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NILPOTENT SADDLES OF LINEAR PLUS CUBIC HOMOGENEOUS POLYNOMIAL REVERSIBLE VECTOR FIELDS

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Abstract. We provide normal forms and the global phase portraits in the Poincaré disk for all planar polynomial vector fields of the form lineal plus cubic homogeneous that are symmetric with respect to the x-axis or to the y-axis and having a nilpotent saddle at the origin.

1. Introduction and statement of the results

Quadratic systems have been widely studied in the last 100 years, and more than 1.000 papers have been published about them. The classification of centers for quadratic polynomial differential systems goes back mainly to Dulac [5], Kapteyn [7, 8] and Bautin [2]. In [9] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center and there are also many partial results for the centers of planar polynomial differential systems of degree larger than two. Mostly of these results are for linear centers but recently Colak, Llibre and Valls [3, 4] provided the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a nilpotent center at the origin, together with their bifurcation diagrams. Despite the fact that there is a large list of works regarding systems which have a center at the origin, mostly nothing is known about systems that have a saddle at the origin. In this paper we are concerned with this last problem and we are interested in providing all global phase portraits on the Poincaré disk of all planar polynomial vector fields having only linear and cubic homogeneous terms which have a nilpotent saddle at the origin. This problem is too hard to study due to the huge number of parameters and so we will restrict to the systems that are $\mathbb{Z}_2$ reversible.

Vector fields with symmetry appear very often in applications, so the study of symmetric vector fields is not old fashioned and nowadays it has been an increasing interest in systems that are $\mathbb{Z}_2$ reversible. We recall that a vector field

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y} \tag{1}$$

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has a $\mathbb{Z}_2$-reversibility if is invariant by the change of variables $(x, y, t) \mapsto (R(x, y), -t)$ in which

$$R(x, y) = (x, -y), \text{ or } R(x, y) = (-x, y) \text{ or } R(x, y) = (-x, -y).$$

We consider that P and Q are linear plus cubic homogeneous. Note that under these assumptions, the vector field (1) is never invariant by the change $(x, y, t) \mapsto (R(x, y), -t)$ with $R(x, y) = (-x, -y)$ and so we will restrict to the cases $R(x, y) = (x, -y)$, or $R(x, y) = (-x, y)$. In these two cases, the vector field must satisfy

$$MX(x, y) = -X(x, -y), \quad MX(x, y) = X(-x, y),$$

where

$$M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

In short, in this paper we classify the global phase portraits of all planar polynomial vector fields of the form linear plus homogeneous of degree three reversible with respect to the $x-$axis or to the y-axis having a nilpotent saddle at the origin. To do this we will use the Poincaré compactification of polynomial vector fields. The Poincaré compactification that we shall use for describing the global phase portraits of our systems is standard. For all the definitions and results on the Poincaré compactification see Chapter 5 of [6]. We say that two vector fields on the Poincaré disk are topologically equivalent if there exists a homeomorphism from one into the other which sends orbits to orbits preserving or reversing the direction of the flow. Our main result is the following one.

Theorem 1. *A planar polynomial vector field of the form linear plus cubic homogeneous of degree three with a nilpotent saddle at the origin and $\mathbb{Z}_2$-reversible with $R(x, y) = (x, -y)$ or $R(x, y) = R(-x, y)$, after a linear change of variables and a rescaling of ten classes:*

$$
\begin{aligned}
(I)\ & x' = y + bx^2y + sy^3,\ y' = ax^3 + xy^2;\\
(II)\ & x' = y + bx^2y,\ y' = x^3 + xy^2;\\
(III)\ & x' = y + bx^2y + sy^3,\ y' = ax^3 - xy^2;\\
(IV)\ & x' = y + bx^2y,\ y' = x^3 - xy^2;\\
(V)\ & x' = y + sx^2y + y^3,\ y' = ax^3;\\
(VI)\ & x' = y + y^3,\ y' = x^3;\\
(VII)\ & x' = y + sx^2y - y^3,\ y' = ax^3;\\
(VIII)\ & x' = y - y^3,\ y' = x^3;\\
(IX)\ & x' = y + sx^2y,\ y' = x^3;\\
(X)\ & x' = y,\ y' = x^3,
\end{aligned}
$$

where $b \in \mathbb{R}$, $a \in \mathbb{R}^+$ and $s \in \{-1, 1\}$. Moreover, the global phase portraits of these twelve families are topologically equivalent to the following of Figure 1:

- *for system (I): 1.1 if $s = 1$; 1.2 if $b > 1 + 2\sqrt{a}$, $s = -1$; 1.3 if $b = 1 + 2\sqrt{a}$, $s = -1$ and 1.4 if $b < 1 + 2\sqrt{a}$, $s = -1$;*
- *for system (II): 1.5 if $b \leq 1$ and 1.6 if $b > 1$;*
- *for system (III): 1.7 if $b < -a$, $s = 1$; 1.1 if $b \geq -1$, $s = 1$; 1.8 if $b < -1 - 2\sqrt{a}$, $s = -1$; 1.9 if $b \in (-1 - 2\sqrt{a}, 1)$, $a \geq 1$, $s = -1$,*

