigma_{N+1} a_{N+1\,1} = -2a_{11} - \sum_{j\neq 1}^{N} a_{1j},\\
&\sigma_1 a_{1\,2} + \sigma_2 a_{2\,2} + \ldots + \sigma_{N+1} a_{N+1\,2} = -2a_{22} - \sum_{j\neq 2}^{N} a_{2j},\\
&\quad\vdots \qquad\quad \vdots \qquad \ldots \qquad\qquad\qquad \vdots \qquad\qquad ,\\
&\sigma_1 a_{1\,N} + \sigma_2 a_{2\,N} + \ldots + \sigma_{N+1} a_{N\,N+1} = -2a_{NN} - \sum_{j\neq N}^{N} a_j,\\
&\sigma_1 a_{1\,N+1} + \sigma_2 a_{2\,N+1} + \ldots + \sigma_{N+1} a_{N+1\,N+1} = -2a_{N+1\,N+1} - \sum_{j\neq N+1}^{N} a_{N+1\,j},
\end{aligned}$$

After the change $\sigma_j = \mu_j - 1$ for $j = 1, \ldots, N+1$ we obtain that

$$\begin{aligned}
\sum_{j=1}^{N+1} \mu_j &= -n + 1,\\
\mu_1 a_{1\,1} + \mu_2 a_{2\,1} + \ldots + \mu_{N+1} a_{N+1\,1} &= \quad -a_{11} + a_{N+1\,1},\\
\mu_1 a_{1\,2} + \mu_2 a_{2\,2} + \ldots + \mu_{N+1} a_{N+1\,2} &= \quad -a_{22} + a_{N+1\,2},\\
\vdots \qquad\quad \vdots \qquad \ldots \qquad\qquad & \qquad\quad \vdots \qquad\qquad ,\\
\mu_1 a_{1\,N} + \mu_2 a_{2\,N} + \ldots + \mu_{N+1} a_{N\,N+1} &= \quad -a_{NN} + a_{N+1\,N},\\
\mu_1 a_{1\,N+1} + \mu_2 a_{2\,N+1} + \ldots + \mu_{N+1} a_{N+1\,N+1} &= \quad a_{N+1\,N+1}
\end{aligned} \tag{32}$$

Consequently in view of Rouché–Frobenius Theorem we get that system (32) has solution if and only if (10) holds. In short the theorem is proved. □

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Proof of Theorem 11. Clearly, by Theorem 9 and (30) it follows that $\operatorname{rank} B_1 = \operatorname{rank} W_1 \leq N$, then there exists the first integral H (see Theorem 1). Hence (22) holds. On the other hand from (32) follows that

$$a_{N+1\,N+1} = 0, \qquad a_{N+1\,j} = a_{jj} \quad \text{for} \quad j = 1, \ldots, N$$

and $\sum_{j=1}^{N} \mu_j = 1 - n$. Since the cofactor of $g_{N+1} = 0$ is (20). From the relation $\operatorname{rank} W_1 = \operatorname{rank} W_2$ it follows that there exist a Jacobi multiplier which in view of Theorem 10 becomes (21). In short the theorem is proved. □

Proof of Corollary 12. It is analogously to the proof of Corollary 5 and by considering that $\sigma_j = -1 + \mu_j$ and $\sum_{j=1}^{N} \mu_j = 1 - n$. □

Remark 21. *It is possible to study the Kolmogorov systems* (16) *with* $\mathrm{rank W}_1 = 2$ *and proved that this system is completely integrable and we can study Kolmogorov systems with an additional exponential factor, analogously the study the Lotka-Volterra systems* (2) *with and additional exponential factor.*

Example 22. *A three dimensional cubic Kolmogorov differential system with complementary invariant plane* $g_4 = x + y + z = 0$ *with cofactor* $s_1x + s_2y + s_3z + \Psi_2(x, y, z)$ *can be written as*

$$\begin{aligned} \dot{x} = &\ x\left(s_1x + (s_2 + s_1 - \beta_1)y + (s_1 + s_3 - \lambda_1)z\right) + x\Psi_2(x, y, z), \\ \dot{y} = &\ y\left(\beta_1 x + s_2y + (s_3 + s_2 - \lambda_2)z\right) + y\Psi_2(x, y, z), \\ \dot{z} = &\ z\left(\lambda_1 x + \lambda_2 y + s_3 z\right) + z\Psi_2(x, y, z), \end{aligned} \tag{33}$$

