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ON THE CONFIGURATIONS OF THE SINGULAR POINTS AND THEIR TOPOLOGICAL INDICES FOR THE SPATIAL QUADRATIC POLYNOMIAL DIFFERENTIAL SYSTEMS

JAUME LLIBRE[1] AND CLAUDIA VALLS[2]

Abstract. Using the Euler-Jacobi formula there is a relation between the singular points of a polynomial vector field and their topological indices. Using this formula we obtain the configuration of the singular points together with their topological indices for the polynomial differential systems $\dot{x} = P(x, y, z)$, $\dot{y} = Q(x, y, z)$, $\dot{z} = R(x, y, z)$ with degrees of P, Q and R equal to two when these systems have the maximum number of isolated singular points, i.e., 8 singular points. In other words we extend the well-known Berlinskii's Theorem for quadratic polynomial differential systems in the plane to the space.

1. Introduction and statement of the main results

Consider in $\mathbb{R}^3$ the polynomial differential systems

$$\dot{x} = P(x, y, z), \quad \dot{y} = Q(x, y, z), \quad \dot{z} = R(x, y, z), \tag{1}$$

where $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ are real polynomials of degrees 2, called spatial quadratic polynomial differential system, or simply *quadratic systems* in what follows.

The motivation of our paper comes from the fact that for the planar quadratic polynomial differential systems the characterization of all configurations of the (topological) indices of the singular points of these systems having four singular points is the well-known Berlinskii's Theorem proved in [2, 7] and reproved in [5] using the Euler-Jacobi formula. More precisely, the Berlinskii's Theorem can be stated as follows: *Assume that a real planar quadratic polynomial differential system has exactly four real singular points. In this case if the quadrilater formed by these points is convex, then two opposite singular points are anti-saddles (i.e. nodes, foci or centers) and the other two are saddles. If this quadrilater is not convex, then either the three exterior vertices are saddles and the interior vertex is an anti-saddle or the exterior vertices are anti-saddles and the interior vertex is a saddle.*

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The corresponding author is Jaume Llibre.

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We recall that four points form a *convex* quadrilater if and only if none of the four points is contained in the convex hull of the other three.

We want to extend the Berlinskii's Theorem from the plane to the space, i.e., we shall obtain all the configurations of the singular points and their topological indices for the quadratic systems in $\mathbb{R}^3$ having 8 singular points.

We recall that if the degree of R is one (that is, if R is a polynomial of degree 1) then the following theorem holds. *Assume that a real spatial polynomial differential system of degrees* $(2, 2, 1)$ *has exactly four real singular points. In this case the four singular points are in a plane of* $\mathbb{R}^3$ *(not necessarily invariant) and on this plane Berlinskii's Theorem holds, that is, if the quadrilater formed by these points is convex, then two opposite singular points have index* 1 *and the other two have index* -1*. If this quadrilater is not convex, then either the three exterior vertices have index* -1 *and the interior vertex has index* 1*, or the exterior vertices have index* 1 *and the interior vertex has index* -1. The proof is essentially the same as Berlinskii's theorem. This result was also observed in [5]. So we will consider the case in which the degrees are $(2, 2, 2)$.

Assuming that the quadratic system (1) has 8 singular points, then using the Euler-Jacobi formula we shall obtain the configuration of the singular points and their topological indices.

If the number of singular points of a spatial quadratic polynomial differential system (1) is finite, then it is at most 8, see for more details the Bézout's Theorem (for a proof of this theorem see [18]). When all the singular points have multiplicity one we can apply the Euler-Jacobi formula (see [1] for a proof of a such formula). If system (1) has exactly 8 singular points, then the Jacobian determinant

$$J = \begin{vmatrix} \partial P/\partial x & \partial P/\partial y & \partial P/\partial z \\ \partial Q/\partial x & \partial Q/\partial y & \partial Q/\partial z \\ \partial R/\partial x & \partial R/\partial y & \partial R/\partial z \end{vmatrix}$$

evaluated at each singular point does not vanish, and for any polynomial S of degree less than or equal to 2 we have

$$\sum_{a \in A} \frac{S(a)}{J(a)} = 0, \tag{2}$$

where A is the set of singular points of system (1). Given a finite subset of points B of $\mathbb{R}^3$, we denote by $\hat{B}$ its convex hull,by $\partial\hat{B}$ the boundary of $\hat{B}$, and by $\#B$ the cardinal of B.

Set $A_0 = A \cap \partial\widehat{A}$ and for $i \geq 1$ $A_i = A \cap \partial(A \setminus (A_0 \widehat{\cup \cdots \cup} A_{i-1}))$ for $i = 1, 2, \ldots$. Clearly since A is finite there is an integer q such that $A_{q+1} = \emptyset$ and $A_q \neq \emptyset$.

We say that A has the *configuration* $(K_0; K_1; K_2; \dots; K_q)$ where $K_i = \#A_i$. We say that the singular points of system (1) which belong to A_i are on the *i-th level*.

We recall that if we assume that $\#A = 8$ then the Jacobian determinant J is non-zero at any singular point of system (1), and that the topological indices of these singular points are 1 (respectively -1) if $J > 0$ (respectively $J < 0$) (for more details see [17, 12]).

Our main result is the extension of Berlinskii's Theorem to the quadratic systems (1) having 8 singular points, see Theorem 5.

2. Preliminary results

First of all we observe that if a configuration exists for a polynomial vector field X with degrees $(2, 2, 2)$ and $\#A = 8$, then it is possible to construct the same configuration but changing the sign of all the indices of the singular points, i.e. the points with index $+1$ become with index -1 and vice versa. For doing that it is enough to take the vector field $Y = (-P, Q, R)$ instead of the vector field $X = (P, Q, R)$.

In the proof of Theorem 5 we will denote by $\{p_1, \dots, p_8\}$ the set of points of A. If the index of p_k is $+1$ we shall denote it by p_k^+, and similarly for p_k^-. Also we will denote by Π_{ijk} the plane $\Pi_{ijk}(x, y, z) = 0$ through the three points p_i, p_j and p_k where $i, j, k \in \{1, \dots, 8\}$. We note that three singular points of a polynomial differential system (1) of degree $(2, 2, 2)$ having 8 singular points cannot be collinear, otherwise the straight line containing them is a factor of the polynomials P, Q and R, in contradiction that system (1) has exactly 8 singular points.

