This is a preprint of: "Many periodic solutions for a second order cubic periodic differential equation", Adriana Buica, Armengol Gasull, *Monatsh. Math.*, 2020.
DOI: [10.1007/s00605-020-01433-4]

MANY PERIODIC SOLUTIONS FOR A SECOND ORDER CUBIC PERIODIC DIFFERENTIAL EQUATION

ADRIANA BUICĂ[1] AND ARMENGOL GASULL[2]

Abstract. The aim of this work is to provide results that assure the existence of many isolated T-periodic solutions for T-periodic second-order differential equations of the form $x'' = a(t)x + b(t)x^2 + c(t)x^3$. We use bifurcation methods, including Malkin functions and results of Fonda, Sabatini and Zanolin. In addition, we give a general result that assures the existence of a T-periodic perturbation of a non-isochronous center with an arbitrary number of T-periodic solutions.

1. Introduction

The aim of this work is to provide results that assure the existence of many T-periodic solutions for the class of T-periodic second-order differential equations

$$x'' = a(t)x + b(t)x^2 + c(t)x^3. \tag{1}$$

We assume that $a, b, c \in C(\mathbb{R})$ are T-periodic functions, and $T > 0$ is a fixed real number.

Recall that, when in the above equation we replace x'' by x', we obtain the classical Abel equation. Hence equation (1) can be seen as a generalized Abel equation, and our goal is similar to the celebrated result of Lins-Neto ([15]) where the author proved that there is no upper bound for the number of isolated T-periodic solutions for the classical T-periodic Abel differential equations.

Another motivation to look for such results came after reading the papers [12] and [1] where sufficient conditions are given in order to assure the existence of 1 and, respectively, 2 non-null T-periodic solutions. We mention that the authors of [1, 12] were motivated to study equations of this form by Austin [2] who proposed a similar equation as a biomathematical model of an aneurysm, and by Cronin [6] who was the first to study the existence of periodic solutions of it.

In the class of equations of the form (1) we distinguish the ones with constant coefficients $a, b, c \in \mathbb{R}$. Here $T > 0$ is not related with the coefficients. Note that, whenever $a \neq 0$ and $b^2 - 4ac > 0$, equation (1) has exactly 2 non-null constant (thus T-periodic for any T) isolated solutions. The equation $x'' = 0$ has any constant function as T-periodic solution (for arbitrary $T > 0$), while the solutions of the equation $x'' = -x$ are all 2π-periodic. Another interesting example of autonomous equation of the form (1) is $x'' = -x^3$ whose set of T-periodic orbits (for arbitrary $T > 0$) is a countably infinite set. This example will be discussed in detail in Section 2.

In addition, the linear Hill equation $x'' = a(t)x$ is in the class (1) when $a \in C(\mathbb{R})$ is T-periodic. An interesting aspect to discuss is how to find $a(t)$ such that all the solutions of this equation are T-periodic. An answer will be given in Section 3.4.1. For other results in this direction one can see [13, 17].

For equation (1) with time-dependent coefficients we obtain different sufficient conditions that guarantee the existence of 2, 3, 8 and finally $2N$ (with arbitrary $N \in \mathbb{N}^*$) isolated T-periodic solutions. Our examples of such equations are small perturbations of one of the equations $x'' = 0$, $x'' = -x$, or $x'' = -x^3$.

To obtain the main result in [1], Araujo-Pedroso used the Mawhin continuation theorem, a result within the topological coincidence degree theory for infinite dimensional operators. The results in [12], which conclude the existence of a nonnull T-periodic solution, where obtained by Grossinho-Sanchez using variational methods, more precisely, a variant of Mountain-Pass Theorem due to Rabinowitz. Here we use bifurcation methods, more precisely, the Malkin bifurcation theorem and another result due to Fonda-Sabatini-Zanolin [9]. These results will be stated in Section 2. For a short history of the Malkin bifurcation theorem and for a modern proof one can see [3, 4]. The main results on the generalized Abel equation (1) are stated and proved in Section 3. In Section 4, the last one, we revisit the Malkin theorem in the planar case and present new results on non-necessarily Hamiltonian systems. One of these results assures the existence of $2N$ different T-periodic solutions, for any arbitrary integer $N \geq 2$. Note that, in Section 3, we will apply the Fonda-Sabatini-Zanolin theorem in order to obtain $2N$ different T-periodic solutions for a planar system with a Hamiltonian structure.

2. Main tools

2.1. **The Malkin bifurcation theorem.** Let $n \geq 1$ be an integer and $T > 0$ be a real. We consider in this subsection the n-dimensional differential system

$$u' = f(u) + \varepsilon g(t, u) \tag{2}$$

where $\varepsilon \in \mathbb{R}$, $f : \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ are sufficiently smooth and T-periodic with respect to the variable $'t'$. Let $k \in \mathbb{Z}$ be such that $1 \leq k \leq n$. We assume that the unperturbed system

$$u' = f(u) \tag{3}$$

has a *k-dimensional T-period manifold* $\mathcal{Z} \subset \mathbb{R}^n$. More precisely, we assume that there exists a C^2 function $\xi : U \to \mathbb{R}^n$, where U is a nonempty, open subset of $\mathbb{R}^k$, such that the Jacobian matrix $D\xi(\theta)$ has full rank for any $\theta \in U$, and $\mathcal{Z} = \xi(U)$. Moreover, for any $\theta \in U$, the solution of (3) with $u(0) = \xi(\theta)$, denoted $u(t, \theta)$, is T-periodic. Note that $T > 0$ is not necessarily the minimal period.

In addition, we assume that $\mathcal{Z} = \xi(U)$ is *normally nondegenerate*, that is, for each $\theta \in U$, the first variational (T-periodic) system

$$v' = Df\left(u(t, \theta)\right) v \tag{4}$$

has exactly k linearly independent T-periodic solutions.

