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Crossing limit cycles for piecewise linear differential centers separated by a reducible cubic curve

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Abstract. As for the general planar differential systems one of the main problems for the piecewise linear differential systems is to determine the existence and the maximum number of crossing limits cycles that these systems can exhibit. But in general to provide a sharp upper bound on the number of crossing limit cycles is a very difficult problem. In this work we study the existence of crossing limit cycles and their distribution for piecewise linear differential systems formed by linear differential centers and separated by a reducible cubic curve, formed either by a circle and a straight line, or by a parabola and a straight line.

Keywords: discontinuous piecewise linear differential centers, limit cycles, conics.

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1 Introduction and statement of the main results

The discontinuous piecewise differential systems arose from the study of nonlinear oscillations by Andronov, Vitt and Khaikin in [1]. And nowadays the qualitative theory of the discontinuous piecewise differential systems is a matter of great interest in the mathematical community because these systems arise naturally in the modeling of several real phenomena and processes for instance in electronics, mechanics, economy, biology, neuroscience etc., see [3, 4, 10, 13, 21, 23] and references quoted therein.

One of the main problems in the qualitative theory of the discontinuous piecewise differential systems is to determine the maximum number of crossing limits cycles that these systems can have and their distribution. In this work we study the *crossing limit cycles* which are periodic orbits isolated in the set of all periodic orbits of the piecewise linear differential system, which only have isolated points of intersection with the discontinuity curve.

We recall that the 16th Hilbert's problem requests for the maximum number of limit cycles that can have a polynomial differential system in $\mathbb{R}^2$ in function of the degree of the system,

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see [11, 12]. Then the problem of establishing a sharp upper bound on the number of crossing limits cycles for the class of planar piecewise linear differential systems can be considered as an extension of the 16th Hilbert's problem to this class and is in general a very difficult problem, because there are few developed techniques. In the plane the class of piecewise linear differential systems separated by a straight line is apparently the simplest class to study, and has been studied in several papers, see [2, 5, 6, 7, 8, 9, 16, 19, 22] but it is still an open problem to know if three is the maximum number of crossing limit cycles that this class can have.

In particular when the class of piecewise linear differential systems separated by a straight line is formed by linear differential centers we know that these systems have no crossing limit cycles, see [15]. However, there are more recent works which study planar discontinuous piecewise linear differential centers where the curve of discontinuity is not a straight line, see [18, 20], there it was proved that there are crossing limit cycles in those systems. Moreover in the paper [14] it was provided the maximum number of crossing limit cycles for piecewise linear differential centers separated by any conic, then the objective of this work is to study the existence of crossing limit cycles of the discontinuous piecewise linear differential centers in $\mathbb{R}^2$ separated a reducible cubic curve, formed either by a circle and a straight line, or by a parabola and a straight line.

We observe that we have three options for crossing limit cycles of discontinuous piecewise linear differential centers separated by such reducible cubic curves here considered. First we have the crossing limit cycles which intersect in two points the discontinuity curve. In [15] was proved that the class of linear differential centers separated by a straight line have no crossing limit cycles, then we can consider that those two intersection points on the discontinuity curve are on the circle or on the parabola and these two options were considered in the paper [14]. Second the crossing limit cycles intersect the discontinuity curve in exactly four points, here we consider that at least one of the four points is on the straight line, because the case which the four points are only on the circle or on the parabola was studied in [14]. Finally we have the crossing limit cycles such that intersect the discontinuity curve in six points.

In this paper we study the crossing limit cycles with four points on discontinuity curve. In subsection 1.1 we consider the piecewise linear differential systems formed by linear differential centers separated by the cubic

$$\Sigma_k = \left\{(x,y) \in \mathbb{R}^2 : (x-k)(x^2+y^2-1) = 0,\ k \in \mathbb{R},\ k \geq 0\right\}.$$

And in subsection 1.2 we consider the piecewise linear differential systems formed by linear differential centers separated by the cubic

$$\tilde{\Sigma}_k = \left\{(x,y) \in \mathbb{R}^2 : (y-k)(y-x^2) = 0,\ k \in \mathbb{R}\right\}.$$

1.1 Crossing limit cycles intersecting the discontinuity curve Σ_k

Let $\mathcal{F}_1$ be the family of piecewise linear differential centers separated by Σ_k with $k > 1$. Let $\mathcal{F}_2$ be the family of piecewise linear differential centers separated by Σ_k with $k = 1$. In these cases we have the following regions in the plane

$$\begin{aligned} R_1 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\},\\ R_2 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1 \text{ and } x < k\},\\ R_3 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1 \text{ and } x > k\}.\end{aligned}$$

And finally let $\mathcal{F}_3$ be the family of piecewise linear differential centers separated by Σ_k with $0 \leq k < 1$. Here we have the following regions in the plane

$$\begin{aligned}
R_1 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1, \text{ and } x > k\},\\
R_2 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1, \text{ and } x > k\},\\
R_3 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 > 1 \text{ and } x < k\},\\
R_4 =& \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1 \text{ and } x < k\}.
\end{aligned}$$

In the family $\mathcal{F}_3$ we have three types of crossing limit cycles. First crossing limit cycles such that are formed by parts of orbits of the four linear differential centers considered, namely crossing limit cycles of type 1, see Figure 2.3, second we have crossing limit cycles which intersect the regions R_1, R_2 and R_4 or crossing limit cycles that intersect the regions R_1, R_4 and R_3, namely crossing limit cycles of type 2^+ or crossing limit cycles of type 2^-, respectively, see Figure 2.4. Without loss of generality we only study the crossing limit cycles of type 2^+ because the analysis for the crossing limit cycles of type 2^- is the same, moreover we observe that these two cases can not occur simultaneously, because the orbits of linear differential system in the region R_4 would not be nested. And finally we have the crossing limit cycles such that are formed by parts of orbits of the three linear differential centers in the regions R_1, R_2 and R_3, or crossing limit cycles formed by parts of orbits of the three linear differential centers in the regions R_2, R_3 and R_4, namely crossing limit cycles of type 3^+ and crossing limit cycles of type 3^-, respectively, see Figure 2.5. Without loss of generality in Theorem 1.1 we study the crossing limit cycles of type 3^+ because the study by the crossing limit cycles of type 3^- is the same. We observe that these types of crossing limit cycles can not appear simultaneously, because the orbits of linear differential system in the region R_3 would not be nested. If we study the piecewise linear differential centers in the family $\mathcal{F}_3$ which have simultaneously two types of crossing limit cycles we observe we would have three possible combinations between the three different crossing limit cycles types, $1, 2^+$ and 3^+, but we observe that the crossing limit cycles of types 2^+ and 3^+ can not appear simultaneously, because the orbits of linear differential system in the region R_1 would not be nested. For this same reason there are no piecewise linear differential centers in $\mathcal{F}_3$ with three types of crossing limit cycles simultaneously. Then in the following Theorem we provide examples of piecewise linear differential centers in $\mathcal{F}_3$ with crossing limit cycles of types 1, 2^+ and 3^+ separately and piecewise linear differential centers in $\mathcal{F}_3$ such that have simultaneously crossing limit cycles of types 1 and 2^+ or of types 1 and 3^+.

Theorem 1.1. *The following statements hold.*

(a) *There are piecewise linear differential systems in $\mathcal{F}_1$ and in $\mathcal{F}_2$ formed by three linear differential centers that have four crossing limit cycles, see Figures 2.1 and 2.2.*

(b) *There are piecewise linear differential systems in $\mathcal{F}_3$ that have five crossing limit cycles of type* 1*, see Figure 2.3.*

(c) *There are piecewise linear differential systems in $\mathcal{F}_3$ that have four crossing limit cycles of type 2^+, see Figure 2.4.*

(d) *There are piecewise linear differential systems in $\mathcal{F}_3$ that have three crossing limit cycles of type 3^+, see Figure 2.5.*

(e) *There are piecewise linear differential systems in $\mathcal{F}_3$ that have four crossing limit cycles of type* 1 *and two crossing limit cycles of type* 2^+, *see Figure* 2.6.

(f) *There are piecewise linear differential systems in $\mathcal{F}_3$ that have four crossing limit cycles of type* 1 *and one crossing limit cycle of type* 3^+, *see Figure* 2.7.

Theorem 1.1 is proved in section 2.

By the numerical computations made for the families $\mathcal{F}_1, \mathcal{F}_2$ and $\mathcal{F}_3$ and the illustrated examples of Theorem 1.1 we propose the following problem.

Open problem 1. *The numbers of crossing limit cycles determined in Theorem* 1.1 *for the families $\mathcal{F}_1$, $\mathcal{F}_2$ and $\mathcal{F}_3$ are the maximum numbers of crossing limit cycles in each family.*

1.2 Crossing limit cycles intersecting the discontinuity curve $\tilde{\Sigma}_k$

Let $\mathcal{F}_4$ be the family of piecewise linear differential centers separated by $\tilde{\Sigma}_k$ with $k < 0$. In this case, we have following three regions in the plane

$$
\begin{aligned}
R_1 &= \{(x,y) \in \mathbb{R}^2 : y > x^2\},\\
R_2 &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k\},\\
R_3 &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < k\}.
\end{aligned}
$$

For this family we have the following Theorem.

Theorem 1.2. *There are piecewise linear differential systems in $\mathcal{F}_4$ that have four crossing limit cycles with four points on $\tilde{\Sigma}_k$, see Figure* 3.1.

Theorem 1.2 is proved in section 3.

Let $\mathcal{F}_5$ be the family of piecewise linear differential centers separated by $\tilde{\Sigma}_k$ with $k = 0$. When the discontinuity curve is $\tilde{\Sigma}_0$ we have following four regions in the plane

$$
\begin{aligned}
R_1 &= \{(x,y) \in \mathbb{R}^2 : y > x^2\},\\
R_2 &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > 0,\ x < 0\},\\
R_3 &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < 0\},\\
R_4 &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > 0,\ x > 0\}.
\end{aligned}
$$

Here we have two types of crossing limit cycles, first crossing limit cycles formed by parts of orbits of the four linear differential centers considered, namely crossing limits cycles of type 4, see Figure 4.1. Second crossing limit cycles of type 5, see Figure 4.2, which intersect only three regions, in this case we have two options, first we have the case where the crossing limit cycles are formed by parts of orbits of the linear differential centers in the regions R_1, R_3 and R_4 and second the crossing limit cycles that intersect only the three regions R_1, R_2 and R_3, without loss of generality we can consider the first case because the study by the second is the same. Here we observe that it is not possible to have crossing limit cycles of type 5 that satisfy those two cases simultaneously, because the orbits of linear differential system in the region R_3 would not be nested. Therefore in the following Theorem we study the piecewise linear differential centers in $\mathcal{F}_5$ which have crossing limit cycles of types 4 and 5 separately, and crossing limit cycles of types 4 and 5 simultaneously.

Theorem 1.3. *The following statements hold.*

(a) *There are piecewise linear differential systems in $\mathcal{F}_5$ that have four crossing limit cycles of type* 4*, see Figure* 4.1.

(b) *There are piecewise linear differential systems in $\mathcal{F}_5$ that have three crossing limit cycles of type* 5*, see Figure* 4.2.

(c) *There are piecewise linear differential systems in $\mathcal{F}_5$ that have simultaneously four crossing limit cycles of type* 4 *and two crossing limit cycles of type* 5*, see Figure* 4.3.

Theorem 1.3 is proved in section 4.

Let $\mathcal{F}_6$ be the family of piecewise linear differential centers separated by $\tilde{\Sigma}_k$ with $k > 0$, in this case we have the following five regions in the plane

$$\begin{aligned}
R_1 =&\{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k,\ x > \sqrt{k}\},\\
R_2 =&\{(x,y) \in \mathbb{R}^2 : y > x^2 \text{ and } y > k\},\\
R_3 =&\{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k,\ x < -\sqrt{k}\},\\
R_4 =&\{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < k\},\\
R_5 =&\{(x,y) \in \mathbb{R}^2 : x^2 < y < k\}.
\end{aligned}$$

Here we have six types of crossing limit cycles. First we have crossing limit cycles such that are formed by parts of orbits of the four linear differential centers in the regions R_1, R_2, R_5 and R_4, or crossing limit cycles formed by parts of orbits of the four linear differential centers in the regions R_2, R_3, R_4 and R_5, namely crossing limit cycles of type 6^+ and crossing limit cycles of type 6^-, respectively, see Figure 6.1. In Theorem 1.4 we study the crossing limit cycles of type 6^+ because the study for the case of crossing limit cycles of type 6^- is the same. Second we have crossing limit cycles type 7, see Figure 5.2, which intersect the three regions R_2, R_5 and R_4. Third we have the crossing limit cycles of type 8, see Figure 5.3, which intersect the regions R_1, R_2, R_3 and R_4. And finally we have the crossing limit cycles such that are formed by parts of orbits of the three linear differential centers in the regions R_1, R_2 and R_4, or crossing limit cycles formed by parts of orbits of the three linear differential centers in the regions R_2, R_3 and R_4, namely crossing limit cycles of type 9^+ and crossing limit cycles of type 9^-, respectively, see Figure 5.4. Without loss of generality in Theorem 1.4 we study the crossing limit cycles of type 9^+ because the study by the crossing limit cycles of type 9^- is the same. Then in Theorem 1.4 we study the crossing limit cycles of types 6^+, 7, 8 and 9^+. In Theorem 1.5 we study the piecewise linear differential centers in the family $\mathcal{F}_6$ which have two types of crossing limit cycles simultaneously. And in Theorem 1.6 we study the piecewise linear differential centers in the family $\mathcal{F}_6$ which have three types of crossing limit cycles simultaneously.

Theorem 1.4. *The following statements hold.*

(a) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have five crossing limit cycles of type* 6^+*, see Figure* 5.1.

(b) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have three crossing limit cycles of type* 7*, see Figure* 5.2.

(c) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have four crossing limit cycles of type* 8*, see Figure* 5.3.

(d) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have three crossing limit cycles of type 9^+, see Figure 5.4.*

Theorem 1.4 is proved in section 5.

In Theorem 1.5 we would have fifteen possible combinations of pairs between the six different crossing limit cycles types, namely types $6^+, 6^-, 7, 8, 9^+$ and 9^-, we will analyze each one. Piecewise linear differential centers with crossing limit cycles of types 6^+ and 6^- are analyzed in statement (a) of Theorem. The study for piecewise linear differential centers with crossing limit cycles of types 6^+ and 7, or 6^+ and 8, or 6^+ and 9^+ is the same for piecewise linear differential centers with crossing limit cycles of types 6^- and 7, or 6^- and 8, or 6^- and 9^-, respectively, and they are in statements (b), (c) and (d) of Theorem 1.5, respectively. The crossing limit cycles of types 6^- and 9^+ can not appear simultaneously because the orientation of these crossing limit cycles in region R_4 would not be well defined, similarly happens with the crossing limit cycles of types 6^+ and 9^-. Piecewise linear differential centers with crossing limit cycles of types 7 and 8 are analyzed in statement (e) of Theorem 1.5. It is not possible to have crossing limit cycles of type 7 and 9^+, or 7 and 9^- simultaneously, because the orbits of linear differential system in the region R_2 would not be nested. Piecewise linear differential centers with crossing limit cycles of types 8 and 9^+ are analyzed in statement (f) of Theorem 1.5, the case where appear crossing limit cycles of types 8 and 9^-, simultaneously is the same. Finally we observe that it is not possible to have simultaneously crossing limit cycles of types 9^+ and 9^-, because the orbits of linear differential system in the region R_4 would not be nested. Then we only have six cases analyzed in the following Theorem.

Theorem 1.5. *The following statements hold.*

(a) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type 6^+ and four crossing limit cycles of type 6^-, see Figure 6.1.*

(b) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type 6^+ and two crossing limit cycles of type 7, see Figure 6.2.*

(c) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously three crossing limit cycles of type 6^+ and four crossing limit cycle of type 8, see Figure 6.3.*

(d) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type 6^+ and two crossing limit cycles of type 9^+, see Figure 6.4.*

(e) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously three crossing limit cycles of type 7 and four crossing limit cycle of type 8, see Figure 6.5.*

(f) *There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type 8 and two crossing limit cycles of type 9^+, see Figure 6.6.*

Theorem 1.5 is proved in section 6.

In Theorem 1.6 we would have twenty possible combinations of triplets between the six different crossing limit cycles types above, but we have fourteen combinations that include couples 6^+ and 9^-, 6^- and 9^+, 7 and $9^\pm$, or 9^+ and 9^- and as it was said before these combinations are not possible. Therefore we have six options, first we observed that crossing limit cycles of types 6^+, 6^- and 7, or 6^+, 6^- and 8 can not appear simultaneously because the orientation of these crossing limit cycles in region R_4 would not be well defined. Second we have that there are piecewise linear differential centers with crossing limit cycles of types 6^+, 7 and 8, this case is in statement (a) of Theorem 1.6, the case where appear crossing limit cycles of types 6^-, 7 and 8 is the same. Finally we have the piecewise linear differential centers with crossing limit cycles of types 6^+, 8 and 9^+, this case is in statement (b) of Theorem 1.6 and the case where appear crossing limit cycles of types 6^-, 7 and 9^- is the same.

Theorem 1.6. *The following statements hold.*

(a) There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously two crossing limit cycles of type 6^+, two crossing limit cycles of type 7 and four crossing limit cycles of type 8, see Figure 7.1.

(b) There are piecewise linear differential systems in $\mathcal{F}_6$ that have simultaneously four crossing limit cycles of type 6^+, three crossing limit cycles of type 8 and two crossing limit cycles of type 9^+, see Figure 7.2.

Theorem 1.6 is proved in section 7.

Similar to the previous case and by the illustrated examples in previous theorems we propose the following problem.

Open problem 2. *The numbers of crossing limit cycles determined in Theorems 1.2, 1.3, 1.4, 1.5 and 1.6 for the families $\mathcal{F}_4$, $\mathcal{F}_5$ and $\mathcal{F}_6$ are the maximum numbers of crossing limit cycles in each family.*

By the previous analysis we observed that it is not possible to have piecewise linear differential centers in $\mathcal{F}_6$ with four, five or six types of crossing limit cycles simultaneously.

2 Proof of Theorem 1.1

In the proof of the theorems will use the following lemma which provides a normal form for an arbitrary linear differential linear differential center, see a proof in [17].