Proposition 1. *Let $X = (P, Q, R)$ be a polynomial differential system in $\mathbb{R}^3$ with finitely many singular points and with $\max(\deg P, \deg Q, \deg R) = n$. Then, any plane cannot contain more than n^2 singular points.*

Proof. Note that the polynomial system $P = Q = R = 0$ restricted to a plane is a polynomial system in $\mathbb{R}^2$ with degree n, and so by Bézout's Theorem it cannot contain more than n^2 singular points in this plane. □

In our case since $n = 2$ we cannot have more than four singular points in any plane of a quadratic system (1).

Proposition 2. *Let $X = (P, Q, R)$ be the vector field associated to the quadratic system (1) with 8 singular points. If four of these singular points are contained in a plane, then the other four singular points are also contained in a plane.*

Proof. By Proposition 1 we know that at most four singular points of the vector field X are contained in a plane. Assume that four singular points

$p_1, \ldots, p_4$ are in a plane and the remaining singular points $p_5, \ldots, p_8$ are not contained in a plane. Without loss of generality we can assume that p_8 is not contained in the plane generated by $\{p_5, p_6, p_7\}$. Then applying the Euler-Jacobi formula (2) with the polynomial $S = \Pi_{123}\Pi_{567}$ we reach to a contradiction. Indeed, we have

$$\sum_{i=1}^{8} \frac{\Pi_{123}(p_i)\Pi_{567}(p_i)}{J(p_i)} = \frac{\Pi_{123}(p_8)\Pi_{567}(p_8)}{J(p_8)} = 0,$$

so $\Pi_{123}(p_8)\Pi_{123}(p_8) = 0$. Since $\Pi_{123}(p_8) \neq 0$, otherwise five singular points will be contained in a plane, we have that $\Pi_{567}(p_8) = 0$ in contradiction that the singular point p_8 was assumed that it is contained in the plane $\Pi_{567}(x, y, z) = 0$ generated by the three singular points $\{p_5, p_6, p_7\}$. □

From Proposition 2 it follows immediately the next corollary.

Corollary 3. *If a quadratic system* (1) *has* 8 *singular points, then they must satisfy one of the following two conditions.*

(i) *Four singular points are contained in a plane and the other four singular points are also contained in another plane.*
(ii) *In any plane there are at most three singular points of the eight possible. Then all the faces of the polyhedron of the convex hull of the eight singular points are triangles.*

The next result follows from the theory of the topological index, see [3].

Proposition 4. *The topological index of an isolated singular point of system* (1) *remains constant under sufficiently small continuous perturbations of the coefficients of system* (1).

2.1. **Polyhedra with 4, 5, 6, 7 and 8 vertices.** A *graph* is a pair (V, E) where V is a set whose elements are called *vertices* and E is a set of two-sets vertices, whose elements are called *edges*.

A *planar graph* is a graph that can be embedded in a plane, i.e. it can be drawn on the plane in such a way that its edges intersect only at the endpoints, which are *vertices*.

A 3-*connected planar graph* is a connected graph having more than 3 vertices whenever fewer than 3 vertices are removed.

An *n-polyhedral graph* is a 3-*connected planar graph on n vertices.* Every convex polyhedron can be represented in the plane by a 3-connected planar graph. This can be done enlarging one of the faces of the convex polyhedron and projecting on it the other faces. Conversely, by a theorem of Steinitz (see [19, 20]) as restated by Grünbaum (see p. 235 of [13]), every 3-connected planar graph can be realized as a convex polyhedron (see [10]).

The number of distinct polyhedral graphs having 4, 5, 6, 7 and 8 vertices are 1, 2, 7, 34 and 257 (see p. 424 of [13] and [8, 9, 10]). In Figure 1 we provide the n-polyhedral graphs for $n = 4, 5, 6, 7$ which provide all the polyedra with vertices $4, 5, 6, 7$. We do not provide the 257 8-polyhedral graphs, but for instance they are given in [13] or in
http://mathworld.wolfram.com/OctahedralGraph.html.

We note that polyhedral graphs are sometimes simply known as polyhedra, as we do in Figure 1 and when it would be convenient.

2.2. **Possible polyhedra for the convex hull of the 8 singular points of a quadratic system.** Note that any configuration of the form $(3;*)$ cannot occur because three points determine a plane and then its convex hull is a triangle so the other 5 points must be in the interior of the triangle, and therefore the 8 points must be on a plane contradicting Proposition 1. Clearly, the configurations $(2;*)$ and $(1;*)$ also cannot occur.

For the configuration of the form $(4;3;1)$ proceeding as we did for the case $(3;*)$ we have that the four points in the first and second levels must be in a plane, but then in view of Proposition 2 the four points in the 0 level must also be in the same plane contradicting Proposition 1. So this case is also not possible. The configuration $(4;2;2)$ is not possible because the four singular points of the first and second level are collinear and we have seen that this is not possible. The configuration $(5;2;1)$ is also non possible because the three singular points in the first and second levels are collinear.

In short the possible configurations are (8), $(7;1)$, $(6;2)$, $(5;3)$ and $(4;4)$. Moreover, in view of Proposition 1 the previous configurations of the form $(K;*)$ cannot have the K points in a plane.

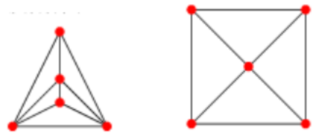
We numerate the 34 convex polyhedra with seven vertices of Figure 1(d) from 1 to 34 starting from the first polyhedron on the left hand side of the first row of Figure 1(d). We also numerate the 7 convex polyhedra with six vertices of Figure 1(c) from 1 to 7 starting from the first polyhedron on the left hand side of the first row of Figure 1(c). In a similar way we numerate the convex polyhedra with five vertices.

Using Corollary 3 we have the following possible polyhedra for the convex hull of the 8 singular points of a quadratic system (1) when these 8 points are in the boundary of the convex hull, i.e. for the configuraion (8):

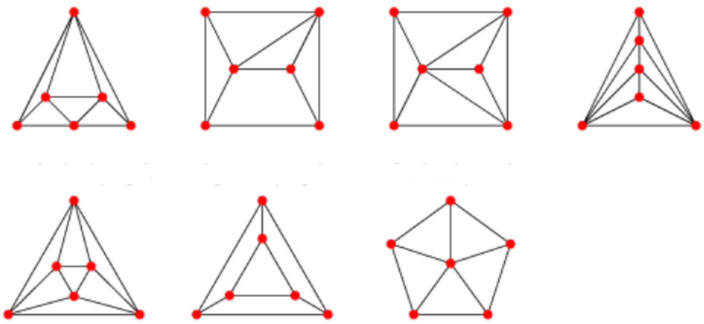
(8)(i) If there is a plane containing four singular points we have the following four possibilities for the polyhedron defined by the convex hull of the eight vertices.

