This is a preprint of: "Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: local setting", David Marín, Jordi Villadelprat, *J. Differential Equations*, vol. 269(10), 8425–8467, 2020.
DOI: [10.1016/j.jde.2020.06.024]

Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: local setting

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Abstract. In this paper we study unfoldings of planar vector fields in a neighbourhood of a hyperbolic resonant saddle. We give a structure theorem for the asymptotic expansion of the local Dulac time (as well as the local Dulac map) with the remainder uniformly flat with respect to the unfolding parameters. Here local means close enough to the saddle in order that the normalizing coordinates provided by a suitable normal form can be used. The principal part of the asymptotic expansion is given in a monomial scale containing a deformation of the logarithm, the so-called Roussarie-Ecalle compensator. Especial attention is paid to the remainder's properties concerning the derivation with respect to the unfolding parameters.

Contents

1 Introduction and statements of the results

In this paper we study unfoldings of planar vector fields in a neighbourhood of a hyperbolic resonant saddle. It can be viewed as the continuation of a previous paper where we give a $\mathscr{C}^K$ normal form for the unfolding with respect to the conjugacy relation, see [10, Theorem A]. By means of this normal form in that paper

2010 *AMS Subject Classification*: 34C07; 34C20; 34C23.

Key words and phrases: Dulac map, Dulac time, asymptotic expansion, uniform flatness.

This work has been partially funded by the Ministry of Science, Innovation and Universities of Spain through the grants MTM2015-66165-P and MTM2017-86795-C3-2-P, the Agency for Management of University and Research Grants of Catalonia through the grants 2017SGR1725 and 2017SGR1617, and by the "María de Maeztu" Programme for Units of Excellence in R&D (MDM-2014-0445).

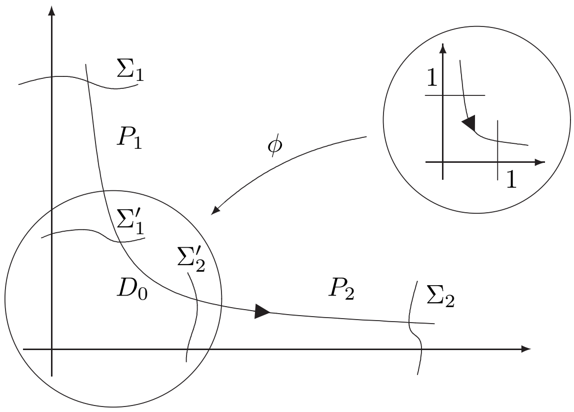

Figure 1: Auxiliary transverse sections in the decomposition of the Dulac map $D = P_2 \circ D_0 \circ P_1$ and the Dulac time $T = T_1 + T_0 \circ P_1 + T_2 \circ D_0 \circ P_1$, where P_1 (respectively, P_2) is the Poincaré map from Σ_1 to Σ_1' (respectively, Σ_2 to Σ_2') and T_1 (respectively T_2) is the time that spends the flow to do this transition. Here the local Dulac map is D_0 and the local Dulac time is T_0.

we also determine an asymptotic expansion, uniform with respect to the parameters, for the local Dulac time of a resonant saddle, see [10, Theorem B]. The *Dulac map* of a saddle is the transition map from a transverse section Σ_1 in the stable separatrix to a transverse section Σ_2 in the unstable separatrix, whereas the *Dulac time* is the time that spends the flow to do this transition, see Figure 1. By *local* we mean that Σ_1 and Σ_2 cannot be at arbitrary distance from the saddle but close enough in order that we can use the normalizing coordinates provided by the normal form. In other words, and more precisely, the local Dulac map (respectively, local Dulac time) is the Dulac map (respectively, Dulac time) of the normal form.

The asymptotic expansion of the Dulac map (see [15, Chapter 5] and references therein) is a key tool to study the *cyclicity* of a polycycle Γ (i.e., the maximum number of limit cycles that bifurcate from Γ) and to this end the remainder in the asymptotic expansion must be uniformly flat. In this respect recall that the second part of Hilbert's 16th problem asks for the maximum number of limit cycles, called $H(n)$, of a polynomial vector field $P(x,y)\partial_x + Q(x,y)\partial_y$ as a function of $n = \max(\deg(P), \deg(Q))$. It is still unknown whether $H(n)$ is finite. In case that the polycycle is monodromic and its return map is the identity then there is an annulus foliated by periodic orbits where the *period function* (i.e., the time of the return map) is defined. In this context the object of study are the so-called *critical periodic orbits*, which are the critical points of the period function. Similarly as with Hilbert's 16th problem, it arises the notion of *criticality* of a polycycle Γ, i.e., the maximum number of critical periodic orbits that bifurcate from Γ, see [7, 9]. In the same way as for the cyclicity, an asymptotic expansion of the Dulac time with remainder uniformly flat constitutes a key tool to study the criticality of a polycycle. Both asymptotic expansions are of similar nature, they are given in a monomial scale containing the so-called Roussarie-Ecalle compensator, which is deformation of the logarithm.

The asymptotic expansion of the local Dulac map (respectively, time) is a basic building block for establishing an asymptotic expansion of the Dulac map (respectively, time) and, in its turn, the latter is essential to study the cyclicity (respectively, criticality) of polycycles. In the present paper we focus on the local setting. Our main result for the local Dulac time is an asymptotic expansion that improves the one we previously obtained, see [10, Theorem B], in two aspects. Firstly because it gives a more precise description of the monomials appearing in the principal part. And secondly, more important, it shows that the remainder can be smoothly extended also with respect to the unfolding parameters. This was in fact our initial motivation to tackle the problem. In order to state our main theorems some results concerning normal forms are needed.

Let V be an open subset of $\mathbb{R}^N$ and consider a $\mathscr{C}^\infty$ unfolding $\{X_\mu\}_{\mu\in V}$ of a hyperbolic saddle point at the origin. More precisely,

$$X_\mu = A(x,y;\mu)x\partial_x + B(x,y;\mu)y\partial_y,$$

where $A,B\in\mathscr{C}^\infty(U\times V)$ for some open neighbourhood U of $(0,0)\in\mathbb{R}^2$ and $A(0,0;\mu)B(0,0;\mu)<0$ for all $\mu\in V$. The hyperbolicity ratio of the saddle is

$$\lambda=\lambda(\mu)=-\frac{B(0,0;\mu)}{A(0,0;\mu)}.$$

Given $m,n\in\mathbb{Z}$ we also consider the collinear family

$$Y_\mu=\frac{1}{x^my^n}X_\mu.$$

The reason why we permit this "polar" factor is because, when dealing with polynomial vector fields, a special attention must be paid to the study of those polycycles with vertices at infinity in the Poincaré disc. The factor can come from the line at infinity in a saddle at infinity or, more generally, appear in a divisor after desingularizing more general singular points at infinity of a polycycle. The case of lines of zeros in at least one of the separatrices is also allowed as it can appear after desingularizing a degenerate singular point at finite distance. It is important to remark that (by means of a reparametrization of time) this factor can be neglected to study the Dulac map but, on the contrary, this cannot be done when dealing with the Dulac time. For the same reason, to study the Dulac time we need normal forms with respect to the conjugacy relation rather than the equivalence relation.