Lemma 2.1. *Through a linear change of variables and a rescaling of the independent variable every center in $\mathbb{R}^2$ can be written*

$$\dot{x} = -bx - \frac{4b^2+\omega^2}{4a}y + d, \qquad \dot{y} = ax + by + c,$$

with $a \neq 0$ and $\omega > 0$. This linear differential system has the first integral

$$H_1(x,y) = 4(ax+by)^2 + 8a(cx-dy) + y^2\omega^2.$$

Proof of statement (a) for the family $\mathcal{F}_1$ of Theorem 1.1. By Lemma 2.1 we can consider the following piecewise linear differential center

$$\begin{aligned}
\dot{x} &= -b_1x - \frac{4b_1^2+\omega_1^2}{4}y + d_1, \qquad \dot{y} = x + b_1y + c_1, \text{ in } R_1,\\
\dot{x} &= -b_2x - \frac{4b_2^2+\omega_2^2}{4}y + d_2, \qquad \dot{y} = x + b_2y + c_2, \text{ in } R_2,\\
\dot{x} &= -b_3x - \frac{4b_3^2+\omega_3^2}{4}y + d_3, \qquad \dot{y} = x + b_3y + c_3, \text{ in } R_3.
\end{aligned} \tag{2.1}$$

And the linear differential centers in (2.1) have the first integrals

$$H_i(x,y) = 4(x+b_iy)^2 + 8(c_ix-d_iy) + y^2\omega_i^2, \text{ with } i = 1,2,3,$$

respectively. In order to have a crossing limit cycle, which intersects Σ_k in four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$ and $p_4 = (k, y_4)$, with $p_2, p_3 \in \mathbb{S}^1$, where $\mathbb{S}^1 = \{(x, y) : x^2 + y^2 = 1\}$. These points must satisfy the *closing equations*

$$\begin{aligned} e_1 = H_2(k,\ y_1) - H_2(x_2, y_2) = 0,\\ e_2 = H_1(x_2, y_2) - H_1(x_3, y_3) = 0,\\ e_3 = H_2(x_3, y_3) - H_2(\ k, y_4) = 0,\\ e_4 = H_3(k,\ y_4) - H_3(\ k, y_1) = 0,\\ x_2^2 + y_2^2 = 1,\\ x_3^2 + y_3^2 = 1. \end{aligned} \tag{2.2}$$

For to build the example, we will impose the existence of periodic solutions and we will determine the parameters in (2.1) with the established conditions. First we fix the constant $k = 2$ and we assume that there is a real solution, namely $q^1 = (y_1^1, x_2^1, y_2^1, x_3^1, y_3^1, y_4^1) = (3, \cos(\pi/2), \sin(\pi/2), \cos(-\pi/3), \sin(-\pi/3), -5/2)$, then by equations e_i with $i = 1, 2, 3, 4$ in (2.2) we have the parameters

$d_2 = 1 + b_2(3 + 2b_2) + c_2 + \dfrac{\omega_2^2}{2}$; $d_1 = -\dfrac{1}{16}(-2 + \sqrt{3})(-4 + 4b_1(2\sqrt{3} + b_1) - 16c_1 + \omega_1^2)$;

$c_2 = \dfrac{70 - 8\sqrt{3} + 4b_2(10 - 5\sqrt{3} + 31b_2 - 4\sqrt{3}b_2) + (31 - 4\sqrt{3})\omega_2^2}{8(-8 + \sqrt{3})}$; $d_3 = \dfrac{1}{16}(4b_3(8 + b_3) + \omega_3^2)$,

respectively. Now by the equation e_4 we have

$$y_4 = \frac{1}{2}(1 - 2y_1),$$

then we suppose that the point $q^2 = (y_1^2, x_2^2, y_2^2, x_3^2, y_3^2, y_4^2) = (3, \cos(\pi/2), \sin(\pi/2), \cos(-\pi/3), \sin(-\pi/3), -5/2)$ is also a real solution of system (2.2), then by the equations e_1, e_2 and e_3 in (2.2) we obtain the following parameters

$$\begin{aligned} \omega_2 = &-\frac{2}{\sqrt{3894 - 523\sqrt{3} + 225\sqrt{15} + 50\sqrt{2(5+\sqrt{5})}}}\sqrt{(-635 + 25\sqrt{3} + 675\sqrt{5} - 75\sqrt{15}}\\ &+75\sqrt{2(5+\sqrt{5})} + 5(1468 + 34\sqrt{3} + 100\sqrt{5} - 50\sqrt{15} + 5\sqrt{2(5+\sqrt{5})}(-68 + \sqrt{3} - 8\sqrt{5} + \sqrt{15}))b_2\\ &+(-3894 + 523\sqrt{3} + 25\sqrt{5}(70 - 9\sqrt{3}) - 50\sqrt{2(5+\sqrt{5})})b_2^2); \end{aligned}$$

$$c_1 = \frac{(-2+\sqrt{3})\sqrt{\frac{1}{2}\left(5+\sqrt{5}\right)}(-4 + 8\sqrt{3}b_1 + 4b_1^2 + \omega_1^2)}{8(-1 + \sqrt{5} - 2\sqrt{2(5+\sqrt{5})}} + \sqrt{6(5+\sqrt{5})}));\quad b_2 = 3.119845..,$$

respectively. Now we fix the points $x_2 = \cos(4\pi/7)$, $y_2 = \sin(4\pi/7)$ and by equation e_6 we have

$$y_3 = -\sqrt{1 - x_3^2},$$

then by the equations e_1, e_2 and e_3 we have

$y_1 = 3.144465..; \omega_1 = -9.702226..\sqrt{0.042492.. + 0.031501..b_1 - 0.042492..b_1^2}$; $x_3 = 0.365470..$, respectively. These conditions generate the real solution $q^3 = (3.144465.., \cos(4\pi/7), \sin(4\pi/7), 0.365470.., -0.930823.., -2.644465..)$. We build a fourth solution fixing the points $x_2 = -0.018219..$; $y_2 = 0.999834..$; therefore by the equations e_1, e_2 and e_3 we obtain $y_1 = 3.012016..$; $x_3 = 0.489429..$; $b_1 = 0.608380..$, respectively. With these conditions we have the real solution $q^4 = (3.012016.., -0.018219.., 0.999834.., 0.489429.., -0.872042.., -2.512016..)$. With these four real solutions we determined all the parameters that appear in system (2.2), even more in this particular case the parameters $b_3, c_3, \omega_3 \in \mathbb{R}$, then we fix them, $b_3 = 1$; $c_3 = 1/4$; $\omega_3 = 1$. Therefore we obtain the following piecewise linear differential center

$$\begin{aligned}
&\dot{x} = 0.977474.. - 0.608380..x - 1.451017..y, && \dot{y} = -3.008357.. + x + 0.608380..y, \text{ in } R_1,\\
&\dot{x} = 9.710162.. - 3.119845..x - 10.075224..y, && \dot{y} = -20.799821.. + x + 3.119845..y, \text{ in } R_2,\\
&\dot{x} = \frac{37}{16} - x - \frac{5}{4}y, && \dot{y} = \frac{1}{4} + x + y, \text{ in } R_3.
\end{aligned} \tag{2.3}$$

The linear differential centers in (2.3) have the first integrals

$$\begin{aligned}
H_1(x,y) =& x^2 + x(-6.016714.. + 1.216760..y) + y(-1.954949.. + 1.451017..y),\\
H_2(x,y) =& x^2 + x(-41.599643.. + 6.239690..y) + y(-19.420324.. + 10.075224..y),\\
H_3(x,y) =& 2x + 4x^2 - \frac{37}{2}y + 8xy + 5y^2,
\end{aligned}$$

respectively.

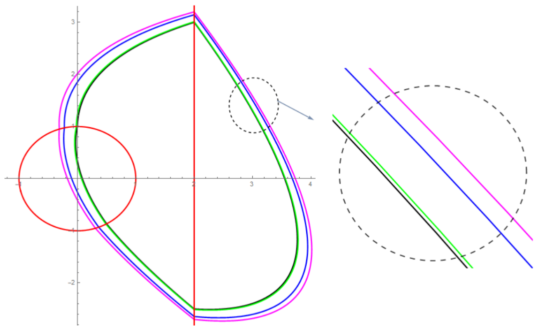

Figure 2.1: Four crossing limit cycles of the discontinuous piecewise linear differential (2.3). These limit cycles are traveled in counterclockwise.

In this case system (2.2) is equivalent to system

$$\begin{aligned}
79.199286.. + x_2^2 + 6.940944..y_1 - 10.075224..y_1^2 - 19.420324..y_2 + 10.075224..y_2^2&\\
+x_2(-41.599643.. + 6.239690..y_2) &= 0,\\
x_2^2 - x_3^2 - 1.954949..y_2 + 1.451017..y_2^2 + x_2(-6.016714.. + 1.216760..y_2)&\\
+x_3(6.016714.. - 1.216760..y_3) + 1.954949..y_3 - 1.451017..y_3^2) &= 0,\\
79.199286.. + x_3^2 - 19.420324..y_3 + 10.075224..y_3^2&\\
+x_3(-41.599643.. + 6.239690..y_3) + 6.940944..y_4 - 10.075224..y_4^2 &= 0,\\
(y_1 - y_4)\left(-\frac{5}{2} + 5y_1 + 5y_4\right) &= 0,\\
x_2^2 + y_2^2 = 1, \quad x_3^2 + y_3^2 &= 1.
\end{aligned} \tag{2.4}$$

Taking into account that the solutions $q^i = (y_1^i, x_2^i, y_2^i, x_3^i, y_3^i, y_4^i)$ of system (2.4) must satisfy $y_4^i < y_1^i$ we have that the unique reals solutions are the points q^1, q^2, q^3 and q^4 which provide four crossing limit cycles of the piecewise linear differential center (2.3). See these crossing limit cycles in Figure 2.1. □

Proof of statement (a) for the family $\mathcal{F}_2$ of Theorem 1.1. Following the steps illustrated in the previous case we obtain a discontinuous piecewise linear differential system which is formed by the following linear differential centers in each region. First in the region R_1 we have

$$\dot{x} = 2.185588.. - \frac{3}{20}x - 6.201094..y, \quad \dot{y} = -6.726549.. + x + \frac{3}{20}y. \tag{2.5}$$

This linear differential center has the first integral $H_1(x,y) = x^2 + x\,(-13.453098.. + 3y/10) + y(-4.371176.. + 6.201094..y)$. In the region R_2 we consider the linear differential center

$$\dot{x} = -0.263120.. - 0.874044..x - 4.914345..y, \qquad \dot{y} = -23.305757.. + x + 0.874044..y, \tag{2.6}$$

which has the first integral $H_2(x,y) = x^2 + x(-46.611514.. + 1.748088..y) + y(0.526241.. + 4.914345..y)$. And in the region R_3 we have the linear differential center

$$\dot{x} = \frac{21}{16} - x - \frac{5}{4}y, \quad \dot{y} = \frac{1}{4} + x + y, \tag{2.7}$$

which has the first integral $H_3(x,y) = 2x + 4x^2 - 21y/2 + 8xy + 5y^2$. In order to have a

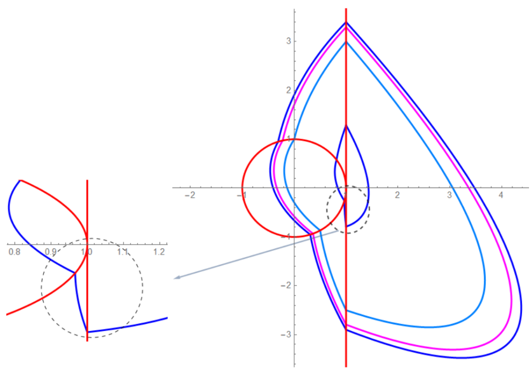

Figure 2.2: Four crossing limit cycles of the discontinuous piecewise linear differential formed by (2.5), (2.6) and (2.7) and separated by Σ_1. These limit cycles are traveled in counterclockwise.

crossing limit cycle, which intersects Σ_1 in four different points $p_1 = (1, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (x_3, y_3)$ and $p_4 = (1, y_4)$, with $p_2, p_3 \in \mathbb{S}^1$, these points must satisfy the *closing equations* given in (2.2). Then for the piecewise linear differential system formed by the centers (2.5), (2.6) and (2.7) we have that system (2.2) is equivalent to system

$$\begin{aligned}
45.611514.. + x_2^2 - 2.274330..y_1 - 4.914345..y_1^2 + 0.526241..y_2 \\
+4.914345..y_2^2 + x_2(-46.611514.. + 1.748088..y_2) = 0, \\
x_2^2 - x_3^2 + x_2\left(-13.453098.. + \frac{3}{10}y_2\right) - 4.371176..y_2 \\
+6.201094..y_2^2 + x_3\left(13.453098.. - \frac{3}{10}y_3\right) + 4.371176..y_3 - 6.201094..y_3^2 = 0, \\
45.611514.. + x_3^2 + 0.526241..y_3 + 4.914345..y_3^2 + x_3(-46.611514.. \\
+1.748088..y_3) - 2.274330..y_4 - 4.914345..y_4^2 = 0, \\
(y_1 - y_4)\left(-\frac{5}{2} + 5y_1 + 5y_4\right) = 0, \\
x_2^2 + y_2^2 = 1, \quad x_3^2 + y_3^2 = 1,
\end{aligned} \tag{2.8}$$

Therefore the unique real solutions $q^i = (y_1^i, x_2^i, y_2^i, x_3^i, y_3^i, y_4^i)$ for system (2.8) that satisfy the condition $y_4^i < y_1^i$, are the points $q^1 = (3, \cos(\pi/2), \sin(\pi/2), \cos(-\pi/3), \sin(-\pi/3), -5/2)$; $q^2 = (17/5, \cos(3\pi/5), \sin(3\pi/5), \cos(-2\pi/5), \sin(-2\pi/5), -29/10)$; $q^3 = (3.294676..,$ $\cos(4\pi/7), \sin(4\pi/7), 0.362651.., -0.931924.., -2.794676..)$ and $q^4 = (1.287554.., 0.814865..,$ $0.579649.., 0.966364.., -0.257177.., -0.787554)$, which generated four crossing limit cycles. See these crossing limit cycles of the piecewise linear differential center formed by (2.5), (2.6) and (2.7) in Figure 2.2. □

Proof of statement (b) of Theorem 1.1. We consider the piecewise linear differential center such that in the region R_1 it has the linear differential center

$$\dot{x} = 0.309248.. - 0.237408..x - 0.439335..y, \qquad \dot{y} = -0.478770.. + x + 0.237408..y, \tag{2.9}$$

this system has the first integral $H_1(x,y) = x^2 + x(-0.957540.. + 0.474817..y) + (-0.618496.. + 0.439335..y)y$. In the region R_2 we have the linear differential center

$$\dot{x} = 0.396090.. - 0.335276..x - 0.180370..y, \qquad \dot{y} = -0.861570.. + x + 0.335276..y, \tag{2.10}$$

which has the first integral $H_2(x,y) = x^2 + x(-1.723140.. + 0.670553..y) + (-0.792181.. + 0.180370..y)y$. In the region R_3 we have the linear differential center

$$\dot{x} = 0.242967.. + 0.112091..x - 0.194871..y, \qquad \dot{y} = 0.375114.. + x - 0.112091..y, \tag{2.11}$$

this system has the first integral $H_3(x,y) = x^2 + x(0.750229.. - 0.224182..y) + (-0.485935.. + 0.194871..y)y$. And in the region R_4 we have the linear differential center

$$\dot{x} = 0.394133.. + 0.278957..x - 0.25146..y, \qquad \dot{y} = 0.516804.. + x - 0.278957..y, \tag{2.12}$$

which has the first integral $H_4(x,y) = x^2 + x(1.033609.. - 0.557914..y) - (0.788267.. - 0.251469..y)y$. In order to have a crossing limit cycle of type 1, which intersects the discontinuity curve Σ_k in

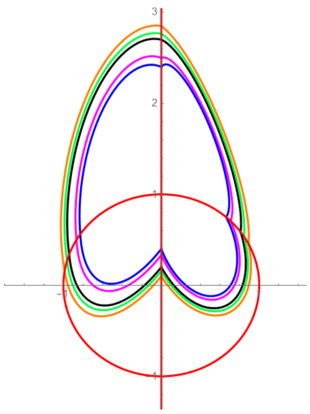

Figure 2.3: Five crossing limit cycles of type 1 of the discontinuous piecewise linear differential system formed by the centers (2.9), (2.10), (2.11) and (2.12). These limit cycles are traveled in counterclockwise.

four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$, with $p_2, p_4 \in \mathbb{S}^1$, then these points must satisfy the system

$$\begin{aligned} H_1(k,\ y_1) &= H_1(x_2, y_2), \\ H_2(x_2, y_2) &= H_2(k,\ y_3), \\ H_3(k,\ y_3) &= H_3(x_4, y_4), \\ H_4(x_4, y_4) &= H_4(k, y_1), \\ x_2^2 + y_2^2 &= 1, \\ x_4^2 + y_4^2 &= 1, \end{aligned} \tag{2.13}$$

Considering $k = 0$ and the previous piecewise linear differential center, system (2.13) is equivalent to system

$$\begin{aligned} x_2^2 + 0.618497..y_1 - 0.439336..y_1^2 + x_2(-0.957541.. + 0.474817..y_2) - 0.618497..y_2 & \\ +0.439336..y_2^2 &= 0, \\ 4x_2^2 - 3.168726..y_2 + 0.721481..y_2^2 + x_2(-6.892562.. + 2.682214..y_2) + 3.168726..y_3 & \\ -0.721481..y_3^2 &= 0, \\ x_4^2 + 0.485936..y_3 - 0.194871..y_3^2 + x_4(0.750229.. - 0.224183..y_4) - 0.485936..y_4 & \\ +0.194871..y_4^2 &= 0, \\ 4x_4^2 + 3.153071..y_1 - 1.005879..y_1^2 + x_4(4.134439.. - 2.231658..y_4) - 3.153071..y_4 & \\ +1.005879..y_4^2 &= 0, \\ x_2^2 + y_2^2 = 1, \quad x_4^2 + y_4^2 &= 1. \end{aligned} \tag{2.14}$$

Therefore discontinuous piecewise differential system formed by the linear differential centers (2.9), (2.10), (2.11) and (2.12) has five crossing limit cycles of type 1, because system (2.14) has five real solutions $q^i = (y_1^i, x_2^i, y_2^i, y_3^i, x_4^i, y_4^i)$, for $i = 1,2,3,4,5$ that satisfy the conditions $-1 < y_1^i < 1 < y_3^i$; $x_2^i > 0$ and $x_4^i < 0$. Where $q^1 = (1/3,\ \cos(\pi/4),\ \sin(\pi/4),\ 5/2,\ \cos(5\pi/6),\ \sin(5\pi/6))$; $q^2 = (\ 2/5,\ \cos(27\pi/10),\ \ \sin(27\pi/10),\ 12/5,\ \cos(81\pi/100),\ \cos(81\pi/100))$; $q^3 = (1/5, \cos(\pi/5),\ \sin(\pi/5), 27/10,\ \cos(89\pi/100), \sin(89\pi/100))$; $q^4 = (1/10, \cos(3\pi/20), \sin(3\pi/20), 57/20,\ \cos(19\pi/20),\ \sin(19\pi/20))$ and $q^5 = (0.157052..,\ 0.843891.., 0.536513.., 2.764619.., -0.962848.., 0.270041..)$. See these five crossing limit cycles of type 1 in Figure 2.3. $\square$

