This is a preprint of: "N-dimensional zero-Hopf Bifurcation of Polynomial Differential Systems via Averaging Theory of Second Order", Sara Kassa, Jaume Llibre, Ammar Makhlouf, *J. Dyn. Control Syst.*, 2020.
DOI: [10.1007/s10883-020-09501-6]

N-DIMENSIONAL ZERO-HOPF BIFURCATION OF POLYNOMIAL DIFFERENTIAL SYSTEMS VIA AVERAGING THEORY OF SECOND ORDER

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Abstract. Using the averaging theory of second order we study the limit cycles which bifurcate from a zero-Hopf equilibrium point of polynomial vector fields with cubic nonlinearities in $\mathbb{R}^n$. We prove that there are at least 3^{n-2} limit cycles bifurcating from such zero-Hopf equilibrium points. Moreover we provide an examples in dimension 6 showing that this number of limit cycles is reached.

1. Introduction and statement of the main result

Our goal is to study the limit cycles that bifurcate from a zero-Hopf equilibrium of polynomial differential systems in $\mathbb{R}^n$ with cubic nonlinearities by using the averaging theory.

In [5] the authors studied the Hopf bifurcation in dimension $n > 2$, by using the first order averaging method. They proved that at least 2^{n-3} limit cycles can bifurcate from one singularity with eigenvalues $\pm bi$ and $n-2$ zeros, i.e. from a *zero-Hopf equilibrium* of $\mathbb{R}^n$. They proved for the first time that the number of bifurcated limit cycles in a Hopf bifurcation can grow exponentially with the dimension of the system. For a general information about Hopf bifurcations see [7].

In [2] the authors studied the occurrence of the limit cycles bifurcating from the origin of a differential system with cubic homogeneous nonlinearities in $\mathbb{R}^4$. The authors proved that there are at most $9 = 3^{4-2}$ limit cycles.

In this paper we investigate the limit cycles bifurcating in a zero-Hopf bifurcation at the origin of coordinates of the following cubic polynomial differential systems in $\mathbb{R}^n$

$$
\begin{aligned}
\dot{x} &= (a_1\varepsilon + a_2\varepsilon^2)x - (b + b_1\varepsilon + b_2\varepsilon^2)y + \sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} a_{j,i_1,\ldots,i_n} x^{i_1} y^{i_2} z_3^{i_3}\cdots z_n^{i_n},\\
\dot{y} &= (b + b_1\varepsilon + b_2\varepsilon^2)x + (a_1\varepsilon + a_2\varepsilon^2)y_2 + \sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} b_{j,i_1,\ldots,i_n} x^{i_1} y^{i_2} z_3^{i_3}\cdots z_n^{i_n},\\
\dot{z_k} &= (c_1^{(k)}\varepsilon + c_2^{(k)}\varepsilon^2)z_k + \sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} c^{(k)}_{j,i_1,\ldots,i_n} x^{i_1} y^{i_2} z_3^{i_3}\ldots z_n^{i_n},
\end{aligned} \tag{1}
$$

where $k = 3, \ldots, n$.

Our main result is the following.

Theorem 1. *Consider the differential systems* (1) *in* $\mathbb{R}^n$ *with* $n \geq 2$. *Applying to these systems the averaging theory of second order they can exhibit at least* 3^{n-2} *limit cycles bifurcating from the zero-Hopf equilibrium point localized at the origin of coordinates when* $\varepsilon = 0$.

In the next corollary we provide a differential system (1) in $\mathbb{R}^6$ exhibiting the maximum number of limit cycles stated in Theorem 1.

2010 *Mathematics Subject Classification.* Primary: 34C07, 34C05, 34C40.
Key words and phrases. Hopf bifurcation, averaging theory, cubic polynomial differential systems.

Corollary 2. *Consider the polynomial differential system*

$$\begin{aligned} \dot{x} &= \tfrac{1}{2}\varepsilon^2 x - y - \tfrac{1}{2}x^3, \\ \dot{y} &= x + \tfrac{1}{2}\varepsilon^2 y + \tfrac{3}{2}x^2 y - y^3, \\ \dot{z_3} &= \tfrac{3}{2}\varepsilon^2(z + x^2 y) - \tfrac{1}{2}z^3, \\ \dot{z_4} &= -\varepsilon^2 u + \tfrac{1}{3}u^3, \\ \dot{z_5} &= \tfrac{3}{2}\varepsilon^2 v - \tfrac{1}{3}v^3, \\ \dot{z_6} &= \tfrac{1}{4}\varepsilon^2 w - w^3 + \varepsilon(-v^3 + x^3 + y^3). \end{aligned}$$

It has 81 *limit cycles bifurcating from the zero–Hopf equilibrium localized at the origin of coordinates when* $\varepsilon = 0$.