where $\Psi_2(x, y, z) = \kappa_1x^2+\kappa_2y^2+\kappa_3z^2+\kappa_4xy+\kappa_5xz+\kappa_6yz$ *is such that* $rank W_1 = 3$. *Hence system* (33) *has a first integral given by*

$$H = \log\left(\left(\frac{|x|}{|g_4|}\right)^{s_2-\lambda_2}\left(\frac{|y|}{|g_4|}\right)^{\lambda_1-s_1}\left(\frac{|z|}{|g_4|}\right)^{s_1-\beta_1}\right).$$

If $s_1 = \dfrac{\lambda_1 s_2 - \beta_1 s_3}{\lambda_2 - s_3}$, *then* $rank W_1 = rank W_2 = 3$. *Therefore system* (33) *is completely integrable with the first integral* H *and the Jacobi multiplier*

$$J = |y|^{\sigma_1}|z|^{\sigma_2}|g_4|^{\sigma_3},$$

where

$$\begin{aligned} \sigma_1 = &\ \frac{(\lambda_1 + \lambda_2 - s_3)s_2 + (\lambda_1 + s_3 - \beta_1)s_3 - \lambda_2(2s_3 + \lambda_1)}{(\lambda_2 - s_2)(\lambda_2 - s_3)}, \\ \sigma_2 = &\ \frac{(\beta_1 + 2s_2 + s_3 - \lambda_2)\lambda_2 - (2s_3 + \lambda_1)s_2}{(\lambda_2 - s_2)(\lambda_2 - s_3)}, \\ \sigma_3 = &\ \frac{(s_3 + 2s_2 + \lambda_1 - \beta_1 - 3\lambda_2)(\lambda_2 - s_3)}{(\lambda_2 - s_2)(\lambda_2 - s_3)}. \end{aligned}$$

5. Integrability of the asymmetric May-Leonard model.

5.1. Asymmetric May-Leonard model with four invariant planes.

A particular 3–dimensional Lotka–Volterra system is

$$\begin{aligned} \dot{x} = &\ x\left(1 - x - a_1y - b_1z\right) = X_1, \\ \dot{y} = &\ y\left(1 - y - a_2z - b_2x\right) = X_2, \\ \dot{z} = &\ z\left(1 - z - a_3x - b_3y\right) = X_3. \end{aligned} \tag{34}$$

This system is known as the *asymmetric May–Leonard model* (see for instance [3]). This model describes the competitions between three species and depending on six non–negative parameters a_j and b_j for $j = 1, 2, 3$. The state space is the set

$$\mathbb{R}^3_+ = \left\{(x, y, z) \in \mathbb{R}^3 : x \geq 0 \quad y \geq 0\text{and} \quad z \geq 0, \right\}.$$

We shall study the integrability of system (34) under the condition that it has an additional invariant plane.

Proposition 23. *Differential system* (34) *under the conditions*

$$b_2 = \frac{a_3(a_2-1)-a_2b_3+1}{1-b_3}, \qquad a_1 = \frac{a_3b_1-b_1b_3+b_3-1}{a_3-1}, \tag{35}$$

i.e. the differential system

$$\begin{aligned}
\dot{x} = &\; x\left(1-x-\frac{a_3b_1-b_1b_3+b_3-1}{a_3-1}y-b_1z\right) = \tilde{X}_1,\\
\dot{y} = &\; y\left(1-y-a_2z-\frac{a_3(a_2-1)-a_2b_3+1}{1-b_3}x\right) = \tilde{X}_2,\\
\dot{z} = &\; z\,(1-z-a_3x-b_3y) = \tilde{X}_3.
\end{aligned}$$

has the additional invariant plane

$$g := g(x,y,z) = (1-a_3)(a_2-1)x+(1-b_1)(b_3-1)y+(a_2-1)(b_1-1)z = 0,$$

with cofactor $K = 1-x-y-z$. Moreover this differential system is completely integrable with the first integral H_1 and the Jacobi multiplier J given by

$$H_1 = \log\left(\left|\frac{x}{g}\right|^{\alpha_1}\left|\frac{y}{g}\right|^{\alpha_2}\left|\frac{z}{g}\right|^{\alpha_3}\right), \qquad J = \left|\frac{y}{g}\right|^{\beta_2}\left|\frac{z}{g}\right|^{\beta_3}|g|^{-3}, \tag{36}$$