Proof of statement (c) of Theorem 1.1. We consider the following discontinuous piecewise linear differential system

$$\begin{aligned} &\dot{x} = -0.045605.. + 0.048166..x - 0.671455..y, && \dot{y} = -0.418364.. + x - 0.048166..y, \text{ in } R_1, \\ &\dot{x} = 0.058276.. + \frac{x}{100} - 0.178664..y, && \dot{y} = -0.763833.. + x - \frac{y}{100}, \text{ in } R_2, \\ &\dot{x} = \frac{901}{50000} - \frac{x}{50} - \frac{901}{2500}y, && \dot{y} = \frac{1}{10} + x + \frac{y}{50}, \text{ in } R_4. \end{aligned} \tag{2.15}$$

The linear differential centers in (2.15) have the first integrals

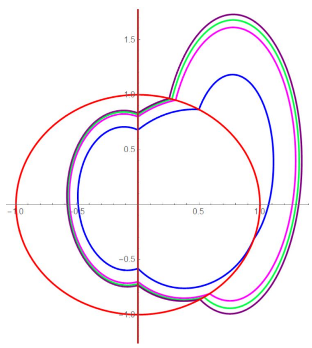

Figure 2.4: Four crossing limit cycles of type 2^+ of the discontinuous piecewise linear differential center (2.15). These limit cycles are traveled in counterclockwise.

$$\begin{aligned}
H_1(x,y) &= x^2 + x(-0.836729.. - 0.096332..y) + (0.091210.. + 0.671455..y)y,\\
H_2(x,y) &= x^2 + x\left(-1.527667.. - \frac{y}{50}\right) + (-0.116553.. + 0.178664..y)y,\\
H_4(x,y) &= 4x^2 + \frac{4}{25}x(5+y) + \frac{901y(-1+10y)}{6250},
\end{aligned}$$

respectively. In order to have a crossing limit cycle of type 2^+, which intersects Σ_k in four different points $p_1 = (x_1, y_1)$, $p_2 = (k, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$, with $p_1, p_4 \in \mathbb{S}^1$, these points must satisfy the system

$$\begin{array}{rl}
H_1(x_1,y_1) = & H_1(\,k,y_2),\\
H_4(\,k,y_2) = & H_4(\,k,y_3),\\
H_1(\,k,y_3) = & H_1(x_4,y_4),\\
H_2(x_4,y_4) = & H_2(x_1,y_1),\\
x_1^2+y_1^2 = & 1,\\
x_4^2+y_4^2 = & 1,
\end{array} \tag{2.16}$$

Then for the piecewise linear differential system (2.15) we have that system (2.16) becomes

$$\begin{aligned}
4x_1^2 + x_1(-3.346917.. - 0.385331..y_1) + y_1(0.364840.. + 2.685822..y_1)&\\
+(-0.364840.. - 2.685822..y_2)y_2 &= 0,\\
(y_2-y_3)\left(-\frac{901}{6250} + \frac{901}{625}(y_2+y_3)\right) &= 0,\\
-4x_4^2 + y_3(0.364840.. + 2.685822..y_3) + x_4(3.346917.. + 0.385331..y_4)&\\
+(-0.364840.. - 2.685822..y_4)y_4 &= 0,\\
-4x_1^2 + 4x_4^2 + x_1\left(6.110671.. + \frac{2}{25}y_1\right) + (0.466214.. - 0.714659..y_1)y_1&\\
+x_4\left(-6.110671.. - \frac{2}{25}y_4\right) + (-0.466214.. + 0.714659..y_4)y_4 &= 0,\\
x_1^2+y_1^2 = 1, \quad x_4^2+y_4^2 &= 1,
\end{aligned} \tag{2.17}$$

where $k = 0$. Therefore the unique real solutions $q^i = (x_1^i, y_1^i, y_2^i, y_3^i, x_4^i, y_4^i)$ for system (2.17) that satisfy the conditions $-1 < y_3^i < y_2^i < 1$; $x_1^i > 0$ and $x_4^i > 0$ are the points $q^1 = (\cos(2\pi/5), \sin(2\pi/5), 8/10, -7/10, \cos(-3\pi/10), \sin(-3\pi/10))$; $q^2 = (\cos(\pi/3), \sin(\pi/3), 17/25, -29/50, \cos(-\pi/10), \sin(-\pi/10))$; $q^3 = (\cos(41\pi/100), \sin(41\pi/100), 0.819235.., -0.719235.., 0.541860.., -0.840468..)$ and $q^4 = (0.256532.., 0.966535.., 0.833667.., -0.733667.., 0.508672.., -0.860960..)$. These four real solutions generated four crossing limit cycles of type 2^+. See these crossing limit cycles of the piecewise linear differential center (2.15) in Figure 2.4. □

Proof of statement (d) of Theorem 1.1. We consider the following discontinuous piecewise linear differential system

$$\begin{array}{ll}
\dot{x} = 1.018312.. + \dfrac{51}{40}x + 9.463668..y, & \dot{y} = -5.008011.. - x - \dfrac{51}{40}y, \text{ in } R_1,\\
\dot{x} = 0.712799.. - 0.278320..x - 0.250791..y, & \dot{y} = -1.026464.. + x + 0.278320..y, \text{ in } R_2,\\
\dot{x} = \dfrac{969}{1280} + \dfrac{x}{8} - \dfrac{17}{64}y, & \dot{y} = \dfrac{1}{8} + x - \dfrac{y}{8}, \text{ in } R_3.
\end{array} \tag{2.18}$$

The linear differential centers in (2.18) have the first integrals

$$
\begin{aligned}
H_1(x,y) =&4x^2 + x\left(40.064090.. + \frac{51}{5}y\right) + y(8.146500.. + 37.854675..y),\\
H_2(x,y) =&x^2 + x(-2.052928.. + 0.556641..y) + (-1.425599.. + 0.250791..y)y,\\
H_3(x,y) =&x + 4x^2 - xy + \frac{17}{160}y(-57 + 10y).
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 3^+, which intersects the discontinuity curve Σ_k in four different points $p_1 = (k, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (k, y_3)$ and $p_4 = (x_4, y_4)$, with $p_2, p_4 \in \mathbb{S}^1$, these points must satisfy the system

$$
\begin{aligned}
H_2(x_1, y_1) =& \ H_2(k,\ y_2),\\
H_3(k,\ y_2) =& \ H_3(k,\ y_3),\\
H_2(k,\ y_3) =& \ H_2(x_4, y_4),\\
H_1(x_4, y_4) =& \ H_1(x_1, y_1),\\
x_1^2 + y_1^2 =& \ 1,\\
x_4^2 + y_4^2 =& \ 1,
\end{aligned}
\tag{2.19}
$$

Considering $k = 0$, system (2.19) is equivalent to system

$$
\begin{aligned}
4x_1^2 + y_1(-5.702397.. + 1.003165..y_1) + x_1(-8.211712.. + 2.226564..y_1)&\\
+5.702397..y_2 - 1.003165..y_2^2 &= 0,\\
(y_2 - y_3)(-57 + 10y_2 + 10y_3) &= 0,\\
x_4^2 + (1.425599.. - 0.250791..y_3)y_3 + x_4(-2.052928.. + 0.556641..y_4)&\\
-1.425599..y_4 + 0.250791..y_4^2 &= 0,\\
x_1^2 - x_4^2 + x_1\left(10.016022.. + \frac{51}{20}y_1\right) + y_1(2.036625.. + 9.463668..y_1)&\\
+x_4\left(-10.016022.. - \frac{51}{20}y_4\right) + (-2.036625.. - 9.463668..y_4)y_4 &= 0,\\
x_1^2 + y_1^2 = 1, \quad x_4^2 + y_4^2 &= 1.
\end{aligned}
\tag{2.20}
$$

Therefore discontinuous piecewise differential (2.18) has three crossing limit cycles of type 3^+, because system (2.20) has three real solutions $q^i = (x_1^i, y_1^i, y_2^i, y_3^i, x_4^i, y_4^i)$, for $i = 1,2,3$ that satisfy the conditions $0 < x_4^i < x_1^i$ and $1 < y_3^i < y_2^i$. Where $q^1 = (\cos(\pi/5), \sin(\pi/5), 43/10, 7/5\cos(2\pi/5), \sin(2\pi/5))$; $q^2 = (\cos(16\pi/125), \sin(16\pi/125), 447/100, 123/100\cos(9\pi/50),$

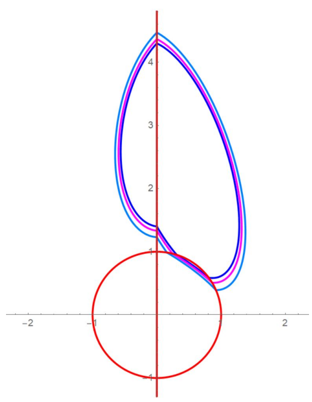

Figure 2.5: Three crossing limit cycles of type 3^+ of the discontinuous piecewise linear differential center (2.18). These limit cycles are traveled in counterclockwise.

$\cos(9\pi/50))$ and $q^3 = (\cos(17\pi/100), \sin(17\pi/100), 4.366812.., 1.333187.., 0.242211.., 0.970223..)$. See these three crossing limit cycles of type 3^+ in Figure 2.5. □

Proof of statement (e) of Theorem 1.1. We consider the following discontinuous piecewise linear differential system

$$\begin{aligned}
\dot{x} &= 0.244909.. - 0.132672..x - 0.724279..y, & \dot{y} &= -0.471887.. + x + 0.132672..y, \text{ in } R_1,\\
\dot{x} &= 0.668802.. - 0.514522..x - 0.636209..y, & \dot{y} &= -0.985653.. + x + 0.514522..y, \text{ in } R_2,\\
\dot{x} &= -0.081198.. - 0.207828..x - 0.061343..y, & \dot{y} &= -0.124956.. + x + 0.207828..y, \text{ in } R_3,\\
\dot{x} &= 0.211524.. - 0.634777..x - 0.705080..y, & \dot{y} &= -0.356652.. + x + 0.634777..y, \text{ in } R_4.
\end{aligned} \tag{2.21}$$

The linear differential centers in (2.21) have the first integrals

$$\begin{aligned}
H_1(x,y) =& x^2 + x(-0.943775.. + 0.265344..y) + (-0.489818.. + 0.724279..y)y,\\
H_2(x,y) =& x^2 + (-1.337605.. + 0.636209..y)y + x(-1.971307.. + 1.029044..y),\\
H_3(x,y) =& x^2 + x(-0.249913.. + 0.415657..y) + (0.162397.. + 0.061343..y)y,\\
H_4(x,y) =& x^2 + (-0.423048.. + 0.705080..y)y + x(-0.713304.. + 1.269555..y),
\end{aligned}$$

respectively. In order to have crossing limit cycles of types 1 and 2^+, simultaneously, such

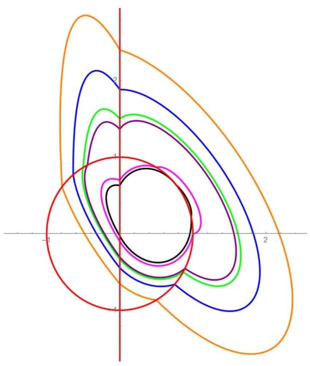

Figure 2.6: Four crossing limit cycles of type 1 and two crossing limit cycles of type 2^+ (black and magenta) of the discontinuous piecewise linear differential center (2.21). These limit cycles are traveled in counterclockwise.

that the crossing limit cycles of type 1 intersect the discontinuity curve Σ_0 in four different points $p_1 = (0, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (0, y_3)$ and $p_4 = (x_4, y_4)$, with $-1 < y_1 < 1 < y_3$ and $x_4 < 0 < x_2$ and $p_2, p_4 \in \mathbb{S}^1$; and the crossing limit cycles of type 2^+ intersect the discontinuity curve Σ_0 in four different points $p_5 = (x_5, y_5)$, $p_6 = (0, y_6)$, $p_7 = (0, y_7)$ and $p_8 = (x_8, y_8)$, with $-1 < y_7 < y_6 < 1$ and $x_5, x_8 > 0$, with $p_5, p_8 \in \mathbb{S}^1$. These points must satisfy systems (2.13) and (2.16), respectively. Considering the piecewise linear differential center (2.21) systems

(2.13) and (2.16) become

$$
\begin{aligned}
x_2^2+x_2(-0.943775..+0.265344..y_2)-0.489818..y_2+0.724279..y_2^2&\\
+0.489818..y_1-0.724279..y_1^2&=0,\\
4x_2^2-5.350421..y_2+2.544838..y_2^2+x_2(-7.885229..+4.116178..y_2)&\\
+5.350421..y_3-2.544838..y_3^2&=0,\\
x_4^2-0.162397..y_3-0.061343..y_3^2+x_4(-0.249913..+0.415657..y_4)&\\
+0.162397..y_4+0.061343..y_4^2&=0,\\
4x_4^2-1.692192..y_4+2.820321..y_4^2+x_4(-2.853217..+5.078222..y_4)&\\
+1.692192..y_1-2.820321..y_1^2&=0,\\
4x_5^2-1.959275..y_5+2.897117..y_5^2+x_5(-3.775101..+1.061377..y_5)&\\
+1.959275..y_6-2.897117..y_6^2&=0,\\
(y_6-y_7)(-1.692192..+2.820321..(y_6+y_7))&=0,\\
x_8^2+0.489818..y_7-0.724279..y_7^2+x_8(-0.943775..+0.265344..y_8)&\\
-0.489818..y_8+0.724279..y_8^2)&=0,\\
x_5^2-x_8^2-1.337605..y_5+0.636209..y_5^2+x_5(-1.971307..+1.029044..y_5)&\\
+x_8(1.971307..-1.029044..y_8)+1.337605..y_8-0.636209..y_8^2&=0,\\
x_2^2+y_2^2=1,\quad x_4^2+y_4^2=1,\quad x_5^2+y_5^2=1,\quad x_8^2+y_8^2&=1.
\end{aligned}
\tag{2.22}
$$

We have four real solutions $q^i=(y_1^i,x_2^i,y_2^i,y_3^i,x_4^i,y_4^i,x_5^i,y_5^i,y_6^i,y_7^i,x_8^i,y_8^i)$ with $i=1,2,3,4$, for system (2.22) that satisfy the above conditions, namely $q^1=(\,-1/3,\ \cos(-\pi/6),\ \sin(-\pi/6),$ $3/2,\ \cos(2\pi/3),\ \sin(2\pi/3),\ \cos(\pi/3),\ \sin(\pi/3),\ 7/10,\ -1/10,\ 1,0)$; $q^2=(\,-0.654342..,$ $\cos(-\pi/3),\ \sin(-\pi/3),\ 12/5,\ \cos(79\pi/100),\ \sin(79\pi/100),\ \cos(11\pi/50),\ \sin(11\pi/50),$ $63/100,\ -3/100,\ 0.975733..,\ 0.216981..)$; $q^3=(\,-0.447098..,\ \cos(-23\pi/100),\ \sin(-23\pi/100),$ $1.882264..,\ \cos(18\pi/25),\ \sin(18\pi/25),\ -0.654342..,\ \cos(11\pi/50),\ \sin(11\pi/50),\ 63/100,$ $-3/100, 0.975733.., 0.216981..)$; $q^4=(\,-0.305568..,\ \cos(-3\pi/20),\ \sin(-3\pi/20),\ 1.365012..,$ $-0.441883..,\ 0.897073..,\ \cos(11\pi/50),\ \sin(11\pi/50), 63/100, -3/100, 0.975733.., 0.216981..)$, these four solutions generated four crossing limit cycles of type 1 and two crossing limit cycles of type 2^+. See these crossing limit cycles of the piecewise linear differential center (2.21) in Figure 2.6.

Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 1 and of type 2^+, moreover these six crossing limit cycles have the configuration $(4,2)$, this is, 4-crossing limit cycle of type 1 and 2-crossing limit cycles of type 2^+. Clearly this lower bound for the maximum number of crossing limit cycles of types 1 and 2^+ simultaneously, could be also obtained with the configurations $(3,3)$ or $(2,4)$. But after several numeric computations we could not build a third limit cycle of type 2^+, previously fixing two limit cycles of type 1, so we only get those lower bound with the configuration $(4,2)$. □

Proof of statement (f) of Theorem 1.1. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x}=0.078341..+0.855624..x+1.571418..y, && \dot{y}=-0.065526..-x-0.855624..y, \text{ in } R_1,\\
&\dot{x}=0.496667..+0.078616..x-0.193136..y, && \dot{y}=-0.471461..+x-0.078616..y, \text{ in } R_2,\\
&\dot{x}=5.276135..+0.212817..x-1.851275..y, && \dot{y}=-5.383865..+x-0.212817..y, \text{ in } R_3,\\
&\dot{x}=0.484115..+0.548314..x-0.303113..y, && \dot{y}=0.569064..+x-0.548314..y, \text{ in } R_4.
\end{aligned}
\tag{2.23}
$$

The linear differential centers in (2.23) have the first integrals

$$\begin{aligned}
H_1(x,y) =& x^2 + y(0.156682.. + 1.571418..y) + x(0.131053.. + 1.711249..y),\\
H_2(x,y) =& x^2 + x(-0.942922.. - 0.157232..y) + (-0.993334.. + 0.193136..y)y,\\
H_3(x,y) =& x^2 + x(-10.767731.. - 0.425635..y) + y(-10.552270.. + 1.851275..y),\\
H_4(x,y) =& x^2 + x(1.138128.. - 1.096628..y) + (-0.968231.. + 0.303113..y)y,
\end{aligned}$$

respectively. In order to have crossing limit cycles of types 1 and 3^+, simultaneously, such that the crossing limit cycles of type 1 intersect the discontinuity curve Σ_0 in four different points $p_1 = (0, y_1)$, $p_2 = (x_2, y_2)$, $p_3 = (0, y_3)$ and $p_4 = (x_4, y_4)$,with $-1 < y_1 < 1 < y_3$ and $x_4 < 0 < x_2$ and $p_2, p_4 \in \mathbb{S}^1$; and the crossing limit cycles of type 3^+ intersect the discontinuity curve Σ_0 in four different points $p_5 = (x_5, y_5)$, $p_6 = (0, y_6)$, $p_7 = (0, y_7)$ and $p_8 = (x_8, y_8)$, with $1 < y_7 < y_6$, $x_5, x_8 > 0$ and $p_5, p_8 \in \mathbb{S}^1$, these points must satisfy systems (2.13) and (2.19), respectively. Considering the piecewise linear differential center (2.23) systems (2.13) and (2.19) become

$$\begin{aligned}
-0.524214..x_2 - 4x_2^2 - 6.844999..x_2y_2 + (y_1 - y_2)(0.626729.. + 6.285673..(y_1 + y_2)) &= 0,\\
-3.771690..x_2 + 4x_2^2 - 0.628929..x_2y_2 + (y_2 - y_3)(-3.973339.. + 0.772547..(y_2 + y_3)) &= 0,\\
43.070926..x_4 - 4x_4^2 + 1.702543..x_4y_4 + (y_3 - y_4)(-42.209082.. + 7.405102..(y_3 + y_4)) &= 0,\\
4.552513..x_4 + 4x_4^2 - 4.386514..x_4y_4 - (y_1 - y_4)(-3.872927.. + 1.212454..(y_1 + y_4)) &= 0,\\
-3.771690..x_5 + 4x_5^2 - 0.628929..x_5y_5 + (y_5 - y_6)(-3.973339.. + 0.772547..(y_5 + y_6)) &= 0,\\
(y_6 - y_7)(-42.209082.. + 7.405102..(y_6 + y_7)) &= 0,\\
3.771690..x_8 - 4x_8^2 + 0.628929..x_8y_8 + (y_7 - y_8)(-3.973339.. + 0.772547..(y_7 + y_8)) &= 0,\\
-4x_5^2 + 4x_8^2 + x_5(-0.524214.. - 6.844999..y_5) + (-0.626729..&\\
-6.285673..y_5)y_5 + y_8(0.626729.. + 6.285673..y_8) + x_8(0.524214.. + 6.844999..y_8) &= 0,\\
x_2^2 + y_2^2 = 1, \quad x_4^2 + y_4^2 = 1, \quad x_5^2 + y_5^2 = 1, \quad x_8^2 + y_8^2 &= 1.
\end{aligned} \tag{2.24}$$