2. The averaging theory of first and second order

The aim of this section is to present the averaging theory of first and second order as it was developed in [1, 3, 4]. The following result is Theorem 4.2 of [1].

Theorem 3. *We consider the following differential system*

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{2}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *are continuous functions, T-periodic in the first variable, and D is an open subset of* $\mathbb{R}^n$. *Assume that the following hypotheses* (*i*) *and* (*ii*) *hold. We assume:*

(i) F_1, F_2, R *are locally Lipschitz with respect to* x, $F_1(t,.) \in C^1(D)$ *for all* $t \in \mathbb{R}$, *and R is differentiable with respect to* ε. *We define* f_1, f_2: $D \longrightarrow \mathbb{R}^n$ *as*

$$\begin{aligned} f_1(z) &= \frac{1}{T}\int_0^T F_1(s,z)ds, \\ f_2(z) &= \frac{1}{T}\int_0^T \left[D_z F_1(s,z)\int_0^s F_1(t,z)dt + F_2(s,z)\right]ds. \end{aligned} \tag{3}$$

(ii) *For* $V \subset D$ *an open and bounded set and for each* $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, *there exists* $a \in V$ *such that* $f_1(a) + \varepsilon f_2(a) = 0$ *and* $d_B(f_1 + \varepsilon f_2, V, a) \neq 0$.

Then for $|\varepsilon| > 0$ *sufficiently small there exists a T-periodic solution* $\varphi(\cdot,\varepsilon)$ *of the system* (4) *such that* $\varphi(0,\varepsilon) \to a$ *when* $\varepsilon \to 0$.

Where $d_B(f_1 + \varepsilon f_2, V, 0)$ denotes the Brouwer degree of the function $f_1 + \varepsilon f_2$ in the neighborhood V of zero. It is known that if the function $f_1 + \varepsilon f_2$ is C^1 then it is sufficient to check that $\det(D(f_1 + \varepsilon f_2(a_\varepsilon))) \neq 0$ in order to have that $d_B(f_1 + \varepsilon f_2, V, 0) \neq 0$, for more details see [6].

For additional information on the averaging theory see the books [8, 10].

3. Proof of Theorem 1

We consider the polynomial differential system (1) with cubic nonlinearities in $\mathbb{R}^n$. By doing the scaling $(x, y, z_3 \dots, z_n) = (\varepsilon X, \varepsilon Y, \varepsilon Z_3 \dots, \varepsilon Z_n)$, system (1) becomes

(4)
$$\begin{aligned}
\dot{X} = &\quad (a_1\varepsilon + a_2\varepsilon^2)X - (b + b_1\varepsilon + b_2\varepsilon^2)Y + \frac{1}{\varepsilon}\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} a_{j,i_1,\ldots,i_n}\varepsilon^{i_1+\ldots+i_n} X^{i_1}Y^{i_2}Z_3^{i_3}\ldots Z_n^{i_n},\\
\dot{Y} = &\quad (b + b_1\varepsilon + b_2\varepsilon^2)X + (a_1\varepsilon + a_2\varepsilon^2)Y + \frac{1}{\varepsilon}\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} \varepsilon^{i_1+\ldots+i_n} b_{j,i_1,\ldots,i_n} X^{i_1}Y^{i_2}Z_3^{i_3}\ldots Z_n^{i_n},\\
\dot{Z}_k = &\quad (c_1^{(k)}\varepsilon + c_2^{(k)}\varepsilon^2)Z_k + \frac{1}{\varepsilon}\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} \varepsilon^{i_1+\ldots+i_n} c_{j,i_1,\ldots,i_n}^{(k)} X^{i_1}Y^{i_2}Z_3^{i_3}\ldots Z_n^{i_n},
\end{aligned}$$