or $b \in (-1-2\sqrt{a}, -1+2\sqrt{a})$, $a \in (0,1)$, $s=-1$*;* 1.10 *if* $b=1$, $a>1$, $s=-1$*;* 1.11 *if* $b \in (1, -1+2\sqrt{a})$, $a>1$, $s=-1$*;* 1.12 *if* $b=-1+2\sqrt{a}$, $a>1$, $s=-1$*;* 1.13 *if* $b \in (-1+2\sqrt{a}, a)$, $a>1$, $s=-1$*;* 1.14 *if* $b \geq a$, $a \neq 1$, $s=-1$, *or* $b>1$, $a=1$, $s=-1$*;* 1.15 *if* $b=a=1$, $s=-1$*;* 1.16 *if* $b=-1+2\sqrt{a}$, $a \in (0,1)$, $s=-1$*;* 1.17 *if* $b \in (-1+2\sqrt{a}, a)$, $a \in (0,1)$, $s=-1$*;*
- *for system* (IV)*:* 1.6 *if* $b \geq 0$*;* 1.19 *if* $b \in (-1,0)$ *and* 1.18 *if* $b \leq -1$*;*
- *for system* (V) *or system* (VI)*:* 1.1*;*
- *for system* (VII)*;* 1.4 *if* $a > 1/4$*;* 1.20 *if* $a=1/4$, $s=1$*;* 1.21 *if* $a<1/4$, $s=1$ *and* 1.22 *if* $a \leq 1/4$, $s=-1$*;*
- *for system* $(VIII)$*:* 1.4*;*
- *for system* (IX)*:* 1.6 *if* $s=1$ *and* 1.5 *if* $s=-1$*;*
- *for system* (X)*:* 1.5.

The proof of Theorem 1 is given in section 3.

2. Normal forms (I)–(X) in Theorem 1

Consider the planar polynomial vector fields of the form linear plus homogeneous of degree three

$$\begin{aligned} \dot{x} &= y + a_1x^3 + a_2x^2y + a_3xy^2 + a_4y^3 \\ \dot{y} &= b_1x^3 + b_2x^2y + b_3xy^2 + b_4y^3. \end{aligned} \tag{2}$$

By imposing that the system is reversible with respect to the $x-$axis or to the y-axis; that is, that is invariant by the symmetry $(x,y,t) \to (-x,y,-t)$ or $(x,y,t) \to (x,-y,t)$ we get

$$a_1 = 0, \quad a_3 = 0, \quad b_2 = 0, \quad b_4 = 0,$$

so system (2) becomes

$$\begin{aligned} \dot{x} &= y + A(x,y) = y + a_2x^2y + a_4y^3 \\ \dot{y} &= B(x,y) = b_1x^3 + b_3xy^2. \end{aligned} \tag{3}$$

Now we impose that (3) has a nilpotent saddle at the origin by applying Theorem 3.5 in [6]. Clearly the origin is a singular point of (3) and $y=0$ is a solution of $y + A(x,y) = 0$. Substituting this solution into $F(x) = B(x,0)$ and $G(x) = (\partial A/\partial x + \partial A/\partial y)(x,0)$ we get $F(x) = b_1x^3$ and $G=0$. Then system (3) has a nilpotent saddle at the origin if and only if the parameter b_1 is positive. From now on we assume that $b_1 > 0$.

We consider the change of variables and the rescaling of time

$$X \to Ax, \qquad Y \to By, \qquad T \to Ct. \tag{4}$$

Case 1. Assume $b_3 > 0$. We consider two different subcases.

Subcase 1.1. If $a_4 \neq 0$, by the change given in (4) with

$$A = \sqrt{b_3}, \qquad B = \sqrt{|a_4|}, \qquad C = \frac{\sqrt{b_3}}{\sqrt{|a_4|}}$$