We recall at this point Theorem A in [10], which generalizes well-known orbital normal forms with respect to the equivalence relation (see [6, 15] and references therein). To this end let us fix $\mu_0\in V$ and denote $\lambda_0=\lambda(\mu_0)$ for shortness. If $\lambda_0\in\mathbb{Q}$, say $\lambda_0=p/q$ with $(p,q)=1$, then that result shows that for any $k\in N$ the family $\{Y_\mu\}_{\mu\in V}$ is $\mathscr{C}^k$ conjugated, by means of a diffeomorphism $\Phi(x,y,\mu)=(\phi(x,y,\mu),\mu)$ defined in a neighbourhood of $(0,0,\mu_0)\in\mathbb{R}^2\times V$, to the normal form

$$Y_\mu^{NF}=\frac{1}{\eta(\mu)x^my^n+u^\ell Q(u;\mu)}\Big(x\partial_x+\big(-\lambda(\mu)+P(u;\mu)\big)y\partial_y\Big),$$

where η is a $\mathscr{C}^\infty$ function, P and Q are polynomials in the resonant monomial $u=x^py^q$ with the coefficients being also $\mathscr{C}^\infty$ functions in μ, and

$$\ell:=\begin{cases}\left\lceil\max\left(\frac{m}{p},\frac{n}{q}\right)\right\rceil & \text{if } mp-nq\neq 0,\\[1ex] \left\lceil\max\left(\frac{m}{p},\frac{n}{q}\right)\right\rceil+1 & \text{if } mp-nq=0.\end{cases}\tag{1}$$

Finally, if $\lambda_0\notin\mathbb{Q}$ then the result shows that we can take $P=Q=0$. (In this paper we use the common notation $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ for the floor and ceiling functions respectively.)

As we already explained, our aim in this paper is to study the Dulac time (as well as the Dulac map) associated to Y_μ^{NF}. (Note in this respect that the only interesting case is the resonant one, i.e., $\lambda_0\in\mathbb{Q}$, because otherwise both maps can be computed explicitly.) More generally, we consider the polynomial normal family

$$Y_{\alpha,\beta}:=\frac{1}{\beta_0x^my^n+u^\ell\sum_{i=1}^M\beta_iu^{i-1}}X_\alpha\tag{2}$$

where

$$X_\alpha:=x\partial_x+\tfrac{1}{q}\left(-p+\textstyle\sum_{i=0}^{N-1}\alpha_{i+1}u^i\right)y\partial_y.\tag{3}$$

In this way, setting $\alpha=(\alpha_1,\ldots,\alpha_N)\in\mathbb{R}^N$ and $\beta=(\beta_0,\ldots,\beta_M)\in\mathbb{R}^{M+1}$, we thus consider the coefficients of the polynomials $P(\,\cdot\,;\mu)$ and $Q(\,\cdot\,;\mu)$ in the normal form Y_μ^{NF} as independent parameters. Naturally we work with $\alpha_1\approx 0$ because

$$\lambda=\lambda(\alpha_1):=\frac{p-\alpha_1}{q}.$$

Note also that, with regard to the Dulac map, we can ignore the time and take X_α instead of $Y_{\alpha,\beta}$. That being said, we denote the Dulac map between $(0,1)\times\{1\}$ and $\{1\}\times(0,1)$ by $D(\,\cdot\,;\alpha)$. Similarly, the Dulac time between the same sections is denoted by $T(\,\cdot\,;\alpha,\beta)$. More explicitly, let $\varphi(t;s,\alpha)$ be the solution of X_α passing through $(s,1)\in\mathbb{R}^2$ with $s>0$ at $t=0$. Then, since this solution reaches $\{y=1\}$ at time $t=-\ln s$ due to $\varphi_1(t;s,\alpha)=se^t$, it turns out that $D(s;\alpha)=\varphi_2(-\ln s;s,\alpha)$. Likewise, if $\phi(t;s,\alpha,\beta)$ is the solution of $Y_{\alpha,\beta}$ passing through $(s,1)\in\mathbb{R}^2$ with $s>0$ then the Dulac time is the function $T(\,\cdot\,;\alpha,\beta)$ verifying

$$\phi_1(t;s,\alpha,\beta)|_{t=T(s;\alpha,\beta)}=1 \text{ for all } s>0 \text{ small enough.}$$

The present paper has two main results, namely: Theorem A, devoted to the Dulac map $D(s;\alpha)$, and Theorem B, addressed to the Dulac time $T(s;\alpha,\beta)$. The idea behind the proof, and also the aim of the result, is the same for both theorems. We show firstly that we can write the function as an infinite series for $s>0$ and α_1 small enough. Secondly, that we can truncate this series in order that the tail is uniformly flat at $s=0$. And, thirdly, that the finite truncation can be expressed in terms of a polynomial in s^p and $s^p\omega(s;\alpha_1)$, where ω is a deformation of the logarithm (see Definition 1.3), the so-called *Ecalle-Roussarie compensator*.

In this paper we use a more general notion of flatness (see Definition 1.2), which constitutes the key point in our approach as well as the main motivation to tackle the problem. Let us advance that it has better properties with respect to parameters and that this enables us to elucidate a delicate point which we think did not received the required attention in the literature (see Remark 1.4).

Definition 1.1. Consider $K\in\mathbb{Z}_{\geq 0}\cup\{+\infty\}$ and an open subset U of $\mathbb{R}^N$. We say that a function $\psi(s;\mu)$ belongs to the class $\mathscr{C}^K_{s>0}(U)$, respectively $\mathscr{C}^K_{s=0}(U)$, if there exist an open neighbourhood V of

$$\{(s,\mu)\in\mathbb{R}^{N+1};s=0,\mu\in U\}=\{0\}\times U$$

in $\mathbb{R}^{N+1}$ such that $(s,\mu)\mapsto\psi(s;\mu)$ is $\mathscr{C}^K$ on $V\cap\big((0,+\infty)\times U\big)$, respectively V. □

More formally, the definition of $\mathscr{C}^K_{s>0}(U)$ and $\mathscr{C}^K_{s=0}(U)$ must be thought in terms of germs with respect to relative neighborhoods of $\{0\}\times U$ in $(0,+\infty)\times U$. In doing so these sets become rings and we have the inclusions $\mathscr{C}^K(U)\subset\mathscr{C}^K_{s=0}(U)\subset\mathscr{C}^K_{s>0}(U)$. These facts are implicitly used in Lemma A.3.

We can now introduce the notion of (finitely) flatness that we shall use in the sequel.

