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PERIODIC ORBITS OF A HAMILTONIAN SYSTEM RELATED WITH THE FRIEDMANN-ROBERTSON-WALKER SYSTEM IN ROTATING COORDINATES

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ABSTRACT. We provide sufficient conditions on the four parameters of a Hamiltonian system, related with the Friedmann-Robertson-Walker Hamiltonian system in a rotating reference frame, which guarantee the existence of 12 continuous families of periodic orbits, parameterized by the values of the Hamiltonian, which born at the equilibrium point localized at the origin of coordinates. The main tool for finding analytically these families of periodic orbits is the averaging theory for computing periodic orbits adapted to the Hamiltonian systems. The technique here used can be applied to arbitrary Hamiltonian systems.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

In astrophysics the study of the dynamics of the universe is an area where the application of the techniques of the dynamical systems provide good results, mainly in galactic dynamics, see the articles [2, 9, 13, 14, 19] and the references cited therein.

Recently it has been detected numerical and analytical existence of chaotic motion in the following simplified version of the Friedmann-Robertson-Walker Hamiltonian

$$H = \frac{1}{2}(p_Y^2 - p_X^2) + \frac{1}{2}(Y^2 - X^2) + \frac{b}{2}\,X^2Y^2, \tag{1}$$

introduced by Calzeta and Hasi in [6]. In fact this model is too simplified in order to be considered realistic, but it is interesting due to its simplicity and for showing the existence of chaos in cosmology, look for more details in [6]. Hawking [7] and Page [12] used analogous models to analyze the relation between the thermodynamic arrow of time and the cosmology.

The usual potentials in galactic dynamics are of the form $V(x^2, y^2)$, see the article [15] and the previous mentioned articles on galactic dynamics. These potentials show a reflection symmetry with respect to both axes. Then in [10] was studied the following generalized version of the Calzeta–Hasi's model

$$H = \frac{1}{2}(p_Y^2 - p_X^2) + \frac{1}{2}(Y^2 - X^2) + \frac{a}{4}\,X^4 + \frac{b}{2}\,X^2Y^2 + \frac{c}{4}\,Y^4. \tag{2}$$

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Following the classical restricted circular three-body problem in which its dynamics is better understand in a rotating frame that in a sideral frame of coordinates, our objective is to study the dynamics of the generalized version of the Calzeta–Hasi's model (2) in rotating coordinates. More precisely, we consider the following generalized version of the Calzeta–Hasi's model in rotating coordinates that itself is a simplified version of the Friedmann-Robertson-Walker Hamiltonian

$$H = \frac{1}{2}\left(y^2 - x^2 + p_y^2 - p_x^2\right) + \frac{1}{4}\left(ax^4 + 2bx^2y^2 + cy^4\right) - \omega\left(xp_y - yp_x\right), \tag{3}$$

where a, b, c, $\omega \in \mathbb{R}$ and $\omega > 0$. Therefore the corresponding Hamiltonian system is

$$\begin{aligned} \dot{x} &= \omega y - p_x, \\ \dot{y} &= -\omega x + p_y, \\ \dot{p}_x &= x + \omega p_y - ax^3 - bxy^2, \\ \dot{p}_y &= -y - \omega p_x - bx^2y - cy^3. \end{aligned} \tag{4}$$

In the qualitative theory of differential equations any orbit or trajectory is homeomorphic either to a straight line, or to a circle, or to a point. The equilibrium points are the orbits homeomorphic to a point and the periodic orbits are the ones homeomorphic to a circle. These two types of orbits are relevant in the study of the dynamics of a differential system, and usually their study is simpler than the study of the orbits homeomorphic to straight lines, that in general exhibit more complicate dynamics. Therefore in order to understand the dynamics of a differential system we must start analyzing its equilibrium points and its periodic solutions.

The objective of this paper is to study analytically the periodic orbits of the Hamiltonian system (4) in each Hamiltonian level $H = h$ varying $h \in \mathbb{R}$. For obtaining the results we shall use the averaging theory for computing periodic solutions. We shall give sufficient conditions on the parameters of the Hamiltonian system (4) implying the existence of continuous families of periodic orbits parameterized by h, and the expression of these families are provided explicitly up to first order in a small parameter.

Our main result is the following one.

Theorem 1. *In section* 3 *we provide sufficient conditions for the existence of twelve families of periodic orbits of the Hamiltonian system* (4) *parametrized by the values of the Hamiltonian* (3)*. Six of these families only exist for positive values of the Hamiltonian, two only exist for negative values of the Hamiltonian, and the remain four families can exist either for positive or negative values of the Hamiltonian depending on the values of the parameters a, b and c. All these twelve families born at the equilibrium point localized at the origin of coordinates of the Hamiltonain system* (4)*.*

2. The averaging theory

In this section we recall the averaging theory of first order for finding periodic solutions. The averaging theory up to third order specifically for studying periodic orbit was developed in [5]. See this paper for a proof of the result stated in this section. The averaging theory of higher order can be found in [8]. Other versions of the averaging theory can also be found in [4] and in Theorems 11.5 and 11.6 of [18]. For a general view on the averaging theory see the book [16].