We have four real solutions $q^i = (y_1^i, x_2^i, y_2^i, y_3^i, x_4^i, y_4^i, x_5, y_5, y_6, y_7, x_8, y_8)$ with $i = 1, 2, 3, 4$, for

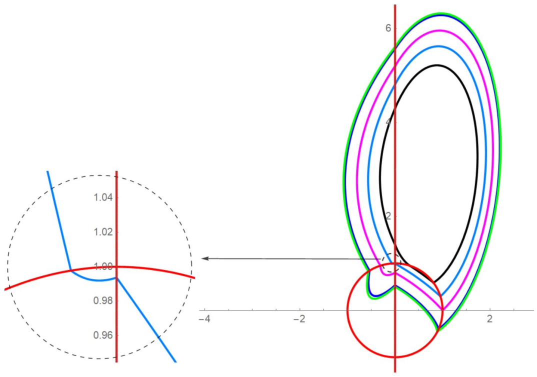

Figure 2.7: Four crossing limit cycles of type 1 and one crossing limit cycle of type 3^+ (black) of the discontinuous piecewise linear differential center (2.23). These limit cycles are traveled in counterclockwise.

system (2.24) that satisfy the above conditions, namely $q^1 = (4/5,\ 1,\ 0,\ 26/5,\ \cos(3\pi/5),$ $\sin(3\pi/5),\ \cos(\pi/5),\ \sin(\pi/5),\ 43/10,\ 7/5,\ \cos(2\pi/5),\ \sin(2\pi/5))$; $q^2 = (53/100,$ $\cos(-13\pi/100), \sin(-13\pi/100),\ 557/100,\ \cos(17\pi/25),\ \sin(17\pi/25),\ \cos(\pi/5),\ \sin(\pi/5),$

$43/10,\ 7/5,\ \cos(2\pi/5),\ \sin(2\pi/5))$; $q^3 = (\ 1/2,\ \cos(-3\pi/20),\ \sin(-3\pi/20),\ 5.611962..,$ $\cos(17239\pi/25000),\ \sin(17239\pi/25000),\ \cos(\pi/5),\ \sin(\pi/5),\ 43/10,\ 7/5,\ \cos(2\pi/5),$ $\sin(2\pi/5))$; $q^4 = (\ 0.993727.., \cos(12\pi/125), \sin(12\pi/125), 4.808026.., -0.066301.., 0.997799..,$ $\cos(\pi/5), \sin(\pi/5), 43/10, 7/5, \cos(2\pi/5), \sin(2\pi/5))$, these four solutions generated four crossing limit cycles of type 1 and one crossing limit cycle of type 3^+. See these crossing limit cycles of the piecewise linear differential center (2.23) in Figure 2.7.

Here we observed that we obtain a total of five crossing limit cycles between limit cycles of type 1 and of type 3^+, moreover these five crossing limit cycles have the configuration $(4,1)$, this is, 4-crossing limit cycle of type 1 and 1-crossing limit cycles of type 3^+. In order to obtain a result similar to the previous statement, this is, an example with a configuration $(4,2)$, we tried to build a second cycle of type 3^+ but when building this second cycle we lost a cycle of type 1, so we only got a configuration $(3,2)$. If we consider the piecewise linear system

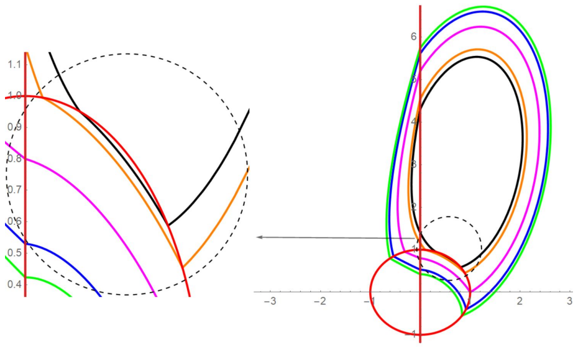

Figure 2.8: Three crossing limit cycles of type 1 and two crossing limit cycle of type 3^+ (black and orange) of the discontinuous piecewise linear differential center (2.25). These limit cycles are traveled in counterclockwise.

$$
\begin{aligned}
\dot{x} &= -0.128852.. - 0.332114..x - 0.791281..y, & \dot{y} &= -0.143708.. + x + 0.332114..y, \text{ in } R_1,\\
\dot{x} &= 0.597908.. + 0.108856..x - 0.227688..y, & \dot{y} &= -0.530777.. + x - 0.108856..y, \text{ in } R_2,\\
\dot{x} &= 0.716356.. + 0.457342..x - 0.251353..y, & \dot{y} &= -0.189975.. + x - 0.457342..y, \text{ in } R_3,\\
\dot{x} &= 1.857676.. - \frac{4}{5}x - 0.688147..y, & \dot{y} &= -1.219907.. + x + \frac{4}{5}y, \text{ in } R_4.
\end{aligned}
\tag{2.25}
$$

It is possible verify that we obtain the configuration $(3,2)$, see Figure 2.8. But after several numeric computations we could not build a third limit cycle of type 3^+, previously fixing two limit cycles of type 1, so we only get those lower bound by the maximum number of types 1 and 3^+, simultaneously, with the configurations $(4,1)$ and $(3,2)$. $\square$

3 Proof of Theorem 1.2

We consider the following piecewise linear differential center

$$\begin{aligned}
&\dot{x} = -124.644504.. + \frac{111}{50}x - 6.045715..y, && \dot{y} = -148.901657.. + x - \frac{111}{50}y, \text{ in } R_1,\\
&\dot{x} = 0.236087.. + 0.003662..x - 0.009243..y, && \dot{y} = -0.402647.. + x - 0.003662..y, \text{ in } R_2,\\
&\dot{x} = 1 + \frac{x}{5} - 0.102500..y, && \dot{y} = -\frac{9}{20} + x - \frac{y}{5}, \text{ in } R_3.
\end{aligned} \tag{3.1}$$

The linear differential centers in (3.1) have the first integrals

$$\begin{aligned}
H_1(x,y) &= x^2 + x\left(-297.803314.. - \frac{111}{25}y\right) + y(249.289008.. + 6.045715..y),\\
H_2(x,y) &= x^2 + x(-0.805295.. - 0.007324..y) + (-0.472175.. + 0.009243..y)y,\\
H_3(x,y) &= x^2 + x\left(-\frac{9}{10} - \frac{2}{5}y\right) + (-2 + 0.102500..y)y,
\end{aligned}$$

respectively.

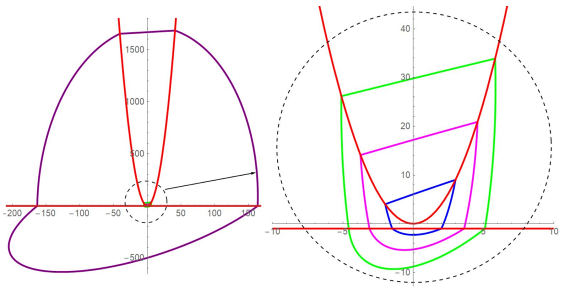

Figure 3.1: Four crossing limit cycles of the discontinuous piecewise linear differential system (3.1). These limit cycles are traveled in counterclockwise.

For piecewise linear differential systems in the family $\mathcal{F}_4$ we have crossing limit cycles which intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, k)$ and $p_4 = (x_4, k)$, if these points satisfy the system

$$\begin{aligned}
H_1(x_1, x_1^2) &= H_1(x_2, x_2^2),\\
H_2(x_2, x_2^2) &= H_2(x_3, k),\\
H_3(x_3,\ k) &= H_3(x_4, k),\\
H_2(x_4,\ k) &= H_2(x_1, x_1^2).
\end{aligned} \tag{3.2}$$

Then for the piecewise linear differential centers (3.1) and $\tilde{\Sigma}_k$ considering $k = -1$, system (3.2) becomes

$$\begin{aligned}
x_1(-1191.213259.. + x_1(1001.156032.. + x_1\,(-44425 + 24.182863..x_1)))&\\
+x_2(1191.213259.. + x_2(-1001.156032.. + (-44425 - 24.182863..x_2)\,x_2)) &= 0,\\
-1.925675.. + x_2(-3.221182.. + x_2(2.111297.. + (-0.029297.. + 0.036973..x_2)x_2))&\\
+(3.191885.. - 4x_3)x_3 &= 0,\\
(x_3 - x_4)\left(-\frac{1}{2} + x_3 + x_4\right) &= 0,\\
1.925675.. + x_1(3.221182.. + x_1(-2.111297.. + (0.029297.. - 0.036973..x_1)x_1))&\\
+x_4(-3.191885.. + 4x_4) &= 0.
\end{aligned} \tag{3.3}$$

Taking into account that the solutions (x_1, x_2, x_3, x_4) must satisfy $x_2 < x_1$ and $x_3 < x_4$, system (3.3) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$, with $i = 1, 2, 3, 4$. Namely, $q^1 = (3, -2, -3/2, 2)$; $q^2 = (457/100, -3.753677.., -3.116713.., 3.616713..)$; $q^3 = (5.820000.., -5.115260.., -4.592690.., 5.092690..)$ and $q^4 = (41.045251.., -40.667957.., -162.945374.., 163.445374..)$. Which provide four crossing limit cycles of the piecewise linear differential center (3.1). See these four crossing limit cycles in Figure 3.1.

Here we observe that there is a duality between the crossing limit cycles that intersect the discontinuity curve $\tilde{\Sigma}_{-1}$ and the crossing limit cycles that intersect the discontinuity curve Σ_2 for the family $\mathcal{F}_1$ studied in statement (a) of Theorem 1.1, where we also got four crossing limit cycles, see Figures 2.1 and 3.1.

4 Proof of Theorem 1.3

Proof of statement (a) of Theorem 1.3. We consider the following piecewise linear differential center

$$
\begin{aligned}
\dot{x} &= \frac{11}{10} + \frac{4}{5}x - \frac{4}{5}y, & \dot{y} &= 1 + x - \frac{4}{5}y, \text{ in } R_1,\\
\dot{x} &= \frac{17}{75} - \frac{3}{10}x - \frac{17}{150}y, & \dot{y} &= -\frac{61}{20} + x + \frac{3}{10}y, \text{ in } R_2,\\
\dot{x} &= \frac{1}{6} + x - \frac{25}{16}y, & \dot{y} &= -\frac{1}{4} + x - y, \text{ in } R_3,\\
\dot{x} &= \frac{133}{36} + \frac{x}{10} - \frac{7}{45}y, & \dot{y} &= \frac{543}{20} + x - \frac{y}{10}, \text{ in } R_4.
\end{aligned}
\tag{4.1}
$$

The linear differential centers in (4.1) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= 5x(2+x) - (11+8x)y + 4y^2,\\
H_2(x,y) &= 150x^2 + 17(-4+y)y + 15x(-61+6y),\\
H_3(x,y) &= 4x^2 - 2x(1+4y) + \frac{y}{12}(-16+75y),\\
H_4(x,y) &= 90x^2 + 9x(543-2y) + 7y(-95+2y),
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 4, which intersects the discontinuity

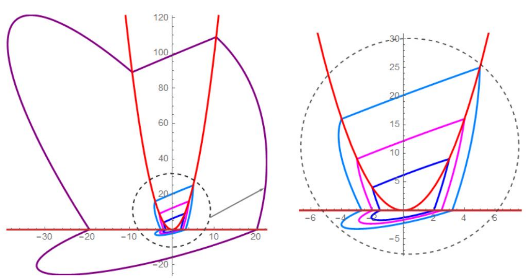

Figure 4.1: Four crossing limit cycles of type 4 of the discontinuous piecewise linear differential system (4.1). These limit cycles are traveled in counterclockwise.

curve $\tilde{\Sigma}_0$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, 0)$ and $p_4 = (x_4, 0)$, these points must satisfy system

$$\begin{aligned} H_1(x_1, x_1^2) &= H_1(x_2, x_2^2), \\ H_2(x_2, x_2^2) &= H_2(x_3, 0), \\ H_3(x_3,\ 0) &= H_3(x_4, 0), \\ H_4(x_4,\ 0) &= H_4(x_1, x_1^2). \end{aligned} \tag{4.2}$$

Considering the piecewise linear differential center (4.1) system (4.2) becomes

$$\begin{aligned} (x_1 - x_2)(-1 + x_1 + x_2)(-5 + 2(-1 + x_1)x_1 + 2(-1 + x_2)x_2) &= 0, \\ 2x_2(-915 + x_2(82 + x_2(90 + 17x_2))) + 30(61 - 10x_3)x_3 &= 0, \\ 4(x_3 - x_4)\left(-\frac{1}{2} + x_3 + x_4\right) &= 0, \\ 2x_1(-4887 + x_1(575 + 2(9 - 7x_1)x_1)) + 18x_4(543 + 10x_4) &= 0. \end{aligned} \tag{4.3}$$

In this case we have that the solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ must satisfy $x_2^i < 0 < x_1^i$ and $x_3^i < 0 < x_4^i$ then we have four real solutions $q^1 = (3, -2, -3/2, 2)$; $q^2 = (4, -3, -2, 5/2)$; $q^3 = (5, -4, 27/10, 16/5)$ and $q^4 = (10.440607.., -9.440607.., -19.555603.., 20.055606..)$ of system (4.3), which provide four crossing limit cycles of type 4 of the piecewise linear differential center (4.1). See these four crossing limit cycles in Figure 4.1.

Here we observe that there is a duality between the crossing limit cycles of type 4 that intersect the discontinuity curve $\tilde{\Sigma}_0$ and the crossing limit cycles that intersect the discontinuity curve Σ_1 for the family $\mathcal{F}_2$ studied in statement (a) of Theorem 1.1, where we also got four crossing limit cycles, see Figures 2.2 and 4.1. □

Proof of statement (b) of Theorem 1.3. In this case we consider the following piecewise linear differential center

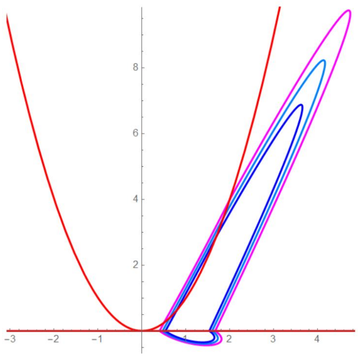

Figure 4.2: Three crossing limit cycles of type 5 of the discontinuous piecewise linear differential system (4.4). These limit cycles are traveled in counterclockwise.

$$\begin{aligned} &\dot{x} = 0.100318.. - \frac{2}{5}x + 0.161744..y && \dot{y} = 0.260062.. - x + \frac{2}{5}y, \text{ in } R_1, \\ &\dot{x} = 1 - x - \frac{13}{4}y, && \dot{y} = -\frac{31}{30} + x + y, \text{ in } R_3, \\ &\dot{x} = -0.399222.. + 0.378090..x - 0.144616..y, && \dot{y} = -1.020635.. + x - 0.378090..y, \text{ in } R_4. \end{aligned} \tag{4.4}$$

The linear differential centers in (4.4) have the first integrals

$$
\begin{aligned}
H_1(x,y) &= x^2 + x\left(-0.520124.. - \frac{4}{5}y\right) + (0.200636.. + 0.161744..y)y,\\
H_3(x,y) &= -\frac{124}{15}x - 8y + 9y^2 + 4(x+y)^2,\\
H_4(x,y) &= 4(x - 0.378090..y)^2 + 8(-1.020635..x + 0.399222..y) + 0.006657..y^2,
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 5, which intersects the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, 0)$ and $p_4 = (x_4, 0)$, with $0 < x_2 < x_1$ and $0 < x_3 < x_4$, these points must satisfy system

$$
\begin{aligned}
H_1(x_1, x_1^2) &= H_1(x_2, x_2^2),\\
H_4(x_2, x_2^2) &= H_4(x_3, 0),\\
H_3(x_3,\ 0) &= H_3(x_4, 0),\\
H_4(x_4,\ 0) &= H_4(x_1, x_1^2).
\end{aligned}
\tag{4.5}
$$

Considering the piecewise linear differential center (4.4) system (4.5) becomes

$$
\begin{aligned}
-2.080498..x_1 + 4.802546..x_1^2 - 3.199999..x_1^3 + 0.646977..x_1^4&\\
+x_2(2.080498.. - 4.802546..x_2 + 3.199999..x_2^2 - 0.646977..x_2^3) &= 0,\\
x_2(-2591625737556 + x_2(2283329836763 + 50x_2(-19201143493 + 3672147700x_2)))&\\
-324x_3(-7998844869 + 3918560960x_3) &= 0,\\
4(x_3 - x_4)\left(-\frac{31}{15} + x_3 + x_4\right) &= 0,\\
x_1(2591625737556 + x_1(-2283329836763 + 50(19201143493 - 3672147700x_1)x_1))&\\
+324x_4(-7998844869 + 3918560960x_4) &= 0.
\end{aligned}
\tag{4.6}
$$

In this case system (4.6) has three real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$, where $q^1 = (2, 1/2, 2/5, 5/3)$; $q^2 = (93/50, 63/100, 47/100, 479/300)$ and $q^3 = (17/10, 0.785691.., 0.534387.., 1.532279..)$ which provide three crossing limit cycles of type 5 of the piecewise linear differential center (4.4). See these three crossing limit cycles in Figure 4.2. □

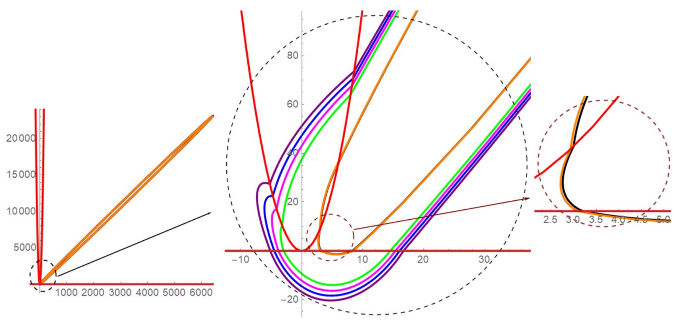

Figure 4.3: Four crossing limit cycles of type 4 and two crossing limit cycles of type 5 (black and orange) of the discontinuous piecewise linear differential system (4.7). These limit cycles are traveled in counterclockwise.