Definition 1.2. Consider $K\in\mathbb{Z}_{\geq 0}\cup\{+\infty\}$ and an open subset U of $\mathbb{R}^N$. Given some $L\in\mathbb{R}$ and $\hat\mu\in U$, we say that $\psi(s;\mu)\in\mathscr{C}^K_{s>0}(U)$ is (L,K)*-flat with respect to s at* $\hat\mu$, and we write $\psi\in\mathcal{F}^K_L(\hat\mu)$, if for each $\nu=(\nu_0,\ldots,\nu_N)\in\mathbb{Z}^{N+1}_{\geq 0}$ with $|\nu|=\nu_0+\cdots+\nu_N\leqslant K$ there exist a neighbourhood V of $\hat\mu$ and $C,s_0>0$ such that

$$\left|\frac{\partial^{|\nu|}\psi(s;\mu)}{\partial s^{\nu_0}\partial\mu_1^{\nu_1}\cdots\partial\mu_N^{\nu_N}}\right|\leqslant Cs^{L-\nu_0} \text{ for all } s\in(0,s_0) \text{ and } \mu\in V. \tag{4}$$

If W is a (not necessarily open) subset of U then define $\mathcal{F}^K_L(W):=\bigcap_{\hat\mu\in W}\mathcal{F}^K_L(\hat\mu)$. □

The class $\mathcal{F}^K_L(W)$ consists in those functions $\psi(s;\mu)$ that are (finitely) flat along $\{0\}\times W$. The usual notion of (finitely) flatness is addressed to functions ψ that are smooth at $s=0$ and not depending on parameters. In that context one simply requires the s derivatives of ψ to vanish at $s=0$ up to order $K-1$. When dealing with functions that are not smooth at $s=0$, the natural and common definition is to require the estimates in (4). In this non-smooth context, and when the function depends on parameters, one can alternatively require (4) to hold for all $\mu \in V$ but only for derivation with respect to s. This is precisely the notion of flatness used in [14, 15] for the remainder of the asymptotic expansion of the Dulac map (cf. Remark 1.4). For instance the function $(s,\mu) \mapsto s^{\mu}$ is obviously L-flat at any $\hat{\mu} > L$ according to this alternative notion whereas to show that $(s,\mu) \mapsto s^{\mu}$ belongs to $\mathcal{F}_L^{\infty}(\{\mu > L\})$ requires some computations (see Lemma A.4). Coming again to Definition 1.2, note that the case $L < K$ is not excluded (and so it may occur that $L - \nu_0$ is negative) and that the case $L = K$ corresponds to the usal notion of (finitely) flatness.

The principal part of $D(\,\cdot\,;\alpha)$ and $T(\,\cdot\,;\alpha,\beta)$ will be expressed in terms of the following deformation of the logarithm.

Definition 1.3. The function defined for $s > 0$ and $\kappa \in \mathbb{R}$ by means of

$$\omega(s;\kappa) = \begin{cases} \frac{s^{-\kappa}-1}{\kappa} & \text{if } \kappa \neq 0, \\ -\ln s & \text{if } \kappa = 0, \end{cases}$$

is called the *Ecalle-Roussarie compensator.* □

Lemma A.4 gives several properties of the Ecalle-Roussarie compensator in relation with the class $\mathcal{F}_L^K(W)$ as introduced in Definition 1.2. It shows in particular that $(s,\kappa) \mapsto \omega(s;\kappa)$ belongs to $\mathcal{F}_{-\varepsilon}^{\infty}(\{\kappa < \varepsilon\})$ for all $\varepsilon > 0$. With regard to the parameter space of the family of vector fields in (3), hereafter we denote

$$U_0 := \{\alpha \in \mathbb{R}^N; \alpha_1 = 0\} = \{0\} \times \mathbb{R}^{N-1}.$$

We can now state our two main results. In both statements we set $\omega = \omega(s;\alpha_1)$ and $\lambda = \lambda(\alpha_1)$ for the sake of shortness. The first one is a structure theorem for the asymptotic expansion of the local Dulac map.

Theorem A. *Let $D(\,\cdot\,;\alpha)$ be the Dulac map of the vector field X_α in (3) between the sections $(0,1)\times\{1\}$ and $\{1\}\times(0,1)$. Then for each $L \in \mathbb{R}$ there exists a unique $\Delta(z,w;\alpha) \in \mathbb{Q}[z,w,\alpha]$, with $\deg_{(z,w)} \Delta \leqslant \frac{L}{p} - \frac{1}{q}$, and $\mathscr{D}_L \in \mathcal{F}_L^{\infty}(U_0)$ such that*

$$D(s;\alpha) = s^{\lambda}\Delta(s^p, s^p\omega;\alpha) + \mathscr{D}_L(s;\alpha).$$

Moreover, $\Delta(0,0;\alpha) = 1$ in case that $L \geqslant \frac{p}{q}$ and $\Delta \equiv 0$ otherwise.

This result has strong connections with the seminal works on the structure of the local Dulac map by A. Mourtada and R. Roussarie. Indeed, we write the principal part along the same lines as Mourtada, see [13, Proposition 2], in the sense that it is the Dulac map of the linear vector field $x\partial_x - \lambda y\partial_y$, i.e., $s \mapsto s^{\lambda}$, multiplied by a unity (that we show is polynomial in s^p and $s^p\omega$). Roussarie (see [14, Theorem F] or [15, Chapter 5]) writes the principal part in a different way and it is difficult to compare since he considers the case $p = q = 1$ only, which does not fit very well for $q \neq 1$. Next we make some further comments about it.

Remark 1.4. The proof of Theorem A (and also the forthcoming Theorem B) relies on some previous results by R. Roussarie in [14] (see also [15, Chapter 5]) that we gather in Lemma 2.1 and constitute our starting point. In that paper the author studies the cyclicity of a saddle loop and to this aim he proves Theorem F, which describes the structure of the local Dulac map $D(s;\alpha)$. That result is very similar to our Theorem A, but important differences exist. Firstly his result is addressed to the case $p = q = 1$ because at that time it was already well-known that the cyclicity of a saddle loop with $\lambda_0 \neq 1$ is at most one. Secondly his result is more precise in the description of the principal part, i.e., $D - \mathscr{D}_L$, since he divides it in the ideal generated by the coefficients $\alpha_1, \alpha_2, \ldots, \alpha_N$. And, thirdly, his proof concerning the remainder consists in showing that it verifies

$$\left|\partial_s^k \mathscr{D}_L(s;\alpha)\right| \leqslant C s^{L-k}.$$

This kind of estimate, similar to (4) but *without derivation with respect to parameters*, behaves well through the so-called derivation-division algorithm that yields to the main result in [14] on the cyclicity of the saddle loop (which in our opinion is perfectly right and, what is more, correctly proved). However it does not enable to assert that $\mathscr{D}_L$ extends to a $\mathscr{C}^L$ function in (s,α) at $s=0$ (see Example A.2 for a counterexample). Sadly enough, this is precisely what the author states in Theorem F with regard to the remainder $\mathscr{D}_L$ (see also Theorem 14 in [15, page 103]). This inexactness yields to a crucial gap in a subsequent paper by the same author [16]. Indeed, in that paper he studies the smoothness property of the bifurcation diagram of a generic saddle loop unfolding of codimension 2, and to prove the main result he appeals to this (unproved) claim in Theorem F. To be more precise, by taking advantage of the smoothness with respect to parameters of the remainder, he is able to apply an ad hoc implicit function theorem to prove Proposition 2.1. In our Theorem A we show that $\mathscr{D}_L \in \mathcal{F}_L^\infty(U_0)$, i.e., that the above bound holds for derivation with respect parameters as well, and on the other hand we prove (see Lemma A.1) that any function in $\mathcal{F}_L^K(U_0)$ with $L>K$ extends to a $\mathscr{C}^K$ function in a neighbourhood of $\{0\}\times U_0$ in $\mathbb{R}^{N+1}$. We can thus fill the gap between the proof of [14, Theorem F] and its statement. This shows in particular the validity of the proof of [16, Proposition 2.1], which constitutes a key step to show the main result in that paper. $\square$

Next result provides the structure of the asymptotic expansion of the local Dulac time and in its statement we assume that $\lceil \max\left(\frac{m}{p},\frac{n}{q}\right)\rceil \geqslant 0$. Let us point out however that we do not need this assumption in any of the previous auxiliary results. In this regard note that this hypothesis is satisfied if m and n are not both negative. From the point of view of the bifurcation of critical periodic orbits, the most interesting situation comes from the Dulac time associated to a saddle placed in the line at infinity, and in this case either $m>0$ or $n>0$.