Theorem 2. *Consider the differential system*

$$\dot{x}(t) = \varepsilon F(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{5}$$

where $F : \mathbb{R} \times D \to \mathbb{R}^n$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions, T-periodic in the first variable and D is an open subset of* $\mathbb{R}^n$. *We define* $f : D \to \mathbb{R}^n$ *as*

$$f(z) = \frac{1}{T} \int_0^T F(s, z)\, ds,$$

and assume that

(i) F *and* R *are locally Lipschitz with respect to* x*;*
(ii) for all $a \in D$ *with* $f(a) = 0$*, there exists a neighborhood* V *of* a *such that* $f(z) \neq 0$ *for all* $z \in \overline{V} \backslash \{a\}$ *and* $d_B(f, V, 0) \neq 0$ *(see its definition later on).*

Then for $|\varepsilon| > 0$ *small enough there exists a* T*-periodic solution* $\varphi(\cdot, \varepsilon)$ *of system* (5) *such that* $\varphi(\cdot, \varepsilon) \to a$ *as* $\varepsilon \to 0$.

We denoted by $d_B(f, V, 0)$ the *Brouwer degree* at the triple $(f, V, 0)$. A sufficient condition for showing that the Brouwer degree is non-zero is that the Jacobian of the function f at a (when it is defined) is non-zero, for a proof see [11]. For more details about the Brouwer degree see [3].

3. Proof of Theorem 1

To prove Theorem 1 we apply Theorem 2 to the Hamiltonian system (4). Generically the periodic orbits of a Hamiltonian system with more than one degree of freedom are on cylinders fulfilled of periodic orbits, see [1]. Therefore, we cannot apply Theorem 2 directly to system (4), because the Jacobian will always be zero. Then we must apply Theorem 2 at each Hamiltonian fixed level where the periodic orbits generically are isolated.

In order to apply Theorem 2 we need a small parameter $\varepsilon > 0$. So in the Hamiltonian system (4) we scaling the variables as follows

$$(x, y, p_x, p_y) = \sqrt{\varepsilon}(X, Y, p_X, p_Y). \tag{6}$$

In these new variables system (4) becomes

$$\begin{array}{rcl} \dot{X} &=& \omega Y - p_X, \\ \dot{Y} &=& -\omega X + p_Y, \\ \dot{p}_X &=& X + \omega p_Y - \varepsilon\left(aX^3 + bXY^2\right), \\ \dot{p}_Y &=& -Y - \omega p_X - \varepsilon\left(bX^2 Y + cY^3\right). \end{array} \tag{7}$$

This system is again Hamiltonian with Hamiltonian

$$\frac{Y^2 - X^2 + p_Y^2 - p_X^2}{2} + \frac{\varepsilon(aX^4 + 2bX^2Y^2 + cY^4)}{4} - \omega\left(Xp_Y - Yp_X\right). \tag{8}$$

Therefore for all $\varepsilon \neq 0$ the original and the transformed systems (4) and (7) have essentially the same phase portrait. The linear part of system (7) at the origin of coordinates is

$$L=\begin{pmatrix} 0 & \omega & -1 & 0\\ -\omega & 0 & 0 & 1\\ 1 & 0 & 0 & \omega\\ 0 & -1 & -\omega & 0\end{pmatrix}.$$

One can see that L has two eigenvalues of multiplicity two, given by $\pm i\sqrt{1+\omega^2}$. Therefore we can apply a linear change of variables (X, Y, p_X, p_Y) to (u, v, p_u, p_v) such that the new system has the linear part

$$J=\begin{pmatrix} 0 & \sqrt{1+\omega^2} & 0 & 0\\ -\sqrt{1+\omega^2} & 0 & 0 & 0\\ 0 & 0 & 0 & \sqrt{1+\omega^2}\\ 0 & 0 & -\sqrt{1+\omega^2} & 0\end{pmatrix}$$

at the origin of coordinates in the real Jordan normal form. A linear change of variables doing this is

$$X=u,\qquad Y=\frac{p_u+\sqrt{1+\omega^2}v}{\omega},\qquad p_X=p_u,\qquad p_Y=\frac{\sqrt{1+\omega^2}p_v-u}{\omega}.$$