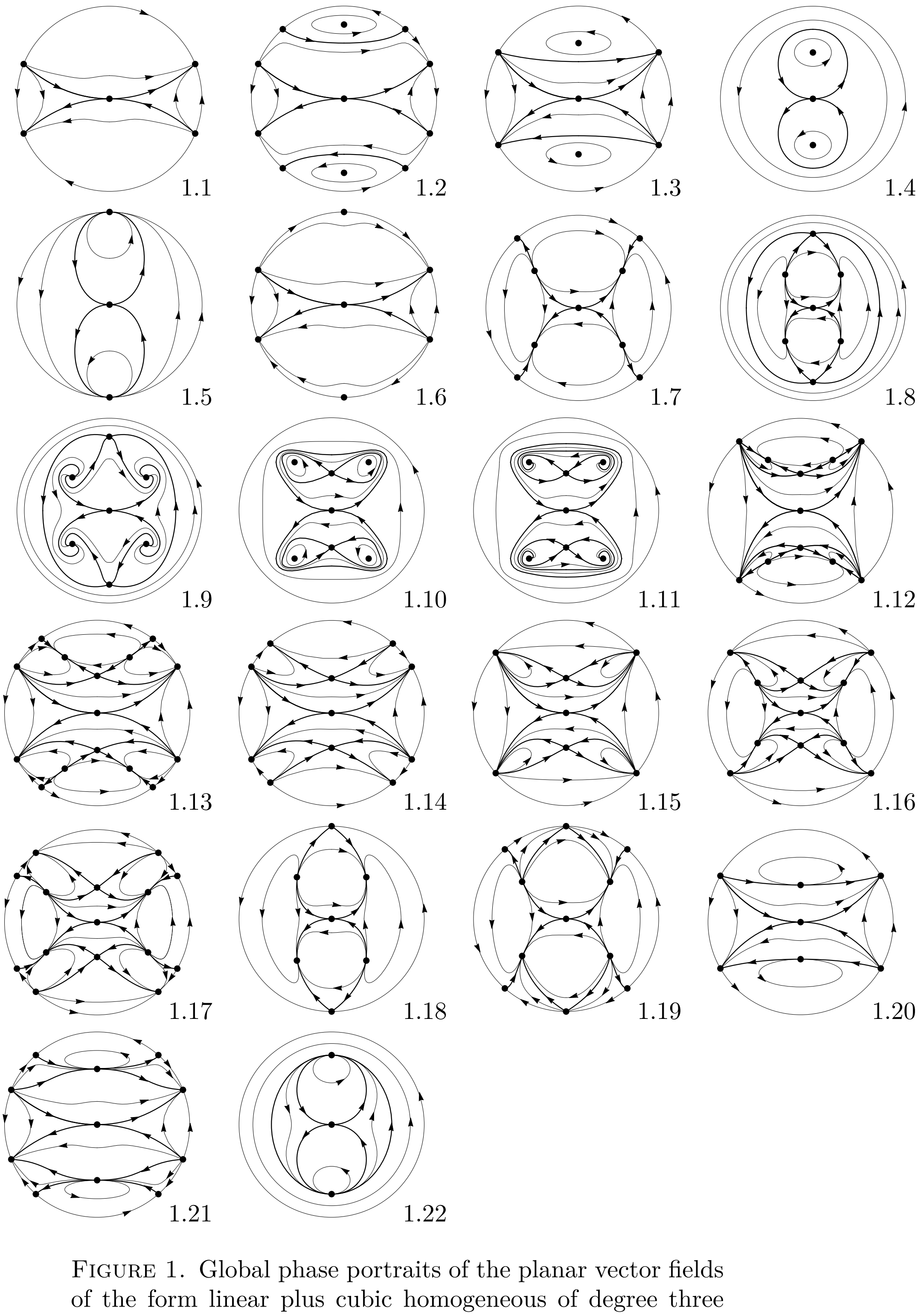

FIGURE 1. Global phase portraits of the planar vector fields of the form linear plus cubic homogeneous of degree three with a nilpotent saddle at the origin and $\mathbb{Z}_2$-reversible with $R(x,y) = (x,-y)$ or $R(x,y) = R(-x,y)$. The separatrices are in bold.

we get system (I) with $b = a_2/b_3$ and $a = a_4b_1/b_3^2$. The case $s = 1$ (resp. $s = -1$) corresponds to $a_4 > 0$ (resp. $a_4 < 0$).

Subcase 1.2. If $a_4 = 0$, by (4) with

$$A = \sqrt{b_3}, \qquad B = \frac{b_3}{\sqrt{b_1}}, \qquad C = \frac{\sqrt{b_1}}{\sqrt{b_3}},$$

we get system (II) with $b = a_2/b_3$.

Case 2. Assume $b_3 < 0$. We consider also two different subcases.

Subcase 2.1. If $a_4 \neq 0$, by the change given in (4) with

$$A = \sqrt{-b_3}, \qquad B = \sqrt{|a_4|}, \qquad C = \frac{\sqrt{-b_3}}{\sqrt{|a_4|}},$$

we get system (III) with $b = -a_2/b_3$ and $a = a_4b_1/b_3^2$. The case $s = 1$ (resp. $s = -1$) corresponds to $a_4 > 0$ (resp. $a_4 < 0$).

Subcase 2.2. If $a_4 = 0$, by (4) with

$$A = \sqrt{-b_3}, \qquad B = \frac{b_3}{\sqrt{b_1}}, \qquad C = -\frac{\sqrt{b_1}}{\sqrt{-b_3}},$$

we get system (IV) with $b = -a_2/b_3$.

Case 3. Assume $b_3 = 0$. We consider different subcases.

Subcase 3.1. If $a_4 > 0$ we consider two subcases.
Subcase 3.1.1. If $a_2 \neq 0$, by the change given in (4) with

$$A = \sqrt{|a_2|}, \qquad B = \sqrt{a_4}, \qquad C = \frac{\sqrt{|a_2|}}{\sqrt{a_4}},$$

we get system (V) with $a = a_4b_1/a_2^2$. The case $s = 1$ (resp. $s = -1$) corresponds to $a_2 > 0$ (resp. $a_2 < 0$).

Subcase 3.1.2. If $a_2 = 0$, by (4) with

$$A = \sqrt[4]{a_4b_1}, \qquad B = \sqrt{a_4}, \qquad C = \sqrt[4]{\frac{b_1}{a_4}},$$

we get system (VI).

Subcase 3.2. If $a_4 < 0$ we consider three subcases.
Subcase 3.2.1. If $a_2 \neq 0$, by the change given in (4) with

$$A = \sqrt{|a_2|}, \qquad B = \sqrt{-a_4}, \qquad C = \frac{\sqrt{|a_2|}}{\sqrt{-a_4}},$$

we get system (VII) with $a = -a_4b_1/a_2^2$. The case $s = 1$ (resp. $s = -1$) corresponds to $a_2 > 0$ (resp. $a_2 < 0$).

Subcase 3.2.2. If $a_2 = 0$, by (4) with

$$A = \sqrt[4]{-a_4b_1}, \qquad B = \sqrt{-a_4}, \qquad C = \sqrt[4]{\frac{b_1}{-a_4}},$$

we get system (VIII).

Subcase 3.3. If $a_4 = 0$ we consider three subcases.
Subcase 3.3.1. If $a_2 \neq 0$, by the change given in (4) with

$$A = \sqrt{|a_2|}, \qquad B = \frac{|a_2|}{\sqrt{b_1}}, \qquad C = \frac{\sqrt{b_1}}{\sqrt{|a_2|}},$$

we get a system (IX). The case $s = 1$ (resp. $s = -1$) corresponds to $a_2 > 0$ (resp. $a_2 < 0$).

Subcase 3.3.2. If $a_2 = 0$, by (4) with

$$A = A, \qquad B = \frac{A^2}{\sqrt{b_1}}, \qquad C = \frac{\sqrt{b_1}}{A},$$

we get system (X).

3. Proof of Theorem 1

3.1. **Global phase portrait of system (I) with $s = 1$.** Consider system (I) with $s = 1$

$$x' = y + bx^2y + y^3, \quad y' = ax^3 + xy^2, \quad a \in \mathbb{R}^+,\ b \in \mathbb{R}.$$

There unique finite singular point is the origin. In the local chart U_1 system (I) with $s = 1$ becomes

$$u' = a - u^2(-1 + b + u^2 + v^2), \quad v' = -uv(b + u^2 + v^2).$$

When $v = 0$, taking into account that $a > 0$ it follows that for all values of $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ there are two infinite singular points on the local chart U_1 which are $(u_\pm, 0)$ where

$$u_\pm = \pm\frac{\sqrt{1 - b + \sqrt{4a + (-1 + b)^2}}}{\sqrt{2}}.$$

Computing the eigenvalues of the Jacobian matrix at the points $(w, 0)$ with $w = u_\pm$ we get that they are $-w(w^2 + b)$ and $-2w(2w^2 + b - 1)$. The quantities $(u_\pm)^2 + b$ and $2(u_\pm)^2 + b - 1$ are both positive. So, $(u_+, 0)$ is a stable node, $(u_-, 0)$ is an unstable node.