Theorem B. *Let $T(\,\cdot\,;\alpha,\beta)$ be the Dulac time of the vector field $Y_{\alpha,\beta}$ in (2) between the sections $(0,1)\times\{1\}$ and $\{1\}\times(0,1)$. Suppose that $\kappa:=\lceil \max\left(\frac{m}{p},\frac{n}{q}\right)\rceil \geqslant 0$. Then for each $L\in\mathbb{R}$ we can write*

$$T(s;\alpha,\beta = T^L(s;\alpha,\beta)+\mathcal{T}_L(s;\alpha,\beta),$$

where

(1) *the principal part is given by*

$$T^L(s;\alpha,\beta):=\tau_0(\beta)\ln s+s^{\lambda n}\tau_1(s^p,s^p\omega;\alpha,\beta)-s^m\tau_1(s^p,0;\alpha,\beta)+s^{\kappa p}\tau_2(s^p,s^p\omega;\alpha,\beta),$$

with $\tau_0(\beta)\in\mathbb{Q}[\beta]$, $\tau_1(z,w;\alpha,\beta)\in\mathbb{Q}(\alpha_1)[z,w,\alpha_2,\ldots,\alpha_N,\beta]$ and $\tau_2(z,w;\alpha,\beta)\in\mathbb{Q}[z,w,\alpha,\beta]$,

(2) *and the remainder $\mathcal{T}_L(s;\alpha,\beta)$ belongs to $\mathcal{F}_L^\infty(U_0\times\mathbb{R}^{M+1})$.*

Moreover the principal part verifies the following:

(a) τ_1 is linear in β and without poles along $\alpha_1=0$.

(b) τ_2 is linear in β and $\tau_2(z,0;\alpha,\beta)=0$.

(c) $\tau_1=0$ if $mp-nq=0$.

(d) $\tau_0=-\beta_0$ if $(m,n)=(0,0)$ and $\tau_0=-\beta_1$ if $\ell=0$, whereas $\tau_0=0$ in any other case.

In a previous paper we already give a structure theorem for the asymptotic expansion of the local Dulac time, see [10, Theorem B]. The main difference between both results is that we can now guarantee that the remainder $\mathcal{T}_L$ is flat along $s=0$, not only for the derivation with respect to s, but also with respect to α and β (cf. Definition 1.2). Consequently, as we explain in Remark 1.4, by applying Lemma A.1 we can assert that if $K<L$ then the remainder $\mathcal{T}_L(s;\alpha,\beta)$ extends to a $\mathscr{C}^K$ function in a neighbourhood of $\{0\}\times U_0\times\mathbb{R}^{M+1}$ in $\mathbb{R}\times\mathbb{R}^N\times R^{M+1}$. We are convinced that this regularity of the remainder will be crucial in future applications, for instance to have a better understanding of the bifurcation diagram of the critical periodic orbits of the Loud's centers, see [9]. In fact this kind of property has already been used to study the

period of the limit cycle appearing in one-parameter saddle loop bifurcations (see [4, Theorem 16]). To this end the authors prove Proposition 23, which corresponds to Theorem B particularized to $m = n = 0$ and $K = L = 1$. (As a matter of fact while trying to extend it we realized that their proof contains a bridgeable mistake that we correct here, see Remark 2.4.) Coming back to our previous result in [10], let us note that the principal part T^L that we provide here is more precise than the one given there.

The paper is organized as follows. In Section 2, taking Roussarie's results in [15, Chapter 5] as starting point, we consider the solution $\varphi(t; s, \alpha)$ of X_α passing through $(s, 1) \in \mathbb{R}^2$ at $t = 0$ and we expand $\varphi_2(t; s, \alpha)$ as a power series in s for each fixed t and α. We obtain sharp uniform estimates for the radius of convergence of this series (see Lemma 2.5) and also for the derivatives of its coefficients (see Lemma 2.7). Next, on account of $D(s; \alpha) = \varphi_2(-\ln s; s, \alpha)$, in Section 3 we use these results to prove Theorem A. Section 4 is devoted to the proof of Theorem B and to this end, see (2), we take advantage of the previous results thanks to the identity

$$T(s; \alpha, \beta) = \int_0^{-\ln s} \left(\beta_0 x^m y^n + \sum_{i=\ell}^{M+\ell-1} \beta_{i+1-\ell} (x^p y^q)^i \right) \Bigg|_{\{(x,y)=\varphi(t;s,\alpha)\}} dt.$$

Some technical but crucial issues about the sets $\mathcal{F}_L^K(W)$ are treated in Appendix A. Among other properties we show that any $g(s; \mu) \in \mathcal{F}_L^K(W)$ with $L > K$ extends to a $\mathscr{C}^K$-function (on s and the parameter μ) along $s = 0$. (This applies in particular to the remainder $\mathscr{D}_L$ in Theorem A, as well as to $\mathcal{T}_L$ in Theorem B.) Finally, in Appendix B we recall some specific results from analysis and calculus, in particular the multivariate Faa di Bruno formula for higher-order derivatives of a composite function (see Theorem B.1) that we use repeatedly all over the paper.

2 Further results on Roussarie's series expansion

Observe that performing the singular change of variables $\{u = x^p y^q, x = x\}$, the differential equation given by the vector field X_α in (3) is brought to the following form:

$$\begin{cases} \dot{x} = x, \\ \dot{u} = P(u; \alpha) := \sum_{i=1}^N \alpha_i u^i. \end{cases}$$

The first equation gives $x(t, x_0) = x_0 e^t$ and we denote by $u(t, u_0; \alpha)$ the solution of the second one with initial condition $u(0, u_0; \alpha) = u_0$. For each fixed t and α, we expand it as a power series in u_0,

$$u(t, u_0; \alpha) = \sum_{i=1}^{+\infty} g_i(t; \alpha) u_0^i. \tag{5}$$

In what follows, for any given $\delta > 0$ we define

$$U_\delta := \{\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{R}^N; |\alpha_1| < \delta\}.$$

Following this notation, Roussarie [15, §5.1.2] shows the next result with regard to the series in (5).