Therefore the new system becomes

$$\begin{aligned}
\dot u &= \sqrt{1+\omega^2}v,\\
\dot v &= -\sqrt{1+\omega^2}u+\varepsilon\frac{a\omega^2u^3+bu\left(p_u^2+2\sqrt{1+\omega^2}p_uv+\left(1+\omega^2\right)v^2\right)}{\omega^2\sqrt{1+\omega^2}},\\
\dot p_u &= \sqrt{1+\omega^2}p_v-\varepsilon\left(au^3+\frac{bu\left(p_u+\sqrt{1+\omega^2}v\right)^2}{\omega^2}\right),\\
\dot p_v &= -\sqrt{1+\omega^2}p_u-\varepsilon\frac{\left(p_u+\sqrt{1+\omega^2}v\right)\left(b\omega^2u^2+c\left(p_u+\sqrt{1+\omega^2}v\right)^2\right)}{\omega^2\sqrt{1+\omega^2}},
\end{aligned}\tag{9}$$

and the old Hamiltonian becomes the first integral

$$\begin{aligned}
&\frac{1+\omega^2}{2\omega^2}\left(u^2+v^2+p_u^2+p_v^2+2\sqrt{1+\omega^2}\left(vp_u-up_v\right)\right)\\
&+\varepsilon\frac{1}{4}\left(au^4+\frac{2bu^2\left(p_u+\sqrt{1+\omega^2}v\right)^2}{\omega^2}+\frac{c\left(p_u+\sqrt{1+\omega^2}v\right)^4}{\omega^4}\right).
\end{aligned}\tag{10}$$

Now we apply a generalized polar change of coordinates given by

$$u=r\cos\theta,\qquad v=r\sin\theta,\qquad p_u=\rho\cos(\theta+\phi),\qquad p_v=\rho\sin(\theta+\phi).$$

We recall that this is a change of variables when $r>0$ and $\rho>0$. Moreover, doing this change of variables, the angular variables θ and ϕ appear in the system. Later on the variable θ will be used for obtaining the periodicity necessary for applying the averaging theory. After this change of variables the first integral writes

$$H=\frac{1+\omega^2}{2\omega^2}\left(r^2+\rho^2-2r\rho\sqrt{1+\omega^2}\sin\phi\right)+\varepsilon W_1,\tag{11}$$

where

$$\begin{aligned}
W_1 &= \frac{1}{4}\left(ar^4\cos^4\theta+\frac{2b}{\omega^2}r^2\cos^2\theta\, W_2^2+\frac{c}{\omega^4}W_2^4\right),\\
W_2 &= \rho\cos(\theta+\phi)+r\sqrt{1+\omega^2}\sin\theta.
\end{aligned}$$

In the new variables system (9) writes

$$
\begin{aligned}
\dot r &= \varepsilon \frac{r\cos\theta\sin\theta}{\omega^2\sqrt{1+\omega^2}} W_3,\\
\dot\theta &= -\sqrt{1+\omega^2} + \varepsilon\frac{\cos^2\theta}{\omega^2\sqrt{1+\omega^2}} W_3,\\
\dot\rho &= -\varepsilon\left(r\cos\theta\cos(\theta+\phi)W_5 + \frac{1}{\sqrt{1+\omega^2}}W_6\right),\\
\dot\phi &= \varepsilon\left(\frac{r\cos\theta}{\rho}W_7 - \frac{\cos^2\theta}{\omega^2\sqrt{1+\omega^2}}W_3 - \frac{\cos(\theta+\phi)}{\rho\sqrt{1+\omega^2}}W_8\right),
\end{aligned} \tag{12}
$$

where

$$
\begin{aligned}
W_3 &= b\rho^2\cos^2(\theta+\phi) + a\omega^2 r^2\cos^2\theta + b\, r\sin\theta\, W_4,\\
W_4 &= 2\rho\sqrt{1+\omega^2}\cos(\theta+\phi) + r(1+\omega^2)\sin\theta,\\
W_5 &= ar^2\cos^2\theta + \frac{b}{\omega^2}W_2^2,\\
W_6 &= W_2\left(br^2\cos^2\theta + \frac{c}{\omega^2}W_2^2\right)\sin(\theta+\phi),\\
W_7 &= \left(ar^2\cos^2\theta + \frac{b}{\omega^2}W_2^2\right)\sin(\theta+\phi),\\
W_8 &= W_2\left(br^2\cos^2\theta + \frac{c}{\omega^2}W_2^2\right).
\end{aligned}
$$