Proof of statement (c) of Theorem 1.3. We consider the following piecewise linear differential center

$$
\begin{aligned}
&\dot{x} = 45.736851.. - \frac{x}{2} - 7.515818..y, && \dot{y} = -1146.321640.. + x + \frac{y}{2}, \text{ in } R_1,\\
&\dot{x} = -0.320594.. - 0.199436..x - 0.051960..y, && \dot{y} = 0.460058.. + x + 0.199436..y, \text{ in } R_2,\\
&\dot{x} = 2 + \frac{x}{20} - \frac{13}{200}y, && \dot{y} = -\frac{23}{4} + x - \frac{y}{20}, \text{ in } R_3,\\
&\dot{x} = -0.457007.. + 0.276952..x - 0.076768..y, && \dot{y} = -4.377702.. + x - 0.276952..y, \text{ in } R_4.
\end{aligned}
\tag{4.7}
$$

The linear differential centers in (4.7) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-2292.643280.. + y) + y(-91.473702.. + 7.515818..y),\\
H_2(x,y) =& x^2 + x(0.920117.. + 0.398872..y) + (0.641188.. + 0.051960..y)yx^2\\
&+ x(0.920117.. + 0.398872..y) + (0.641188.. + 0.051960..y)y,\\
H_3(x,y) =& 2x(-23 + 2x) - \frac{2}{5}(40 + x)y + \frac{13}{50}y^2,\\
H_4(x,y) =& x^2 + x(-8.755405.. - 0.553904..y) + (0.914014.. + 0.076768..y)y,
\end{aligned}
$$

respectively In order to have crossing limit cycles of type 4 and 5, simultaneously, such that the crossing limit cycles of type 4 intersect the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, 0)$ and $p_4 = (x_4, 0)$, with $x_2 < 0 < x_1$ and $x_3 < 0 < x_4$, and the crossing limit cycles of type 5 intersect the discontinuity curve $\tilde{\Sigma}_0$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, 0)$ and $p_8 = (x_8, 0)$, with $0 < x_6 < x_5$ and $0 < x_7 < x_8$, these points must satisfy systems (4.2) and (4.5), respectively. Considering the piecewise linear differential center (4.7) systems (4.2) and (4.5) become

$$
\begin{aligned}
x_1(-9170.573120.. + x_1(-361.894811.. + x_1(3.999999.. + 30.063275..x_1)))&\\
+x_2(9170.573120.. + x_2(361.894811.. + (-3.999999.. - 30.063275..x_2)x_2)) &= 0,\\
x_2(3.680468.. + x_2(6.564754.. + (1.595489.. + 0.207843..x_2)x_2)) - 3.680468..x_3 - 4x_3^2 &= 0,\\
(x_3 - x_4)(-23 + 2x_3 + 2x_4) &= 0,\\
x_1(35.021620.. + x_1(-7.656056.. + (2.215618.. - 0.307072..x_1)x_1)) - 35.021620..x_4&\\
+4x_4^2 &= 0,\\
x_5(-9170.573120.. + x_5(-361.894811.. + x_5(3.999999.. + 30.063275..x_5)))&\\
+x_6(9170.573120.. + x_6(361.894811.. + (-3.999999.. - 30.063275..x_6)x_6)) &= 0,\\
x_6(-35.021620.. + x_6(7.656056.. + (-2.215618.. + 0.307072..x_6)x_6)) + 35.021620..x_7&\\
-4x_7^2 &= 0,\\
(x_7 - x_8)(-23 + 2x_7 + 2x_8) &= 0,\\
x_5(35.021620.. + x_5(-7.656056.. + (2.215618.. - 0.307072..x_5)x_5)) - 35.021620..x_8&\\
+4x_8^2 &= 0.
\end{aligned}
\tag{4.8}
$$

In this case system (4.8) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5, x_6, x_7, x_8)$, that satisfy the necessary conditions to have crossing limit cycles of types 4 and 5. Namely, $q^1 = (8, -16/5, -3, 29/2, 6, 3, 16/5, 83/10)$; $q^2 = (823/100, -413/100, -96/25, 767/50, 6, 3, 16/5, 83/10)$; $q^3 = (841/100, -4.737905.., -4.516438.., 16.016438.., 6.040228.., 2.934482.., 3.093430.., 8.406569..)$ and $q^4 = (429/50, -5.236369.., -5.170738.., 16.670738.., 6.040228.., 2.934482.., 3.093430.., 8.406569..)$. These solutions provide four crossing limit cycles of type 4 and two crossing limit cycles of type 5 of the piecewise linear differential center (4.7). See these crossing limit cycles in Figure 4.3. Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 4 and of type 5, moreover these six crossing limit cycles have the configuration $(4,2)$, this is, 4-crossing limit cycle of type 4 and 2-crossing limit cycles of type 5. We know that this lower bound for the maximum number of crossing limit cycles of types 4 and 5 simultaneously, could be also obtained with the configuration $(3,3)$. But if we previously fixing two limit cycles of each type after several numeric computations we could not build a third limit cycle of type 5, then we only get those lower bound with the configuration $(4,2)$. □

5 Proof of Theorem 1.4

Proof of statement (a) of Theorem 1.4. We consider the following piecewise linear differential center

$$\begin{aligned}
&\dot{x} = -0.678037.. + 0.111302..x - 0.025436..y, && \dot{y} = -3.106005.. + x - 0.111302..y, \text{ in } R_1,\\
&\dot{x} = -0.133244.. + 0.232759..x - 0.058573..y, && \dot{y} = -0.290609.. + x - 0.232759..y, \text{ in } R_2,\\
&\dot{x} = 3.074032.. + 0.434135..x - 2.713559..y, && \dot{y} = -3.035258.. + x - 0.434135..y, \text{ in } R_4,\\
&\dot{x} = 1.427543.. + 0.059092..x - 0.651180..y, && \dot{y} = -1.450367.. + x - 0.059092..y, \text{ in } R_5.
\end{aligned} \tag{5.1}$$

The linear differential centers in (5.1) have the first integrals

$$\begin{aligned}
H_1(x,y) =&x^2 + x(-6.212010.. - 0.222604..y) + (1.356074.. + 0.025436..y)y,\\
H_2(x,y) =&x^2 + x(-0.581218.. - 0.465518..y) + (0.266488.. + 0.058573..y)y,\\
H_4(x,y) =&x^2 + x(-6.070516.. - 0.868271..y) + y(-6.148064.. + 2.713559..y),\\
H_5(x,y) =&x^2 + x(-2.900734.. - 0.118185..y) + (-2.855087.. + 0.651180..y)y,
\end{aligned}$$

respectively. In order to have a crossing limit cycle of type 6^+, which intersects the discontinu-

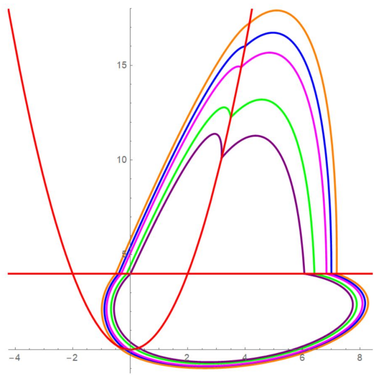

Figure 5.1: Five crossing limit cycles of type 6^+ of the discontinuous piecewise linear differential system (5.1). These limit cycles are traveled in counterclockwise.

ity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, these points must satisfy system

$$
\begin{aligned}
H_1(x_4,\ k) = &\ \ H_1(x_1, x_1^2),\\
H_2(x_1, x_1^2) = &\ \ H_2(x_2, k),\\
H_5(x_2,\ k) = &\ \ H_5(x_3, x_3^2),\\
H_4(x_3, x_3^2) = &\ \ H_4(x_4, k).
\end{aligned}
\tag{5.2}
$$

Considering piecewise linear differential center (5.1) and $k = 4$, system (5.2) becomes

$$
\begin{aligned}
&-8.012495.. + x_1(-2.324875.. + x_1(5.065954.. + (-1.862075.. + 0.234292..x_1)x_1))\\
&\qquad\qquad + (9.773178.. - 3.999999..x_2)x_2 = 0,\\
&-1.001459.. + (-3.373476.. + x_2)x_2 + x_3(2.900734.. + x_3(1.855087.. + (0.118185..\\
&\qquad\qquad -0.651180..x_3)x_3)) = 0,\\
&-75.298768.. + x_3(-24.282066.. + x_3(-20.592258.. + x_3(-3.473086..\\
&\qquad\qquad +10.854237..x_3))) + (38.174413.. - 4x_4)x_4 = 0,\\
&23.325149.. + x_1(24.848040.. + x_1(-9.424297.. + (0.890418.. - 0.101747..x_1)x_1))\\
&\qquad\qquad + x_4(-28.409714.. + 4x_4) = 0,
\end{aligned}
\tag{5.3}
$$

In this case system (5.3) has five real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ that satisfy the conditions $-2 < x_2^i < 2 < x_1^i$ and $-2 < x_3^i < 2 < x_4^i$. We have $q^1 = (4, -2/5, -1/5, 7)$; $q^2 = (193/50, -31/100, -1/20, 683/100)$; $q^3 = (7/2, -3/25, 9/50, 641/100)$; $q^4 = (159/50, 1/100, 3/10, 303/50)$ and $q^5 = (4.149236.., -0.507154.., -0.449658.., 7.185104..)$, which provide five crossing limit cycles of type 6^+ of the piecewise linear differential center (5.1). See these crossing limit cycles in Figure 5.1.

Here we observe that there is a duality between the crossing limit cycles of type 6^+ that intersect the discontinuity curve $\tilde{\Sigma}_4$ and the crossing limit cycles of type 1 for the family $\mathcal{F}_3$ that intersect the discontinuity curve Σ_0 studied in statement (b) of Theorem 1.1, where we also got five crossing limit cycles, see Figures 2.3 and 5.1. □

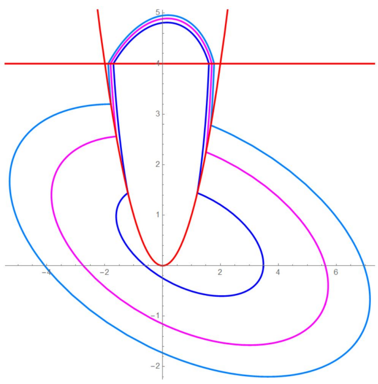

Figure 5.2: Three crossing limit cycles of type 7 of the discontinuous piecewise linear differential system (5.4). These limit cycles are traveled in counterclockwise.

Proof of statement (b) of Theorem 1.4. We consider the following piecewise linear differential center

$$
\begin{aligned}
\dot{x} &= 3 + \frac{x}{4} - \frac{17}{16}y, & \dot{y} &= \frac{21}{20} + x - \frac{y}{4}, \text{ in } R_2,\\
\dot{x} &= 3.601959.. - x - 5.323060..y, & \dot{y} &= -\frac{36}{25} + x + y, \text{ in } R_4,\\
\dot{x} &= \frac{11827667}{24434928} - \frac{91445}{6205696}x - \frac{8433175}{97739712}y, & \dot{y} &= \frac{26369}{1108160} + x + \frac{91445}{6205696}y, \text{ in } R_5.
\end{aligned} \tag{5.4}
$$

The linear differential centers in (5.4) have the first integrals

$$
\begin{aligned}
H_2(x,y) &= \frac{2}{5}x(21+10x) - 2(12+x)y + \frac{17}{4}y^2,\\
H_4(x,y) &= x^2 + x\left(-\frac{72}{25} + 2y\right) + y(-7.203918.. + 5.323060..y),\\
H_5(x,y) &= 977397120x^2 + 63x(738332 + 457225y) + 10y(-94621336 + 8433175y),
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 7, which intersects the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, k)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, x_4^2)$, these points must satisfy system

$$
\begin{aligned}
H_2(x_1,\ k) &= H_2(x_2,\ k),\\
H_5(x_2,\ k) &= H_5(x_3, x_3^2),\\
H_4(x_3, x_3^2) &= H_4(x_4, x_4^2),\\
H_5(x_4, x_4^2) &= H_5(x_1,\ k).
\end{aligned} \tag{5.5}
$$

In this case considering $k = 4$, system (5.5) becomes

$$
\begin{aligned}
4(x_1 - x_2)\left(\frac{1}{10} + x_1 + x_2\right) &= 0,\\
-2435545440 + 4032x_2(40113 + 242410x_2) - x_3(46514916 + 5x_3(6236752&\\
+5x_3(1152207 + 3373270x_3))) &= 0,\\
x_3\left(-\frac{288}{25} + x_3(-24.815674.. + x_3(8 + 21.292240..x_3))\right)&\\
+x_4\left(-\frac{288}{25} + x_4(24.815674.. + (-8 - 21.292240..x_4)x_4)\right) &= 0,\\
2435545440 - 4032x_1(40113 + 242410x_1) + x_4(46514916 + 5x_4(6236752&\\
+5x_4(1152207 + 3373270x_4))) &= 0.
\end{aligned} \tag{5.6}
$$

System (5.6) has three real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ that satisfy the conditions $-2 < x_2^i < x_1^i < 2$ and $-2 < x_3^i < x_4^i < 2$. They are $q^1 = (17/10, -9/5, -8/5, 3/2)$; $q^2 = (8/5, -17/10, -6/5, 6/5)$ and $q^3 = (89/50, -47/25, -1.788665.., 1.667136..)$, which provide three crossing limit cycles of type 7 of the piecewise linear differential center (5.4). See these three crossing limit cycles in Figure 5.2. □

Proof of statement (c) of Theorem 1.4. We consider the following piecewise linear differential center

$$\begin{aligned}
&\dot{x} = -0.228658.. + 0.153388..x - 0.043263..y, && \dot{y} = -1.233713.. + x - 0.153388..y, \text{ in } R_1,\\
&\dot{x} = \frac{52}{5} + x - 5y, && \dot{y} = 2 + x - y, \text{ in } R_2,\\
&\dot{x} = -0.208786.. - 0.135584..x - 0.040106..y, && \dot{y} = 1.549735 + x + 0.135584..y, \text{ in } R_3,\\
&\dot{x} = 2 - \frac{x}{2} - \frac{5}{4}y, && \dot{y} = -\frac{41}{20} + x + \frac{y}{2}, \text{ in } R_4.
\end{aligned} \tag{5.7}$$

The linear differential centers in (5.7) have the first integrals

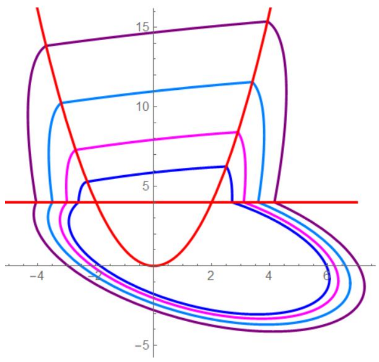

Figure 5.3: Four crossing limit cycles of type 8 of the discontinuous piecewise linear differential system (5.7). These limit cycles are traveled in counterclockwise.

$$\begin{aligned}
H_1(x,y) =& 15298879995x^2 + 5y(1399284923 + 132375500y) - 6x(6291478429 + 782226050y),\\
H_2(x,y) =& 4x(4+x) - \frac{8}{5}(52+5x)y + 20y^2,\\
H_3(x,y) =& 57070082030x^2 + 15y(1588730299 + 152593500y) + x(176887019081\\
&+ 15475638300y),\\
H_4(x,y) =& 4x^2 + x\left(-\frac{82}{5} + 4y\right) + y(-16+5y),
\end{aligned}$$

respectively. In order to have a crossing limit cycle of type 8, which intersects the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, k)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, these points must satisfy system

$$\begin{aligned}
H_2(x_1, x_1^2) =& \ H_2(x_2, x_2^2),\\
H_3(x_2, x_2^2) =& \ H_3(x_3,\ k),\\
H_4(x_3,\ k) =& \ H_4(x_4,\ k),\\
H_1(x_4,\ k) =& \ H_1(x_1, x_1^2).
\end{aligned} \tag{5.8}$$

In this case considering $k = 4$, system (5.8) becomes

$$\begin{aligned}
(x_1-x_2)(-1+5x_1+5x_2)(-20+x_1(-1+5x_1)+x_2(-1+5x_2))&=0,\\
\frac{2}{28535041015}(-131946257940+x_2(176887019081+5x_2(16180207303&\\
+60x_2(51585461+7629675x_2))))-\frac{854345518x_3}{51046585}-4x_3^2&=0,\\
4(x_3-x_4)\left(-\frac{1}{10}+x_3+x_4\right)&=0,\\
\frac{8}{15298879995}(19287869230+x_1(18874435287-5x_1(2229530461&\\
+10x_1(-46933563+6618775x_1))))-\frac{1196239064x_4}{80946455}+4x_4^2&=0.
\end{aligned} \tag{5.9}$$

System (5.9) has four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i)$ that satisfy the conditions $x_3^i < -2 < 2 < x_4^i$ and $x_2^i < -2 < 2 < x_1^i$. They are $q^1 = (5/2, -23/10, -13/5, 27/10)$; $q^2 = (29/10, -27/10, -3, 31/10)$; $q^3 = (17/5, -16/5, -7/2, 18/5)$ and $q^4 = (98/25, -93/25, -203/50, 104/25)$ which provide four crossing limit cycles of type 8 of the piecewise linear differential center (5.7). See these four crossing limit cycles in Figure 5.3.