for $k = 3, \dots, n$. Since we have $i_1 + \ldots + i_n = 3$, then $\varepsilon^{i_1+\cdots+i_n} = \varepsilon^3$. We write system (4) as

$$\begin{aligned}
\dot{X} = &\quad (a_1\varepsilon + a_2\varepsilon^2)X - (b + b_1\varepsilon + b_2\varepsilon^2)Y + \varepsilon^2\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} a_{j,i_1,\ldots,i_n} X^{i_1}Y^{i_2}Z_3^{i_3}\ldots Z_n^{i_n},\\
\dot{Y} = &\quad (b + b_1\varepsilon + b_2\varepsilon^2)X + (a_1\varepsilon + a_2\varepsilon^2)Y + \varepsilon^2\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} b_{j,i_1,\ldots,i_n} X^{i_1}Y^{i_2}Z_3^{i_3}\ldots Z_n^{i_n},\\
\dot{Z}_k = &\quad (c_1^{(k)}\varepsilon + c_2^{(k)}\varepsilon^2)Z_k + \varepsilon^2\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} c_{j,i_1,\ldots,i_n}^{(k)} X^{i_1}Y^{i_2}Z_3^{i_3}\ldots Z_n^{i_n},
\end{aligned} \tag{5}$$

for $k = 3, \dots, n$. We pass now to the cylindric coordinates $(X, Y, Z_3, \dots, Z_n) = (\rho\cos\theta, \rho\sin\theta, \eta_3, \dots, \eta_n)$, system (5) becomes

$$\begin{aligned}
\dot{\rho} = &\quad a_1\rho\varepsilon + \varepsilon^2\Bigg(\cos\theta\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} a_{j,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n}\\
&\quad + \sin\theta\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} b_{j,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n} + a_2\rho\Bigg),\\
\dot{\theta} = &\quad \tfrac{1}{\rho}\Bigg(b\rho + b_1\rho\varepsilon + \varepsilon^2\cos\theta\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} b_{j,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n}\\
&\quad - \sin\theta\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} a_{j,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n} + b_2\Bigg),\\
\dot{\eta}_k = &\quad c_1^{(k)}\varepsilon\eta_k + \varepsilon^2\Bigg(\sum_{j=0}^{2}\varepsilon^j \sum_{i_1+\ldots+i_n=3} c_{j,i_1,\ldots,i_n}^{(k)}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n} + c_2^{(k)}\eta_k\Bigg),
\end{aligned} \tag{6}$$

for $k = 3, \dots, n$. We take θ as the new independent variable in the neighborhood of $(\rho, z_3, ..., z_n) = (0, 0, ..., 0)$, and system (6) writes

$$\begin{aligned}
\frac{d\rho}{d\theta} = &\quad \frac{\varepsilon a_1}{b}\rho + \frac{\varepsilon^2}{b}\Bigg(\cos\theta\sum_{i_1+\ldots+i_n=3} a_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n}\\
&\quad + \sin\theta\sum_{i_1+\ldots+i_n=3} b_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n} + a_2\rho - \frac{a_1 b_1}{b}\Bigg) + O(\varepsilon^3),\\
\frac{d\eta_k}{d\theta} = &\quad \frac{\varepsilon c_1^{(k)}}{b}\eta_k + \frac{\varepsilon^2}{b}\Bigg(\sum_{i_1+\ldots+i_n=3} c_{j,i_1,\ldots,i_n}^{(k)}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n} + c_2^{(k)}\eta_k - \frac{c_1^{(k)}b_1}{b}\Bigg) + O(\varepsilon^3),
\end{aligned}$$

where $k = 3, \dots, n$.

By using the notation of Theorem 3, i.e.

$$\begin{aligned} x = z &= (\theta, \rho, \eta_3, \ldots, \eta_n),\\ t &= \theta,\\ F_1(t,x) &= (F_{11}(\theta, \rho, \eta_3, \ldots, \eta_n), F_{12}(\theta, \rho, \eta_3, \ldots, \eta_n), F_{13}(\theta, \rho, \eta_3, \ldots, \eta_n), \ldots, F_{1n}(\theta, \rho, \eta_3, \ldots, \eta_n)),\\ F_2(t,x) &= (F_{21}(\theta, \rho, \eta_3, \ldots, \eta_n), F_{22}(\theta, \rho, \eta_3, \ldots, \eta_n), F_{23}(\theta, \rho, \eta_3, \ldots, \eta_n), \ldots, F_{2n}(\theta, \rho, \eta_3, \ldots, \eta_n)),\\ T &= 2\pi, \end{aligned}$$