In order to apply the averaging theory we take θ as the new independent variable, and denote by a prime the derivative with respect to θ. With this change of independent variable system (12) goes over to

$$
\begin{aligned}
r' &= -\varepsilon\frac{r\cos\theta\sin\theta}{\omega^2(1+\omega^2)}W_3 + O(\varepsilon^2),\\
\rho' &= \frac{\varepsilon}{\sqrt{1+\omega^2}}\left(r\cos\theta\cos(\theta+\phi)W_5 + \frac{1}{\sqrt{1+\omega^2}}W_6\right) + O(\varepsilon^2),\\
\phi' &= \frac{-\varepsilon}{\sqrt{1+\omega^2}}\left(\frac{r\cos\theta}{\rho}W_7 - \frac{\cos^2\theta}{\omega^2\sqrt{1+\omega^2}}W_3 - \frac{\cos(\theta+\phi)}{\rho\sqrt{1+\omega^2}}W_8\right) + O(\varepsilon^2).
\end{aligned} \tag{13}
$$

This system has only three equations because we do not need the $\dot\theta$ equation of (12). Observe that system (13) is 2π-periodic in the variable θ. To apply Theorem 2 we must fix the value of the first integral at $h \in \mathbb{R}$. By solving equation (11) in ρ we obtain

$$
\rho = r\sqrt{1+\omega^2}\sin\phi + \sqrt{\frac{2h\omega^2 - r^2(1+\omega^2) + r^2(1+\omega^2)^2\sin^2\phi}{1+\omega^2}} + O(\varepsilon).
$$

Substituting ρ into equations (13) we obtain the two differential equations

$$
\begin{aligned}
r' &= -\varepsilon\frac{r\cos\theta\sin\theta}{\omega^2(1+\omega^2)}\overline{W}_3 + O(\varepsilon^2),\\
\phi' &= \frac{-\varepsilon}{\sqrt{1+\omega^2}}\left(\frac{r\cos\theta}{\rho}\overline{W}_7 - \frac{\cos^2\theta}{\omega^2\sqrt{1+\omega^2}}\overline{W}_3 - \frac{\cos(\theta+\phi)}{\rho\sqrt{1+\omega^2}}\overline{W}_8\right) + O(\varepsilon^2),
\end{aligned} \tag{14}
$$

where $\overline{W}_i = W_i(\theta, r, \rho(r,\phi), \phi)$ with

$$\rho(r,\phi) = r\sqrt{1+\omega^2}\sin\phi + \sqrt{\frac{2h\omega^2 - r^2(1+\omega^2) + r^2(1+\omega^2)^2\sin^2\phi}{1+\omega^2}}. \tag{15}$$

We observe that in order to apply the first order averaging theory it is not necessary to have information about the terms in $O(\varepsilon^2)$.

One can now see that system (14) satisfies the assumptions of Theorem 2, and it has the form (5) with $T = \pi$ and $F = (F_1, F_2)$ analytic where

$$F_1 = -\frac{r\cos\theta\sin\theta}{\omega^2(1+\omega^2)}\overline{W}_3,$$
$$F_2 = -\frac{1}{\sqrt{1+\omega^2}}\left(\frac{r\cos\theta}{\rho}\overline{W}_7 - \frac{\cos^2\theta}{\omega^2\sqrt{1+\omega^2}}\overline{W}_3 - \frac{\cos(\theta+\phi)}{\rho\sqrt{1+\omega^2}}\overline{W}_8{}^{oo}\right).$$

The averaging function of first order is

$$f(r,\phi) = (f_1(r,\phi), f_2(r,\phi)) = \frac{1}{\pi}\int_0^\pi (F_1(\theta,r,\phi), F_2(\theta,r,\phi))\ d\theta,$$

becomes

$$f_1(r,\phi) = \frac{br\cos\phi\ \rho(r,\phi)\left(\sin\phi\ \rho(r,\phi) - r\sqrt{1+\omega^2}\right)}{4\omega^2(1+\omega^2)},$$
$$f_2(r,\phi) = -\frac{Ar^3\sin\phi + Br^2\rho(r,\phi) + Cr\sin\phi\rho(r,\phi)^2 + D\rho(r,\phi)^3}{8\omega^2\sqrt{1+\omega^2}\rho(r,\phi)}, \tag{16}$$

where

$$A = (1+\omega^2)\left(b + 2b\omega^2 + 3\left(c + (a+c)\omega^2\right)\right),$$
$$B = -\sqrt{1+\omega^2}(b + 6c + 3(a+b+2c)\omega^2 - \left(2b + 3c + (b+3c)\omega^2\right)\cos(2\phi)),$$
$$C = 3(1+\omega^2)(b+3c),$$
$$D = -\sqrt{1+\omega^2}(2b + 3c + b\cos(2\phi)).$$

According with Theorem 2 we must find the zeros (r_0, ϕ_0) of the function $f = (f_1, f_2)$ and check that the Jacobian determinant of f at these points is not zero.