Here we observe that there is a duality between the crossing limit cycles for family $\mathcal{F}_4$ studied in Theorem 1.2, the crossing limit cycles of type 4 for the family $\mathcal{F}_5$ studied in statement (a) of Theorem 1.3 and crossing limit cycles of type 8 for the family $\mathcal{F}_6$ studied in statement (c) of Theorem 1.4. In these three cases we got four crossing limit cycles. See Figures 3.1, 4.1 and 5.3. □

Proof of statement (d) of Theorem 1.4. We consider the following piecewise linear differential center

$$\begin{aligned}
\dot{x} &= \frac{243469}{1620885}+\frac{1826}{77185}x-\frac{9088}{324177}y, & \dot{y} &= -\frac{614289}{154370}+x-\frac{1826}{77185}y, \text{ in } R_1,\\
\dot{x} &= -0.229652..+\frac{7}{5}x-0.020472..y, & \dot{y} &= -1.718896..+x-\frac{7}{5}y, \text{ in } R_2,\\
\dot{x} &= 1+\frac{9}{10}x-\frac{53}{50}y, & \dot{y} &= -\frac{1}{2}+x-\frac{9}{10}y, \text{ in } R_4.
\end{aligned} \tag{5.10}$$

The linear differential centers in (5.10) have the first integrals

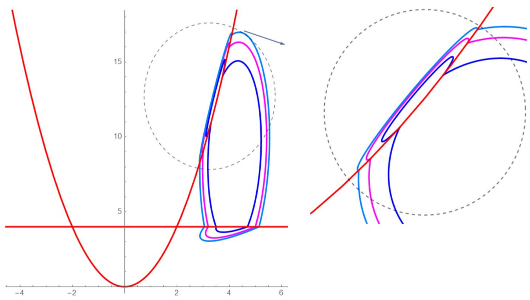

Figure 5.4: Three crossing limit cycles of type 9^+ of the discontinuous piecewise linear differential system (5.10). These limit cycles are traveled in counterclockwise.

$$
\begin{aligned}
H_1(x,y) =& 21x(-614289+77185x)-2(243469+38346x)y+45440y^2,\\
H_2(x,y) =& x^2+x\left(-3.437793..-\frac{14}{5}y\right)+(0.459305..+0.020472..y)y,\\
H_4(x,y) =& 4\left(x-\frac{9}{10}y\right)^2+y^2-4(x+2y),
\end{aligned}
$$

respectively. In order to have a crossing limit cycle of type 9^+, which intersects the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1=(x_1,x_1^2)$, $p_2=(x_2,x_2^2)$, $p_3=(x_3,k)$ and $p_4=(x_4,k)$, these points must satisfy system

$$
\begin{aligned}
H_2(x_1,x_1^2) =& \ H_2(x_2,x_2^2),\\
H_1(x_2,x_2^2) =& \ H_1(x_3,\ k),\\
H_4(x_3,\ k) =& \ H_4(x_4,\ k),\\
H_1(x_4,\ k) =& \ H_1(x_1,x_1^2).
\end{aligned}
\tag{5.11}
$$

Considering $k=4$, system (5.11) becomes

$$
\begin{aligned}
x_1\left(-13.751172..+x_1\left(5.837222..+\left(-\frac{23}{25}+0.081889..x_1\right)x_1\right)\right)+x_2\,(13.751172..&\\
+x_2\left(-5.837222..+\left(-\frac{23}{25}-0.081889..x_2\right)x_2\right)\right) &= 0,\\
x_2(12900069-x_2(1133947-76692x_2+45440x_2^2))-3(406904+7(628897&\\
-77185x_3)x_3) &= 0,\\
4(x_3-x_4)\left(-\frac{41}{5}+x_3+x_4\right) &= 0,\\
x_1(12900069-x_1(1133947-76692x_1+45440x_1^2))-3(406904+7(628897&\\
-77185x_4)x_4) &= 0,
\end{aligned}
\tag{5.12}
$$

And we have that system (5.12) has three real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i)$ that satisfy the conditions $2<x_2^i<x_1^i$ and $2<x_3^i<x_4^i$. They are $q^1=(4,3,16/5,5)$; $q^2=(15/4,33/10,7/2,47/10)$ and $q^3=(41/10,2.879320..,3.058075..,5.141924..)$ which provide three crossing limit cycles of type 9^+ of the piecewise linear differential center (5.10). See these three crossing limit cycles in Figure 5.4.

Here we observe that there is a duality between the crossing limit cycles of type 3^+ for family $\mathcal{F}_3$ studied in statement (d) of Theorem 1.1, the crossing limit cycles of type 5 for the family $\mathcal{F}_5$ studied in statement (b) of Theorem 1.3 and crossing limit cycles of type 9^+ for the family $\mathcal{F}_6$ studied in statement (d) of Theorem 1.4. In these three cases we got three crossing limit cycles. See Figures 2.5, 4.2 and 5.4. □

6 Proof of Theorem 1.5

Proof of statement (a) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$\begin{array}{ll}
\dot{x} = 0.751960.. - 0.008805..x - 0.043938..y, & \dot{y} = -1.117055.. + x + 0.008805..y, \text{ in } R_1,\\
\dot{x} = -\dfrac{4701043}{7161144} - \dfrac{122761}{156650025}x + \dfrac{91946}{31330005}y, & \dot{y} = -\dfrac{42715283}{313300050} - x + \dfrac{122761}{156650025}y, \text{ in } R_2,\\
\dot{x} = 0.041424.. - 0.228644..x - 0.115044..y, & \dot{y} = 2.030027.. + x + 0.228644..y, \text{ in } R_3,\\
\dot{x} = 6.094659.. - 0.970562..x - 1.475325..y, & \dot{y} = -4.066695 + x + 0.970562..y, \text{ in } R_4,\\
\dot{x} = -0.014046.. - 0.011408..x + 0.000796..y, & \dot{y} = -0.900270.. - x + 0.011408..y, \text{ in } R_5.
\end{array} \tag{6.1}$$

The linear differential centers in (6.1) have the first integrals

$$\begin{aligned}
H_1(x,y) =& x^2 + x(-2.234111.. + 0.017610..y) + (-1.503920.. + 0.043938..y)y,\\
H_2(x,y) =& 626600100x^2 + x(170861132 - 982088y) + 5y(-164536505 + 367784y),\\
H_3(x,y) =& x^2 + x(4.060055.. + 0.457288..y) + (-0.082848.. + 0.115044..y)y,\\
H_4(x,y) =& x(-54480047924280068901 83 + 6698319382773302134 20x) - 160y\\
& (51029434834312436627 - 8126422570764957500x) + 988220002292252000000y^2,\\
H_5(x,y) =& 17172023317192110696x^2 + x(309189342506522332 87 - 391817091205831000y)\\
& + 6y(-80400672913407451 + 2279188834700000y),
\end{aligned}$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 6^-, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, and the crossing limit cycles of type 6^- intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, k)$, with $x_5 < -2 < x_7 < 2$ and $x_6 < -2 < x_8 < 2$, these points must satisfy systems (5.2) and

$$\begin{aligned}
H_3(x_5, x_5^2) =& \; H_3(x_6,\ k),\\
H_4(x_6,\ k) =& \; H_4(x_7, x_7^2),\\
H_5(x_7, x_7^2) =& \; H_5(x_8,\ k),\\
H_2(x_8,\ k) =& \; H_2(x_5, x_5^2),
\end{aligned} \tag{6.2}$$

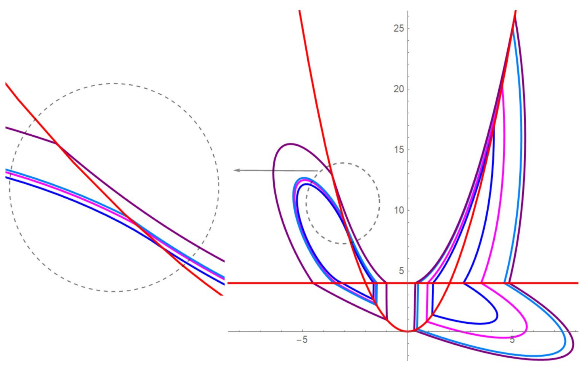

Figure 6.1: Four crossing limit cycles of type 6^+ in the right hand side and four crossing limit cycles of type 6^- in the left hand side, of the discontinuous piecewise linear differential system (6.1). These limit cycles are traveled in counterclockwise.

respectively. Considering the piecewise linear differential center (6.1) and $k = 4$, systems (5.2) and (6.2) become

$$
\begin{aligned}
&170861132x_1 - 196082425x_1^2 - 982088x_1^3 + 1838920x_1^4 - 60(-54355123 + 2782213x_2 \\
&\qquad +10443335x_2^2) = 0,\\
&-1710814021790578824 + 2935166588582890287x_2 + 1717202331719211069 6x_2^2 \\
&-3091893425065223328 7x_3 - 1668961927971166599 0x_3^2 + 391817091205831000x_3^3 \\
&\qquad -136751330082000 00x_3^4 = 0,\\
&-544800479242800689 0183x_3 - 749487763521265964 6900x_3^2+ \\
&1300227611322393200000x_3^3 + 988220002292252000000x_3^4 + 21(802253250346853687680 \\
&\qquad -117663974827825757 23x_4 + 318967589655871530 20x_4^2) = 0,\\
&-21.250638.. + 8.936444..x_1 + 2.015680..x_1^2 - 0.070440..x_1^3 \\
&\qquad -0.175755..x_1^4 - 8.654682..x_4 + 4x_4^2 = 0,\\
&-6.037269.. + 16.240221..x_5 + 3.668606..x_5^2 + 1.829154..x_5^3 \\
&\qquad +0.460177..x_5^4 - 23.556840..x_6 - 4x_6^2 = 0,\\
&168473182572839274412 80 + 247094347138434090183x_6 - 669831938277330213420x_6^2 \\
&-544800479242800689 0183x_7 - 749487763521265964 6900x_7^2 \\
&\qquad +1300227611322393200000x_7^3 + 988220002292252000000x_7^4 = 0,\\
&309189342506522332 87x_7 + 166896192797116659 90x_7^2 - 391817091205831000x_7^3 \\
&+136751330082000 00x_7^4 - 21(-814673343709799 44 + 139769837551566234 7x_8 \\
&\qquad +817715396056767176x_8^2) = 0,\\
&-170861132x_5 + 196082425x_5^2 + 982088x_5^3 - 1838920x_5^4 + 60(-54355123 + 2782213x_8 \\
&\qquad +10443335x_8^2) = 0.
\end{aligned}
\tag{6.3}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i)$ with $i = 1, 2, 3, 4$, for system (6.3) that satisfy the above conditions namely $q^1 = (5, 1/2, 9/50, 23/5, -18/5, -9/2, -49/50, -1)$; $q^2 = (9/2,\ 19/20,\ 91/100,\ 7/5, 3,\ -17/5,\ -303/200,\ -3/2)$; $q^3 = (41/10,\ 1.208958.., 1.176604.., 2.657283.., -2.816357..,\ -31/10, -1.626433..,\ -1.613770..)$, and $q^4 = (51/10, 0.368157.., 0.315951.., 4.829311.., -3.059352.., -7/2, -1.475955.., -1.460360..)$, these four solutions generated four crossing limit cycles of type 6^+ and four crossing limit cycles of type 6^-. See these crossing limit cycles of the piecewise linear differential center (6.1) in Figure 6.1.

Here we obtain a total of eight crossing limit cycles of types 6^+ and 6^- simultaneously, with a configuration $(4, 4)$. And observed that it is possible obtain this lower bound with the configurations $(5, 3)$ or $(3, 5)$, but here we only present the example with the configuration $(4, 4)$. □

Proof of statement (b) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
\dot{x} &= 1.717686.. + 0.650612..x - 0.423688..y, & \dot{y} &= 0.850546.. + x - 0.650612..y, \text{ in } R_1,\\
\dot{x} &= 0.516832.. + 0.082481..x - 0.038759..y, & \dot{y} &= 0.179926.. + x - 0.082481..y, \text{ in } R_2,\\
\dot{x} &= 1.470269.. + 0.406982..x - 3.640154..y, & \dot{y} &= -0.122065.. + x - 0.406982..y, \text{ in } R_4,\\
\dot{x} &= 0.685228.. + 0.043300..x - 0.293631..y, & \dot{y} &= 0.017396.. + x - 0.043300..y, \text{ in } R_5.
\end{aligned}
\tag{6.4}
$$

The linear differential centers in (6.4) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(1.701093.. - 1.301224..y) + (-3.435373.. + 0.423688..y)y,\\
H_2(x,y) =& x^2 + x(0.359853.. - 0.164963..y) + (-1.033664.. + 0.038759..y)y,\\
H_4(x,y) =& x^2 + x(-0.244130.. - 0.813965..y) + y(-2.940538.. + 3.640154..y),\\
H_5(x,y) =& x^2 + x(0.034792.. - 0.086601..y) + (-1.370456.. + 0.293631..y)y,
\end{aligned}
$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 7, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, and the crossing limit cycles of type 7 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, k)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, x_8^2)$, with $-2 < x_6 < x_5 < 2$ and $-2 < x_7 < x_8 < 2$ these points must satisfy systems (5.2) and (5.5), respectively. Considering the piecewise linear differential center (6.4) and $k = 4$, systems (5.2) and (5.5) become

$$
\begin{aligned}
14.058034.. + 1.439414..x_1 - 0.134656..x_1^2 - 0.659853..x_1^3 + 0.155036..x_1^4&\\
+\frac{6}{5}x_2 - 4x_2^2 &= 0,\\
-0.783728.. - 0.311613..x_2 + x_2^2 - 0.034792..x_3 + 0.370456..x_3^2 + 0.086601..x_3^3&\\
-0.293631..x_3^4 &= 0,\\
-185.921253.. - 0.976522..x_3 - 7.762153..x_3^2 - 3.255860..x_3^3 + 14.560616..x_3^4&\\
+13.999964..x_4 - 4x_4^2 &= 0,\\
-27.849933.. - 6.804375..x_1 + 9.741494..x_1^2 + 5.204898..x_1^3 - 1.694752..x_1^4&\\
-14.015217..x_4 + 4x_4^2 &= 0,\\
4(x_5 - x_6)\left(-\frac{3}{10} + x_5 + x_6\right) &= 0,\\
-0.783728.. - 0.311613..x_6 + x_6^2 - 0.034792..x_7 + 0.370456..x_7^2&\\
+0.086601..x_7^3 - 0.293631..x_7^4 &= 0,\\
-0.976522..x_7 - 7.762153..x_7^2 - 3.255860..x_7^3 + 14.560616..x_7^4&\\
+x_8(0.976522.. + 7.762153..x_8 + 3.255860..x_8^2 - 14.560616..x_8^3) &= 0,\\
-0.783728.. - 0.311613..x_5 + x_5^2 - 0.034792..x_8 + 0.370456..x_8^2&\\
+0.086601..x_8^3 - 0.293631..x_8^4 &= 0.
\end{aligned}
\tag{6.5}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5, x_6, x_7, x_8)$ with $i = 1,2,3,4$, for system (6.5) that satisfy the above conditions. We have $q^1 = (4, -9/5, -19/10, 7/2, 1, -7/10, -9/10, 11/10)$; $q^2 = (106/25, -39/20, -1.975633.., 51/10, 1, -7/10, -9/10, 11/10)$; $q^3 = (413/100, -469/250, -1.938820.., 4.420122.., 101/100, -71/100, -941/1000, 1.132764..)$ and $q^4 = (401/100, -1.805407.., -1.902798.., 3.579564.., 101/100, -71/100, -941/1000, 1.132764..)$. These four real solutions generated four crossing limit cycles of type 6^+ and two crossing limit cycles of type 7. See these crossing limit cycles of the piecewise linear differential center (6.4) in Figure 6.2.

Here we observed that we obtain a total of six crossing limit cycles between limit cycles of type 6^+ and of type 7, moreover these six crossing limit cycles have the configuration $(4,2)$. We observe that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 7 simultaneously, could be also obtained with the configuration $(3,3)$. But if we previously fixing two limit cycles of type 6^+ after several numeric computations we could not

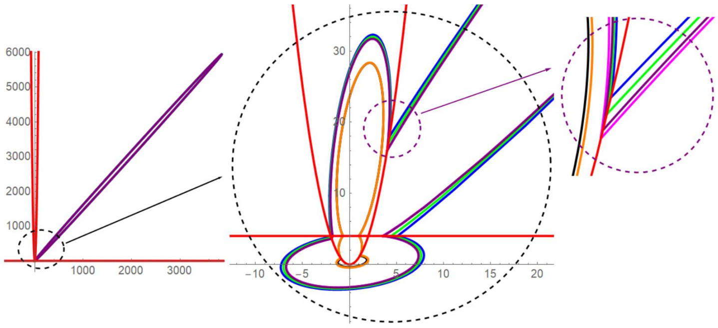

Figure 6.2: Four crossing limit cycles of type 6^+ and two crossing limit cycles of type 7 (black and orange) of the discontinuous piecewise linear differential system (6.4). These limit cycles are traveled in counterclockwise.

build a third limit cycle of type 7, then we only get those lower bound with the configuration $(4, 2)$.

We can also observe that there is a duality between the case studied in statement (e) of Theorem 1.1, where we have studied simultaneously crossing limit cycles of types 1 and 2^+ and this case, where study the crossing limit cycles of types 6^+ and 7, simultaneously. In these two cases we got the configuration $(4, 2)$. See Figures 2.6 and 6.2. $\square$

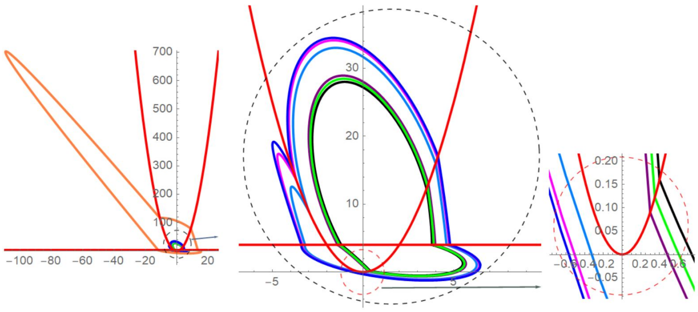

Figure 6.3: Three crossing limit cycles of type 6^+ (purple, green and black) and four crossing limit cycles of type 8 (orange, blue, magenta and light blue) of the discontinuous piecewise linear differential system (6.6). These limit cycles are traveled in counterclockwise.