(8)(i.1) A *quadrilateral-quadrilateral prism* is a prism with two convex quadrilater bases which do not share any edge, the lateral faces can be triangles or quadrilaters, these polyhedra have eight vertices.

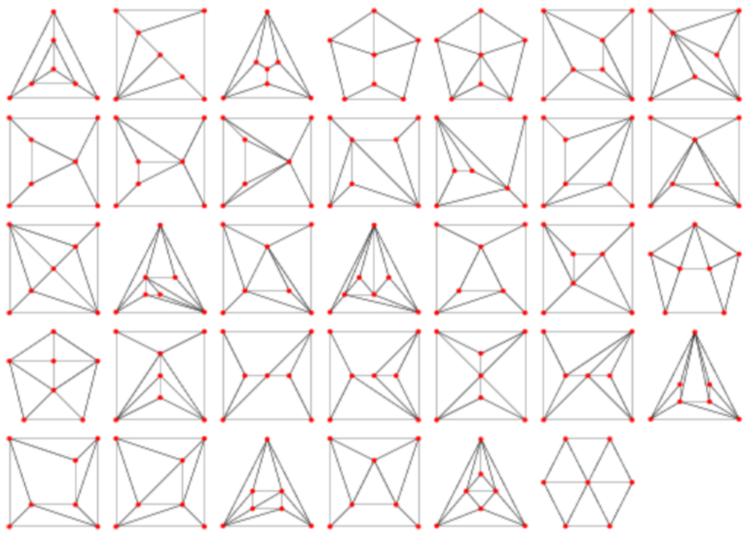
(a) The polyhedron with 4 vertices.



(b) The polyhedra with 5 vertices.



(c) The polyhedra with 6 vertices.



(d) The polyhedra with 7 vertices.

Figure 1. All convex polyedra with $4, 5, 6$ and 7 vertices.

(8)(i.2) A *quadrilateral-triangular prism* is a prism with a convex quadrilater and a triangle as bases (the triangle of the bases has a singular point in its interior), the lateral faces are triangles or quadrilaters, these polyhedra have seven vertices. Then, from Figure 1(d) these polyhedra are one of the following $8, 9, 10, 14, 17, 19, 20$, 30 and 32 of Figure 1(d).

(8)(i.3) A *triangular-triangular prism* is a prism with two triangular bases containing each base a singular point in its interior, the lateral faces are triangles or quadrilaters, these polyhedra have six vertices, see the polyhedra 5 and 6 of Figure 1(c).

(8)(i.4) A *tetrahedron* with a point in the interior of the four triangles of its faces, of course this tetrahedron has four vertices.

(8)(ii) If there is not a plane containing four singular points, then it is a polyhedron with twelve triangular faces, eighteen edges and eight vertices.

Applying the Euler-Jacobi formula (2) with the polynomial S equal to the product of a plane containing four vertices of a face of the polyhedron with 7 vertices, with the plane determined by the remainder 3 vertices, we obtain that the polyhedra 1, 2, 3, 4, 6, 7, 8, 9, 10, 11 ,12 ,13 ,14 ,15 ,17 , 19, 20, 21, 23, 24, 25, 26, 27, 29, 30 and 32 of Figure 1(d) cannot be a convex hull of type $(7; 1)$. Since the polyhedra 5, 22 and 34 of Figure 1(d) have a face with more than 4 vertices by Proposition 1 they cannot be the convex hull of type $(7; 1)$. Therefore the unique possible polyhedra of type (7;1) are

(7;1) the polyhedra $16, 18, 28, 31$ and 33 of Figure 1(d), which have ten triangular faces, fifteen edges and seven vertices. The singular point which is not a vertex of this polyhedra is contained in the interior of the polyhedra.

We analyze the convex hulls of type (6;2).

(6;2)(i) If there is a plane containing four singular points, then we have four possibilities.

(6;2)(i.1) A *quadrilateral bipyramid* is formed by two pyramids glued by the same quadrilateral base in such a way that the two vertices of the two pyramids and the two points in the 1st level are in the same plane, again by Proposition 2. See the polyhedron 5 of Figure 1(c). Note that in the polyhedron 5 we have projected the quadrilateral bipyramid in a lateral triangular face.

(6;2)(i.2) A *quadrilateral-edge pyramid* is a pyramid with a convex quadrilateral as base and instead of a vertex the pyramid has an edge E, see the polyhedra 2 and 3 of Figure 1(c). The two singular points in the interior of these polyhedra and the edge E are contained in a plane due to Proposition 2.

(6;2)(i.3) A *triangular-edge pyramid* is a pyramid with a triangular base having in the interior an additional singular point and instead of a vertex the pyramid has an edge E, see the polyhedron 1 of Figure 1(c). The two singular points in the interior of this polyhedron and the edge E are contained in a plane due to Proposition 2.

(6;2)(i.4) A *truncated triangular pyramid* it has two triangular bases with one singular point in the interior of each triangle, see the convex polyhedron 6 of Figure 1(c). The two singular points in the interior of this polyhedron must be in the same plane than each one of the lateral edges of this polyhedron due to Proposition 2.

(6;2)(ii) If there is no a plane containing four singular points, then it must be a polyhedron with eight triangular faces, twelve edges and six vertices, see the polyhedra 4 and 5 of Figure 1(c).

We note that the polyhedra 7 cannot be a convex hull of type (6;2) due to the existence of a pentagon.

Now we study the polyhedra which are convex hull of type (5;3).

(5;3)(i) If there is a plane containing four singular points, then a *quadrilateral pyramid* is a pyramid with a quadrilateral base in such a way that the vertex of the pyramid which is not contained in the base is in the plane generated by the three points in the 1st level, see the polyhedron 2 of Figure 1(b).

(5;3)(ii) If there is not a plane containing four singular, then it must be a *triangular bipyramid* which is formed by two pyramids glued by the same triangular base in such a way that the two vertices of the two pyramids and the three points in the 1st level never are more than three in the same plane. This polyhedron has six triangular faces, nine edges and five vertices. See the polyhedron 1 of Figure 1(b).

Finally the polyhedron which is a convex hull of type (4;4) is:

(4;4) A *tetrahedron* such that the four points in the 1st level are not contained in a plane, and never four of the eight singular points are contained in a plane.