¿From $f_1(r,\phi) = 0$ we obtain $r = r(\phi)$, and in order that $\rho(r(\phi),\phi) \neq 0$ (otherwise $f_2(r,\phi)$ is not defined), we get that

$$r(\phi) = \begin{cases} R_0 & \text{with } h > 0, \\ R_1 & \text{with } h > 0 \text{ and } \sin\phi > 0, \\ R_2(\phi) & \text{with } h(1+\omega^2 - (1+2\omega^2)\sin^2\phi) > 0, \end{cases}$$

where

$$R_0 = 0, \quad R_1 = \sqrt{\frac{2h\omega^2}{1+\omega^2}}, \quad R_2(\phi) = \sqrt{\frac{2h\omega^2}{(1+\omega^2)(1+\omega^2-(1+2\omega^2)\sin^2\phi)}}\sin\phi.$$

Substituting $r = R_0$ into $f_2(r,\phi) = 0$, and solving with respect to ϕ we obtain the following two zeros of the averaged function $f(r,\phi)$

$$(r_1,\phi_1) = \left(0, \arccos\sqrt{-\frac{b+3c}{2b}}\right),$$

$$(r_2,\phi_2) = \left(0, -\arccos\sqrt{-\frac{b+3c}{2b}}\right).$$

Since the value of

$$\rho(r_i,\phi_i) = \frac{\omega\sqrt{2h\,(\omega^2+1)}}{1+\omega^2} \quad \text{for } i = 1,2,$$

and the determinant of the Jacobian matrix of f at these two zeros is

$$\frac{3h^2(b+c)(b+3c)}{8\,(1+\omega^2)^4},$$

it follows from the averaging theory (Theorem 2) that if

$$h > 0, \quad 0 < -\frac{b+3c}{2b} \le 1, \quad \text{and } (b+c)(b+3c) \neq 0,$$

then the zeros (r_i,ϕ_i) provide two periodic solutions of the differential system (14), and consequently of the Hamiltonian system (4) in every level $H = h > 0$.

Substituting $r = R_1$ into $f_2(r,\phi) = 0$, and solving with respect to ϕ we obtain the following six zeros of the averaged function $f(r,\phi)$

$$(r_3,\phi_3) = \left(\sqrt{\tfrac{2h\omega^2}{1+\omega^2}}, \pi\right),$$

$$(r_4,\phi_4) = \left(\sqrt{\tfrac{2h\omega^2}{1+\omega^2}}, 0\right),$$

$$(r_5,\phi_5) = \left(\sqrt{\tfrac{2h\omega^2}{1+\omega^2}}, -\arccos\sqrt{\tfrac{b(4\omega^2+3)-\sqrt{b(12(\omega^2+1)(\omega^2(a+c)+c)+b(24\omega^4+36\omega^2+13))}}{8b(1+\omega^2)}}\right),$$

$$(r_6,\phi_6) = \left(\sqrt{\tfrac{2h\omega^2}{1+\omega^2}}, \arccos\sqrt{\tfrac{b(4\omega^2+3)-\sqrt{b(12(\omega^2+1)(\omega^2(a+c)+c)+b(24\omega^4+36\omega^2+13))}}{8b(1+\omega^2)}}\right),$$

$$(r_7,\phi_7) = \left(\sqrt{\tfrac{2h\omega^2}{1+\omega^2}}, -\arccos\sqrt{\tfrac{b(4\omega^2+3)+\sqrt{b(12(\omega^2+1)(\omega^2(a+c)+c)+b(24\omega^4+36\omega^2+13))}}{8b(1+\omega^2)}}\right),$$

$$(r_8,\phi_8) = \left(\sqrt{\tfrac{2h\omega^2}{1+\omega^2}}, \arccos\sqrt{\tfrac{b(4\omega^2+3)+\sqrt{b(12(\omega^2+1)(\omega^2(a+c)+c)+b(24\omega^4+36\omega^2+13))}}{8b(1+\omega^2)}}\right).$$

Since the value of

$$\rho(r_3,\phi_3) = \rho(r_4,\phi_4) = 0,$$

$$\rho(r_5,\phi_5) = \rho(r_6,\phi_6) = \omega\sqrt{\tfrac{h(b(4\omega^2+5)+\sqrt{b(12(\omega^2+1)(\omega^2(a+c)+c)+b(24\omega^4+36\omega^2+13))})}{b(1+\omega^2)}},$$

$$\rho(r_7,\phi_7) = \rho(r_8,\phi_8) = \omega\sqrt{\tfrac{h(b(4\omega^2+5)-\sqrt{b(12(\omega^2+1)(\omega^2(a+c)+c)+b(24\omega^4+36\omega^2+13))})}{b(1+\omega^2)}}.$$

Since $\rho(r_i,\phi_i)$ cannot be zero, otherwise f_2 is not defined, for the zeros $\rho(r_i,\phi_i)$ with $i=3,4$ the averaging theory does not provide any information about if these zeros produce or not periodic solutions of the differential system (14).