These hypotheses assure the existence (see [14]), for each $\theta \in U$, of exactly k linearly independent T-periodic solutions of the adjoint system

$$w' = -\left[Df(u(t,\theta))\right]^{\mathrm{tr}} w,$$

denoted $w_i(t,\theta)$, $i = \overline{1,k}$. The Malkin bifurcation function

$$M : U \to \mathbb{R}^k$$

is defined componentwise by

$$\theta \mapsto \int_0^T \langle\, w_i(t,\theta), g(t,u(t,\theta))\rangle dt, \quad i = \overline{1,k}.$$

Here $[\cdot]^{\mathrm{tr}}$ denotes the transpose of a matrix, while $\langle\cdot,\cdot\rangle$ denotes the scalar product in $\mathbb{R}^n$. We remark that in the literature there are other representations of the bifurcation function, that eventually differ by a change of variable and/or a factor without zeros.

In the hypotheses and notations of this section we have the following result known as the Malkin theorem [16] (for a proof see also [3, 4]).

Theorem 1. *Assume that there exists $\theta^* \in U$ such that $M(\theta^*) = 0$ and the Jacobian determinant $\det DM(\theta^*) \neq 0$. Then there exists $\overline{\varepsilon} > 0$ such that for each $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$, system (2) has a T-periodic solution u_ε with $\lim_{\varepsilon\to 0} u_\varepsilon(0) = \xi(\theta^*)$, and with the property that it is isolated in the set of T-periodic solutions of (2).*

We say that a T-periodic solution $u(t)$ of a non-autonomous differential equation is *isolated in the set of T-periodic solutions* when there exists a neighborhood of $u(0) \in \mathbb{R}^n$ where $u(0)$ is the only initial value of a T-periodic solution. Note that this definition is meaningful only for non-autonomous differential equations since the only isolated T-periodic solutions of an autonomous differential equations are its isolated critical points.

In the end of this subsection we discuss, for further use, different characterizations of the condition that the k-dimensional T-period manifold $\mathcal{Z}$ is *normally nondegenerate.*

For the case when the unperturbed system (3) is linear, we have the following result.

Proposition 2. *Assume that the unperturbed system (3) is linear and has exactly k linearly independent T-periodic solutions. Let $\mathcal{Z}$ be the linear space spanned by the initial values of the k linearly independent T-periodic solutions. Then $\mathcal{Z}$ is a k–dimensional normally nondegenerate T-period manifold.*

For the case when the unperturbed system (3) is nonlinear, the hypothesis that $\mathcal{Z}$ is a k-dimensional T-period manifold assures that the Jacobian matrix $D_\theta u(t,\theta)$ is an $n \times k$ matrix whose columns are k linearly independent T-periodic solutions of the variational system (4). Other solutions of the variational system are derivatives of the general solution of the unperturbed system (3) with respect to other directions in $\mathbb{R}^n$. Thus, one has to decide whether one of these other solutions is T-periodic.

2.2. **The Fonda-Sabatini-Zanolin theorem.** We consider in this subsection the planar Hamiltonian system

$$u' = J\nabla H(u) + \varepsilon J\nabla \mathcal{H}(t,u) \tag{5}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ denotes the symplectic matrix, $H : \mathbb{R}^2 \to \mathbb{R}$ and $\mathcal{H} : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ are sufficiently smooth and T-periodic with respect to the variable $'t'$ and in $\nabla\mathcal{H}(t,u)$ the gradient is only with respect the variable u. We assume that the unperturbed autonomous planar Hamiltonian system

$$u' = J\nabla H(u) \tag{6}$$

has a period annulus $A \subset \mathbb{R}^2$ such that the inner and outer components of its boundary are Jordan curves. Assume, in addition, that A is not isochronous, that is, the period of the periodic orbits in A covers an interval $[T_1, T_2]$, with $T_1 < T_2$. Without loosing the generality, assume that the origin is a singularity of (6) and the periodic orbits in A encircle the origin. Under these assumptions, the authors in [9] proved the following result.

Theorem 3. *Given a positive integer j satisfying*

$$T_1 < T/j < T_2$$

there is an $\overline{\varepsilon} > 0$ such that, if $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$, then system (5) *has at least two T-periodic solutions, whose orbits are contained in A, which make exactly j rotations around the origin in the period time T.*

Note that this result does not provide any information whether the T-periodic solutions are isolated or not.

3. Results on the generalized Abel equation

3.1. Examples with two non-null periodic solutions.

Theorem 4. *Let $T > 0$ and $a, b, c \in C(\mathbb{R})$ be T-periodic functions. Denote*

$$\widehat{a} = \int_0^T a(t)dt, \quad \widehat{b} = \int_0^T b(t)dt, \quad \widehat{c} = \int_0^T c(t)dt$$

and assume that

$$\widehat{a} \neq 0, \quad \widehat{b}^2 - 4\,\widehat{a}\,\widehat{c} > 0.$$

Then there exists $\overline{\varepsilon} > 0$ such that, for each $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$, the equation

$$x'' = \varepsilon a(t)x + \varepsilon b(t)x^2 + \varepsilon c(t)x^3$$

has at least two isolated non-null T-periodic solutions.