If one of the fourteen previous described possible convex hulls of the 8 singular points of a quadratic system (1) is realizable, in what follows we will only say that the corresponding polyhedron is *realizable*.

3. The main result

We denote by $i(a)$ the (topological) index of a singular point $a \in A$ of the quadratic system (1). It was proved in [15] that for a quadratic system (1) either $\sum_{a\in A} i_X(a) = 0$, or $|\sum_{a\in A} i_X(a)| = 2$. Our main result is the following.

Theorem 5. *For quadratic systems* (1) *having* 8 *singular points the following polyhedra are realizable.*

(8)(i.1) *If a quadrilateral-quadrilateral prism is realizable, then the indices in the two quadrilaters of the bases are* $1, -1, 1, -1$ *in counterclockwise sense.*

(8)(i.3) *If a triangular-triangular prism is realizable, then the indices of the vertices of the two triangular bases are* $1, 1, 1$ *and* $-1, -1, -1$*, and the indices of the interior singular point at every one of these triangles is different from the ones of the vertices of the triangle.*

(8)(ii) *The configuration of the indices of the vertices of a polyhedron with twelve triangular faces, eighteen edges and eight vertices realizable by a quadratic system can be obtained as follows. Take two triangular faces without intersection, then the two remaining vertices have different index. Repeating this for every pair of triangular faces without intersection the configuration of the indices of this polyhedron is obtained.*

(6;2)(i.1) *If a quadrilateral bipyramid is realizable, then the indices of the quadrilater of the basis are* $1, 1, -1 - 1$ *in counterclockwise sense, the indices of the two vertices outside the quadrilateral are different, and the indices of the two interior singular points are different.*

(6;2)(i.2) *If a quadrilateral-edge pyramid is realizable, then the indices of the quadrilater of the basis are* $1, 1, -1, -1$ *in counterclockwise sense, the indices of the two vertices of the edge* E *are different and the indices of the two interior points are also different.*

(6;2)(i.3) *If a truncated triangular pyramid is realizable, then the indices of the vertices of the two triangular bases are* $1, 1, 1$ *and* $-1, -1, -1$*, and the indices of the two interior points of the pyramid are different.*

(6;2)(ii) *If the polyhedra* 4 *and* 5 *of Figure 1(c) are realizable, then the indices of two triangular faces without intersection are* $1, 1, 1$ *and* $-1, -1, -1$*, and the indices of the two interior singular points of the polyhedra are different.*

(5;3)(i) *If a quadrilateral pyramid is realizable then it has the following two possible configurations of indices. First the indices of the vertices of the quadrilateral basis are* $1, 1, 1, -1$*, the vertex of the pyramid which is not contained in the basis has index* -1*, and the three interior singular points have indices* $1, -1, -1$*; of course we can reverse all the signs of these indices. Second the indices of the vertices of the quadrilateral basis are* $1, 1, -1, -1$ *in counterclockwise, the vertex of the pyramid which is not contained in the basis has index* 1*, and the three interior singular points have indices* $1, -1, -1$*; of course we can reverse all the signs of these indices.*

(5;3)(ii) *If a triangular bipyramid is realizable, then it has the following three possible configurations of indices. First the indices of the three vertices of the triangular base (where both pyramids are glued) are* $1, 1,$

1, the indices of the two vertices of the bipyramid which are not in the triangular base are $-1, -1$*, and the three interior singular points have indices* $1, -1, -1$*; of course we can reverse all the signs of these indices. Second the indices of the three vertices of the triangular base (where both pyramids are glued) are* $1, 1, -1$*, the indices of the two vertices of the bipyramid which are not in the triangular base are* $1, -1$*, and the three interior singular points have indices* $1, -1, -1$*; of course we can reverse all the signs of these indices. Third the indices of the triangular base are* $1, -1, -1$*, the indices of the two vertices are* $1, 1$ *and the three interior singular points have indices* $1, -1, -1$*; of course we can reverse all the signs of these indices.*

(a) *There are quadratic systems* (1) *realizing the configurations of the indices of all the previous statements.*
(b) *The polyhedra of* $(8)(i.2)$*,* $(8)(i.4)$*,* $(7;1)$*,* $(6;2)(i.3)$ *and* $(4;4)$ *are not realizable.*

Proof of statement $(8)(i.1)$*.* Let Q_1 and Q_2 be the two quadrilaters of the bases of the quadrilateral-quadrilateral prism. Let Π_1 be a plane containing Q_1. Let Π_2 be a plane through an edge of Q_2 containing only this edge of the quadrilateral-quadrilateral prism. Applying the Euler-Jacobi formula (2) with $S = \Pi_1\Pi_2$ we get that the two remaing vertices of Q_2 must have different index. Repeating this process for every edge of Q_1 and Q_2 statement (8)(i.1) follows. □

Take $P = x^2 - x$, $Q = y^2 - y$ and $R = z^2 - z$. Then the quadratic system (1) has the singular points $(0, 0, 0)$, $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, $(1, 0, 1)$, $(1, 1, 1)$ and $(0, 1, 1)$, and their indices are $-1, 1, -1, 1, 1, -1, 1$ and -1, respectively. So the configuration (8)(i.1) is realizable.

Proof of statement $(8)(i.3)$*.* Let Q_1 and Q_2 be the two triangles of the bases of the triangular-triangular prism, and let p_1 and p_2 be the singular points at the interior of the triangles Q_1 and Q_2 respectively. Let Π_1 be the plane containing Q_1. Let Π_2 be a plane defined by an edge of Q_2 containing only this edge of the triangular-triangular prism. Applying the Euler-Jacobi formula (2) with $S = \Pi_1\Pi_2$ we get that the remaining vertex of the triangle Q_2 and p_2 must have different index. Repeating this process for every edge of Q_2 we obtain that the three vertices of Q_2 have the same index different from the index of p_2. Doing a similar process for Q_1 we obtain that the three vertices of Q_1 have the same index different from the index of p_1. Taking into account that the absolute value of the sum of all the indices of the singular points is 0 or 2, the indices of the vertices of Q_1 are different from the indices of the vertices of Q_2. □

Take $P = -2x - 2xy + xz$, $Q = -4x^2 - 2y + 2y^2 + yz$ and $R = -z + z^2$. Then the quadratic system (1) has the singular points $(-1, -1, 0)$, $(1, -1, 0)$,

$(0,1,0)$, $(-1/2,-1/2,1)$, $(1/2,-1/2,1)$, $(0,1/2,1)$, $(0,0,0)$ and $(0,0,1)$, and their indices are $1,1,1,-1,-1,-1,-1$ and 1, respectively. So the configuration (8)(i.3) is realizable.