In the local chart U_2 we get

$$u' = 1 + (-1 + b)u^2 - au^4 + v^2, \quad v' = -(u + au^3)v.$$

The origin is not an infinite singular point of the system.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system (I) with $s = 1$ is topologically equivalent to 1.1 of Figure 1.

3.2. **Global phase portrait of system (I) with $s = -1$.** Consider system (I) with $s = -1$

$$x' = y + bx^2y - y^3, \quad y' = ax^3 + xy^2, \quad a \in \mathbb{R}^+,\ b \in \mathbb{R}.$$

There are three finite singular points: the origin which is a nilpotent saddle and the points $(0, \pm 1)$. Computing the eigenvalues of the Jacobian matrix at these two points we get that they are $\pm i\sqrt{2}$ for both points. Taking into account that the system is reversible we get that they are both centers.

In the local chart U_1 system (I) with $s = -1$ becomes

$$u' = a + u^2(1 - b + u^2 - v^2), \quad v' = -uv(b - u^2 + v^2).$$

When $v = 0$, taking into account that $a > 0$ it follows that if $b < 1 - 2\sqrt{a}$ there are no infinite singular points in the local chart U_1, if $b > 1 + 2\sqrt{a}$ there are four infinite singular points on the local chart U_1 which are $(u^0_{\pm}, 0)$ and $(u^1_{\pm}, 0)$ where

$$u^0_{\pm} = \pm \frac{\sqrt{b - 1 + \sqrt{-4a + (-1 + b)^2}}}{\sqrt{2}},\ u^1_{\pm} = \pm \frac{\sqrt{b - 1 - \sqrt{-4a + (-1 + b)^2}}}{\sqrt{2}}$$

and if $b = 1 + 2\sqrt{a}$ there are two infinite singular points on the local chart U_1 which are $(u^0_+, 0)$ and $(u^0_-, 0)$ (in this case the points $(u^0_+, 0)$ and $(u^1_+, 0)$ collide and the same happens with $(u^0_-, 0)$ and $(u^1_-, 0)$).

Computing the eigenvalues of the Jacobian matrix at the points $(w, 0)$ with w either $u^0_{\pm}$ or $u^1_{\pm}$ we get that they are $w(w^2 - b)$ and $2w(2w^2 - b + 1)$.

If $b > 1 + 2\sqrt{a}$ the quantities $(u^1_{\pm})^2 - b$ and $2(u^1_{\pm})^2 - b + 1$ are both negative and the quantities $(u^0_{\pm})^2 - b$ and $2(u^0_{\pm})^2 - b + 1$ are negative and positive, respectively. So, $(u^1_+, 0)$ is a stable node, $(u^1_-, 0)$ is an unstable node and $(u^0_{\pm}, 0)$ are saddles.

If $b = 1 + 2\sqrt{a}$ the points $(u^0_+, 0)$ and $(u^0_-, 0)$ are both semihyperbolic. Using Theorem 2.19 in [6] we get that they are both saddle-nodes.

In the local chart U_2 we get

$$u' = -1 + (-1 + b)u^2 - au^4 + v^2, \quad v' = -(u + au^3)v.$$

The origin is not an infinite singular point of the system.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system (I) with $s = -1$ is topologically equivalent to: 1.2 of Figure 1 if $b > 1 + 2\sqrt{a}$; 1.3 of Figure 1 if $b = 1 + 2\sqrt{a}$ and 1.4 of Figure 1 if $b < 1 + 2\sqrt{a}$.

3.3. **Global phase portrait of system (II).** Consider system (II)

$$x' = y + bx^2y, \quad y' = x^3 + xy^2, \quad b \in \mathbb{R}.$$

The origin is the unique finite singular point.

In the local chart U_1 system (II) becomes

$$u' = 1 - u^2(-1 + b + v^2), \quad v' = -uv(b + v^2).$$

When $v = 0$ and $b \le 1$ there are no infinite singular points on the local chart U_1 and when $b > 1$ there are two infinite singular points on the local chart U_1 which are $(u^0_\pm, 0)$ with $u^0_\pm = \pm 1/\sqrt{b-1}$. The eigenvalues of the Jacobian matrix at these points are $\mp 2\sqrt{b-1}$ and $\mp b/\sqrt{b-1}$, respectively. Hence, $(u^0_+, 0)$ is a stable node and $(u^0_-, 0)$ is an unstable node.

In the local chart U_2 we get

$$u' = (-1 + b)u^2 - u^4 + v^2, \quad v' = -(u + u^3)v.$$

The origin is an infinite singular point of the system, whose linear part is zero. Applying the blow-up technique (see [1] for more details) we obtain that the origin of U_2 is formed by two hyperbolic sectors when $b > 1$; by two elliptic sectors when $b \in [0, 1]$ and by two elliptic and four parabolic sectors when $b < 0$.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system (II) is topologically equivalent to: 1.5 of Figure 1 if $b \le 1$ and 1.6 of Figure 1 if $b > 1$.