Proof of statement (c) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x} = 0.212208.. - 0.051128..x - 0.004724..y, && \dot{y} = -3.713538.. + x + 0.051128..y, \text{ in } R_1,\\
&\dot{x} = 0.592855.. - 0.098217..x - 0.044462..y, && \dot{y} = -1.739750.. + x + 0.098217..y, \text{ in } R_2,\\
&\dot{x} = -0.324307.. - 0.152006..x - 0.023227..y, && \dot{y} = 2.010345.. + x + 0.152006y, \text{ in } R_3,\\
&\dot{x} = 5.173755.. - 0.530837..x - 1.789344..y, && \dot{y} = -2.823348.. + x + 0.530837..y, \text{ in } R_4,\\
&\dot{x} = 0.905547.. + \frac{9}{50}x + 0.037591..y, && \dot{y} = -2.213772.. - x - \frac{9}{50}y, \text{ in } R_5.
\end{aligned}
\tag{6.6}
$$

The linear differential centers in (6.6) have the first integrals

$$
\begin{aligned}
H_1(x,y) =&92350000x^2+2y(-19597489+218145y)+x(-685890524+9443461y),\\
H_2(x,y) =&x(-2350427721+675507095x)+2(-400478067+66346510x)y+30034700y^2,\\
H_3(x,y) =&x^2+x(4.020691..+0.304014..y)+(0.648615..+0.023227..y)y,\\
H_4(x,y) =&2.248715..\times 10^{16}x^2-5x(2.539563..\times 10^{16}-4.774807..\times 10^{15}y)\\
&+y(-2.326860..\times 10^{17}+4.023727..\times 10^{16}y),\\
H_5(x,y) =&-5.437818..\times 10^{22}x^2+6x(-4.012698..\times 10^{22}-3.262691..\times 10^{21}y)\\
&+5(-1.969681..\times 10^{22}-4.088345..\times 10^{20}y)y,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 6^+ and 8, simultaneously, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1=(x_1,x_1^2)$, $p_2=(x_2,k)$, $p_3=(x_3,x_3^2)$ and $p_4=(x_4,k)$, with $-2<x_2<2<x_4$ and $-2<x_3<2<x_1$, and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5=(x_5,x_5^2)$, $p_6=(x_6,x_6^2)$, $p_7=(x_7,k)$ and $p_8=(x_8,k)$, with $x_7<-2<2<x_8$ and $x_6<-2<2<x_5$, these points must satisfy systems (5.2) and (5.8), respectively. Considering the piecewise linear differential center (6.6) and $k=4$, systems (5.2) and (5.8) become

$$
\begin{aligned}
&16.125777..-13.918004..x_1-0.742843..x_1^2+0.785738..x_1^3+0.177849..x_1^4\\
&\qquad +10.775049..x_2-4x_2^2=0,\\
&31.383400..+23.470181..x_2+4x_2^2-17.710181..x_3-11.244381..x_3^2-\frac{36}{25}x_3^3\\
&\qquad -0.150367..x_3^4=0,\\
&51.042105..-22.586789..x_3-37.390043..x_3^2+4.246697..x_3^3+7.157379..x_3^4\\
&\qquad +5.599999..x_4-4x_4^2=0,\\
&-6.488327..+29.708306..x_1-2.302329..x_1^2-0.409029..x_1^3-0.018897..x_1^4\\
&\qquad -28.072189..x_4+4x_4^2=0,\\
&-149799272-648116680x_8+92350000x_8^2+685890524x_5-53155022x_5^2\\
&\qquad -9443461x_5^3-436290x_5^4=0,\\
&-2350427721x_5-125449039x_5^2+132693020x_5^3+30034700x_5^4\\
&\qquad +x_6(2350427721+125449039x_6-132693020x_6^2-30034700x_6^3)=0,\\
&-11.864396..+16.082766..x_6+6.594461..x_6^2+1.216054..x_6^3+0.092909..x_6^4\\
&\qquad -20.946982..x_7-4x_7^2=0,\\
&(x_8-x_7)(-7+5x_8+5x_7)=0.
\end{aligned}
\tag{6.7}
$$

We have four real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i,x_5^i,x_6^i,x_7^i,x_8^i)$ with $i=1,2,3,4$, for system (6.7) that satisfy the above conditions. We have $q^1=(7/2,-6/5,2/5,19/5,4,-3,-16/5,23/5)$; $q^2=(18/5,\ -7/5,\ 3/10,\ 199/50,\ 41/10,\ -37/10,\ -3351/1000,\ 4751/1000)$; $q^3=(71/20,\ -1.299400..,\ 7/20,\ 3.893976..,\ 4.132430..,\ -3.871790..,\ -17/5,\ 24/5)$ and $q^4=(71/20,\ -1.299400..,7/20,3.893976..,178349/20000,108083/10000,-119/10,133/10)$. These four real solutions generated three crossing limit cycles of type 6^+ and four crossing limit cycle of type 8. See these crossing limit cycles of the piecewise linear differential center (6.6) in Figure 6.3.

Here we observed that we obtain a total of seven crossing limit cycles between limit cycles of type 6^+ and of type 8, moreover in this example, the seven crossing limit cycles have the configuration $(3,4)$. We observe that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 8 simultaneously, could be also obtained with the configurations $(4,3)$. And we obtain a example with this configuration in the proof of statement (b) of Theorem 1.6 with piecewise linear differential center (7.3), see Figure 7.2. $\square$

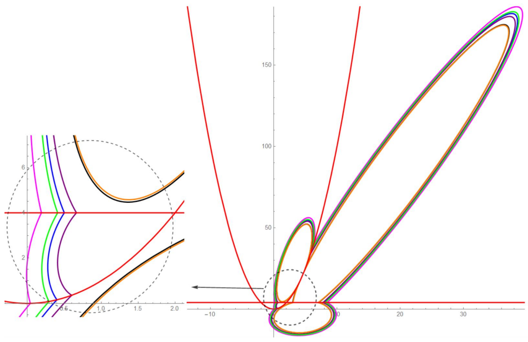

Figure 6.4: Four crossing limit cycles of type 6^+ and two crossing limit cycles of type 9^+ (black and orange) of the discontinuous piecewise linear differential system (6.8). These limit cycles are traveled in counterclockwise.

Proof of statement (d) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
\dot{x} &= -0.478750.. + 0.183274..x - 0.037189..y, & \dot{y} &= -4.300673.. + x - 0.183274..y, \text{ in } R_1,\\
\dot{x} &= 0.122511.. + 0.079715..x - 0.013506..y, & \dot{y} &= -1.007263.. + x - 0.079715..y, \text{ in } R_2,\\
\dot{x} &= -1.261810.. + 0.053348..x - 0.212413..y, & \dot{y} &= -4.836606.. + x - 0.053348..y, \text{ in } R_4,\\
\dot{x} &= 0.060157.. + 0.062627..x - 0.047729..y, & \dot{y} &= -0.739728.. + x - 0.062627..y, \text{ in } R_5.
\end{aligned}
\tag{6.8}
$$

The linear differential centers in (6.8) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-8.601346.. - 0.366548..y) + (0.957501401147845' + 0.037189..y)y,\\
H_2(x,y) =& x^2 + x(-2.014527.. - 0.159430..y) + (-0.245022.. + 0.013506..y)y,\\
H_4(x,y) =& x^2 + x(-9.673213.. - 0.106696..y) + (2.523620.. + 0.212413..y)y,\\
H_5(x,y) =& x^2 + x(-1.479456.. - 0.125255..y) + (-0.120314.. + 0.047729..y)y,
\end{aligned}
$$

respectively. In order to have simultaneously crossing limit cycles of types 6^+ and 9^+, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_4$ and $-2 < x_3 < 2 < x_1$, and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $2 < x_6 < x_5$ and $2 < x_7 < x_8$, these points must satisfy systems (5.2) and (5.11), respectively. Considering the piecewise linear differential center (6.8) and $k = 4$, systems (5.2) and (5.11) become

$$
\begin{aligned}
3.055923..+x_1(-8.058108..+x_1(3.019909..+(-0.637722..+0.054027..x_1)x_1))&\\
+(10.608997..-4x_2)x_2&=0,\\
0.282412..+(-1.980480..+x_2)x_2+x_3(1.479456..+x_3(-0.879685..+(0.125255..&\\
-0.047729..x_3)x_3))&=0,\\
-53.972411..+x_3(-38.692854..+x_3(14.094480..+(-0.426786..+0.849655..x_3)x_3))&\\
+(40.4000000..-3.999999..x_4)x_4&=0,\\
17.700131..+x_1(34.405384..+x_1(-7.8300056..+(1.466193..-0.148756..x_1)x_1))&\\
+x_4(-40.270159..+4x_4)&=0,\\
-8.058108..x_5+3.019909..x_5^2-0.637722..x_5^3+0.054027..x_5^4+x_6(8.058108..&\\
-3.019909..x_6+0.637722..x_6^2-0.054027..x_6^3)&=0,\\
-17.700131..-34.405384..x_6+7.830005..x_6^2-1.466193..x_6^3+0.148756..x_6^4&\\
+40.270159..x_7-4x_7^2&=0,\\
4(x_7-x_8)(-10.100000..+x_7+x_8)&=0,\\
17.700131..+34.405384..x_5-7.830005..x_5^2+1.466193..x_5^3-0.148756..x_5^4&\\
-40.270159..x_8+4x_8^2&=0.
\end{aligned}
\tag{6.9}
$$

We have four real solutions $q^i=(x_1^i,x_2^i,x_3^i,x_4^i,x_5,x_6,x_7,x_8)$ with $i=1,2,3,4$, for system (6.9) that satisfy the above conditions. We have $q^1=(6,1/2,4/10,8,5,14/5,3,71/10)$; $q^2=(317/50,\ 19/100,\ 1/25,\ 423/50,\ 5,\ 14/5,\ 3,\ 71/10)$; $q^3=(291/50,\ 0.664193..,\ 3/5,\ 7.554404..,487/100,3.986608..,3.058022..,7.041977..)$ and $q^4=(61/10,0.409425..,0.293958..,\ 8.128324..,487/100,3.986608..,3.058022..,\ 7.041977..)$ These four real solutions generated four crossing limit cycles of type 6^+ and two crossing limit cycles of type 9^+. See these crossing limit cycles of the piecewise linear differential center (6.8) in Figure 6.4.

Here we obtain a total of six crossing limit cycles between limit cycles of type 6^+ and of type 9^+, moreover these six crossing limit cycles have the configuration $(4,2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 6^+ and 9^+ simultaneously, could be also obtained with the configuration $(3,3)$. But if we build two crossing limit cycles of type 6^+ and two of type 9^+, simultaneously, we have that all the parameters that appear in system (5.11) are determined, where this system is such that generated limit cycles of type 9^+, then it is no possible to build a third crossing limit cycle of type 9^+ and therefore we can not obtain the configuration $(3,3)$. $\square$

Proof of statement (e) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x}=-0.147861..+0.083875..x-0.018000..y, && \dot{y}=-3.106437..+x-0.083875..y, \text{ in } R_1,\\
&\dot{x}=\frac{7769951}{9492348}+\frac{176465}{2373087}x-\frac{204250}{2373087}y, && \dot{y}=\frac{6997939}{47461740}+x-\frac{176465}{2373087}y, \text{ in } R_2,\\
&\dot{x}=-0.284659..-0.174915..x-0.046689..y, && \dot{y}=1.660380..+x+0.174915..y, \text{ in } R_3,\\
&\dot{x}=-\frac{3871251}{31913000}+\frac{3}{10}x-\frac{4335}{31913}y, && \dot{y}=-\frac{19}{20}+x-\frac{3}{10}y, \text{ in } R_4,\\
&\dot{x}=0.206531..+0.150466..x-0.054352..y, && \dot{y}=0.451143..+x-0.150466..y, \text{ in } R_5.
\end{aligned}
\tag{6.10}
$$

The linear differential centers in (6.10) have the first integrals

$$
\begin{aligned}
H_1(x,y) =&(58546435625x^2 + 4y(4328392296 + 263466775y) - 15x(24249448597 \\
&+ 654747306y),\\
H_2(x,y) =&x(6997939 + 23730870x) - 5(7769951 + 705860x)y + 2042500y^2,\\
H_3(x,y) =&1.054579.. \times 10^{-58}(3.792980.. \times 10^{58}x^2 + y(2.159417.. \times 10^{58}\\
&+ 1.770939.. \times 10^{57}y) + x(1.259558..10^{59} + 1.326899.. \times 10^{58}y)),\\
H_4(x,y) =&4x^2 + \frac{2}{5}x(19-6y) + \frac{3y(1290417 + 722500y)}{3989125},\\
H_5(x,y) =&16x(472818597 + 524021995x) - 75(46176919 + 33641680x)y + 455712500y^2,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 7 and 8, simultaneously, such that the crossing limit cycles of type 7 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, k)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, x_4^2)$, with $-2 < x_2 < x_1 < 2$ and $-2 < x_3 < x_4 < 2$, and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $x_6 < -2 < 2 < x_5$ and $x_7 < -2 < 2 < x_8$, these points must satisfy systems (5.5) and (5.8), respectively. Considering the piecewise linear differential center (6.10) and $k = 4$, systems (5.5) and (5.8) become

$$
\begin{aligned}
4(x_1 - x_2)\left(-\frac{3}{10} + x_1 + x_2\right) &= 0,\\
-6561675700 - 2527406448x_2 + 8384351920x_2^2 - 7565097552x_3&\\
-4921082995x_3^2 + 2523126000x_3^3 - 455712500x_3^4 &= 0,\\
30317350x_3 + 19827751x_3^2 - 9573900x_3^3 + 2167500x_3^4&\\
-x_4(30317350 + 19827751x_4 - 9573900x_4^2 + 2167500x_4^3) &= 0,\\
6561675700 + 2527406448x_1 - 8384351920x_1^2 + 7565097552x_4&\\
+4921082995x_4^2 - 2523126000x_4^3 + 455712500x_4^4 &= 0,\\
86116150336 - 403026567315x_8 + 58546435625x_8^2 + 363741728955x_5&\\
-75860004809x_5^2 + 9821209590x_5^3 - 1053867100x_5^4 &= 0,\\
6997939x_5 - 15118885x_5^2 - 3529300x_5^3 + 2042500x_5^4 + x_6(-6997939&\\
+15118885x_6 + 3529300x_6^2 - 2042500x_6^3) &= 0,\\
-1.030050.. + 8(1.660379.. + 0.284660..x_6)x_6 + 4(1 + 0.174915..x_6)^2x_6^2&\\
+0.064378..x_6^4 + 8(-1.138640.. - 1.660379..x_7) - 4(0.699661.. + x_7)^2 &= 0,\\
-4(x_8 - x_7)\left(-\frac{1}{2} + x_8 + x_7\right) &= 0.
\end{aligned}
\tag{6.11}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i)$ with $i = 1,2,3,4$, for system (6.11) that satisfy the above conditions. We have $q^1 = (1, -7/10, -9/10, -1/10, 37/10, -5/2, -3, 7/2)$; $q^2 = (1, -7/10, -9/10, -1/10, 4, -29/10, -33/10, 19/5)$; $q^3 = (11/10, -8/10, -26/25, 1/10, 21/5, -157/100, -7/2, 4)$ and $q^4 = (1.194602.., -0.894602.., -1.147986.., 0.273096.., 87/20, -3.312719.., -3653/1000, 4153/1000)$. These four real solutions generated three crossing limit cycles of type 7 and four crossing limit cycle of type 8. See these crossing limit cycles of the piecewise linear differential center (6.10) in Figure 6.5.

Here we obtain a total of seven crossing limit cycles between limit cycles of type 7 and of type 8, moreover these seven crossing limit cycles have the configuration $(3,4)$. By our numer-

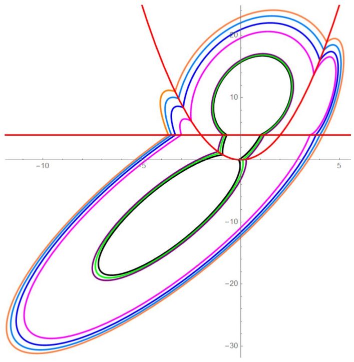

Figure 6.5: Three crossing limit cycles of type 7 (purple, green and black) and four crossing limit cycles of type 8 of the discontinuous piecewise linear differential system (6.10). These limit cycles are traveled in counterclockwise.

ical computations we observed that this lower bound for the maximum number of crossing limit cycles of types 7 and 8 simultaneously, could not be obtained with the configuration $(4,3)$, because in the statement (b) of Theorem 1.4 we only got three crossing limit cycle of type 7. □

Proof of statement (f) of Theorem 1.5. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x} = -0.224106.. + 0.256615..x - 0.075244..y, && \dot{y} = -3.489877.. + x - 0.256615..y, \text{ in } R_1,\\
&\dot{x} = 33.031408.. - \frac{x}{2} - 5.321982..y, && \dot{y} = -816.418879.. + x + \frac{y}{2}, \text{ in } R_2,\\
&\dot{x} = -0.151463.. - 0.173662..x - 0.047290..y, && \dot{y} = 0.297861.. + x + 0.173662..y, \text{ in } R_3,\\
&\dot{x} = 2 + \frac{x}{20} - \frac{13}{200}y, && \dot{y} = -\frac{111}{20} + x - \frac{y}{20}, \text{ in } R_4.
\end{aligned}
\tag{6.12}
$$

The linear differential centers in (6.12) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-6.979755.. - 0.513231..y) + (0.448213.. + 0.075244..y)y,\\
H_2(x,y) =& x^2 + x(-1632.837759.. + y) + y(-66.062816.. + 5.321982..y),\\
H_3(x,y) =& x^2 + x(0.595723.. + 0.347324..y) + (0.302926.. + 0.047290..y)y,\\
H_4(x,y) =& 4x^2 - 16y + \frac{13}{50}y^2 - \frac{2}{5}x(111 + y),
\end{aligned}
$$

respectively. In order to have simultaneously crossing limit cycles of types 8 and 9^+, such that the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, x_2^2)$, $p_3 = (x_3, k)$ and $p_4 = (x_4, k)$, with $x_2 < -2 < 2 < x_1$ and $x_3 < -2 < 2 < x_4$, and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $2 < x_6 < x_5$ and $2 < x_7 < x_8$, these points must satisfy systems (5.8) and (5.11), respectively. Considering the piecewise linear differential center (6.12) and $k = 4$, systems (5.8) and (5.11) become

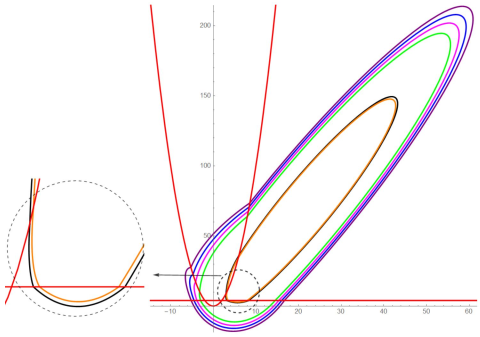

Figure 6.6: Four crossing limit cycles of type 8 and two crossing limit cycles of type 9^+ (black and orange) of the discontinuous piecewise linear differential system (6.12). These limit cycles are traveled in counterclockwise.

$$\begin{aligned}
&-6531.351039..x_1 - 260.251264..x_1^2 + 4x_1^3 + 21.287931..x_1^4 + x_2(6531.351039.. \\
&\qquad +260.251264..x_2 - 4x_2^2 - 21.287931..x_2^3) = 0, \\
&-7.873414.. + 2.382895..x_2 + 5.211706..x_2^2 + 1.389297..x_2^3 + 0.189161..x_2^4 \\
&\qquad -7.940084..x_3 - 4x_3^2 = 0, \\
&4(x_3 - x_4)\left(-\frac{23}{2} + x_3 + x_4\right) = 0, \\
&11.987037.. + 27.919023..x_1 - 5.792854..x_1^2 + 2.052924..x_1^3 - 0.300976..x_1^4 \\
&\qquad -36.130722..x_4 + 4x_4^2 = 0 \\
&x_5(-6531.351039.. + x_5(-260.251264.. + x_5(4 + 21.287931..x_5))) + x_6(6531.351039.. \\
&\qquad +x_6(260.251264.. + (-4 - 21.287931..x_6)x_6)) = 0, \\
&-11.987037.. + x_6(-27.919023.. + x_6(5.792854.. + (-2.052924.. + 0.300976..x_6)x_6)) \\
&\qquad +(36.130722.. - 4x_7)x_7 = 0, \\
&4(x_7 - x_8)\left(-\frac{23}{2} + x_7 + x_8\right) = 0, \\
&11.987037.. + x_5(27.919023.. + x_5(-5.792854.. + (2.052924.. - 0.300976..x_5)x_5)) \\
&\qquad +x_8(-36.130722.. + 4x_8) = 0,
\end{aligned} \tag{6.13}$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5, x_6, x_7, x_8)$ with $i = 1, 2, 3, 4$, for system (6.13) that satisfy the above conditions. We have $q^1 = (8,\ -16/5,\ -3,\ 29/2,\ 6,\ 3, 16/5, 83/10)$; $q^2 = (823/100,\ -4.136449..,\ -3.840062..,\ 15.340062..,\ 6,\ 3,\ 16/5,\ 83/10)$; $q^3 = (\ 841/100, -4.748093.., -4.516514.., 16.016514..587/100, 3.203924.., 177/50, 199/25)$ and $q^4 = (429/50, -5.249123.., -5.170790.., 16.670790.., 587/100, 3.203924.., 177/50, 199/25)$. These four real solutions generated four crossing limit cycles of type 8 and two crossing limit cycles of type 9^+. See these crossing limit cycles of the piecewise linear differential center (6.12) in Figure 6.6.