Proof of statement $(8)(ii)$. Let Π_1 and Π_2 the two planes defined by two triangular faces without intersection, then applying the Euler-Jacobi formula with $S = \Pi_1\Pi_2$ we get that the two remaining vertices of the polyhedron have different index. Repeating this for every pair of triangular faces without intersection the configuration of the indices of this polyhedron is obtained. Note that in this way we can describe the configuration of the indices of the vertices of all the polyhedra with twelve triangular faces, eighteen edges and eight vertices realizable by the quadratic systems. In what follows we provide an example. □

Take $P = 6 + z - 6x^2 - 6y^2 - z^2$, $Q = y - 2y^2/\sqrt{3} - 2yz$ and $R = 3\sqrt{3}x^2 - \sqrt{3}y^2 + x(6y + 3\sqrt{3}(-1+2z))$. Then the quadratic system (1) has the singular points $(1,0,0)$, $(1/2,\sqrt{3}/2,0)$, $(-1/2,\sqrt{3}/2,0)$, $(-1,0,1)$, $(-1/2,-\sqrt{3}/2,1)$, $(1/2,-\sqrt{3}/2,1)$, $(0,0,3)$ and $(0,0,-2)$, and their indices are $-1,1,-1,1,-1,1,-1$ and 1, respectively. So this quadratic system realize a configuration (8)(ii).

Proof of statement $(6;2)(i.1)$. Let p_7 and p_8 be two interior points of a quadrilateral bipyramid, and p_k for $k = 1,2,3,4$ the vertices of the quadrilateral base of the bipyramid ordered in counterclockwise sense. The two vertices of the bipyramid which are not in the quadrilateral base are denoted by p_5 and p_6. Applying the Euler-Jacobi formula to $\Pi_{145}\Pi_{236}$ we get that p_7 and p_8 must have opposite index. Let L be the straight line defined by the points p_7 and p_8.

By Proposition 4 we can assume that L intersects the interior of the triangular face with vertices $p_ip_jp_k$ being i,j,k distinct and $\{i,j,k\} \subset \{1,2,3,4,5,6\}$. In this case the indices of p_i, p_j and p_k are equal. Indeed, we can choose the vertex p_i such that the plane Π_{78i} separates the points p_j and p_k and then applying the Euler-Jacobi formula to $C = \Pi_{rst}\Pi_{78i}$ with r,s,t distinct, $\{r,s,t\} \subset \{1,2,3,4,5,6\}$ and $\{r,s,t\} \cap \{i,j,k\} = \emptyset$, we get that p_j and p_k have the same indices. In a similar way p_i and p_k, and p_j and p_k have the same indices.

Again by Proposition 4 we can assume that L intersects the interior of the triangular faces T_1 and T_2. First assume that the two triangular faces T_1 and T_2 intersected by L have no edges in common. Since the absolute value of the sum of the indices of all singular points must be either zero or two, the indices of the vertices of T_1 are different from the indices of the vertices of T_2, and this configurations of indices is realizable (see below).

Suppose now that the two triangular faces T_1 and T_2 intersected by L have an edge in common. Then the indices of the vertices of $T_1 \cup T_2$ are equal. Since the indices of p_7 and p_8 are different, and again since the absolute value of the sum of the indices of all singular points must be either zero or two, the indices of the remaining singular points must be different from the ones in $T_1 \cup T_2$. So we have three points with an index and five with the opposite index. Two of the singular points of the three having the same index are vertices of the quadrilateral base and the other is in the interior of the bipyramid. Let Π_0 be the plane defined by the three points with the same index. This plane separates the other five singular points with the same index either in four points in one side and the other in the other side, or three points in one side and two in the other side. In the first case the unique point p in a side of the plane Π_0 must be a vertex of the quadrilateral bipyramid. Let Π_1 be a plane through p whose intersection with the bypiramid is only p. Applying the Euler-Jacobi formula to $S = \Pi_0\Pi_1$ we reach a contradiction, because all the singular points which are not in $S = 0$ are in the same connected component of $\mathbb{R}^3 \setminus S$, and all of them have the same index. In the second case let Π_2 be the parallel plane to Π_0 containing the two points in a side of the plane Π_0. Then applying the Euler-jacobi formula to $S = \Pi_0\Pi_2$ we obtain again a contradiction. $\square$

Take $P = yz$, $Q = x^2-y^2+5xz$ and $R = -4+4y^2-399xz/5+z^2$. Then the quadratic system (1) has the singular points $(1,-1,0)$, $(1,1,0)$, $(-1,1,0)$, $(-1,-1,0)$, $(0,0,2)$, $(0,0,-2)$, $(1/2,0,-1/10)$ and $(-1/2,0,1/10)$, and their indices are $1,1,-1,-1,-1,1,-1$ and 1, respectively. So the configuration (6;2)(i.1) described in Theorem 5 is realizable.

Proof of statement $(6;2)(i.2)$. Let Q_1 be the quadrilateral basis of the quadrilateral-edge pyramid, and let p_1 and p_2 be the singular points at the interior of pyramid. Let Π_1 be the plane containing Q_1, and let Π_2 be a plane which intersection with the pyramid be only the edge E. Applying the Euler-Jacobi formula with $S = \Pi_1\Pi_2$ we get that p_1 and p_2 have different index. Let Π_3 be the plane containing the edge E and p_1 and p_2. The plane Π_3 intersects either two opposite edges of Q_1 or two consecutive edges of Q_1.

If Π_3 intersects two opposite (respectively consecutive) edges of Q_1 let E_4 and E_5 be the two edges of Q_1 which are not intersected by Π_3. If Π_k is a plane whose intersection with the pyramid only contains E_k for $k = 4,5$, then applying the Euler-Jacobi formula with $S = \Pi_3\Pi_k$ we get that the two vertices of E_j with $j \in \{4,5\}$ and $j \neq k$ have different index (respectively the same index) for $k = 4,5$. Let E_6 and E_7 be the two edges of Q_1 which are intersected by Π_3. If Π_k is a plane whose intersection with the pyramid only contains E_k for $k = 6,7$, then applying the Euler-Jacobi formula with $S = \Pi_3\Pi_k$ we get that the two vertices of E_j with $j \in \{6,7\}$ and $j \neq k$ have the same index for $k = 6,7$. Let Π_8 be a plane containing the points p_1

and p_2 without intersection with the edge E, then applying the Euler-Jacobi formula with $S = \Pi_1\Pi_8$ we get that the two vertices of the edge E have different index.