3.4. **Global phase portrait of system (III) with $s = 1$.** Consider system (III) with $s = 1$,

$$x' = y + bx^2y + y^3, \quad y' = ax^3 - xy^2, \quad a \in \mathbb{R}^+, \ b \in \mathbb{R}.$$

The origin is always a singular point (which is a nilpotent saddle). If $a+b < 0$, there are four additional singular points $(x_\pm, y_\pm)$ with

$$x_\pm = \pm\frac{1}{\sqrt{-a-b}}, \quad y_\pm = \pm\frac{\sqrt{a}}{\sqrt{-a-b}}.$$

Computing the eigenvalues of the Jacobian matrix at the points (x_-, y_-) and (x_+, y_+) we get that they are

$$\lambda_1 = -\frac{\sqrt{a}\left(b - 1 + \sqrt{4a + (b+1)^2}\right)}{a + b}, \quad \lambda_2 = -\frac{\sqrt{a}\left(b - 1 - \sqrt{4a + (b+1)^2}\right)}{a + b},$$

and at the points (x_-, y_+) and (x_+, y_-) they are $-\lambda_1$ and $-\lambda_2$. We see that $\lambda_1, \lambda_2 < 0$ for all $b < -a$. Hence (x_-, y_-) and (x_+, y_+) are stable nodes whereas (x_-, y_+) and (x_+, y_-) are unstable nodes.

In the local chart U_1 system (III) with $s = 1$ becomes

$$u' = a - u^2(1 + b + u^2 + v^2), \quad v' = -uv(b + u^2 + v^2).$$

When $v = 0$, taking into account that $a > 0$, it follows that for all values of $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ there are two infinite singular points on the local chart U_1 which are $(u_\pm, 0)$ where

$$u_\pm = \pm\frac{\sqrt{-1 - b + \sqrt{4a + (1+b)^2}}}{\sqrt{2}}.$$

Computing the eigenvalues of the Jacobian matrix at the points $(w, 0)$ with $w = u_\pm$ we get that they are $-w(w^2+b)$ and $-2w(2w^2+b+1)$. The quantity $2w^2+b+1$ is always positive. The quantity $(u_\pm)^2+b$ is positive for $b > -a$, negative for $b < -a$ and zero for $b = -a$. So, if $b > -a$, $(u_-, 0)$ is an unstable node and $(u_+, 0)$ is a stable node; if $b < -a$, both $(u_-, 0)$ and $(u_+, 0)$ are saddles.

If $b = -a$, the points $(u_\pm, 0)$ are both semihyperbolic. Using Theorem 2.19 in [6] we get that $(u_+, 0)$ is a stable node and $(u_-, 0)$ is an unstable node.

In the local chart U_2 we get

$$u' = 1 + (1+b)u^2 - au^4 + v^2, \quad v' = (u - au^3)v.$$

The origin is not an infinite singular point of the system.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system (III) with $s = 1$ is topologically equivalent to: 1.1 of Figure 1 if $b \geq -a$ and to 1.7 of Figure 1 if $b < -a$.

3.5. **Global phase portrait of system (III) with $s = -1$.** Consider system (III) with $s = -1$

$$x' = y + bx^2y - y^3, \quad y' = ax^3 - xy^2, \quad a \in \mathbb{R}^+, \; b \in \mathbb{R}.$$

If $b \geq a$ there are three finite singular points: the origin which is a nilpotent saddle and the points $(0, \pm 1)$. Computing the eigenvalues of the Jacobian matrix at these two points we get that they are $\pm\sqrt{2}$. Hence, both points are saddles. If $b < a$ among the three above mentioned points $(0, 0)$ and the saddles $(0, \pm 1)$, we have the four finite singular points $(x_\pm, y_\pm)$ where

$$x_\pm = \pm\frac{1}{\sqrt{a-b}}, \quad y_\pm = \frac{1}{\sqrt{a-b}}.$$

The eigenvalues of the Jacobian matrix at the points (x_-, y_-) and (x_+, y_+) are

$$\lambda_1 = \frac{\sqrt{a}(-1+b-\sqrt{(1+b)^2-4a})}{a-b}, \quad \lambda_2 = \frac{\sqrt{a}(-1+b+\sqrt{(1+b)^2-4a})}{a-b}$$

while the eigenvalues of the Jacobian matrix at the points (x_+, y_-) and (x_-, y_+) are $-\lambda_1$, $-\lambda_2$. So, we will study only the behavior of the finite singular point (x_-, y_-) because the behavior of the other remaining three points will follow from this one.

The eigenvalues of the Jacobian matrix at the points (x_-, y_-) are real for $b \in \{(-\infty, -1-2\sqrt{a}] \cup [-1+2\sqrt{a}, a)\}$ and otherwise they are complex. Moreover, if $b \in (-\infty, -1-2\sqrt{a}]$ the eigenvalues are both real and negative, so (x_-, y_-) is a stable node. On the other hand, if $b > -1-2\sqrt{a}$ we need to distinguish between the three cases $a > 1$, $a = 1$ and $a \in (0, 1)$.

Case 1*:* $a > 1$. In this case, we get that if $b \in (-1-2\sqrt{a}, 1)$ the eigenvalues are complex with negative real part. So, the point (x_-, y_-) is an stable focus. If $b = 1$, the eigenvalues are purely imaginary and taking into account that system (III) with $s = -1$ is reversible, the point (x_-, y_-) is a center. If

$b \in (1, -1+2\sqrt{a})$ the eigenvalues are complex with positive real part and if $b \in [-1+2\sqrt{a}, a)$ they are real and positive. So, the point (x_-, y_-) is an unstable focus for $b \in (1, -1+2\sqrt{a})$ and an unstable node for $b \in [-1+2\sqrt{a}, a)$.

Case 2*:* $a = 1$. In this case, we get that if $b \in (-1-2\sqrt{a}, 1)$ the eigenvalues are complex with negative real part, so (x_-, y_-) is a stable focus.

Case 3*:* $a \in (0, 1)$. In this case, we get that the eigenvalues are complex with negative real part if $b \in (-1-2\sqrt{a}, -1+2\sqrt{a})$ and are real and negative real part if $b \in [-1+2\sqrt{a}, a)$. So, the point (x_-, y_-) is an stable focus if $b \in (-1-2\sqrt{a}, -1+2\sqrt{a})$ and a stable node if $b \in [-1+2\sqrt{a}, a)$.