Proof. We write the given second order equation as the first order system,

$$\begin{aligned} x' &= y, \\ y' &= \varepsilon[a(t)x + b(t)x^2 + c(t)x^3], \end{aligned}$$

which has the form (2) with

$$n = 2,\ \ u = \begin{pmatrix} x \\ y \end{pmatrix},\ \ f(t,x,y) = \begin{pmatrix} y \\ 0 \end{pmatrix},\ \ g(t,x,y) = \begin{pmatrix} 0 \\ a(t)x + b(t)x^2 + c(t)x^3 \end{pmatrix}.$$

The unperturbed system is

$$x' = y, \quad y' = 0,$$

which is linear, and whose adjoint system is

$$x' = 0, \quad y' = -x.$$

Let

$$k = 1,\ \ \xi : \mathbb{R} \to \mathbb{R}^2,\ \ \xi(\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix},\ \ \text{for any}\ \ \theta \in \mathbb{R}.$$

It is easy to see that, for all $t, \theta \in \mathbb{R}$,

$$u(t,\theta) = \begin{pmatrix} \theta \\ 0 \end{pmatrix},\ \ w_1(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

which are constant functions, thus, in particular, T-periodic.
Hence $\mathcal{Z} = \xi(\mathbb{R})$ is an 1-dimensional T-period manifold, which, by Proposition 2, is normally non-degenerate.

In this case, the Malkin bifurcation function is

$$M : \mathbb{R} \to \mathbb{R}, \quad M(\theta) = \int_0^T [a(t)\theta + b(t)\theta^2 + c(t)\theta^3]dt.$$

Then

$$M(\theta) = \widehat{a}\,\theta + \widehat{b}\,\theta^2 + \widehat{c}\,\theta^3.$$

The conditions $\widehat{a} \neq 0$, $\widehat{b}^2 - 4\,\widehat{a}\,\widehat{c} > 0$ from the hypothesis, assure that M has two non-null simple zeros. From here, by Theorem 1, the conclusion follows. □

Remark 5. *Let $m \geq 1$ be an integer. One can check that Theorem 4 remains valid for the differential equation of arbitrary order*

$$x^{(m)} = \varepsilon a(t)x + \varepsilon b(t)x^2 + \varepsilon c(t)x^3,$$

since the Malkin bifurcation function is the same.

3.2. **Other examples with two non-null periodic solutions.** For completeness and because it looks similar to our previous result, we state without proof the main result of [1], that also concludes the existence of at least two non-null periodic solutions for some equations of the form (1).

Theorem 6. [1] *Assume that there exist four positive constants* $\underline{a}, \overline{a}, \underline{b}, \overline{b}, \underline{c}, \overline{c} > 0$ *such that*

$$\underline{a} \leq a(t) \leq \overline{a}, \quad \underline{b} \leq -b(t) \leq \overline{b}, \quad \underline{c} \leq -c(t) \leq \overline{c},$$

and

$$\overline{b}^2 - \underline{b}^2 < 4\underline{ac}, \quad \overline{b}^2 + 4\overline{ac} > (\underline{b} - \underline{c})^2.$$

Then there exists $\overline{T} > 0$*, depending on* $\underline{a}, \overline{a}, \underline{b}, \overline{b}, \underline{c}, \overline{c}$*, such that if*

$$T < \overline{T},$$

the equation (1) *has at least two non-null* T*- periodic solutions.*

3.3. **Examples with three non-null periodic solutions.**

Theorem 7. *Let* $T > 0$ *and* $\varphi_i \in C^2(\mathbb{R})$*,* $i \in 1, 2, 3$ *be three* T*-periodic functions. Assume that* $0 < \varphi_1(t) < \varphi_2(t) < \varphi_3(t)$ *for all* $t \in \mathbb{R}$ *and consider*

$$a(t) = \frac{\Delta_1(t)}{\Delta(t)}, \quad b(t) = \frac{\Delta_2(t)}{\Delta(t)}, \quad c(t) = \frac{\Delta_3(t)}{\Delta(t)},$$

where

$$\Delta = \varphi_1\varphi_2\varphi_3(\varphi_3 - \varphi_1)(\varphi_2 - \varphi_1)(\varphi_3 - \varphi_2),$$

$$\Delta_1 = \begin{vmatrix} \varphi_1'' & \varphi_1^2 & \varphi_1^3 \\ \varphi_2'' & \varphi_2^2 & \varphi_2^3 \\ \varphi_3'' & \varphi_3^2 & \varphi_3^3 \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} \varphi_1 & \varphi_1'' & \varphi_1^3 \\ \varphi_2 & \varphi_2'' & \varphi_2^3 \\ \varphi_3 & \varphi_3'' & \varphi_3^3 \end{vmatrix}, \quad \Delta_3 = \begin{vmatrix} \varphi_1 & \varphi_1^2 & \varphi_1'' \\ \varphi_2 & \varphi_2^2 & \varphi_2'' \\ \varphi_3 & \varphi_3^2 & \varphi_3'' \end{vmatrix}.$$

Then $a, b, c \in C(\mathbb{R})$ *are* T*-periodic and the differential equation*

$$x'' = a(t)x + b(t)x^2 + c(t)x^3$$

has at least three non-null T*-periodic solutions, namely* $\varphi_1, \varphi_2, \varphi_3$*.*

Proof. Given $\varphi_1, \varphi_2, \varphi_3$ as in the hypothesis, it is easy to see that a, b, c are chosen such that

$$\begin{aligned} a(t)\varphi_1(t) + b(t)\varphi_1^2(t) + c(t)\varphi_1^3(t) &= \varphi_1''(t) \\ a(t)\varphi_2(t) + b(t)\varphi_2^2(t) + c(t)\varphi_2^3(t) &= \varphi_2''(t) \\ a(t)\varphi_3(t) + b(t)\varphi_3^2(t) + c(t)\varphi_3^3(t) &= \varphi_3''(t). \end{aligned}$$

Thus $\varphi_1, \varphi_2, \varphi_3$ are the three solutions of the given equation. □

Notice that although in most cases the above construction gives a differential equation for which the three functions $\varphi_j(t)$, $j = 1, 2, 3$ are isolated T-periodic solutions, in some cases they can belong to a continuum of periodic solutions.