In the case that Π_3 intersects two consecutive edges of Q_1 the indices of the quadrilater of the basis are $1, 1, 1-1$, the indices of the edge E are different and the indices of the interior points are also different. Let Π_9 be the plane defined by the three points with the same index. This plane separates the other five points with the same index either in four points in one side and the other point in the other side, or three points in one side and two points in the other side. Now we arrive to a contradiction using the same arguments as in the last part o the proof of the case $(6.2)(i.1)$. $\square$

Take $P = -1/3 + z - 2x^2/3 + y^2$, $Q = yz$ and $R = -x^2 + y^2 + z^2$. Then the quadratic system (1) has the singular points $(1, -1, 0)$, $(1, 1, 0)$, $(-1, 1, 0)$, $(-1, -1, 0)$, $(1, 0, 1)$, $(-1, 0, 1)$, $(1/2, 0, 1/2)$ and $(-1/2, 0, 1/2)$, and their indices are $-1, -1, 1, 1, -1, 1, 1$ and -1, respectively. So the configuration (6;2)(i.2) is realizable.

Proof of statement $(6; 2)(i.3)$. Let Q_1 and Q_2 be the two triangles of the bases of the truncated triangular pyramid, and let p_1 and p_2 be the singular points at the interior of pyramid. Let Π_k be the plane containing Q_k for $k = 1, 2$. Applying the Euler-Jacobi formula with $S = \Pi_1\Pi_2$ we get that p_1 and p_2 have different index. Let E be a lateral edge of the pyramid. Let Π_3 the plane containing the edge E and p_1 and p_2. Let E_i^* be the edge of the triangle Q_i of the truncated pyramid which has no intersection with E. Let Π_i^* be a plane such that its intersection with the truncated pyramid is E_i^*. Applying the Euler-Jacobi formula with $S = \Pi_3\Pi_i^*$ we get that the two vertices of Q_j with $j \neq i$ which are not contained in E must have the same index. Repeating this process for every lateral edge of the truncated pyramid we obtain that the three vertices of Q_k have the same index and since the sum of the absolute value of the indices is zero or two, the vertices of Q_1 and Q_2 must have different index. $\square$

Take $P = 4 - 4y - 13z - 8x^2 + 2yz + 10z^2$, $Q = 5x(2 + 2y - z)$ and $R = -4x^2 + y(-2 + 2y + z)$. Then the quadratic system (1) has the singular points $(-1, -1, 0)$, $(1, -1, 0)$, $(0, 1, 0)$, $(-1/2, -1/2, 1)$, $(1/2, -1/2, 1)$, $(0, 1/2, 1)$, $(0, 0, 1/2)$ and $(0, 0, 4/5)$, and their indices are $-1, -1, -1, 1, 1, 1, 1$ and -1, respectively. So the configuration (6;2)(i.3) is realizable.

Proof of statement $(6; 2)(ii)$. The proof of this statement follows using the same arguments of the proof of statement (6;2)(i.1). $\square$

Take

$$P = 180x(y-z+1) - \sqrt{5\left(3\sqrt{1041}-88\right)}(z-1)z,$$
$$Q = 9000\left(2x^2+y-1\right) + \left(-13500y - 3\sqrt{1041}+25213\right)z + \left(3\sqrt{1041}-17338\right)z^2,$$
$$R = y^2 + \tfrac{1}{4}yz - \tfrac{1}{18}(z-1)(25z-18).$$

Then the quadratic system (1) has the singular points $(-1,-1,0)$, $(1,-1,0)$, $(0,-1/4,1)$, $(-1/4,0,1)$, $(1/4,0,1)$, $(0,1,0)$,

$$\left(\frac{1}{100}\sqrt{\frac{1}{30}\left(3023+137\sqrt{1041}\right)}, -\frac{1}{120}\left(9+\sqrt{1041}\right), \frac{3}{5}\right)$$

and

$$\left(-\frac{1}{120}\sqrt{\frac{1}{5}\left(3\sqrt{1041}-88\right)}, \frac{1}{3}, \frac{1}{2}\right),$$

and their indices are $-1,-1,-1,1,1,1,-1$ and 1, respectively. So the configuration (6;2)(ii) is realizable.

Proof of statement $(5;3)(i)$. Let Q_1 be the quadrilateral basis of the quadrilateral pyramid, q the vertex of the pyramid that is not contained in the base, and let p_1, p_2 and p_3 be the singular points in the interior of the pyramid. Let Π_1 be the plane containing Q_1, and let Π_2 be a plane whose intersection with the pyramid is only the vertex q. Applying the Euler-Jacobi formula to $S = \Pi_1\Pi_2$ we get that the three singular points p_1, p_2, p_3 cannot have the same index. Let Π_3 be the plane containing p_1, p_2, p_3 (which also contains q). Note that the plane Π_3 separates the points in the quadrilateral of the basis as follows:

(I) one point p_4 in one side of the plane Π_3 and the remaining three points on the other side of this plane;
(II) two points in one side of the plane Π_3 and the remaining two points on the other side of this plane.

In case (I) let Π_4 be a plane defined by an edge of Q_1 (which does not contain p_4) containing only this edge of the quadrilateral pyramid. Applying the Euler-Jacobi formula to $S = \Pi_3\Pi_4$ we obtain that the two remaining vertices of the quadrilateral must have the same index. Take now Π^* the plane containing an edge of Q_1 having as vertex p_4. Then the Euler-Jacobi formula applied to $S = \Pi_3\Pi^*$ implies that the index of p_5 is different from the index of p_4. In short we get that three vertices of the quadrilateral have the same index and the other that we denote by p_5 has different index, and $p_5 \neq p_4$. Now we claim that the index of q must be the same than the index of p_5. Indeed, assume that it is not the case. Using that the absolute value of the sum of the indices is either 0 or 2 there are two points in $\{p_1, p_2, p_3\}$ having the same index than p_5 denote them by p_{i_1} and p_{i_2}. Now let Π_5 be the plane determined by the points p_5, p_{i_1} and p_{i_2}. The plane Π_5 separates the other five singular points with the same index either in four points in one side and one point r in the other side, or three points in one side and two points in the other side. In the first case the point r is one of the vertices of the pyramid. Let Π_6 be a plane through r whose intersection with the pyramid is only the point r. Applying the Euler-Jacobi formula to $S = \Pi_5\Pi_6$ we reach a contradiction, because all the singular points which are not in $S = 0$ are in the same connected component of $\mathbb{R}^3 \setminus S$ and all of them have the same index. In the second case the two unique points in a side of the plane Π_5 are r and the remaining interior point of the pyramid. Let Π_7 be a plane containing the two points on one side of the plane Π_5 and such that the other three points separated by Π_5 remain in the same side of Π_7. Applying the Euler-Jacobi formula to $S = \Pi_5\Pi_7$ we obtain again a contradiction. Hence the claim is proved. If only one p_{i_1} of the interior points of the polyhedron has the same index than p_5 and q, taking the plane passing through p_5, q, p_{i_1} and using similar arguments to the ones used for proving the claim we reach a contradiction. Hence, the indices of the interior singular points are two equal to the index of p_5 and the other different. This completes the characterization of the indices in case (I).