where

$$\begin{aligned} F_1 &= \left(\frac{a_1}{b}\rho, \frac{c_1^{(3)}}{b}\eta_3, \ldots, \frac{c_1^{(n)}}{b}\eta_n\right)\\ F_2 &= \left(\frac{1}{b}(\cos\theta \sum_{i_1+\ldots+i_n=3} a_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n}\right.\\ &\left. + \sin\theta \sum_{i_1+\ldots+i_n=3} b_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n} + a_2\rho - \frac{a_1 b_1}{b}\right),\\ &\frac{1}{b}\left(\sum_{i_1+\ldots+i_n=3} c^{(3)}_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n} + c_2^{(3)} - \frac{c_1^{(3)} b_1}{b}\right), \ldots,\\ &\frac{1}{b}\left(\sum_{i_1+\ldots+i_n=3} c^{(n)}_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\ldots\eta_n^{i_n} + c_2^{(n)} - \frac{c_1^{(n)} b_1}{b}\right), \end{aligned}$$

where $k = 3, \ldots, n$.

We calculate the averaged function of the first order

$$f_1(\rho, \eta_3, \ldots, \eta_n) = \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, \rho, \eta_3, \ldots, \eta_n)d\theta,$$

and we get

$$f_1(\rho, \eta_3, \ldots, \eta_n) = \begin{pmatrix} \frac{a_1}{b}\rho \\ \frac{c_1^{(3)}}{b}\eta_3 \\ . \\ . \\ . \\ \frac{c_1^{(n)}}{b}\eta_n \end{pmatrix},$$

The unique solution of $f_1(\rho, \eta_3, \ldots, \eta_n) = (0, 0, \ldots, 0)$ with respect to $\rho, \eta_3, \ldots, \eta_n$ is $(\rho, \eta_3, \ldots, \eta_n) = (0, 0, \ldots, 0)$. Then the averaging theory of the first order can not provide information about the existence of the periodic solutions. To pass to the second order, we make the first averaged function identically null, i.e. we take $a_1 = c_1^{(k)} = 0$ for $k = 3, \ldots, n$.

We calculate the averaged function of the second order using the formula (3). We get

$$f_2(\rho, \eta_3, \ldots, \eta_n) = \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta, \rho, \eta_3, \ldots, \eta_n)d\theta,$$

because $F_1(\theta,\rho,\eta_3,\dots,\eta_n)=(0,0,\dots,0)$, where

$$\begin{aligned}
f_{21}(\rho,\eta_3,\dots,\eta_n) &= \frac{1}{2\pi b}\int_0^{2\pi}\Bigg(\cos\theta\sum_{i_1+\ldots+i_n=3} a_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\dots\eta_n^{i_n}\\
&\quad+\sin\theta\sum_{i_1+\ldots+i_n=3} b_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\dots\eta_n^{i_n}+a_2\rho\Bigg)d\theta\\
&= \frac{1}{2\pi b}I_1,\\
f_{2k}(\rho,\eta_3,\dots,\eta_n) &= \frac{1}{2\pi b}\int_0^{2\pi}\Bigg(\sum_{i_1+\ldots+i_n=3} c^{(k)}_{0,i_1,\ldots,i_n}(\rho\cos\theta)^{i_1}(\rho\sin\theta)^{i_2}\eta_3^{i_3}\dots\eta_n^{i_n}+c_2^{(k)}\eta_k\Bigg)d\theta\\
&= \frac{1}{2\pi b}I_2,
\end{aligned}$$