Remark 8. *Let $m \geq 1$ be an integer. One can check that Theorem 7 remains valid for the differential equation of arbitrary order*

$$x^{(m)} = a(t)x + b(t)x^2 + c(t)x^3,$$

by changing each φ_j'' by $\varphi_j^{(m)}$.

3.4. **Examples with eight non-null periodic solutions.**

Theorem 9. *There exist $a_0, a_3, a_4, b_1, b_2, b_5, b_6, c_0, c_3, c_4 \in \mathbb{R}$ and $\overline{\varepsilon} > 0$ such that, for each $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$, the differential equation*

$$x'' = -x + \varepsilon a(t)x + \varepsilon b(t)x^2 + \varepsilon c(t)x^3,$$

where

$$\begin{array}{rcl} a(t) & = & a_0 + a_3\cos(2t) + a_4\sin(2t) \\ b(t) & = & b_1\cos(t) + b_2\sin(t) + b_5\cos(3t) + b_6\sin(3t) \\ c(t) & = & c_0 + c_3\cos(2t) + c_4\sin(2t) + \cos(4t), \end{array}$$

has eight isolated non-null 2π-periodic solutions.

Proof. We write the given second order equation as the first order system,

$$\begin{array}{rcl} x' & = & y, \\ y' & = & -x + \varepsilon[a(t)x + b(t)x^2 + c(t)x^3], \end{array}$$

which has the form (2) with $n = 2$, $T = 2\pi$,

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad f(x, y) = \begin{pmatrix} y \\ -x \end{pmatrix}, \quad g(t, x, y) = \begin{pmatrix} 0 \\ a(t)x + b(t)x^2 + c(t)x^3 \end{pmatrix}.$$

The unperturbed system is

$$x' = y, \quad y' = -x,$$

which is linear, and whose adjoint is again

$$x' = y, \quad y' = -x.$$

Let

$$k = 2, \quad \xi : \mathbb{R}^2 \to \mathbb{R}^2, \quad \xi(\theta) = \theta, \quad \text{for any} \quad \theta \in \mathbb{R}^2.$$

It is easy to see that,

$$u(t, \theta) = \begin{pmatrix} \theta_1\cos t + \theta_2\sin t \\ -\theta_1\sin t + \theta_2\cos t \end{pmatrix}, w_1(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}, \quad w_2(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}.$$

Hence $\mathcal{Z} = \mathbb{R}^2$ is a 2-dimensional T-period manifold which, by Proposition 2, is normally non-degenerate.

In this case, the Malkin bifurcation function is

$$M : \mathbb{R}^2 \to \mathbb{R}^2, \quad M_1(\theta) = -\int_0^{2\pi} m(t, \theta)\sin t dt, \quad M_2(\theta) = \int_0^{2\pi} m(t, \theta)\cos t dt.$$

where $\theta = (\theta_1, \theta_2)$, $M = (M_1, M_2)$ and

$$m(t, \theta) = a(t)(\theta_1\cos t + \theta_2\sin t) + b(t)(\theta_1\cos t + \theta_2\sin t)^2 + c(t)(\theta_1\cos t + \theta_2\sin t)^3.$$

We remark that the above bifurcation function can also be obtained as a classical averaged function. Indeed, through the linear change of variables $(x, y) \mapsto (\varphi, \psi)$, given by

$$x = \varphi \cos t + \psi \sin t, \quad y = -\varphi \sin t + \psi \cos t,$$

our system is transformed into

$$\varphi' = -\varepsilon\, m(t, \varphi, \psi) \sin t, \quad \psi' = \varepsilon\, m(t, \varphi, \psi) \cos t.$$

After some tedious but straightforward computations we obtain:

$$\begin{aligned}
-\frac{8}{\pi} M_1(\theta) &= 4\theta_1 a_4 + (8a_0 - 4a_3)\theta_2 \\
&\quad + (2b_2 + 2b_6)\theta_1^2 + (4b_1 - 4b_5)\theta_1\theta_2 + (6b_2 - 2b_6)\theta_2^2 \\
&\quad + 2c_4\theta_1^3 + (6c_0 - 3)\theta_1^2\theta_2 + 6c_4\theta_1\theta_2^2 + (6c_0 - 4c_3 + 1)\theta_2^3, \\
\frac{8}{\pi} M_2(\theta) &= (8a_0 + 4a_3)\theta_1 + 4a_4\theta_2 \\
&\quad + (6b_1 + 2b_5)\theta_1^2 + (4b_2 + 4b_6)\theta_1\theta_2 + (2b_1 - 2b_5)\theta_2^2 \\
&\quad + (6c_0 + 4c_3 + 1)\theta_1^3 + 6c_4\theta_1^2\theta_2 + (6c_0 - 3)\theta_1\theta_2^2 + 2c_4\theta_2^3.
\end{aligned}$$

Since both components of M are polynomials of degree 3, we deduce, by Bézout Theorem, that M can have either a continuum of roots, or at most $3 \times 3 = 9$ simple roots. In the sequel our work is focused on finding the coefficients such that M has exactly 9 roots: the $(0, 0)$ and 8 more simple zeros.