In case (II) let Π_4 be a plane defined by an edge E of Q_1 containing only this edge of the quadrilateral pyramid and such that E has no intersection with the plane Π_3. Applying the Euler-Jacobi formula to $S = \Pi_3\Pi_4$ we obtain that the two remaining vertices of the quadrilateral must have different index. Repeating these arguments for the other three edges of Q_1 we get that indices of the vertices of the quadrilateral are $1, 1, -1, -1$ in counterclokwise. Now similar arguments to the case (I) show that if the index of q is 1, then the indices of the three interior singular points are $1, -1, -1$, and if the index of q is -1, then the indices of the three interior singular points must be $1, 1, -1$. This completes the configurations of the indices in case (II). □

In the first realization of the index configurations for the quadrilateral pyramids the two planes containing four singular points are the planes $z = 0$ and $x - \sqrt{3}y = 0$. Consider the quadratic system (1) given by $P = (\sqrt{3}y - x)z$, $Q = -6 - x + x^2 + 13z/2 + \sqrt{3}yz - 3z^2/2$, $R = -2 + y^2 + y(-1 + z) + 13z/6 - z^2/2$. This quadratic system has the singular points $(-2, -1, 0)$, $(3, -1, 0)$, $(3, 2, 0)$, $(-2, 2, 0)$, $(0, 0, 4/3)$, $(1, 1/\sqrt{3}, 1)$, $(0, 0, 3)$ and $(-1, -1/\sqrt{3}, 1)$, these singular points have indices $1, 1, 1, -1, 1, -1, -1$ and -1, respectively.

Now we realize the second index configuration for the quadrilateral pyramids of type (5;3)(i).The two planes containing four singular points are the planes $z = 0$ and $y = 0$. Consider the quadratic system (1) given by $P = yz$, $Q = x^2 - y^2 - xz/10$, $R = 3 - 3y^2 - 4z + 5xz/2 + z^2$. This quadratic system has the singular points $(-1, 1, 0)$, $(-1, -1, 0)$, $(1, -1, 0)$,

$(1,1,0)$, $(0,0,1)$, $(1/5,0,2)$, $(0,0,3)$ and $(3/25,0,6/5)$, these singular points have indices $1,1,-1,-1,-1,-1,1$ and 1, respectively.

Proof of statement $(5;3)(ii)$. Let T_1 be the base of the triangular bipyramid, and p_1, p_2 and p_3 be the singular points in the interior of the bipyramid. Let Π_1 be the plane defined by a face of the bipyramid and Π_2 a plane defined by another face of the bipyramid such that these two faces only share one point. Applying the Euler-Jacobi formula to $S = \Pi_1\Pi_2$ we get that the three singular points p_1, p_2, p_3 cannot have the same index.

We claim that the sum of the indices of the singular points is zero. Assume that the sum of the indices of the singular points is 2 (we can assume that it is positive). Then there are three points with negative index that we denote by q_1, q_2, q_3. Note that at least one of them is in the 1st level. Now let Π_5 be the plane determined by the points q_1, q_2, q_3. The plane Π_5 separates the other five singular points with the same index either in four points in one side and one point r in the other side, or three points in one side and two points in the other side. In the first case the point r is one of the vertices. Let Π_6 be a plane through r whose intersection with the bipyramid is only the point r. Applying the Euler-Jacobi formula to $S = \Pi_5\Pi_6$ we reach a contradiction, because all the singular points which are not in $S = 0$ are in the same connected component of $\mathbb{R}^3 \setminus S$ and all of them have the same index. In the second case the two unique points in a side of the plane Π_5 are denoted by r and s. Let Π_7 be a plane containing the two points r and s and such that leave the other three points separated by the plane Π_5 in one side of Π_7. Applying the Euler-Jacobi formula to $S = \Pi_5\Pi_7$ we obtain again a contradiction. Hence the claim is proved.

In view of the claim, taking into account that p_1, p_2, p_3 cannot have the same index, we get the following three configurations:

(I) The indices of the three vertices of the triangular base are $1,1,1$; the indices of the two vertices of the bipyramid which are not in the triangular base are $-1,-1$ and the three interior points must be of the form $1,-1,-1$.

(II) The indices of the three vertices of the triangular base are $1,1,-1$; the indices of the two vertices of the bipyramid which are not in the triangular base are $1,-1$ and the three interior points must be of the form $1,-1,-1$.

(III) The indices of the three vertices of the triangular base are $1,-1,-1$; the indices of the two vertices of the bipyramid which are not in the triangular base are $1,1$ and the three interior points must be of the form $1,-1,-1$.

□

Now we realize the three index configurations for the quadrilateral pyramids of type (5;3)(ii).

For the first realization take $P = (42+91x+252y-14z+35x^2+126xy+157xz+244yz)/42$, $Q = (-56x-168y-21z-28x^2-84xy-101xz-146yz+7z^2)/7$ and $R = (-56y-14x^2-14xy-31xz+28y^2-6yz)/28$. Then the quadratic system (1) has the singular points $(-2,-1,0)$, $(-2,2,0)$, $(3,-1,0)$ in the vertices of the triangular bases of the bipyramide, $(0,0,3)$ and $(3,2,-2)$ in the two vertices of the bipyramide which are not in the triangular basis, and $(1,-1/2,1)$, $(-1,1/2,1)$ and $(1/4,-1/4,-1/4)$ in the interior of the bipyramide, and their indices are $1,1,1,-1,-1,1,-1$ and -1, respectively. So the first realization of the configuration (5;3)(ii) is done.