The determinant $D(r,\phi)$ of the Jacobian matrix of f at the other four zeros is

$$D(r_5,\phi_5)=D(r_6,\phi_6)=\frac{h^{7/2}\omega^7\left(\omega^2+1\right)^6\sin(2\phi)(bF+eD)\left(2A\sqrt{h}\omega\left(\omega^2+1\right)-\sqrt{2}BC\right)}{8b^2},$$

$$D(r_7,\phi_7)=D(r_8,\phi_8)=\frac{h^{7/2}\omega^7\left(\omega^2+1\right)^6\sin(2\phi)(bF-eD)\left(2A\sqrt{h}\omega\left(\omega^2+1\right)-\sqrt{2}BC\right)}{8b^2},$$

where

$$\begin{aligned}
A=&\ \sqrt{\frac{-2\omega^4(3a+2b+3c)-2\omega^2(3a+b+6c)+b-6c-D}{b\left(\omega^2+1\right)^2}},\\
B=&\ \sqrt{\frac{h\omega^2\left(1+\omega^2\right)\left(4b\omega^2+5b+D\right)}{b}},\\
C=&\ \sqrt{\frac{b\left(4\omega^2+3\right)-D}{b\left(1+\omega^2\right)}},\\
D=&\ \sqrt{b\left(12\left(\omega^2+1\right)\left(\omega^2(a+c)+c\right)+b\left(24\omega^4+36\omega^2+13\right)\right)},\\
E=&\ 3\omega^4\left(15ab-9ac+34b^2-15bc-9c^2\right)+\omega^2\left(42ab-27ac+145b^2-66bc-54c^2\right)\\
&+9(b-c)(5b+3c),\\
F=&\ 18\omega^6(3a+14b-9c)(a+2b+c)+\omega^4(54a^2+579ab-225ac+1082b^2\\
&-147bc-495c^2)+3\omega^2\left(72ab-39ac+247b^2-24bc-168c^2\right)\\
&+3(b-c)(55b+57c).
\end{aligned}$$

¿From Theorem 2 if for $i=5,6$ we have that

$$0\le\frac{b(4\omega^2+3)-\sqrt{b\left(12\left(\omega^2+1\right)\left(\omega^2(a+c)+c\right)+b\left(24\omega^4+36\omega^2+13\right)\right)}}{b(1+\omega^2)}\le 1,$$

$$\frac{b(4\omega^2+5)+\sqrt{b\left(12\left(\omega^2+1\right)\left(\omega^2(a+c)+c\right)+b\left(24\omega^4+36\omega^2+13\right)\right)}}{b(1+\omega^2)}\ge 0,$$

$$h>0,\quad \sin\phi_i>0,\quad \rho(r_i,\phi_i)>0\quad\text{and } D(r_i,\phi_i)\neq 0,$$

then these two zeros (r_i,ϕ_i) provide two periodic solutions of the differential system (14), and consequently of the Hamiltonian system (4) in every level $H=h>0$.

¿From Theorem 2 if for $i=7,8$ we have that

$$0\le\frac{b(4\omega^2+3)+\sqrt{b\left(12\left(\omega^2+1\right)\left(\omega^2(a+c)+c\right)+b\left(24\omega^4+36\omega^2+13\right)\right)}}{b(1+\omega^2)}\le 1,$$

$$\frac{b(4\omega^2+5)-\sqrt{b\left(12\left(\omega^2+1\right)\left(\omega^2(a+c)+c\right)+b\left(24\omega^4+36\omega^2+13\right)\right)}}{b(1+\omega^2)}\ge 0,$$

$$h>0,\quad \sin\phi_i>0,\quad \rho(r_i,\phi_i)>0\quad\text{and } D(r_i,\phi_i)\neq 0,$$

then these two zeros (r_i,ϕ_i) provide two periodic solutions of the differential system (14), and consequently of the Hamiltonian system (4) in every level $H=h>0$.

Substituting $r = R_2$ into $f_2(r, \phi) = 0$, and solving with respect to ϕ we obtain the following

$$\phi = \pm \arccos\left(\sqrt{\frac{-(a+b)\omega^2}{b+c+(c-a)\omega^2}}\right).$$

Substituting these values of ϕ into R_2 we get the following two zeros of the averaged function $f(r, \phi)$

$$(r_9, \phi_9) = \left(\sqrt{\frac{-2(b+c)h}{(a+2b+c)(1+\omega^2)}}, -\arccos\sqrt{\frac{-(a+b)\omega^2}{b+c+(c-a)\omega^2}}\right),$$
$$(r_{10}, \phi_{10}) = \left(\sqrt{\frac{-2(b+c)h}{(a+2b+c)(1+\omega^2)}}, \arccos\sqrt{\frac{-(a+b)\omega^2}{b+c+(c-a)\omega^2}}\right).$$

We cannot guarantee that these last two solutions are all the solutions for $r = R_2$, these are the ones that we can obtain explicitly.