Lemma 2.1. *The following assertions hold:*

(a) *For all $i \in \mathbb{N}$, $g_i(t; \alpha) = e^{\alpha_1 t} \bar{g}_{i-1}(t; \alpha)$ with $\bar{g}_i(t; \alpha) \in \mathbb{Q}[\alpha, \Omega]$ where $\Omega := \frac{e^{\alpha_1 t} - 1}{\alpha_1}$ and $\deg_\Omega \bar{g}_i \leqslant i$.*

(b) *For each compact set $\mathcal{C} \subset U_\delta$ with $\delta \in (0, \frac{1}{2}]$ there exist $K_0, C_0 > 0$ such that if $t \geqslant 0$, $|u_0| < C_0^{-1} e^{-\delta t}$ and $\alpha \in \mathcal{C}$ then the series (5) is absolutely convergent and $|u(t, u_0; \alpha)| \leqslant K_0 |u_0| e^{\delta t}$.*

(c) *For all $i \in \mathbb{N}$, $t \geqslant 0$ and $\alpha \in \mathcal{C}$, $|g_i(t; \alpha)| \leqslant K_0 C_0^{-1} (C_0 e^{\delta t})^i$.*

Proof. Assertion (a) is proved in Proposition 10, whereas (b) follows from the proofs of Lemmas 18 and 19 because, using the author's notation,

$$|u(t,u_0;\alpha)| \leqslant \sum_{i=1}^{+\infty} |g_i(t;\alpha)||u_0|^i \leqslant \sum_{i=1}^{+\infty} G_i(t)|u_0|^i = U(t,|u_0|).$$

Finally (c) follows from (b) by applying the Cauchy's estimates (see for instance [18, Theorem 10.26]). ∎

Corollary 2.2. *For each compact set $\mathcal{C} \subset U_\delta$ with $\delta \in (0,\frac{1}{2}]$ there exist $C_0 > 0$ such that the function $u(t,u_0;\alpha)$ is analytic on an open set containing*

$$\{(t,u_0,\alpha) \in \mathbb{R}^{N+2}; t \geqslant 0,\ |u_0| < C_0^{-1}e^{-\delta t}, \alpha \in \mathcal{C}\}.$$

Proof. Recall that $u(t,u_0;\alpha)$ is the solution of $\dot{u} = P(u;\alpha)$ with initial condition $u(0,u_0;\alpha) = u_0$. Let us denote its maximal interval of existence by (ω_-,ω_+), where $\omega_\pm = \omega_\pm(u_0,\alpha)$. Since P is analytic on $\mathbb{R}^{N+1}$,

$$D = \{(t,u_0,\alpha) \in \mathbb{R}^{N+2}; \omega_-(u_0,\alpha) < t < \omega_+(u_0,\alpha)\}$$

is an open set in $\mathbb{R}^{N+2}$ and $u(t,u_0;\alpha)$ is analytic in D (see [5, Theorem 1.1] and [19, page 34]). Moreover, for the same reason, if ω_+ is finite then $|u(t,u_0;\alpha)|$ tends to $+\infty$ as $t \nearrow \omega_+$ (see [1, Theorem 1.263] or [19, page 17] for instance). Note on the other hand that, by Lemma 2.1, if $|u_0| < C_0^{-1}e^{-\delta t}$ then $|u(t,u_0;\alpha)| \leqslant K_0 C_0^{-1}$ for all $t \in (0,\omega_+)$ and $\alpha \in \mathcal{C}$. Arguing by contradiction this implies that $\omega_+ > -\frac{1}{\delta}\ln(C_0|u_0|)$ and concludes the proof of the result. ∎

Given $\nu = (\nu_0,\nu_1,\ldots,\nu_N) \in \mathbb{Z}_{\geq 0}^{N+1}$, we write

$$\partial_{t,\alpha}^{\nu} = \frac{\partial^{|\nu|}}{\partial t^{\nu_0}\partial\alpha_1^{\nu_1}\cdots\partial\alpha_N^{\nu_N}}$$

and, following this notation, we expand $\partial_{t,\alpha}^{\nu}u(t,u_0;\alpha)$ as a power series in u_0,

$$\partial_{t,\alpha}^{\nu}u(t,u_0;\alpha) = \sum_{i=1}^{+\infty} h_i(t,\alpha)u_0^i. \tag{6}$$

Similarly as in Lemma 2.1, we want to estimate the functions h_i and the convergence of the above series in terms of t and α. This is the aim of the next result, where we also write $\nu = (\nu_0,\bar{\nu})$ with $\bar{\nu} = (\nu_1,\ldots,\nu_N)$ for the sake of convenience.

Theorem 2.3. *For all $\nu \in \mathbb{Z}_{\geq 0}^{N+1}$ there exists a real number $\rho_{\bar{\nu}}$, satisfying $1 \leqslant \rho_{\bar{\nu}} \leqslant \max(|\bar{\nu}|,1)$ and independent from ν_0, such that for each compact set $\mathcal{C} \subset U_\delta$ with $\delta \in (0,\frac{1}{2}]$ there exist $C_{\bar{\nu}} > 0$, independent from ν_0, and $K_\nu > 0$ such that if $t \geqslant 0$, $\alpha \in \mathcal{C}$ and $|u_0| < C_{\bar{\nu}}^{-1}e^{-\delta t}$ then*

(i) $|\partial_{t,\alpha}^{\nu}u(t,u_0;\alpha)| \leqslant K_\nu|u_0|e^{\rho_{\bar{\nu}}\delta t}$, *and*

(ii) the series in (6) is absolutely convergent.

Moreover, for all $i \in \mathbb{N}$, $\alpha \in \mathcal{C}$ and $t \geqslant 0$, $h_i(t,\alpha) = \partial_{t,\alpha}^{\nu}g_i(t;\alpha)$ and

$$\left|\partial_{t,\alpha}^{\nu}g_i(t;\alpha)\right| \leqslant K_\nu e^{\rho_{\bar{\nu}}\delta t}\left(C_{\bar{\nu}}e^{\delta t}\right)^{i-1}.$$

Finally there exists $M > 0$ such that if $|u_0| < (2C_0)^{-1}e^{-\delta t}$ then

$$|\partial_{u_0}^2 u(t,u_0;\alpha)| \leqslant Me^{\alpha_1 t}\Omega(t,\alpha_1) \text{ where } \Omega(t,\alpha_1) := \tfrac{e^{\alpha_1 t}-1}{\alpha_1}.$$

for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$.