where

$$\begin{aligned}
\alpha_1 = &\; (a_3-1)(b_3-1)(a_2-1), \quad \alpha_2 = (1-a_3)(b_3-1)(b_1-1),\\
\alpha_3 = &\; (1-a_2)(b_1-1)(a_3-b_3), \;\; \beta_2 = \frac{1-a_2-b_1}{a_2-1}, \;\; \beta_3 = \frac{a_3(b_1+b_3-1)-b_1b_3}{(1-b_3)(a_3-1)},
\end{aligned}$$

Proof. After some computations we can check that $\mathcal{X}(g) = (1-x-y-z)g$, thus $g = 0$ is an invariant plane with cofactor $K = 1-x-y-z$. The existence of the first integral and of the Jacobi multiplier given in (36) follows from Theorem 3 by considering that in this case (see condition (11))

$$\operatorname{rank}(\,(B_1)|_\sigma) = \operatorname{rank}(\,(B_2)|_\sigma) = 3,$$

where by σ we denote conditions (35), and

$$B_1 = \begin{pmatrix} -1 & -b_2 & -a_3 & -1 \\ -a_1 & -1 & -b_3 & -1 \\ -b_1 & -a_2 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} -1 & -b_2 & -a_3 & -1 & 0 \\ -a_1 & -1 & -b_3 & -1 & 0 \\ -b_1 & -a_2 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

□

5.2. **Asymmetric May-Leonard model with five invariant planes.**

Proposition 24. *Differential system* (34) *under the conditions*

$$a_1 = 2-b_2, \;\; a_3 = 2-b_1, \;\; a_2 = 2-b_3 \tag{37}$$

i.e. the differential system

$$\begin{aligned} \dot{x} = & \quad x\,(1 - x - (2-b_2)y - b_1 z)\,, \\ \dot{y} = & \quad y\,(1 - y - (2-b_3)z - b_2 x)\,, \\ \dot{z} = & \quad z\,(1 - z - (2-b_1)x - b_3 y)\,, \end{aligned} \tag{38}$$

has the five invariant planes namely

$$\begin{aligned} g_1 = &\quad x, \qquad K_1 = 1 - x - (2-b_2)y - b_1 z, \\ g_2 = &\quad y, \qquad K_2 = 1 - y - (2-b_3)z - b_2 x, \\ g_3 = &\quad z, \qquad K_3 = 1 - z - (2-b_1)x - b_3 y, \\ g_4 = &\quad 1 - x - y - z, \qquad K_4 = x + y + z, \\ g_5 = &\quad x + y + z, \qquad K_5 = 1 - x - y - z. \end{aligned}$$

Proof. It is follows from the relations

$$\frac{d}{dt}(x+y+z) = (1-x-y-z)(x+y+z), \qquad \frac{d}{dt}(1-x-y-z) = (x+y+z)(1-x-y-z). \tag{39}$$

□

Remark 25. *From* (39) *it follows that*

$$x(t) + y(t) + z(t) = 1 + \frac{1}{1 - ce^{-t}}.$$

Hence we get that

$$\lim_{t \longrightarrow +\infty} (x(t) + y(t) + z(t)) = 1, \qquad \lim_{t \longrightarrow -\infty} (x(t) + y(t) + z(t)) = 0.$$

Thus the α-limits are in the plane $x + y + z = 0$ and the ω- limits are in the plane $x + y + z = 1$.

Proposition 26. *Differential system* (38) *having the invariant plane $\nu_4 - x - y - z = 0$ with cofactor $x + y + z + s_4$, where ν_4 and σ_4 are constants such that $\nu_4 s_4 = 0$, is completely integrable if and only if $(s_4^2 + (b_1 + b_2 + b_3 - 3)^2)\,(1 - s_4) = 0$.*

(i) *If $b_1 + b_2 + b_3 = 3$ and $s_4 = 0$, then system* (38) *has two independent first integrals*

$$H_1 = \log\left(\frac{|x|}{|y(1-x-y-z)^{1-b_2}|}\right), \qquad H_2 = \log\left(\frac{|z|}{|y(1-x-y-z)^{b_3-1}|}\right). \tag{40}$$