For the second realization take $P = (2016+3500x+28000y-672z+406x^2+3500xy+12751xz+4502yz)/2016$, $Q = (-112x-896y-63z-14x^2-386xz-100yz+21z^2)/21$ and $R = (196x+1232y+14x^2+196xy+647xz+336y^2+454yz)/336$. Then the quadratic system (1) has the singular points $(-8,-1,0)$, $(8,-1,0)$, $(-8,2,0)$ in the vertices of the triangular bases of the bipyramide, $(0,0,3)$ and

$$(-125455790514, 1122399166369, -1469697563719)/172072592267$$

in the two vertices of the bipyramide which are not in the triangular basis, and $(1,-1/2,1)$, $(-1,1/2,1)$ and $(3,2,-2)$ in the interior of the bipyramide, and their indices are $1,1,-1,-1,1,-1,1$ and -1, respectively. So the second realization of the configuration (5;3)(ii) is done.

For the fifth realization take $P = x+xy+4xz+yz$, $Q = 4x-2x^2-y+4xy+y^2+15xz$ and $R = -9-36x+18x^2+9y-36xy-269xz+z^2$. Then the quadratic system (1) has the singular points $(0,1,0)$, $(-1,-1,0)$, $(1,-1,0)$, $(0,0,3)$, $(0,0,-3)$, $(1/4,-1/4,-1/4)$, $(1/4,1/4,-1/4)$ and $(9/53,-19/53,-18/53)$, and their indices are $1,1,-1,-1,1,1,-1$ and -1, respectively. So the third realization of the configuration (5;3)(ii) done.

Proof that the polyhedra (8)(i.2) *are not realizable.* Let Q_1 and Q_2 be the quadrilateral and the triangle of the bases of the quadrilateral-triangular prism respectively, and let p be the singular point at the interior of triangle of the bases. Let Π_1 be a plane containing Q_1. Let Π_2 be a plane defined by an edge of Q_2 containing only this edge of the quadrilateral-triangular prism. Applying the Euler-Jacobi formula (2) with $S = \Pi_1\Pi_2$ we get that the remaining vertex of Q_2 and p must have different index. Repeating this process for every edge of Q_2 we obtain that the three vertices of Q_2 have the same index different from the index of p. Let Π_3 be a plane containing one edge E of Q_1 and no other points of the polyhedron, and let Π_4 be the plane defined by Q_2. Applying the Euler-Jacobi formula (2) with $S = \Pi_3\Pi_4$ we get that the two vertices of Q_1 not contained in E must have different index. Repeating these arguments we obtain that the index configuration on Q_1 is $1,-1,1,-1$ in counterclockwise sense.

From the polyhedra 8, 9, 10, 14, 17, 19, 20, 30 and 32 of Figure 1(d) it follows that there exists an edge E of Q_1 such that the plane Π_5 determined by E and p leaves an edge E* of Q_2 on one side of the plane. Let Π_6 be the plane determined by the three vertices of the polyhedron which are neither in the plane Π_5 nor in the edge E*. Then applying the Euler-Jacobi formula (2) with $S = \Pi_5\Pi_6$ we get that the two vertices of E* have different index, a contradiction. □

Proof that the polyhedron (8)(i.4) *is not realizable.* Let Q_k for $k = 1, 2, 3, 4$ be the triangles of the faces of the tetrahedron, and let p_k be the singular point in the interior of Q_k. Let Π_k be the plane determined by Q_k. Then applying the Euler-Jacobi formula (2) with $S = \Pi_1\Pi_2$ we get that the singular points p_3 and p_4 have different index. Repeating this argument we get that the indices of p_i a p_j are different if $i \neq j$, and this provides a contradiction. □

Proof that the polyhedra $(7; 1)$ *are not realizable.* Let q be the point of A in the 1st-level. Let $\Pi_{i_1i_2i_3}$ be the plane determined one face of a polyhedron of type (7;1), and let $\Pi_{j_1j_2j_3}$ be the plane determined by another face which has any poin in the plane $\Pi_{i_1i_2i_3}$. Applying the Euler-Jacobi formula to $S = \Pi_{i_1i_2i_3}\Pi_{j_1j_2j_3}$ we get that the vertex of the polyhedron not contained in these two planes must have different index than the index of q. In this way we obtain that all the vertices of the polyhedron have the same index, in contradiction with the fact that the sum of the indices of the eight singular points is zero or two. □

Proof that the polyhedra $(6; 2)(i.3)$ *are not realizable.* Consider the plane Π_1 containing the edge E and the two interior points, and the plane Π_2 containing the edge of the triangle of the base that contains three singular points. The Euler-Jacobi formula applied with $S = \Pi_1\Pi_2$ provides a contradiction, because it remains in the Euler-Jacobi formula a unique term which cannot be zero. □

Proof that the tetrahedra $(4; 4)$ *are not realizable.* The four points in the 0 level are contained in the vertices of a tetrahedron. We denote them by p_1, p_2, p_3, p_4. By Proposition 2 the four points in the 1st level, that we denote them by p_5, p_6, p_7, p_8, cannot be contained in a plane and so they are also contained in the vertices of a tetrahedron.

Given $l_1, l_2 \in \{5, 6, 7, 8\}$ there exists $k_0 \in \{1, 2, 3, 4\}$ so that Π_{k_0,l_1,l_2} leaves the rest of the points in the 1-st level in the same side of the plane. We denote by p_{l_3}, p_{l_4} the points in the 1-st level different from p_{l_1}, p_{l_2} and by $p_{k_1}, p_{k_2}, p_{k_3}$ the points in the 0 level different from p_{k_0}. Then applying the Euler-Jacobi formula to $\Pi_{k_0l_1l_2}\Pi_{k_1k_2k_3}$ we obtain that p_{l_3} and p_{l_4} have different signs.

We repeat this procedure successively for all pairs of points in the 1-st level and we reach to a contradiction. Indeed, taking $\{l_3, l_4\} = \{5, 6\}$, we get that p_7 and p_8 have different indices; taking $\{l_3, l_4\} = \{6, 7\}$ we get that p_8 and p_5 have different indices and taking $\{l_3, l_4\} = \{7, 8\}$ we get that p_5 and p_6 have different indices. So, p_5 and p_7 have the same index and p_6 and p_8 have the same index. However taking now $\{l_3, l_4\} = \{5, 7\}$ we get that p_6 and p_8 must have different index which is not possible. In short, configuration (4;4) is not possible. $\square$

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[1] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: jllibre@mat.uab.cat

[2] Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1049–001, Lisboa, Portugal

Email address: cvalls@math.ist.utl.pt