Proof. We begin by proving assertions (i) and (ii) in case that $\nu_0 = 0$, i.e., only derivation with respect to parameters. To this end for the sake of shortness we use the compact notation

$$\partial_{t,\alpha}^{(0,\nu_1,\dots,\nu_N)} = \partial_\alpha^{\bar\nu} = \frac{\partial^{|\bar\nu|}}{\partial\alpha_1^{\nu_1}\cdots\partial\alpha_N^{\nu_N}} \text{ where } \bar\nu = (\nu_1,\dots,\nu_N).$$

The proof follows by induction on $|\bar\nu|$. The base case $|\bar\nu| = 0$ is (b) in Lemma 2.1. To show the induction step we first perform the partial derivation $\partial_\alpha^{\bar\nu}$ on both sides of the equality $\partial_t u = P(u;\alpha)$ and then apply Theorem B.1 to obtain

$$\begin{aligned}\partial_t\partial_\alpha^{\bar\nu} u(t,u_0;\alpha) =& \partial_\alpha^{\bar\nu} P(u(t,u_0;\alpha);\alpha)\\ =& \sum_{1\leqslant|\lambda|\leqslant|\bar\nu|} \partial^\lambda P(u;\alpha) \sum_{p(\bar\nu,\lambda)} \bar\nu! \prod_{j=1}^q \left.\frac{(\partial_\alpha^{\ell_j} u)^{k_{j0}} \prod_{i=1}^N (\partial_\alpha^{\ell_j}\alpha_i)^{k_{ji}}}{k_j!(\ell_j!)^{|k_j|}}\right|_{u=u(t,u_0;\alpha)}.\end{aligned} \tag{7}$$

Here, for $\lambda = (\lambda_0,\dots,\lambda_N) \in \mathbb{Z}_{\geq 0}^{N+1}$, we use the notation $\partial^\lambda P(u;\alpha) = \frac{\partial^{|\lambda|}P(u;\alpha)}{\partial u^{\lambda_0}\partial\alpha_1^{\lambda_1}\cdots\partial\alpha_n^{\lambda_N}}$ and, for $k_j \in \mathbb{Z}_{\geq 0}^{N+1}$, we write $k_j = (k_{j0},\dots,k_{jN})$. Note also that both summations are multidimensional and the second one is subject to the coupling conditions given in (37), namely $\sum_{i=1}^q k_i = \lambda$ and $\sum_{i=1}^q |k_i|\ell_i = \bar\nu$. In this respect we observe the following:

(a) The only summand in (7) that contains a factor $\partial_\alpha^{\ell_j} u$ with $|\ell_j| = |\bar\nu|$ is $\partial_u P(u;\alpha)\partial_\alpha^{\bar\nu} u$. Indeed, this is so because if $\ell_j = \bar\nu$ and $k_{j0} \neq 0$ then $|k_i| = 0$ for $i \neq j$ and $\lambda = k_j = (1,0,\dots,0)$.

(b) If $k_{j0} = 0$ for all j then $\lambda_0 = 0$. Consequently any summand in (7) not containing a factor $\partial^\ell u$ with $|\ell| > 0$ has the factor $\partial^{(0,\ell)}P(u;\alpha) = \sum_{i=1}^N (\partial_\alpha^\ell \alpha_i)u^i$, which is a polynomial vanishing at $u = 0$.

Accordingly we can split the right hand side of the equation (7) so that it writes as

$$\partial_t\partial_\alpha^{\bar\nu} u(t,u_0;\alpha) = \partial_u P(u(t,u_0;\alpha);\alpha)\partial_\alpha^{\bar\nu} u(t,u_0;\alpha) + R_{\bar\nu}^1(t,u_0,\alpha) + R_{\bar\nu}^2(t,u_0,\alpha),$$

where we define $R_{\bar\nu}^1$ to be the sum of those summands with $k_{j0} = 0$ for all $j = 1,2,\dots,q$ while $R_{\bar\nu}^2$ is the sum of the remaining summands. Note then that

$$R_{\bar\nu}^1(t,u_0,\alpha) = uS(u;\alpha)|_{u=u(t,u_0;\alpha)}$$

for some polynomial $S(u;\alpha)$ with $\deg_u S = N-2$. The above equality is a first order linear differential equation for $\partial_\alpha^{\bar\nu} u(t,u_0;\alpha)$ that, setting $R_{\bar\nu} = R_{\bar\nu}^1 + R_{\bar\nu}^2$ and

$$B(t,u_0,\alpha) := \exp\left(\int_0^t \partial_u P(u(s,u_0;\alpha);\alpha)ds\right) \tag{8}$$

for the sake of shortness, yields to

$$\partial_\alpha^{\bar\nu} u(t,u_0;\alpha) = B(t,u_0,\alpha)\int_0^t \frac{R_{\bar\nu}(s,u_0,\alpha)}{B(s,u_0,\alpha)}ds. \tag{9}$$

Note that we can write $\partial_u P(u(s,u_0;\alpha);\alpha) = \sum_{i=0}^\infty p_i(s,\alpha)u_0^i$ with the same radius of convergence as (5) because $\partial_u P(\,\cdot\,;\alpha)$ is polynomial. In addition we have $p_0(s;\alpha) \equiv \alpha_1$. Thus, applying (b) in Lemma 2.1 and setting $K_1 := \sup\{|\partial_u P(u;\alpha) - \alpha_1|; |u| \leqslant K_0C_0^{-1}, \alpha \in \mathcal{C}\}$, if $|u_0| < C_0^{-1}e^{-\delta t}$ then

$$\begin{aligned}\left|\int_0^t (\partial_u P(u(s,u_0;\alpha);\alpha) - \alpha_1)\,ds\right| &\leqslant \sum_{i=1}^\infty \int_0^t |p_i(s;\alpha)||u_0|^i ds \leqslant K_1\sum_{i=1}^\infty \int_0^t (C_0^{-1}e^{\delta s})^i|u_0|^i ds\\ &= K_1\sum_{i=1}^\infty C_0^i|u_0|^i\frac{e^{\delta it}-1}{\delta i} \leqslant \frac{K_1}{\delta}\sum_{i=1}^\infty (C_0|u_0|e^{\delta t})^i < +\infty,\end{aligned}$$

where in the second inequality we use Cauchy's estimates (see [18, Theorem 10.26]). Thus, by Lemma B.5,

$$\int_0^t \partial_u P(u(s,u_0;\alpha);\alpha)ds = \sum_{i=0}^{+\infty}\left(\int_0^t p_i(s;\alpha)ds\right)u_0^i$$

and the series converges absolutely for $|u_0| < C_0^{-1}e^{-\delta t}$. Furthermore, setting $K_2 := \frac{K_1}{\delta}$, if $|u_0| \leqslant (2C_0)^{-1}e^{-\delta t}$ then

$$\alpha_1 t - K_2 \leqslant \int_0^t \partial_u P(u(s,u_0;\alpha);\alpha)ds \leqslant \alpha_1 t + K_2 \text{ for all } t \geqslant 0 \text{ and } \alpha \in \mathcal{C}.$$

Consequently, recall (8), if $|u_0| \leqslant (2C_0)^{-1}e^{-\delta t}$ then

$$e^{-K_2}e^{\alpha_1 t} \leqslant B(t,u_0,\alpha) \leqslant e^{K_2}e^{\alpha_1 t} \text{ for all } t \geqslant 0 \text{ and } \alpha \in \mathcal{C}. \tag{10}$$

On the other hand, since $x \mapsto e^{\pm x}$ are entire functions, the Taylor series of $B(t,u_0,\alpha)$ and $1/B(t,u_0,\alpha)$ at $u_0 = 0$ converge absolutely for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$ provided that $|u_0| < C_0^{-1}e^{-\delta t}$. Therefore, from (9) and taking the previous bounds into account, we get that if $|u_0| \leqslant (2C_0)^{-1}e^{-\delta t}$ then

$$|\partial_\alpha^{\bar\nu} u(t,u_0;\alpha)| \leqslant e^{2K_2}e^{\alpha_1 t}\int_0^t |R_{\bar\nu}(s,u_0,\alpha)|\, e^{-\alpha_1 s}ds \text{ for all } t \geqslant 0 \text{ and } \alpha \in \mathcal{C}. \tag{11}$$