Since the value of

$$\rho(r_i, \phi_i) = \left(\omega^2|a+b| + (1+\omega^2)(b+c)\right)\sqrt{\frac{-2h}{(1+\omega^2)\,(a+2b+c)(b+c+(c-a)\omega^2)}},$$

for $i = 9, 10$, and we denote the determinant of the Jacobian matrix of f at these two zeros by $D(r_i, \phi_i)$, we do not give its huge expression here.

It follows from the averaging theory (Theorem 2) that if

$$0 \le \frac{-(a+b)\omega^2}{b+c+(c-a)\omega^2} \le 1, \quad \frac{-2(b+c)h}{a+2b+c} > 0, \quad \rho(r_i, \phi_i) > 0, \quad \text{and} \quad D(r_i, \phi_i) \neq 0,$$

then the zeros (r_i, ϕ_i) for $i = 9, 10$ provide two periodic solutions of the differential system (14), and consequently of the Hamiltonian system (4) in every level $H = h$.

¿From $f_1(r, \phi) = 0$ we obtain $\phi = \phi(r)$, and in order that $\rho(r(\phi), \phi) \neq 0$ (otherwise $f_2(r, \phi)$ is not defined), we get that

$$\phi(r) = \begin{cases} \Phi_1 & \text{with } h < 0, \\ \Phi_2(\phi) & \text{with } \dfrac{(b+3c)h}{(3a+2b+3c)} < 0, \end{cases}$$

where

$$\Phi_1 = \pm\frac{\pi}{2}, \quad \Phi_2 = \pm \arcsin\left(\frac{r(1+\omega^2)}{\sqrt{r^2(1+3\omega^2+2\omega^4)+2h\omega^2}}\right).$$

Substituting $\phi = \Phi_1$ into $f_2(r, \phi) = 0$, and solving with respect to r we obtain the following four zeros of the averaged function $f(r, \phi)$

$$
\begin{aligned}
(r_{11}, \phi_{11}) &= \left(\sqrt{-\frac{2h}{1+\omega^2}}, -\frac{\pi}{2} \right), \\
(r_{12}, \phi_{12}) &= \left(\sqrt{-\frac{2h}{1+\omega^2}}, \frac{\pi}{2} \right), \\
(r_{13}, \phi_{13}) &= \left(\sqrt{-\frac{2(b+3c)h}{(3a+2b+3c)(1+\omega^2)}}, -\frac{\pi}{2} \right), \\
(r_{14}, \phi_{14}) &= \left(\sqrt{-\frac{2(b+3c)h}{(3a+2b+3c)(1+\omega^2)}}, -\frac{\pi}{2} \right).
\end{aligned}
$$

Since the value of

$$
\begin{aligned}
\rho(r_i, \phi_i) = & \quad \sqrt{-2h} \quad \text{for } i = 11, 12, \\
\rho(r_{13}, \phi_{13}) = & \quad \sqrt{\frac{2h(3a+b)\omega^2}{(1+\omega^2)(3a+2b+3c)}} + \sqrt{-\frac{2h(b+3c)}{3a+2b+3c}}, \\
\rho(r_{14}, \phi_{14}) = & \quad \sqrt{\frac{2h(3a+b)\omega^2}{(1+\omega^2)(3a+2b+3c)}} - \sqrt{-\frac{2h(b+3c)}{3a+2b+3c}}.
\end{aligned}
$$

and the determinant of the Jacobian matrix of f at these two zeros is

$$
\begin{aligned}
D(r_i, \phi_i) &= \frac{bh^2(3a+b)}{8\,(1+\omega^2)^4} \quad \text{for } i = 11, 12, \\
D(r_i, \phi_i) &= -\frac{bh^2(3a+b)(b+3c)}{4(1+\omega^2)^4(3a+2b+3c)} \quad \text{for } i = 13, 14,
\end{aligned}
$$

Again from Theorem 2 we obtain that

$$h < 0, \quad \text{and} \quad D(r_i, \phi_i) \neq 0,$$

then the two zeros (r_i, ϕ_i) for $i = 11, 12$ provide two periodic solutions of the differential system (14), and consequently of the Hamiltonian system (4) in every level $H = h < 0$.

Also from Theorem 2 we get that

$$r_i > 0, \quad \rho_i > 0, \quad D(r_i, \phi_i) \neq 0,$$

then the two zeros (r_i, ϕ_i) for $i = 13, 14$ provide two periodic solutions of the differential system (14), and consequently of the Hamiltonian system (4) in every level $H = h$.

Substituting $\phi = \Phi_2$ into $f_2(r, \phi) = 0$, and solving with respect to r we obtain again the solutions (r_i, ϕ_i) for $i = 9, 10, 11, 12$. Again we cannot guarantee that these last four solutions are all the solutions for $\phi = \Phi_2$, because these four solutions are the ones that we can obtain explicitly.