(ii) *If $s_4 = 1$, then system* (38) *has the following first integral and the Jacobi multiplier*

$$\begin{aligned} H = &\quad \log\left(|x|^{b_3-1}|y|^{b_1-1}|z|^{b_2-1}|x+y+z|^{3-b_1-b_2-b_3}\right) \\ = &\quad \log\left(\left(\frac{|x|}{|x+y+z|}\right)^{b_3-1}\left(\frac{|y|}{|x+y+z|}\right)^{b_1-1}\left(\frac{|z|}{|x+y+z|}\right)^{b_2-1}\right), \\ J = &\quad \frac{1}{|xyz(x+y+z-1)|}, \end{aligned} \tag{41}$$

Proof. First we observe that the matrix B_1 and B_2 in this case are

$$B_1|_{(37)} = \begin{pmatrix} -1 & -b_2 & -2+b_1 & -1 \\ -2+b_2 & -1 & -b_3 & -1 \\ -b_1 & -2+b_3 & -1 & -1 \\ 1 & 1 & 1 & s_4 \end{pmatrix},$$

$$B_2|_{(37)} = \begin{pmatrix} -1 & -b_2 & -2+b_1 & -1 & 0 \\ -2+b_2 & -1 & -b_3 & -1 & 0 \\ -b_1 & -2+b_3 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

After some computations we obtain that

(42)
$$\operatorname{rank}\begin{pmatrix} -1 & -b_2 & -2+b_1 & -1 \\ -2+b_2 & -1 & -b_3 & -1 \\ -b_1 & -2+b_3 & -1 & -1 \\ 1 & 1 & 1 & s_4 \end{pmatrix} = \begin{cases} 2, & \text{if } s_4 = 0 \text{ and } b_1+b_2+b_3 = 3 \\ 3, & \text{if } \ s_4 = 1. \end{cases}$$

Hence from Theorem 2 we obtain the proof of the first statement. The independent first integrals are given by formula (40). Differential system (38) under the conditions $b_1+b_2+b_3 = 3$, and $s_4 = 0$, can be written as

$$\dot{x} = \{H_1, H_2, x\}/J, \quad \dot{y} = \{H_1, H_2, y\}/J, \quad \dot{z} = \{H_1, H_2, z\}/J,$$

where $J = 1/|xyz(x+y+z-1)|$.

Now we prove statement (ii). From (42) we get that if $s_4 = 1$, then $\operatorname{rank}\left(B_1|_{(37)}\right) = 3$. On the other hand

$$\operatorname{rank}\left(B_2|_{(37)}\right) = \operatorname{rank}\begin{pmatrix} -1 & -b_2 & -2+b_1 & -1 & 0 \\ -2+b_2 & -1 & -b_3 & -1 & 0 \\ -b_1 & -2+b_3 & -1 & -1 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = 3.$$

Thus in view of Theorem 5 we get that there exits a Darboux first integral and a Jacobi multiplier which in this case are given in (41). In short the proposition is proved. □

Remark 27. *Differential system* (34) *under the conditions*

$$a_1 = a_2 = a_3 = a, \quad b_1 = b_2 = b_3 = b,$$

becomes the so called symmetric May-Leonard model

$$\begin{aligned} \dot{x} &= x\,(1-x-ay-bz)\,, \\ \dot{y} &= y\,(1-y-az-bx)\,, \\ \dot{z} &= z\,(1-z-ax-by)\,. \end{aligned} \tag{43}$$

The integrability of system (43) *was study in* [2] . *Condition* (35) *in this case becomes* $a^2+b^2-ab-a-b+1=0$ *with* $a \neq 1$ *and* $b \neq 1$, *which have not solution in* $\mathbb{R}$. *Condition* (37) *becomes* $a+b=2$ *then if* $s_4 = 0$ *and* $b = a = 1$, *then the first integrals* (40) *can be written as* $H_1 = \log\dfrac{|x|}{|y|}$ *and* $H_2 = \log\dfrac{|z|}{|y|}$. *If* $a+b=2$ *and*

$s_4 = 1$ *then first integral* (41) *is* $H = \log\left|\dfrac{xyz}{x+y+z}\right|$ *and the Jacobi multiplier is* $J = \dfrac{1}{|xyz(1-x-y-z)|}$.

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