In the local chart U_1 system (III) with $s = -1$ becomes

$$u' = a + u^2(-1 - b + u^2 - v^2), \quad v' = -uv(b - u^2 + v^2).$$

When $v = 0$, taking into account that $a > 0$ it follows that if $b < -1+2\sqrt{a}$ there are no infinite singular points in the local chart U_1, if $b > -1+2\sqrt{a}$ there are four infinite singular points on the local chart U_1 which are $(u^0_\pm, 0)$ and $(u^1_\pm, 0)$ with

$$u^0_\pm = \pm\frac{\sqrt{b+1+\sqrt{-4a+(-1+b)^2}}}{\sqrt{2}}, \; u^1_\pm = \pm\frac{\sqrt{b+1-\sqrt{-4a+(-1+b)^2}}}{\sqrt{2}}$$

and if $b = -1+2\sqrt{a}$ there are two infinite singular points on the local chart U_1 which are $(u^0_+, 0)$ and $(u^0_-, 0)$ (in this case the points $(u^0_+, 0)$ and $(u^1_+, 0)$ collide and the same happens with $(u^0_-, 0)$ and $(u^1_-, 0)$).

If $b = -1+2\sqrt{a}$ and $a \neq 1$ both infinite singular points are semihyperbolic. Applying Theorem 2.19 in [6] we obtain that both points are saddle-nodes.

If $b \in (-1+2\sqrt{a}, a)$ then when $a \in (0, 1)$ the points $(u^0_\pm, 0)$ are an unstable node and a stable node, respectively while the points $(u^1_\pm, 0)$ are both saddles, and if $a > 1$, the points $(u^0_\pm, 0)$ are saddles while the points $(u^1_\pm, 0)$ are a stable and an unstable node, respectively.

If $b = a$ and $a \in (0, 1)$, we get that the points $(u^0_\pm, 0)$ are an unstable node and a stable node, respectively. The points $(u^1_\pm, 0)$ are semihyperbolic. Using Theorem 2.19 in [6] we obtain that they are a stable and an unstable node, respectively.

If $b = a$ and $a > 1$, we get that the points $(u^1_\pm, 0)$ are a stable node and an unstable node, respectively. The points $(u^0_\pm, 0)$ are semihyperbolic. Using Theorem 2.19 in [6] we obtain that they are an unstable and a stable node, respectively.

If $b = a = 1$, then there are two infinite singular points on the local chart U_1 which are $(u^0_+, 0)$ and $(u^0_-, 0)$ (in this case the points $(u^0_+, 0)$ and $(u^1_+, 0)$ collide and the same happens with $(u^0_-, 0)$ and $(u^1_-, 0)$). Theses points are linearly zero and applying blow-up techniques we get that they are formed by two elliptic and four parabolic sectors.

Finally, if $b > a$ with $a \in \mathbb{R}^+$ we get that the points $(u^0_\pm, 0)$ are an unstable and a stable node, respectively, while the points $(u^1_\pm, 0)$ are a stable and an unstable node, respectively.

In the local chart U_2 we get

$$u' = -1 + (1 + b)u^2 - au^4 + v^2, \quad v' = uv(1 - au^2).$$

The origin is not an infinite singular point of the system.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get the following for system (III) with $s = -1$:

When $b \in (-\infty, -1 - 2\sqrt{a}]$ among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that (x_-, y_-) and (x_+, y_+) are stable nodes while (x_-, y_+) and (x_+, y_-) are unstable nodes. In the local charts U_1 and U_2 there are no infinite singular points. The global phase portrait is topologically equivalent to 1.8 of Figure 1.

When $b \in (-1-2\sqrt{a}, 1)$ and $a > 1$ among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that (x_-, y_-) and (x_+, y_+) are stable foci while (x_-, y_+) and (x_+, y_-) are unstable foci. In the local charts U_1 and U_2 there are no infinite singular points. The global phase portrait is topologically equivalent to 1.9 of Figure 1.

When $b = 1$ and $a > 1$ among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that $(x_\pm, y_\pm)$ are centers. In the local charts U_1 and U_2 there are no infinite singular points. The global phase portrait is topologically equivalent to 1.10 of Figure 1.

When $b \in (1, -1 + 2\sqrt{a})$ and $a > 1$, among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that (x_-, y_-) and (x_+, y_+) are unstable foci while (x_-, y_+) and (x_+, y_-) are stable foci. In the local charts U_1 and U_2 there are no infinite singular points. The global phase portrait is topologically equivalent to 1.11 of Figure 1.

When $b = -1 + 2\sqrt{a}$ and $a > 1$ among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that (x_-, y_-) and (x_+, y_+) are unstable nodes while (x_-, y_+) and (x_+, y_-) are stable nodes. In the local chart U_1 we have two infinite singular points $(u^0_\pm, 0)$ which are saddle nodes and on the local chart U_2 the origin is not a singular point. The global phase portrait is topologically equivalent to 1.12 of Figure 1.

When $b \in (-1 + 2\sqrt{a}, a)$ and $a > 1$ among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that (x_-, y_-) and (x_+, y_+) are unstable nodes while (x_-, y_+) and (x_+, y_-) are stable nodes. On the local chart U_1 we have four infinite singular points $(u^0_\pm, 0)$ and $(u^1_\pm, 0)$ which are two saddles, a stable node and an unstable node, respectively. Moreover, on the local chart U_2 the origin is not a singular point. The global phase portrait is topologically equivalent to 1.13 of Figure 1.

When $b \geq a$ and $a \neq 1$ or $b > a$ and $a = 1$, the only singular points are the origin and $(0, \pm 1)$ which are all saddles. On the local chart U_1 we have four infinite singular points $(u^0_\pm, 0)$ and $(u^1_\pm, 0)$ which are an unstable node, two stable nodes, and an unstable node, respectively. On the local chart U_2 the origin is not a singular point. The global phase portrait is topologically equivalent to 1.14 of Figure 1.