We are now in position to prove the validity of assertions (i) and (ii) for the case $\nu_0 = 0$ and to this end recall that $R_{\bar\nu} = R_{\bar\nu}^1 + R_{\bar\nu}^2$. Let us begin with the study of $R_{\bar\nu}^2$ by noting that in each one of its summands we have that $(\partial^{\ell_j}u(t,u_0;\alpha))^{k_{j0}}$ verifies $|\ell_j| < |\bar\nu|$ for $j = 1,2,\ldots,q$ and that there is at least one exponent k_{j0} strictly positive. Accordingly, thanks to the induction hypothesis, for each $j = 1,2,\ldots,q$ we know that if $|u_0| < C_{\ell_j}^{-1}e^{-\delta t}$ then $|(\partial^{\ell_j}u(t,u_0;\alpha))^{k_{j0}}| \leqslant (K_{\ell_j}|u_0|e^{\rho_{\ell_j}\delta t})^{k_{j0}}$ for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$. We define

$$p_\star(\bar\nu) := \bigcup_{1\leqslant|\lambda|\leqslant|\bar\nu|}\{(k_1,\ldots,k_q;\ell_1,\ldots,\ell_q) \in p(\bar\nu,\lambda)\,;\, |\ell_j| < |\bar\nu|\},$$

which is nonempty if and only if $|\bar\nu| > 1$. Taking this into account, if $|\bar\nu| > 1$ then we set

$$C_{\bar\nu} := \max\Big(2C_0, \max\big(C_{\ell_j};\, (k_1,\ldots,k_q;\ell_1,\ldots,\ell_q) \in p_\star(\bar\nu)\big)\Big),$$

$$\rho_{\bar\nu} := \max\Big(1, \max\big(\textstyle\sum_{j=1}^q k_{j0}\rho_{\ell_j};\, (k_1,\ldots,k_q;\ell_1,\ldots,\ell_q) \in p_\star(\bar\nu)\big)\Big)$$

and

$$K_3 := \max\Big(\textstyle\prod_{j=1}^q (K_{\ell_j})^{k_{j0}};\, (k_1,\ldots,k_q;\ell_1,\ldots,\ell_q) \in p_\star(\bar\nu)\Big),$$

whereas if $|\bar\nu| = 1$ then we define $C_{\bar\nu} = 2C_0$, $\rho_{\bar\nu} = 1$ and $K_3 = 1$. Furthermore we define

$$K_4 := \sup\left\{|\partial^\lambda P(u;\alpha)|;\, |u| \leqslant K_0C_0^{-1}, \alpha \in \mathcal{C}, 1 \leqslant |\lambda| \leqslant |\bar\nu|\right\}$$

and

$$K_5 := \sup\left\{\textstyle\sum_{1\leqslant|\lambda|\leqslant|\bar\nu|}\sum_{p(\bar\nu,\lambda)}\nu!\prod_{j=1}^q \frac{\prod_{i=1}^N(\partial^{\ell_j}\alpha_i)^{k_{ji}}}{k_j!(\ell_j!)^{|k_j|}};\, \alpha \in \mathcal{C}\right\}$$

Note moreover that $|u_0|^{\sum_{j=1}^q k_{j0}} \leqslant |u_0|$ due to $\sum_{j=1}^q k_{j0} \geqslant 1$ and $|u_0| \leqslant 1$. On account of these definitions and applying (b) in Lemma 2.1, from (7) it follows that if $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$ then $|R_{\bar\nu}^2(t,u_0,\alpha)| \leqslant K_6|u_0|e^{\rho_{\bar\nu}\delta t}$ for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$, where we set $K_6 := K_3K_4K_5$. Let us proceed next with the study of $R_{\bar\nu}^1$. In this case, due to $R_{\bar\nu}^1(t,u_0,\alpha) = uS(u;\alpha)|_{u=u(t,u_0;\alpha)}$, we define

$$K_7 := \sup\left\{|S(u;\alpha)|; |u_0| \leqslant K_0C_0^{-1}, \alpha \in \mathcal{C}\right\}.$$

Thus, by applying Lemma 2.1, if $|u_0| < C_0^{-1}e^{-\delta t}$ then $|R^1_{\bar\nu}(t,u_0,\alpha)| \leqslant K_0K_7|u_0|e^{\delta t}$ for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$. Finally, taking $C_{\bar\nu} \geqslant C_0$ into account, we can assert that if $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$ then

$$|R_{\bar\nu}(t,u_0,\alpha)| \leqslant |R^1_{\bar\nu}(t,u_0,\alpha)| + |R^2_{\bar\nu}(t,u_0,\alpha)| \leqslant K_0K_7|u_0|e^{\delta t} + K_6|u_0|e^{\rho_{\bar\nu}\delta t} \leqslant K_8|u_0|e^{\rho_{\bar\nu}\delta t} \tag{12}$$

for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$, where we set $K_8 := \max(K_6, K_0K_7)$ and we use that $\rho_{\bar\nu} \geqslant 1$. We can now plug this inequality in (11) to obtain that if $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$ then

$$|\partial_\alpha^{\bar\nu}u(t,u_0;\alpha)| \leqslant K_8e^{2K_2}|u_0|e^{\alpha_1 t}\int_0^t e^{(\rho_{\bar\nu}\delta-\alpha_1)s}ds = K_8e^{2K_2}|u_0|e^{\alpha_1 t}\frac{e^{(\rho_{\bar\nu}\delta-\alpha_1)t}-1}{\rho_{\bar\nu}\delta-\alpha_1} \leqslant K_\nu|u_0|e^{\rho_{\bar\nu}\delta t}$$

for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$, where $K_\nu := \frac{K_8e^{2K_2}}{K_9}$ with $K_9 := \inf\{\rho_{\bar\nu}\delta - \alpha_1;\, \alpha \in \mathcal{C}\}$, which is strictly positive because $|\alpha_1| < \delta \leqslant \rho_{\bar\nu}\delta$. This proves the inductive step with regard to assertion (i). Let us turn now to assertion (ii). Since $R_{\bar\nu}$ is a polynomial of $\partial^\ell u$ with $0 \leqslant |\ell| < |\bar\nu|$ on account of property (a), by the induction hypothesis we get that the Taylor series of $R_{\bar\nu}(t,u_0,\alpha)$ at $u_0 = 0$ is absolutely convergent for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$ provided that $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$. Furthermore, from (12), if $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$ then

$$|R_{\bar\nu}(t,u_0,\alpha)| \leqslant K_8C_{\bar\nu}^{-1}e^{(\rho_{\bar\nu}-1)\delta t} \text{ for all } t \geqslant 0 \text{ and } \alpha \in \mathcal{C}.$$