For $i = 1, 2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14$ and according with Theorem 2 the zero (r_i, ϕ_i) provides a periodic solution $(\bar{r}_i(\theta, \varepsilon), \bar{\phi}_i(\theta, \varepsilon))$ of the differential system (14) such that

$$(\bar{r}_i(0,\varepsilon), \bar{\phi}_i(0,\varepsilon)) \to (r_i, \phi_i) \quad \text{when } \varepsilon \to 0.$$

Going back to the differential system (13) we obtain for this system a periodic solution $(\bar{r}_i(\theta,\varepsilon), \bar{\rho}_i(\theta,\varepsilon), \bar{\phi}_i(\theta,\varepsilon))$ such that

$$(\bar{r}_i(0,\varepsilon), \bar{\rho}_i(0,\varepsilon), \bar{\phi}_i(0,\varepsilon)) \to (r_i, \rho_i, \phi_i) \quad \text{when } \varepsilon \to 0,$$

where $\rho_i = \rho(r_i, \phi_i)$. Now going back to the differential system (12) we get for this system a periodic solution $(\bar{r}_i(t,\varepsilon), \bar{\theta}(t,\varepsilon), \bar{\rho}_i(t,\varepsilon), \bar{\phi}_i(t,\varepsilon))$ such that

$$(\bar{r}_i(0,\varepsilon), \bar{\theta}(0,\varepsilon), \bar{\rho}_i(0,\varepsilon), \bar{\phi}_i(0,\varepsilon)) \to (r_i, 0, \rho_i, \phi_i) \quad \text{when } \varepsilon \to 0.$$

Again going back to the differential system (9) we have for this system a periodic solution $(\bar{u}(t,\varepsilon), \bar{v}(t,\varepsilon), \bar{p}_u(t,\varepsilon), \bar{p}_v(t,\varepsilon))$ such that

$$(\bar{u}(0,\varepsilon), \bar{v}(0,\varepsilon), \bar{p}_u(0,\varepsilon), \bar{p}_v(0,\varepsilon)) \to (r_i, 0, \rho_i\cos\phi_i, \rho_i\sin\phi_i) \quad \text{when } \varepsilon \to 0.$$

Going back to the Hamiltonian system (7) we have for this system a periodic solution $(\bar{X}(t,\varepsilon), \bar{Y}(t,\varepsilon), \bar{p}_X(t,\varepsilon), \bar{p}_Y(t,\varepsilon))$ such that

$$(\bar{X}(0,\varepsilon), \bar{Y}(0,\varepsilon), \bar{p}_X(0,\varepsilon), \bar{p}_Y(0,\varepsilon)) \to \left(r_i, \frac{\rho_i\cos\phi_i}{\omega}, \rho_i\cos\phi_i, \frac{\sqrt{1+\omega^2}\rho_i\sin\phi_i - r_i}{\omega}\right)$$

when $\varepsilon \to 0$. Finally going back to the Hamiltonian system (4) we have for this system a periodic solution $(\bar{x}(t,\varepsilon), \bar{y}(t,\varepsilon), \bar{p}_x(t,\varepsilon), \bar{p}_y(t,\varepsilon))$ such that

$$(\bar{x}(0,\varepsilon), \bar{y}(0,\varepsilon), \bar{p}_x(0,\varepsilon), \bar{p}_y(0,\varepsilon)) \to \sqrt{\varepsilon}\left(r_i, \frac{\rho_i\cos\phi_i}{\omega}, \rho_i\cos\phi_i, \frac{\sqrt{1+\omega^2}\rho_i\sin\phi_i - r_i}{\omega}\right)$$
$$\to (0,0,0,0),$$

when $\varepsilon \to 0$. In summary, these 12 families of periodic orbits of the Hamiltonian system (4) born at the equilibrium localized at the origin of coordinates.

This completes the proof of Theorem 1.

4. Conclusions

Under different conditions on the four parameters of the system a, b, c and ω, we have find analytically 12 families of periodic orbits for the Hamiltonian system (4). These families are associated to the given zeros (r_i, ϕ_i) of the averaged function $f(r,\phi) = (f_1(r,\phi), f_2(r,\phi))$ given in (16) for $i = 1,2,5,6,7,8,9,10,11,12,13,14$.

We remark that the families of the periodic orbits associated to the zeros (r_i, ϕ_i) of the averaged function with $i = 1,2,5,6,7,8$ only exist in the levels $H = h > 0$, with $i = 11,12$ only exist in the levels $H = h < 0$, while with $i = 9,10,13,14$ can exist either in the levels $H = h > 0$, or $H = h < 0$, depending on the values of the parameters a, b and c.

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