where $k=3,\dots,n$, and

$$\begin{aligned}
I_1 &= \int_0^{2\pi}\Bigg[\rho^3((a_{0,1,2,0,\ldots,0}+b_{0,2,1,0,\ldots,0})\cos^2\theta\sin^2\theta+a_{0,3,0,\ldots,0}\cos^4\theta+b_{0,0,3,0,\ldots,0}\sin^4\theta)\\
&+\rho\Bigg(\sum_{i_1+\ldots+i_n=2} a_{0,1,0,i_3..i_n}\cos^2\theta\eta_3^{i_3}..\eta_n^{i_n}+\sum_{i_1+\ldots+i_n=2} b_{0,0,1,i_3..i_n}\sin^2\theta\eta_3^{i_3}..\eta_n^{i_n}+a_2\Bigg)\Bigg]d\theta\\
&= \frac{1}{4}\pi(a_{0,1,2,0,\ldots,0}+b_{0,2,1,0,\ldots,0}+3(a_{0,3,0,\ldots,0}+b_{0,0,3,0,\ldots,0}))\rho^3\\
&+\pi\Bigg(2a_2+\sum_{i_1+\ldots+i_n=2} a_{0,1,0,i_3..i_n}\eta_3^{i_3}..\eta_n^{i_n}+\sum_{i_1+\ldots+i_n=2} b_{0,0,1,i_3..i_n}\eta_3^{i_3}..\eta_n^{i_n}\Bigg)\rho,\\
I_2 &= \int_0^{2\pi}\Bigg[\rho^2\Bigg(\sum_{i_3+..+i_n=1} c^{(k)}_{0,2,0,i_3..,i_n}\cos^2\theta\eta_3^{i_3}..\eta_n^{i_n}+\sum_{i_3+..+i_n=1} c^{(k)}_{0,0,2,i_3,..,i_n}\eta_3^{i_3}..\eta_n^{i_n}\sin^2\theta\Bigg)\\
&+\sum_{i_3+..+i_n=3} c^{(k)}_{0,0,0,i_3..,i_n}\eta_3^{i_3}..\eta_n^{i_n}+c_2^{(k)}\eta_k\Bigg]d\theta\\
&= \pi\Bigg(\rho^2\Bigg(\sum_{i_3+..+i_n=1} c^{(k)}_{0,2,0,i_3..,i_n}\eta_3^{i_3}..\eta_n^{i_n}+\sum_{i_3+..+i_n=1} c^{(k)}_{0,0,2,i_3,..,i_n}\eta_3^{i_3}..\eta_n^{i_n}\Bigg)\\
&+2\sum_{i_3+..+i_n=3} c^{(k)}_{0,0,0,i_3..,i_n}\eta_3^{i_3}..\eta_n^{i_n}+c_2^{(k)}\eta_k\Bigg),
\end{aligned}$$

where $k=3,\dots,n$. Then the averaged function of the second order is

$$\begin{aligned}
f_{21} &= \frac{\rho}{8b}\Bigg[8a_2+(a_{0,1,2,0,..,0}+b_{0,2,1,0,..,0}+3(a_{0,3,0,..,0}+b_{0,0,3,0,..,0}))\rho^2\\
&\quad+\sum_{i_1+\ldots+i_n=2} a_{0,1,0,i_3..i_n}\eta_3^{i_3}..\eta_n^{i_n}+\sum_{i_1+\ldots+i_n=2} b_{0,0,1,i_3..i_n}\eta_3^{i_3}..\eta_n^{i_n}\Bigg],\\
f_{2k} &= \frac{1}{2b}\Bigg(\rho^2\Bigg(\sum_{i_3+..+i_n=1} c^{(k)}_{0,2,0,i_3..,i_n}\eta_3^{i_3}\dots\eta_n^{i_n}+\sum_{i_3+..+i_n=1} c^{(k)}_{0,0,2,i_3,..,i_n}\eta_3^{i_3}..\eta_n^{i_n}\Bigg)\\
&\quad+2\sum_{i_3+..+i_n=3} c^{(k)}_{0,0,0,i_3..,i_n}\eta_3^{i_3}..\eta_n^{i_n}+c_2^{(k)}\eta_k\Bigg),
\end{aligned}$$

where $k=3,\dots,n$.