Here we obtain a total of six crossing limit cycles between limit cycles of type 8 and of type 9^+, moreover these six crossing limit cycles have the configuration $(4, 2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 8 and 9^+ simultaneously, could be also obtained with the configurations $(3, 3)$. But if we build two crossing limit cycles of type 8 and two of type 9^+, simultaneously, we have that all the parameters that appear in system (5.11) are determined, where this system is such that generated limit cycles of type 9^+, then it is no possible to build a third crossing limit cycle of type 9^+ and therefore we can not obtain the configurations $(3, 3)$.

We can also observe that there is a duality between the case studied in statement (c) of Theorem 1.3, where we have studied simultaneously crossing limit cycles of types 4 and 5 and this case, where study the crossing limit cycles of types 8 and 9^+, simultaneously. In these two cases we got the configuration $(4, 2)$. See Figures 4.3 and 6.6. □

7 Proof of Theorem 1.6

Proof of statement (a) of Theorem 1.6. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x} = -0.107128.. + 0.268308..x - 0.095415..y, && \dot{y} = -2.390037.. + x - 0.268308..y, \text{ in } R_1,\\
&\dot{x} = 0.492346.. + 0.144928..x - 0.061289..y, && \dot{y} = 0.429713.. + x - 0.144928..y, \text{ in } R_2,\\
&\dot{x} = 1.394400.. + 0.300769..x - 0.091362..y, && \dot{y} = 2.707746.. + x - 0.300769..y, \text{ in } R_3,\\
&\dot{x} = 0.976917.. + 0.400189..x - 4.241691..y, && \dot{y} = -0.349243.. + x - 0.400189..y, \text{ in } R_4,\\
&\dot{x} = 0.685228.. + 0.043300..x - 0.293631..y, && \dot{y} = 0.017396.. + x - 0.043300..y, \text{ in } R_5.
\end{aligned} \tag{7.1}
$$

The linear differential centers in (7.1) have the first integrals

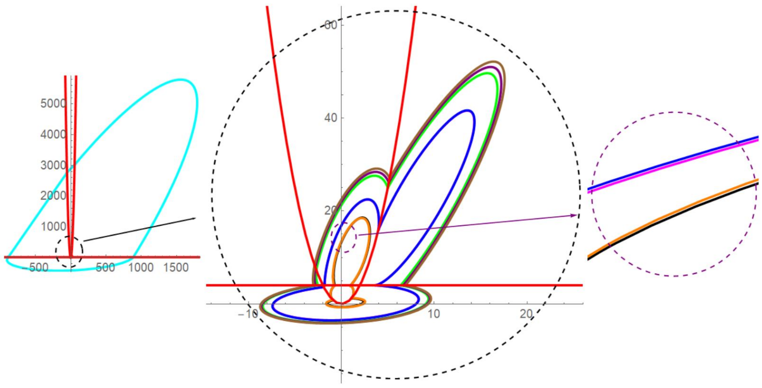

Figure 7.1: Two crossing limit cycle of type 6^+ (magenta and blue), two crossing limit cycles of type 7 (black and orange) and four crossing limit cycles of type 8 (green, purple, brown and cyan) of the discontinuous piecewise linear differential system (7.1). These limit cycles are traveled in counterclockwise.

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-4.780074.. - 0.536616..y) + (0.214257.. + 0.095415..y)y,\\
H_2(x,y) =& x^2 + x(0.859427.. - 0.289856..y) + (-0.984693.. + 0.061289..y)y,\\
H_3(x,y) =& x^2 + x(5.415492.. - 0.601538..y) + (-2.788801.. + 0.091362..y)y,\\
H_4(x,y) =& x^2 + x(-0.698486.. - 0.800378..y) + y(-1.953834.. + 4.241691..y),\\
H_5(x,y) =& x^2 + x(0.034792.. - 0.086601..y) + (-1.370456.. + 0.293631..y)y,
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 6^+, 7 and 8 simultaneously, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, the crossing limit cycles of type 7 intersect the discontinuity curve

$\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, k)$, $p_6 = (x_6, k)$, $p_7 = (x_7, x_7^2)$ and $p_8 = (x_8, x_8^2)$, with $x_5 < -2 < x_7 < 2$ and $x_6 < -2 < x_8 < 2$ and the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_9 = (x_9, x_9^2)$, $p_{10} = (x_{10}, x_{10}^2)$, $p_{11} = (x_{11}, k)$ and $p_{12} = (x_{12}, k)$, with $x_{10} < -2 < 2 < x_9$ and $x_{11} < -2 < 2 < x_{12}$ these points must satisfy systems (5.2), (5.5) and (5.8) respectively. Considering the piecewise linear differential center (7.1) and $k = 4$, systems (5.2), (5.5) and (5.8) become

$$
\begin{aligned}
11.832571.. + 3.437710..x_1 + 0.061227..x_1^2 - 1.159427..x_1^3 + 0.245157..x_1^4 &\\
+1.200000..x_2 - 3.999999..x_2^2 &= 0,\\
-0.783728.. - 0.311613..x_2 + x_2^2 - 0.034792..x_3 + 0.370456..x_3^2 + 0.086601..x_3^3 &\\
-0.293631..x_3^4 &= 0,\\
-240.206876.. - 2.793946..x_3 - 3.815339..x_3^2 - 3.201513..x_3^3 + 16.966764..x_3^4 &\\
+15.600000..x_4 - 4x_4^2 &= 0,\\
9.534728.. + 19.120296..x_1 - 4.857030..x_1^2 + 2.146465..x_1^3 - 0.381662..x_1^4 &\\
-27.706159..x_4 + 4x_4^2 &= 0,\\
4(x_5 - x_6)(-0.300000.. + x_5 + x_6) &= 0,\\
-0.783728.. - 0.311613..x_6 + x_6^2 - 0.034792..x_7 + 0.370456..x_7^2 + 0.086601..x_7^3 &\\
-0.293631..x_7^4 &= 0,\\
-2.793946..x_7 - 3.815339..x_7^2 - 3.201513..x_7^3 + 16.966764..x_7^4 + x_8(2.793946.. &\\
+3.815339..x_8 + 3.201513..x_8^2 - 16.966764..x_8^3) &= 0,\\
-0.783728.. - 0.311613..x_5 + x5^2 - 0.034792..x_8 + 0.370456..x_8^2 &\\
+0.086601..x_8^3 - 0.293631..x_8^4 &= 0,\\
-3.437710..x_{10} - 0.061227..x_{10}^2 + 1.159427..x_{10}^3 - 0.245157..x_{10}^4 &\\
+x_9(3.437710.. + 0.061227..x_9 - 1.159427..x_9^2 + 0.245157..x_9^3) &= 0,\\
38.773655.. + 21.661968..x_{10} - 7.155207..x_{10}^2 - 2.406152..x_{10}^3 + 0.365448..x_{10}^4 &\\
-12.037359..x_{11} - 4x_{11}^2 &= 0,\\
4(x_{11} - x_{12})(-3.900000.. + x_{11} + x_{12}) &= 0,\\
2.383682.. - 6.926539..x_{12} + x_{12}^2 + 4.780074..x_9 - 1.214257..x_9^2 &\\
+0.536616..x_9^3 - 0.095415..x_9^4 &= 0.
\end{aligned}
\tag{7.2}
$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i, x_9^i, x_{10}^i, x_{11}^i, x_{12}^i)$ with $i = 1, 2, 3, 4$, for system (7.2) that satisfy the above conditions, namely $q^1 = (4, -9/5, -19/10, 7/2,$ $1,\ -7/10,\ -9/10, 11/10, 5,\ -27/10, -5/2,\ 32/5)$; $q^2 = (2007/500,\ -181/100,\ -1.905170..,$ $3.692535.., 101/100, -71/100, -941/1000, 1.132764.., 511/100, -2.805313.., -139/50, 167/25)$; $q^3 = (2007/500, -181/100, -1.905170.., 3.692535.., 101/100, -71/100, -941/1000, 1.132764..,$ $26/5,\ -2.891869.., -3.012824..,\ 6.912824..)$ and $q^4 = (\ 2007/500,\ -181/100,\ -1.905170..,$ $3.692535..,\ 101/100,\ -71/100, -941/1000,\ 1.132764..,\ 549/10,\ -52.535582.., -883.528310..,$ $887.428310..)$. These four real solutions generated two crossing limit cycles of type 6^+, two crossing limit cycles of type 7 and four crossing limit cycles of type 8. See these crossing limit cycles of the piecewise linear differential center (7.1) in Figure 7.1.

Here we obtain a total of eight crossing limit cycles between limit cycles of types 6^+, 7 and 8, moreover these eight crossing limit cycles have the configuration $(2, 2, 4)$, this is 2-crossing limit cycles of type 6^+, 2-crossing limit cycles of type 7 and 4-crossing limit of type 8. We observed that this lower bound for the maximum number of crossing limit cycles of types 6^+, 7 and 8 simultaneously, could be also obtained with other configurations. But if we build two crossing limit cycles of each type we obtain that all parameters of systems (5.2) and (5.5) are determined, and these systems are such that generated the limit cycles of types 6^+ and 7, then we can not build more than two crossing limit cycles of types 6^+ or 7 when we have previously fixed two crossing limit cycles of each type. Then we only obtain the configuration obtained here, namely $(2, 2, 4)$. □

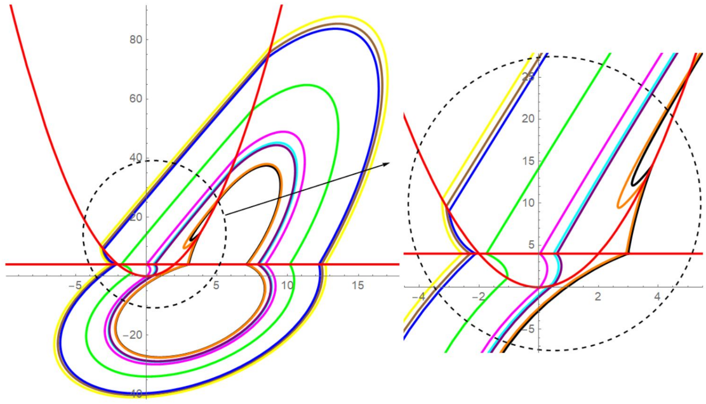

Figure 7.2: Four crossing limit cycles of type 6^+ (green, magenta, cyan and purple), three crossing limit cycles of type 8 (yellow, brown and blue) and two crossing limit cycles of type 9^+ (black and orange) of the discontinuous piecewise linear differential system (7.3). These limit cycles are traveled in counterclockwise.

Proof of statement (b) of Theorem 1.6. We consider the following discontinuous piecewise linear differential system

$$
\begin{aligned}
&\dot{x} = -0.312756.. + 0.105676..x - 0.022483..y, \quad &&\dot{y} = -4.523476.. + x - 0.105676..y, \text{ in } R_1,\\
&\dot{x} = -0.158662.. + 0.176712..x - 0.031977..y, \quad &&\dot{y} = -1.018470.. + x - 0.176712..y, \text{ in } R_2,\\
&\dot{x} = 0.893671.. + \frac{x}{10} - 0.055338..y, \quad &&\dot{y} = 1.647781.. + x - \frac{y}{10}, \text{ in } R_3,\\
&\dot{x} = -1.521810.. + 0.129660..x - 0.102089..y, \quad &&\dot{y} = -4.531357.. + x - 0.129660..y, \text{ in } R_4,\\
&\dot{x} = 2.392166.. + 0.863445..x - 1.210282..y, \quad &&\dot{y} = 11.457801.. + x - 0.863445..y, \text{ in } R_5.
\end{aligned} \tag{7.3}
$$

The linear differential centers in (7.3) have the first integrals

$$
\begin{aligned}
H_1(x,y) =& x^2 + x(-9.046952.. - 0.211353..y) + (0.625512.. + 0.022483..y)y,\\
H_2(x,y) =& x^2 + x(-2.03694.. - 0.353424..y) + (0.317325.. + 0.031977..y)y,\\
H_3(x,y) =& x^2 + x\left(3.295563.. - \frac{y}{5}\right) + (-1.787342.. + 0.055338..y)y,\\
H_4(x,y) =& x^2 + x(-9.062715.. - 0.259321..y) + (3.043621.. + 0.102089..y)y,\\
H_5(x,y) =& x^2 + x(22.915603.. - 1.726890..y) + y(-4.784333.. + 1.210282..y),
\end{aligned}
$$

respectively. In order to have crossing limit cycles of types 6^+, 8 and 9^+ simultaneously, such that the crossing limit cycles of type 6^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_1 = (x_1, x_1^2)$, $p_2 = (x_2, k)$, $p_3 = (x_3, x_3^2)$ and $p_4 = (x_4, k)$, with $-2 < x_2 < 2 < x_1$ and $-2 < x_3 < 2 < x_4$, the crossing limit cycles of type 8 intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_5 = (x_5, x_5^2)$, $p_6 = (x_6, x_6^2)$, $p_7 = (x_7, k)$ and $p_8 = (x_8, k)$, with $x_6 < -2 < 2 < x_5$ and $x_7 < -2 < 2 < x_8$ and the crossing limit cycles of type 9^+ intersect the discontinuity curve $\tilde{\Sigma}_k$ in four different points $p_9 = (x_9, x_9^2)$, $p_{10} = (x_{10}, x_{10}^2)$, $p_{11} = (x_{11}, k)$ and $p_{12} = (x_{12}, k)$, with $2 < x_{10} < x_9$ and $2 < x_{11} < x_{12}$ these points must satisfy systems (5.2), (5.8) and (5.11) respectively. Considering the piecewise linear differential center (7.3) and $k = 4$, systems (5.2), (5.8) and (5.11) become

$$\begin{aligned}
&-7.123782.. + x_1(-8.147767.. + x_1(5.269300.. + (-1.413698.. + 0.127911..x_1)x_1)) \\
&\qquad +(13.802561.. - 4x_2)x_2 = 0, \\
&0.227189.. + x_2(16.008041.. + x_2) + x_3(-22.915603.. + x_3(3.784333.. + (1.726890.. \\
&\qquad -1.210282..x_3)x_3)) = 0, \\
&-55.231640.. + x_3(-36.250863.. + x_3(16.174485.. + (-1.037284.. + 0.408356..x_3)x_3)) \\
&\qquad + \left(\frac{202}{5} - 4x_4\right) x_4 = 0, \\
&11.447141.. + x_1(36.187810.. + x_1(-6.502051.. + (0.845414.. - 0.089933..x_1)x_1)) \\
&\qquad +x_4(-39.569467.. + 4x_4) = 0, \\
&x_5(-8.147767.. + x_5(5.269300.. + (-1.413698.. + 0.127911..x_5)x_5)) + x_6(8.147767.. \\
&\qquad +x_6(-5.269300.. + (1.413698.. - 0.127911..x_6)x_6)) = 0, \\
&25.055786.. + x_6(13.182255.. + x_6(-3.149369.. + \left(-\frac{4}{5} + 0.221355..x_6\right) x_6)) \\
&\qquad +(-9.982255.. - 4x_7)x_7 = 0, \\
&4(x_7 - x_8)\left(-\frac{101}{10} + x_7 + x_8\right) = 0, \\
&11.447141.. + x_5(36.187810.. + x_5(-6.502051.. + (0.845414.. - 0.089933..x_5)x_5)) \\
&\qquad +x_8(-39.569467.. + 4x_8) = 0, \\
&x_{10}(8.147767.. + x_{10}(-5.269300.. + (1.413698.. - 0.127911..x_{10})x_{10})) \\
&\qquad +x_9(-8.147767.. + x_9(5.269300.. + (-1.413698.. + 0.127911..x_9)x_9)) = 0, \\
&-11.447141.. + x_{10}(-36.187810.. + x_{10}(6.502051.. + (-0.845414.. \\
&\qquad +0.089933..x_{10})x_{10})) + (39.569467.. - 4x_{11})x_{11} = 0, \\
&4(x_{11} - x_{12})\left(-\frac{101}{10} + x_{11} + x_{12}\right) = 0, \\
&2.861785.. + (-9.892366.. + x_{12})x_{12} + x_9(9.046952.. + x_9(-1.625512.. + (0.211353.. \\
&\qquad -0.022483..x_9)x_9)) = 0.
\end{aligned} \tag{7.4}$$

We have four real solutions $q^i = (x_1^i, x_2^i, x_3^i, x_4^i, x_5^i, x_6^i, x_7^i, x_8^i, x_9^i, x_{10}^i, x_{11}^i, x_{12}^i)$ with $i = 1, 2, 3, 4$, for system (7.4) that satisfy the above conditions, namely $q^1 = (6, 1/2, 2/5, 8, 87/10, -31/10, -23/10, 62/5, 5, 19/5, 3, 71/10)$; $q^2 = (317/50, 0.042569.., 1/25, 8.417274.., 861/100, -3.007479.., -2.117234.., 12.217234.., 5, 19/5, 3, 71/10)$; $q^3 = (1479/250, -0.610424.., -1/2, 7.904488.., 883/100, -3.233408.., -2.568105.., 12.668105.., 51/10, 3.582979.., 2.936322.., 7.163677..)$, and $q^4 = (15/2, -1.752776.., -1.049779.., 10.157706.., 883/100, -3.233408.., -2.568105.., 12.668105.., 51/10, 3.582979.., 2.936322.., 7.163677..)$ these four solutions generated four crossing limit cycles of type 6^+, three crossing limit cycles of type 8 and two crossing limit cycle of type 9^+. See these crossing limit cycles of the piecewise linear differential center (7.3) in Figure 7.2.

Here we obtain a total of nine crossing limit cycles between limit cycles of types 6^+, 8 and 9^+, moreover these nine crossing limit cycles have the configuration $(4, 3, 2)$. We observed that this lower bound for the maximum number of crossing limit cycles of types 6^+, 8 and

9^+ simultaneously, could be also obtained with other configurations. When we build two crossing limit cycles of each type we obtain that system (5.11) has all parameters determined, and therefore we can not build a third crossing limit cycle of type 9^+. Systems (5.2), (5.8) which generated the limit cycles of types 8 and 9^+ would still have free parameters and it is possible verify that we can have the configurations $(4,3,2)$ or $(3,4,2)$. Here we have illustrated the configuration $(4,3,2)$. □

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