When $b = 1$ and $a = 1$ the only singular points are the origin and $(0, \pm 1)$ which are saddles. On the local chart U_1 we have the two infinite singular points $(u^0_\pm, 0)$ that are formed by two elliptic and four parabolic sectors. On the local chart U_2 the origin is not a singular point. The global phase portrait is topologically equivalent to 1.15 of Figure 1.

When $b \in (-1 - 2\sqrt{a}, -1 + 2\sqrt{a})$ and $a \in (0, 1]$, among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that (x_-, y_-) and (x_+, y_+) are stable foci while (x_-, y_+) and (x_+, y_-) are unstable foci. On the local charts U_1 and U_2 there are no infinite singular points. The global phase portrait is topologically equivalent to 1.9 of Figure 1.

When $b = -1 + 2\sqrt{a}$ and $a \in (0, 1)$ among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that (x_-, y_-) and (x_+, y_+) are stable nodes while (x_-, y_+) and (x_+, y_-) are unstable nodes. On the local chart U_1 we have two infinite singular points $(u^0_\pm, 0)$ which are saddle-nodes and on the local chart U_2 the origin is not a singular point. The global phase portrait is topologically equivalent to 1.16 of Figure 1.

Finally, when $b \in (-1 + 2\sqrt{a}, a)$ and $a \in (0, 1)$ among the finite saddles $(0, 0)$, $(0, \pm 1)$ we have that (x_-, y_-) and (x_+, y_+) are stable nodes while (x_-, y_+) and (x_+, y_-) are unstable nodes. On the local chart U_1 we have four infinite singular points $(u^0_\pm, 0)$ and $(u^1_\pm, 0)$ which are an unstable node, a stable node and two saddles, respectively. Moreover, on the local chart U_2 the origin is not a singular point. Then the global phase portrait of system (V) is topologically equivalent to the phase portrait 1.17 of Figure 1.

3.6. **Global phase portrait of system (IV).** Consider system (IV)

$$x' = y + bx^2 y, \quad y' = x^3 - xy^2, \quad b \in \mathbb{R}.$$

The origin is always a finite singular point (a nilpotent saddle). If $b < 0$, then there are four additional singular points $(x_\pm, y_\pm)$ where

$$x_\pm = \pm\sqrt{\frac{-1}{b}}, \quad y_\pm = \pm\sqrt{\frac{-1}{b}}.$$

Computing the eigenvalues of the Jacobian matrix at the points (x_-, y_-) and (x_+, y_+) we get that they are $\lambda_1 = 2/b$, $\lambda_2 = -2$, and at the points (x_-, y_+) and (x_+, y_-) are $-\lambda_1$ and $-\lambda_2$. We see that $\lambda_1, \lambda_2 < 0$ for all $b < 0$. Hence (x_-, y_-) and (x_+, y_+) are stable nodes whereas (x_-, y_+) and (x_+, y_-) are unstable nodes.

In the local chart U_1 system (IV) becomes

$$u' = 1 - u^2(1 + b + v^2), \quad v' = -uv(b + v^2).$$

When $v = 0$, it follows that if $b > -1$ there are two infinite singular points on the local chart U_1 which are $(u_\pm, 0)$ where $u_\pm = \pm 1/\sqrt{1+b}$. Computing the eigenvalues of the Jacobian matrix at the points $(u_\pm, 0)$ we get that if $b \in (-1, 0)$ they are saddles; whereas if $b > 0$, $(u_-, 0)$ is an unstable node and $(u_+, 0)$ is a stable node.

If $b = 0$, the points $(u_\pm, 0)$ are both semihyperbolic. Using Theorem 2.19 in [6] we get that $(u_+, 0)$ is a stable node and $(u_-, 0)$ is an unstable node.

In the local chart U_2 we get

$$u' = (1 + b)u^2 - u^4 + v^2, \quad v' = (u - u^3)v.$$

The origin is an infinite singular point whose linear part is zero. Applying blow-up techniques we obtain that it is formed by six hyperbolic sectors (three stable and three unstable) if $b \le -1$; by two hyperbolic sectors (one stable and one unstable) and four parabolic sectors (two stable and two unstable) if $b \in (-1, 0)$ and by two hyperbolic sectors (one stable and one unstable) if $b \ge 0$.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that if $b \le -1$ the global phase portrait of system (IV) is topologically equivalent to: 1.18 of Figure 1 if $b \le -1$; to 1.19 of Figure 1 if $b \in (-1, 0)$ and to 1.6 of Figure 1 if $b \ge 0$.

3.7. **Global phase portrait of system (V).** Consider system (V)

$$x' = y + sx^2y + y^3, \quad y' = ax^3, \quad a \in \mathbb{R}^+.$$

The origin is the unique finite singular point. In the local chart U_1 system (V) with $s = 1$ becomes

$$u' = a - u^2(s + u^2 + v^2), \quad v' = -uv(s + u^2 + v^2).$$

Taking into account that $a > 0$, there are two unique infinite singular points on the local chart U_1 defined for all $a > 0$ which are $(u_\pm, 0)$ where

$$u_\pm = \pm \frac{\sqrt{-s + \sqrt{1 + 4a}}}{\sqrt{2}}.$$

Computing the eigenvalues of the Jacobian matrix at the points $(u_\pm, 0)$ we get that $(u_+, 0)$ is a stable node and $(u_-, 0)$ is an unstable node.

In the local chart U_2 we get

$$u' = 1 + su^2 - au^4 + v^2, \quad v' = -au^3v.$$

So the origin is not an infinite singular point.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system (V) is topologically equivalent to 1.1 of Figure 1.