The 10 parameters appearing in the expression of M are uniquely defined after imposing that M has, for instance, the following 5 zeros:

$$(1, 1), \quad (2, 7), \quad (3, 5), \quad (-2, 6), \quad (-3, 3).$$

Now the expression of M is

$$\begin{aligned}
kM_1(\theta) &= 24411173\theta_1^3 + 160141171\theta_1^2\theta_2 + 73235519\theta_1\theta_2^2 + 7188869\theta_2^3 \\
&\quad - 650857731\theta_1^2 - 542824494\theta_1\theta_2 - 203799971\theta_2^2 \\
&\quad + 412079986\theta_1 + 720424478\theta_2, \\
-kM_2(\theta) &= 173308805\theta_1^3 + 73235519\theta_1^2\theta_2 + 160141171\theta_1\theta_2^2 + 24411173\theta_2^3 \\
&\quad - 162702255\theta_1^2 - 1301714620\theta_1\theta_2 - 271412470\theta_2^2 \\
&\quad + 892653100\theta_1 + 412079986\theta_2,
\end{aligned}$$

where $k = 16/(3494667\pi)$. So, $M(\theta) = 0$ is a system of two polynomial equations with two unknowns. Moreover, this system has integer coefficients. For solving such a system we used an algorithm based on resultants, see for instance [18, 11]. We will prove that, beside the 5 solutions that we initially imposed and the $(0, 0)$ solution, the system has other three real solutions. Finally, due to Bézout Theorem we deduce that all these 9 roots are simple.

It holds that

$$\begin{aligned}
S(\theta_1) = \mathrm{Res}_{\theta_2}(M_1, M_2) &= \theta_1(\theta_1 + 3)(\theta_1 + 2)(\theta_1 - 1)(\theta_1 - 2)(\theta_1 - 3)P_3(\theta_1), \\
T(\theta_2) = \mathrm{Res}_{\theta_1}(M_1, M_2) &= \theta_2(\theta_2 - 1)(\theta_2 - 3)(\theta_2 - 5)(\theta_2 - 6)(\theta_2 - 7)Q_3(\theta_2),
\end{aligned}$$

where $\mathrm{Res}_y(U(y), V(y))$ denotes the resultant of two polynomials U and V respect to y, and P_3 and Q_3 are two cubic polynomials with rational coefficients. Since the resultants are not identically zero, it holds that the maximum number of solutions of the system $M = 0$, taking into account their multiplicities, is 9. In particular, if we prove that there are 9 solutions, all of them must be simple.

It is easy to prove that P_3 has 3 real roots, say $u_1 < u_2 < u_3$ and Q_3 also, say $v_1 < v_2 < v_3$. In fact

$$u_1 \approx -1.037593, \quad u_2 \approx -0.874090, \quad u_3 \approx 38.784916,$$
$$v_1 \approx -240.095677, \quad v_2 \approx 0.786412, \quad v_3 \approx 5.035431.$$

Hence, each solution (θ_1, θ_2) of $M = 0$ is contained in the set of $9 \times 9 = 81$ couples formed by one solution of S and one of T. In this case, because P_3 and Q_3 are cubic polynomials, it is not difficult to check that the system has exactly 9 solutions: the 6 ones that we already know, together with

$$(u_1, v_3), (u_2, v_2), (u_3, v_1).$$

For higher degree polynomials, the use of Poincaré-Miranda Theorem helps to decide which of the couples, candidate to be a solution of the system, is an actual solution for it, see [10, 11].

Finally, we give now the exact expressions of the coefficients of the equation that has eight non-null 2π-periodic solutions

$$\begin{aligned}
a(t) &= -\frac{40326947}{6989334} - \frac{130476}{105899}\cos(2t) - \frac{20603993}{3494667}\sin(2t)\\
b(t) &= \frac{21705751}{6989334}\cos t + \frac{42732851}{6989334}\sin t - \frac{32576743}{6989334}\cos(3t) + \frac{87438611}{6989334}\sin(3t)\\
c(t) &= -\frac{7181447}{6989334} - \frac{125848}{105899}\cos(2t) - \frac{2441173}{3494667}\sin(2t) + \cos(4t).
\end{aligned}$$

□

Remark 10. *Let $m \geq 1$ be an integer. One can check that Theorem 9 remains valid for the differential equation of arbitrary order*

$$x^{(2m)} = (-1)^m x + \varepsilon a(t)x + \varepsilon b(t)x^2 + \varepsilon c(t)x^3,$$

since the Malkin bifurcation function is the same.

3.4.1. **The equation $x'' = \tilde{a}(t)x$.** In this subsection we will consider the linear equation $x'' = \tilde{a}(t)x$, assuming that $\tilde{a} \in C(\mathbb{R})$ is T-periodic and such that all the solutions of this equation are T-periodic. Also, consider its perturbation

$$x'' = \tilde{a}(t)x + \varepsilon a(t)x + \varepsilon b(t)x^2 + \varepsilon c(t)x^3,$$

where $a, b, c \in C(\mathbb{R})$ are T-periodic. The Malkin bifurcation theorem can be applied to this equation in the same way as it was applied in the proof of Theorem 9.

Just note that, denoting $C(t)$ and $S(t)$ the two linearly independent solutions of $x'' = \tilde{a}(t)x$, that satisfy $S(0) = C'(0) = 0$ and $C(0) = S'(0) = 1$, we have

$$u(t,\theta) = \begin{pmatrix} \theta_1 C(t) + \theta_2 S(t) \\ \theta_1 C'(t) + \theta_2 S'(t) \end{pmatrix}, \; w_1(t) = \begin{pmatrix} -S'(t) \\ S(t) \end{pmatrix}, \quad w_2(t) = \begin{pmatrix} -C'(t) \\ C(t) \end{pmatrix}.$$

Both components of the Malkin bifurcation function will be, like in Theorem 9, polynomials of degree 3. Thus M can have at most 8 non-null simple zeros. Of course, the realization of this number of non-null simple zeros cannot be decided in this general situation.

An interesting aspect to discuss is how to find $\tilde{a}(t)$ such that all the solutions of $x'' = \tilde{a}(t)x$ are T-periodic. An answer is given in the next result.