Now we solve the system of the averaged functions of the second order with respect to $\rho, \eta_3, \ldots, \eta_n$. First we isolate the expression of ρ^2 from $f_{21}=0$, and we obtain

$$\rho^2 = -\frac{8a_2 + \displaystyle\sum_{i_1+\ldots+i_n=2} a_{0,1,0,i_3..i_n}\eta_3^{i_3}..\eta_n^{i_n} + \sum_{i_1+\ldots+i_n=2} b_{0,0,1,i_3..i_n}\eta_3^{i_3}..\eta_n^{i_n}}{a_{0,1,2,0,..,0}+b_{0,2,1,0,..,0}+3(a_{0,3,0,..,0}+b_{0,0,3,0,..,0})}.$$

After we substitute the expression of ρ^2 in $f_{2k}=0$ for $k=3,\ldots,n$. By using the Bezout Theorem (see [9]) we obtain that these functions admit at most 3^{n-2} real zeros (ρ^*, η_k^*) for $k=1,\ldots,3^{n-2}$. Since the coefficients of the system $f_{2k}=0$ are independent, we can take these 3^{n-2} real zeros with the coordinate ρ positive. Therefore, going back through the changes of coordinates, these zeros provide at least 3^{n-2} periodic solutions bifurcating from the zero-Hopf equilibrium at the origin of coordinates. Note that since the number of zeros are the maximum number provided by the Bezout Theorem the determinants

$$\det\left(\left.\frac{\partial(f_{21},f_{2k})}{\partial(\rho,\eta_k)}\right|_{(\rho,\eta_k)=(\rho^*,\eta_k^*)}\right)$$

are non-zero for $k=1,\ldots,3^{n-2}$.

This completes the proof of Theorem 1.

4. Proof of the Corollary 2

We consider the cubic polynomial differential system (2). By doing the scaling, passing to the cylindrical coordinates $(\rho\cos\theta, \rho\sin\theta, \eta_3, \eta_4, \eta_5, \eta_6)$ and taking θ as the new independent variable, we get that the functions $F_{2j}(\theta,\rho,\eta_3,\eta_4,\eta_5,\eta_6)$ for $j=1,\ldots,5$ are

$$\begin{aligned}
F_{21}(\theta,\rho,\eta_3,\eta_4,\eta_5,\eta_6) &= -\frac{1}{2}\rho(\rho^2(6\cos\theta^4-7\cos\theta^2+2)-1),\\
F_{22}(\theta,\rho,\eta_3,\eta_4,\eta_5,\eta_6) &= -\frac{1}{3}\eta_3(2\eta_3^2-9),\\
F_{23}(\theta,\rho,\eta_3,\eta_4,\eta_5,\eta_6) &= \frac{1}{3}\eta_4(\eta_4^2-3),\\
F_{24}(\theta,\rho,\eta_3,\eta_4,\eta_5,\eta_6) &= -\frac{1}{6}\eta_5(2\eta_5^2-9),\\
F_{25}(\theta,\rho,\eta_3,\eta_4,\eta_5,\eta_6) &= -\eta_6^3+\frac{1}{4}\eta_6.
\end{aligned}$$

We integrate these last functions from 0 to 2π, and we get the averaged functions of the second order $f_2(\rho,\eta_3,\eta_4,\eta_5,\eta_6)=f_{2j}$ for $j=0,\ldots,5$

$$\begin{aligned}
f_{21} &= -\frac{1}{8}\rho(3\rho^2-4),\\
f_{22} &= -\frac{1}{6}\eta_3(2\eta_3^2-9),\\
f_{23} &= \frac{1}{3}\eta_4(\eta_4^2-3),\\
f_{24} &= -\frac{1}{6}\eta_5(2\eta_5^2-9),\\
f_{25} &= -\frac{1}{4}\eta_6(4\eta_6^2-1).
\end{aligned}$$

We solve the system of the averaged functions of the second order $(f_{21}, f_{22}, f_{23}, f_{24}, f_{25}) = (0,0,0,0,0)$ with respect to $\rho, \eta_3, \eta_4, \eta_5$ and η_6, we get 81 solutions $z_i =$

(ρ_i^*, η_i^{j*}) with $\rho_i^* > 0$ for $i = 1, \ldots, 81$ and $j = 3, \ldots, 6$