3.8. **Global phase portrait of system (VI).** Consider system (VI)

$$x' = y + y^3, \quad y' = x^3.$$

The origin is the unique finite singular point. In the local chart U_1 system (VI) becomes

$$u' = 1 - u^4 - u^2v^2, \quad v' = -uv(u^2 + v^2).$$

There are two unique infinite singular points on the local chart U_1 which are $(\pm 1, 0)$. Computing the eigenvalues of the Jacobian matrix we get that $(1, 0)$ is a stable node and $(-1, 0)$ is an unstable node.

In the local chart U_2 we get

$$u' = 1 - u^4 + v^2, \quad v' = -u^3v.$$

So the origin is not an infinite singular point.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system (VI) is topologically equivalent to 1.1 of Figure 1.

3.9. **Global phase portrait of system (VII).** Consider system (VII)

$$x' = y + sx^2y - y^3, \quad y' = ax^3, \quad a \in \mathbb{R}^+.$$

There are three singular points: the origin and $(0, \pm 1)$. Computing the eigenvalues of the Jacobian matrix at the points $(0, \pm 1)$ we see that both points are nilpotent. Using Theorem 3.5 in [6] and the reversibility of the system we conclude that: both of them are centers if $a > 1/4$ and both of them consist in one hyperbolic and one elliptic sector if $a \leq 1/4$.

In the local chart U_1 system (VII) becomes

$$u' = a + u^2(-s + u^2 - v^2), \quad v' = -uv(s - u^2 + v^2).$$

Computing the infinite singular points in the local chart U_1 we conclude that: if either $s = -1$ or $s = 1$ and $a > 1/4$ there are no infinite singular points, if $s = 1$ and $0 < a < 1/4$ there are four infinite singular points which are $(u^0_\pm, 0)$ and $(u^1_\pm, 0)$ where

$$u^0_\pm = \pm\frac{\sqrt{1 - \sqrt{1 - 4a}}}{\sqrt{2}}, \; u^1_\pm = \pm\frac{\sqrt{1 + \sqrt{1 - 4a}}}{\sqrt{2}}$$

and if $s = 1$ and $a = 1/4$ there are two infinite singular points which are $(u^0_+, 0)$ and $(u^0_-, 0)$ (in this case the points $(u^0_+, 0)$ and $(u^1_+, 0)$ collide and the same happens with $(u^0_-, 0)$ and $(u^1_-, 0)$).

Computing the eigenvalues of the Jacobian matrix at the points $(u^0_\pm, 0)$ or $(u^1_\pm, 0)$ we get that if $0 < a < 1/4$, $(u^0_+, 0)$ is a stable node, $(u^0_-, 0)$ is a unstable node and $(u^1_\pm, 0)$ are saddles; if $a = 1/4$, $(u^0_+, 0)$ and $(u^0_-, 0)$ are semihyperbolic. Using Theorem 2.19 in [6] we get that they are both saddle-nodes.

In the local chart U_2 we get

$$u' = -1 + su^2 - au^4 + v^2, \quad v' = -au^3v.$$

So the origin is not an infinite singular point.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system (VII) is topologically equivalent to: 1.4 of Figure 1 if $a > 1/4$; to 1.20 of Figure 1 if $s = 1$ and $a = 1/4$, to 1.21 of Figure 1 if $s = 1$ and $0 < a < 1/4$; and to 1.22 of Figure 1 if $s = -1$ and $0 < a \leq 1/4$.

3.10. **Global phase portrait of system $(VIII)$.** Consider system $(VIII)$

$$x' = y - y^3, \quad y' = x^3.$$

There are three nilpotent singular point: the origin (which is a saddle) and $(0, \pm 1)$. Applying Theorem 3.5 in [6] together with the reversibility of the system we can see that $(0, \pm 1)$ are both centers.

In the local chart U_1 system $(VIII)$ becomes

$$u' = 1 + u^4 - u^2 v^2, \quad v' = uv(u^2 - v^2).$$

So there are no infinite singular points in the local chart U_1. In the local chart U_2 we get

$$u' = -1 - u^4 + v^2, \quad v' = -u^3 v.$$

So the origin is not an infinite singular point.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system $(VIII)$ is topologically equivalent to 1.4 of Figure 1.

3.11. **Global phase portrait of system (IX).** Consider system (IX)

$$x' = y + sx^2 y, \quad y' = x^3.$$

The origin is the unique finite singular point.

In the local chart U_1 system (IX) with $s = 1$ becomes

$$u' = 1 - u^2(1 + v^2), \quad v' = -uv(1 + v^2).$$

So if $s = 1$ there are two singular points at infinity in the local chart U_1, the point $(1, 0)$ which is a stable node and the point $(-1, 0)$ which is a unstable node. If $s = -1$ there are no singular points at infinity in the local chart U_1.

In the local chart U_2 we get

$$u' = su^2 - u^4 + v^2, \quad v' = -u^3 v.$$

So the origin is a infinite singular point which is linearly zero. Applying blow-up techniques we get that it is formed by two hyperbolic (one stable and one unstable) sectors when $s = 1$ and it is formed by two elliptic and four parabolic sectors when $s = -1$.

Gluing all this information (on the finite and infinite singular points) together with the symmetries of the system we get that the global phase portrait of system (IX) is topologically equivalent to 1.6 of Figure 1 when $s = 1$ and to 1.5 of Figure 1 when $s = -1$.

3.12. **Global phase portrait of system (X).** Consider system (X)

$$x' = y, \quad y' = x^3.$$

The origin is the unique finite singular point. In the local chart U_1 system (X) becomes

$$u' = 1 - u^2v^2, \quad v' = -uv^3.$$

So, there are no infinite singular points on the local chart U_1. In the local chart U_2 we get

$$u' = -u^4 + v^2, \quad v' = -u^3v.$$

The origin is an infinite singular point of the system, whose linear part is zero. Applying blow-up techniques we obtain that it is formed by the union of two elliptic and four parabolic sectors. The global phase portrait of system (X) is topologically equivalent to 1.5 of Figure 1.

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