Theorem 11. *Let $C, S \in C^2(\mathbb{R})$ be two T-periodic functions with the property that, for all $t \in \mathbb{R}$,*

$$W(t) = \begin{vmatrix} C(t) & S(t) \\ C'(t) & S'(t) \end{vmatrix} > 0.$$

Define

$$\tilde{a} = \frac{3}{4}\left(\frac{W'}{W}\right)^2 - \frac{W''}{2W} - \frac{\widetilde{W}}{W}, \quad \textit{where} \quad \widetilde{W} = \begin{vmatrix} C' & S' \\ C'' & S'' \end{vmatrix}.$$

Then the functions

$$\frac{C(t)}{\sqrt{W(t)}} \quad \textit{and} \quad \frac{S(t)}{\sqrt{W(t)}}$$

are two linearly independent T-periodic solutions of

$$x'' = \tilde{a}(t)x.$$

Proof. First we make the following notations.

$$\varphi = \frac{C}{\sqrt{W}}, \quad \psi = \frac{S}{\sqrt{W}}, \quad W_\nu = \begin{vmatrix} \varphi & \psi \\ \varphi' & \psi' \end{vmatrix}, \quad \widetilde{W}_\nu = \begin{vmatrix} \varphi' & \psi' \\ \varphi'' & \psi'' \end{vmatrix}.$$

By direct but cumbersome computations we obtain that

$$W_\nu = 1, \quad \widetilde{W}_\nu = -\tilde{a}.$$

The second order differential equation whose solutions are φ and ψ is

$$\begin{vmatrix} \varphi & \psi & x \\ \varphi' & \psi' & x' \\ \varphi'' & \psi'' & x'' \end{vmatrix} = 0,$$

which is equivalent to $W_\nu x'' - W_\nu' x' + \widetilde{W}_\nu x = 0$ and, further, to $x'' = \tilde{a}(t)x$. $\square$

3.5. **Examples with an arbitrary number of non-null periodic solutions.**

Theorem 12. *Let $N \geq 1$ be an arbitrary integer. We have that there exists $\overline{\varepsilon} > 0$ such that for each $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$ the differential equation*

$$x'' = -x^3 + \varepsilon[a(t)x + b(t)x^2 + c(t)x^3],$$

has $2N$ non-null T-periodic solutions.

Proof. We intend to apply Theorem 3. We write the given second order equation as the first order system,

$$\begin{aligned} x' &= -y, \\ y' &= x^3 - \varepsilon[a(t)x + b(t)x^2 + c(t)x^3], \end{aligned}$$

which is T-periodic and has the form (5) with

$$u = \begin{pmatrix} x \\ y \end{pmatrix}, \quad H(x,y) = x^4/4 + y^2/2, \quad \mathcal{H}(t,x,y) = -a(t)x^2/2 - b(t)x^3/3 - c(t)x^4/4.$$

The unperturbed system is

$$x' = -y, \quad y' = x^3. \tag{7}$$

It has a unique singularity $(0,0)$ and all its nontrivial orbits are the level curves of H defined above, and they are Jordan curves. For every $h > 0$ denote by $\tau(h)$ the main period of the orbit $H = h$. It is known that the period function $\tau : (0,\infty) \to (0,\infty)$ is such that $\tau'(h) < 0$ for all $h > 0$ and that the image of τ is the whole interval $(0,\infty)$. For these results see [5]. In fact, when the origin is a degenerated center (the eigenvalues of the linear part at the critical point are not purely imaginary) it is known that the period of the orbits tend to infinity when the orbits approach the critical point, see [7].

Consider $A \subset \mathbb{R}^2$ the period annulus of (7) whose boundaries are the orbits of periods $T_2 = T+1$ and, respectively, $T_1 = T/(N+1)$. Note that $T/j \in (T_1, T_2)$ for each $j = \overline{1,N}$. Theorem 3 assures that, for each $j = \overline{1,N}$, system (7) has at least two T-periodic solutions, which make exactly j rotations around the origin in the period time T. Thus, system (7) has at least $2N$ different T-periodic solutions. □

Remark 13. *The conclusion of Theorem 12 is also valid when, instead of the unperturbed equation $x'' = -x^3$ we consider, for example, $x'' = -x + x^2$ or $x'' = -x + x^3$.*

4. Consequences of the Malkin theorem in the planar case

In this section we use the notations of subsection 2.1, considering the particular case when $n = 2$.

We assume that the planar autonomous system

$$u' = f(u) \tag{8}$$

has a period annulus $\mathcal{P} \subset \mathbb{R}^2$, i.e. $\mathcal{P}$ is nonempty, open and connected, is invariant under the flow of (8) and any orbit of (8) in $\mathcal{P}$ is a nontrivial closed curve. Thus, in particular, $f(u) \neq 0$ for all $u \in \mathcal{P}$. In these hypotheses it is known that there exists a C^1 first integral $H : \mathcal{P} \to \mathbb{R}$ without critical points. Recall that the C^1, locally nonconstant function $H : \mathcal{P} \to \mathbb{R}$ is a first integral of (8) if and only if

$$\langle \nabla H, f \rangle = 0 \ \text{ in } \ \mathcal{P}.$$

Let $I \subset \mathbb{R}$ be an open nonempty interval such that $\mathcal{P} = \bigcup_{h \in I} \Gamma_h$, where Γ_h is the closed orbit of (8) in $\mathcal{P}$ contained in $H = h$. Denote by $\sigma(t, h)$ the solution of (8) whose orbit is Γ_h. Define also the period function $\tau : I \to (0, \infty)$, $\tau(h)$ being the minimal period of $\sigma(t, h)$. Fix $T > 0$ and assume that there exist $h^* \in I$ and an integer $j \geq 1$ such that $\tau(h^*) = T/j$. Denote by $\xi(t) = \sigma(t, h^*)$. We have that the cycle $\mathcal{Z} = \xi(\mathbb{R})$ is an 1–dimensional T-period manifold. It is known that its nondegeneracy condition can be given in terms of the period function. More precisely, we have the following result, whose proof will be included for completeness. The proof in the particular case of a Hamiltonian system can be found in [9].