$$
\begin{aligned}
&z_1 = (\tfrac{2}{3}\sqrt{3}, 0, 0, 0, 0), && z_2 = (\tfrac{2}{3}\sqrt{3}, 0, 0, \tfrac{3}{2}\sqrt{2}, 0),\\
&z_3 = (\tfrac{2}{3}\sqrt{3}, 0, 0, -\tfrac{3}{2}\sqrt{2}, 0), && z_4 = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, 0, 0),\\
&z_5 = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, 0, 0), && z_6 = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0),\\
&z_7 = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0), && z_8 = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0),\\
&z_9 = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0), && z_{10} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, 0, 0),\\
&z_{11} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, 0, 0), && z_{12} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, \tfrac{3}{2}\sqrt{2}, 0),\\
&z_{13} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, \tfrac{3}{2}\sqrt{2}, 0), && z_{14} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, -\tfrac{3}{2}\sqrt{2}, 0),\\
&z_{15} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, -\tfrac{3}{2}\sqrt{2}, 0), && z_{16} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, 0, 0),\\
&z_{17} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, 0, 0), && z_{18} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, 0, 0),\\
&z_{19} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, 0, 0), && z_{20} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0),\\
&z_{21} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0), && z_{22} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0),\\
&z_{23} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0), && z_{24} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0),\\
&z_{25} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0), && z_{26} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0),\\
&z_{27} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0), && z_{28} = (\tfrac{2}{3}\sqrt{3}, 0, 0, 0, \tfrac{1}{2}),\\
&z_{29} = (\tfrac{2}{3}\sqrt{3}, 0, 0, 0, -\tfrac{1}{2}), && z_{30} = (\tfrac{2}{3}\sqrt{3}, 0, 0, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{31} = (\tfrac{2}{3}\sqrt{3}, 0, 0, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{32} = (\tfrac{2}{3}\sqrt{3}, 0, 0, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{33} = (\tfrac{2}{3}\sqrt{3}, 0, 0, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}), && z_{34} = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, 0, \tfrac{1}{2}),\\
&z_{35} = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, 0, \tfrac{1}{2}), && z_{36} = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, 0, -\tfrac{1}{2}),\\
&z_{37} = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, 0, -\tfrac{1}{2}), && z_{38} = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{39} = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{40} = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{41} = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{42} = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{43} = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}), && z_{44} = (\tfrac{2}{3}\sqrt{3}, 0, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{45} = (\tfrac{2}{3}\sqrt{3}, 0, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}), && z_{46} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, 0, \tfrac{1}{2}),\\
&z_{47} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, 0, \tfrac{1}{2}), && z_{48} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, 0, -\tfrac{1}{2}),\\
&z_{49} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, 0, -\tfrac{1}{2}), && z_{50} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{51} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{52} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{53} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{54} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{55} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}), && z_{56} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, 0, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{57} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, 0, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}), && z_{58} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, 0, \tfrac{1}{2}),\\
&z_{59} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, 0, \tfrac{1}{2}), && z_{60} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, 0, \tfrac{1}{2}),\\
&z_{61} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, 0, \tfrac{1}{2}), && z_{62} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, 0, -\tfrac{1}{2}),\\
&z_{63} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, 0, -\tfrac{1}{2}), && z_{64} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, 0, -\tfrac{1}{2}),\\
&z_{65} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, 0, -\tfrac{1}{2}), && z_{66} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{67} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{68} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{69} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{70} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{71} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{72} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}),\\
&z_{73} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \tfrac{1}{2}), && z_{74} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{75} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}), && z_{76} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{77} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}), && z_{78} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{79} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, \sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}), && z_{80} = (\tfrac{2}{3}\sqrt{3}, \tfrac{3}{2}\sqrt{2}, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}),\\
&z_{81} = (\tfrac{2}{3}\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\sqrt{3}, -\tfrac{3}{2}\sqrt{2}, -\tfrac{1}{2}).
\end{aligned}
$$

The determinants

$$
\det\left(\left.\frac{\partial(f_{21}, f_{22}, f_{23}, f_{24})}{\partial(\rho, \eta^j)}\right|_{(\varrho, \eta^j) = (\rho_i^*, \eta_i^{j*})} \right)
$$

evaluated at the zeros are given by $\frac{9}{16}, -\frac{9}{8}, \frac{9}{4}, -\frac{9}{8}, -\frac{9}{2}$ and 9. All of these determinants are non-zero. So there are 81 limit cycles bifurcating from the zero Hopf-equilibrium localized at the origin of coordinates.

5. Conclusion

Using the averaging theory of the second order we show that the number of the limit cycles bifurcating from a zero-Hopf equilibrium point of a polynomial differential systems with cubic nonlinearities increases at least exponentially as 3^{n-2} if n is the dimension of the differential system.

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