Proposition 14. *The cycle $\mathcal{Z}$ is an 1–dimensional T-period manifold which is normally nondegenerate if and only if $\tau'(h^*) \neq 0$.*

Proof. One can check that $u(t, \theta) = \xi(t + \theta)$, which, of course, is T-periodic. It is clear that the number of independent T-periodic solutions of the variational system

$$v' = Df(\xi(t + \theta))v$$

does not depend on θ. Thus, we will study only the system

$$v' = Df(\xi(t))v. \tag{9}$$

Let us start proving that $f(\xi(t))$ and $\frac{\partial \sigma}{\partial h}(t, h^*)$ are both solutions of (9). Recall that $\xi'(t) = f(\xi(t))$. Hence, derivating with respect to t this equality we get

$$\big(f(\xi(t))\big)' = \big(\xi'(t)\big)' = Df(\xi(t))\xi'(t) = Df(\xi(t))f(\xi(t)),$$

as desired. By using that $\sigma(t, h) = f(\sigma(t, h))$ and derivating with respect to h, after some manipulations we get that

$$\Big(\frac{\partial \sigma}{\partial h}(t, h)\Big)' = Df(\sigma(t, h))\frac{\partial \sigma}{\partial h}(t, h).$$

Plugging $h = h^*$ we arrive to

$$\Big(\frac{\partial \sigma}{\partial h}(t, h^*)\Big)' = Df(\xi(t))\frac{\partial \sigma}{\partial h}(t, h^*),$$

as we wanted to prove.

We claim that $f(\xi(t))$ and $\frac{\partial \sigma}{\partial h}(t, h^*)$ are linearly independent. Indeed, taking the derivative with respect to h in the relation

$$H(\sigma(t, h)) = h, \quad h \in I, \ t \in \mathbb{R},$$

one gets $\langle \nabla H(\sigma(t, h)), \frac{\partial \sigma}{\partial h}(t, h) \rangle = 1$ for all $h \in I$, $t \in \mathbb{R}$. Assuming, by contradiction, that $\frac{\partial \sigma}{\partial h}(t, h) = \alpha f(\xi(t))$ (with $\alpha \in \mathbb{R}^*$), from the previous relation we obtain $\langle \nabla H(\xi(t)), f(\xi(t)) \rangle = 1/\alpha$. This contradicts the fact that H is a first integral of (8).

Notice also that $f(\xi(t))$ is a T-periodic solution. Thus, $\mathcal{Z}$ is normally degenerate if and only if $\frac{\partial\sigma}{\partial h}(t,h^*)$ is also T-periodic, which is equivalent to the condition

$$\frac{\partial\sigma}{\partial h}(T,h^*) = \frac{\partial\sigma}{\partial h}(0,h^*). \tag{10}$$

By the definition of $\tau(h)$, we have

$$\sigma(\tau(h),h) = \sigma(0,h), \quad h \in I.$$

Since the minimal period of ξ is T/j and we are interested in studying T-periodic solutions, we need to work with the relation

$$\sigma(j\tau(h),h) = \sigma(0,h), \quad h \in I.$$

After taking the derivative with respect to h in the above relation, and using that $j\tau(h^*) = T$ and $\frac{\partial\sigma}{\partial t}(T,h^*) = \xi'(T) = \xi'(0) = f(\xi(0))$, we obtain

$$j\tau'(h^*)\, f(\xi(0)) + \frac{\partial\sigma}{\partial h}(T,h^*) = \frac{\partial\sigma}{\partial h}(0,h^*).$$

Since $f(\xi(0))$ is not the null vector, using the above relation we deduce that the degeneracy condition (10) is equivalent to $\tau'(h^*) = 0$. $\square$

The next result will be useful in the construction of the Malkin bifurcation function.

Proposition 15. $[\nabla H(\xi(t))]^{\mathrm{tr}}$ *is a solution of the linear system*

$$v' = -[Df(\xi(t))]^{\mathrm{tr}} v.$$

Proof. As usual, ∇H is a line vector. Note that, in fact, we have to prove the validity of the following relation for all $t \in \mathbb{R}$,

$$\frac{d}{dt}\nabla H(\xi(t)) = -\nabla H(\xi(t)) Df(\xi(t)).$$

This will follow, on one hand, from

$$\frac{d}{dt}\nabla H(\xi(t)) = D(\nabla H)(\xi(t))\xi'(t) = D(\nabla H)(\xi(t)) f(\xi(t)),$$

and on the other hand, from the identity

$$D(\nabla H)f = -\nabla H Df, \text{ valid in } \mathcal{P},$$

which is obtained by differentiating $\langle \nabla H, f\rangle = 0$ in $\mathcal{P}$. The proof is done. $\square$

We deduce that for the planar system

$$u' = f(u) + \varepsilon g(t,u) \tag{11}$$

the Malkin bifurcation function, $M : \mathbb{R} \to \mathbb{R}$, has the following expression

$$M(\theta) = \int_0^T \langle \nabla H(\xi(t+\theta)), g(t,\xi(t+\theta))\rangle dt.$$

Note that M is T/j-periodic. Note also that it can be written as

$$M(\theta) = \int_0^T \langle \nabla H(\xi(t)), g(t-\theta, \xi(t))\rangle dt. \tag{12}$$

A consequence of Theorem 1 in this case is the following result.

Theorem 16. *Assume that $\tau'(h^*) \neq 0$ and that there exists $\theta^* \in [0, T/j)$ such that $M(\theta^*) = 0$ and $M'(\theta^*) \neq 0$, where M is given in* (12)*. Then there exists $\overline{\varepsilon} > 0$ such that for each $\varepsilon \in (-\overline{\varepsilon}, \overline{\varepsilon})$, equation* (11) *has a T-periodic solution u_ε with $\lim_{\varepsilon\to 0} u_\varepsilon(0) = \xi(\theta^*)$ and with the property that it is isolated in the set of T-periodic solutions of* (11)*.*

Without loosing the generality, suppose that the origin is a singularity of $u' = f(u)$ and that the closed orbits in $\mathcal{P}$ encircle the origin. Note that, since $u_\varepsilon(t)$ and $\xi(t+\theta^*)$ are "close enough" on the compact interval $[0, T]$, we deduce that the number of rotations of $u_\varepsilon(t)$ around the origin in the interval $[0, T]$ is equal to the number of rotations of $\xi(t+\theta^*)$ around the origin in the same interval. Thus, u_ε makes exactly j rotations around the origin in the period time T.

We present a consequence of Theorem 7 from [4] for the case considered in this section.

Theorem 17. *Assume that $\tau'(h^*) \neq 0$ and M is given in* (12)*. Then*

(i) For any sequences $(u_m)_{m\geq 1}$ from $C(\mathbb{R}, \mathbb{R}^2)$ and $(\varepsilon_m)_{m\geq 1}$ from $[0, 1]$ such that $u_m(0) \to \xi(\theta^*) \in \xi(\mathbb{R})$, $\varepsilon_m \to 0$ as $m \to \infty$ and u_m is a T-periodic solution of (11) *with $\varepsilon = \varepsilon_m$, we have that $M(\theta^*) = 0$.*

(ii) Assume that there exist $0 \leq \theta_1 < \theta_2 < T/j$ such that $M(\theta_1)M(\theta_2) < 0$. Then M has at least two zeros $\theta_1^* \in (\theta_1, \theta_2)$, $\theta_2^* \in (\theta_2, \theta_1 + T/j)$, while system (11) *has at least two T-periodic solutions, denoted $u_{1,\varepsilon}$, $u_{2,\varepsilon}$, which make exactly j rotations around the origin in the period time T, and such that the distance between $\varphi_\varepsilon(0)$ and $\xi(\mathbb{R})$ goes to 0 as $\varepsilon \to 0$.*

Remarks. Hypothesis (ii) in the theorem above assures that the Brouwer degree of M on both intervals (θ_1, θ_2) and $(\theta_2, \theta_1 + T/j)$, respectively, is not null. This remark is to justify that the hypotheses of Theorem 7 from [4] are satisfied. For the Brouwer degree theory one can see the book of Dincă-Mawhin [8].

In general, Theorem 17 does not assure that the periodic solutions obtained are isolated in the set of T-periodic solutions.

If M is not identically null and $\int_0^T M(\theta)d\theta = 0$, then there exist $0 \leq \theta_1 < \theta_2 < T/j$ such that $M(\theta_1)M(\theta_2) < 0$.

In the sequel we provide an example of perturbation g in system (11) such that the Malkin bifurcation function associated satisfies the hypothesis in (ii) of Theorem 17. In addition, we provide an example of such system whose Malkin bifurcation function does not have any zero, thus the system does not have T-periodic solutions like in (i) of Theorem 17. The last result assures the existence of $2N$ periodic solutions for an arbitrary integer $N \geq 1$. We will skip the proofs of the next results, since they are not difficult.

Theorem 18. *Assume that $\tau'(h^*) \neq 0$. Let $A \in C(\mathbb{R})$ be T-periodic and consider $g(t,u) = A(t)\nabla H(u)$. For the planar system*

$$u' = f(u) + \varepsilon A(t)\nabla H(u) \tag{13}$$

the Malkin bifurcation function can be computed as

$$M(\theta) = \int_0^T A(t-\theta)||\nabla H(\xi(t))||^2 dt.$$

(i) If M is not identically null and $\int_0^T A(s)ds = 0$, then there exist $0 \leq \theta_1 < \theta_2 < T/j$ such that $M(\theta_1)M(\theta_2) < 0$.

(ii) If $A(t) \neq 0$ for all $t \in \mathbb{R}$, then $M(\theta) \neq 0$ for all $\theta \in \mathbb{R}$.

Theorem 19. *Let $N \geq 1$ be fixed. Assume that $\tau'(h) \neq 0$ for all $h \in I$ and that $\tau(I) = (0,\infty)$. For any $j \geq 1$ denote by $\xi_j(t)$ the solution of $u' = f(u)$ of minimal period T/j and*

$$M_j(\theta) = \int_0^T A(t-\theta)||\nabla H(\xi_j(t))||^2 dt.$$

(i) If there exist N functions from $\{M_j \; : \; j \geq 1\}$ that are not identically null and $\int_0^T A(s)ds = 0$, then system (13) *has at least $2N$ different T-periodic solutions.*

(ii) If $A(t) \neq 0$ for all $t \in \mathbb{R}$, then $M_j(\theta) \neq 0$ for all $\theta \in \mathbb{R}$, for all $j \geq 1$.

Acknowledgements

This work was supported by Ministerio de Ciencia, Innovación y Universidades of the Spanish Government by grants MTM2016-77278-P (MINECO/AEI/FEDER, UE) and 2017-SGR-1617 from AGAUR, Generalitat de Catalunya.

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[1] Departamentul de Matematică, Universitatea Babeş–Bolyai, Str. Kogălniceanu 1, 400084 Cluj–Napoca, Romania
E-mail address: abuica@math.ubbcluj.ro

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Barcelona, Spain
E-mail address: gasull@mat.uab.cat