Recall on the other hand that the Taylor series of $\frac{1}{B(t,u_0,\alpha)}$ at $u_0 = 0$ is absolutely convergent for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$ provided that $|u_0| < C_0^{-1}e^{-\delta t}$. In addition (10) shows that $\frac{1}{|B(t,u_0,\alpha)|} \leqslant e^{K_2}e^{-\alpha_1 t}$ for this range of values. Hence, due to $C_0 < 2C_0 \leqslant C_{\bar\nu}$, if $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$ then the series $\frac{R_{\bar\nu}(t,u_0,\alpha)}{B(t,u_0,\alpha)} = \sum_{i=0}^\infty r_i(t;\alpha)u_0^i$ is absolutely convergent and the upper bound

$$\left|\frac{R_{\bar\nu}(t,u_0,\alpha)}{B(t,u_0,\alpha)}\right| \leqslant K_8C_{\bar\nu}^{-1}e^{K_2}e^{((\rho_{\bar\nu}-1)\delta-\alpha_1)t} \leqslant K_8C_{\bar\nu}^{-1}e^{K_2}e^{\rho_{\bar\nu}\delta t}$$

holds for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$ due to $|\alpha_1| < \delta$. Note also that $r_0(t;\alpha) \equiv 0$. As we did before, the Cauchy's estimates show that

$$|r_i(t;\alpha)| \leqslant K_8C_{\bar\nu}^{-1}e^{K_2}C_{\bar\nu}^i e^{(\rho_{\bar\nu}+i)\delta t} \text{ for all } i \in \mathbb{N},\, t \geqslant 0 \text{ and } \alpha \in \mathcal{C}.$$

Consequently, for all $i \in \mathbb{N}$, $t \geqslant 0$ and $\alpha \in \mathcal{C}$, we get

$$\int_0^t |r_i(s;\alpha)|ds \leqslant K_8C_{\bar\nu}^{-1}e^{K_2}C_{\bar\nu}^i\frac{e^{(\rho_{\bar\nu}+i)\delta t}-1}{(\rho_{\bar\nu}+i)\delta} \leqslant K_9C_{\bar\nu}^i e^{(\rho_{\bar\nu}+i)\delta t},$$

where $K_9 = \frac{K_8C_{\bar\nu}^{-1}e^{K_2}}{\rho_{\bar\nu}\delta}$. Thanks to these estimates we can assert that if $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$ then

$$\left|\int_0^t \frac{R_{\bar\nu}(s,u_0,\alpha)}{B(s,u_0,\alpha)}ds\right| \leqslant \sum_{i=1}^{+\infty}|u_0|^i\int_0^t |r_i(s;\alpha)|ds \leqslant K_9e^{\rho_{\bar\nu}\delta t}\sum_{i=1}^{+\infty}\left(C_{\bar\nu}|u_0|e^{\delta t}\right)^i < \infty$$

for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$. Therefore, by Lemma B.5,

$$\int_0^t \frac{R_{\bar\nu}(s,u_0,\alpha)}{B(s,u_0,\alpha)}ds = \sum_{i=1}^{+\infty}\left(\int_0^t r_i(s;\alpha)ds\right)u_0^i$$

and the series converges absolutely for $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$. Since we already prove this fact for the Taylor series of $B(t,u_0,\alpha)$ at $u_0 = 0$, from (9) it follows that the Taylor series of $\partial_\alpha^{\bar\nu}u(t,u_0;\alpha)$ at $u_0 = 0$ converges absolutely for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$ provided that $|u_0| < C_{\bar\nu}^{-1}e^{-\delta t}$. This shows the inductive step concerning assertion (ii) for the case $\nu_0 = 0$.

Let us turn now to the proof of the case $\nu_0 > 0$. Since $\partial_t u = P(u;\alpha)$ we deduce that $\partial_t^n u = P_n(u;\alpha)$ for all $n \in \mathbb{N}$, where $P_n := P\partial_u P_{n-1}$ with $P_0(u;\alpha) := u$. The application of Faa di Bruno formula given by Theorem B.1 yields to

$$
\begin{aligned}
\partial_{t,\alpha}^{(\nu_0,\nu_1,\ldots,\nu_N)} u(t,u_0;\alpha) &= \partial_\alpha^{\bar{\nu}} P_{\nu_0}(u(t,u_0;\alpha)) \\
&= \sum_{1\leqslant|\lambda|\leqslant|\bar{\nu}|} \partial^\lambda P_{\nu_0}(u;\alpha) \sum_{p(\bar{\nu},\lambda)} \bar{\nu}! \prod_{j=1}^q \frac{(\partial_\alpha^{\ell_j} u)^{k_{j0}} \prod_{i=1}^N (\partial_\alpha^{\ell_j}\alpha_i)^{k_{ji}}}{k_j!(\ell_j!)^{|k_j|}} \Bigg|_{u=u(t,u_0;\alpha)} \\
&= \hat{R}_\nu^1(t,u_0,\alpha) + \hat{R}_\nu^2(t,u_0,\alpha),
\end{aligned}
\tag{13}
$$

where $\hat{R}_\nu^1$ consists in all the summands with $k_{j0} = 0$ for all j. Accordingly, due to $P_{\nu_0}(0;\alpha) \equiv 0$, exactly as we did to show (b), we can write $\hat{R}_\nu^1(t,u_0,\alpha) = uS_\nu(u;\alpha)|_{u=u(t,u_0;\alpha)}$ for some polynomial S_ν. Thus, setting

$$
K_{10} := \sup\left\{|S_\nu(u;\alpha)|; |u| \leqslant K_0 C_0^{-1}, \alpha \in \mathcal{C}\right\}
$$

and applying (b) in Lemma 2.1, if $|u_0| < C_0^{-1}e^{-\delta t}$ then $|R_\nu^1(t,u_0,\alpha)| \leqslant K_0 K_{10}|u_0|e^{\delta t}$ for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$. On the other hand, since we have already proved the validity of (i) for the particular case $\nu_0 = 0$, it follows that for each j there exist $K_{\ell_j}, C_{\ell_j} > 0$ and $\rho_{\ell_j} \geqslant 1$ such that $|\partial_\alpha^{\ell_j} u(t,u_0;\alpha)| \leqslant K_{\ell_j}|u_0|e^{t\rho_{\ell_j}}$ for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$ provided that $|u_0| \leqslant C_{\ell_j}^{-1}e^{-\delta t}$. Thus, taking upper bounds in (13) as we did before with $R_{\bar{\nu}}^2$, it follows that there exists $K_\nu > 0$ (which we take satisfying $K_\nu \geqslant 2K_0K_{10}$ for convenience) such that if $|u_0| < C_{\bar{\nu}}^{-1}e^{-\delta t}$ then $|\hat{R}_\nu^2(t,u_0,\alpha)| \leqslant \frac{1}{2}K_\nu|u_0|e^{\rho_{\bar{\nu}}\delta t}$ for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$. (Here we remark that $\rho_{\bar{\nu}} \geqslant 1$ and $C_{\bar{\nu}} \geqslant C_0$ are the ones previously defined when we tackle the case $\nu_0 = 0$.) Hence, if $|u_0| < C_{\bar{\nu}}^{-1}e^{-\delta t}$ then

$$
|\partial_{t,\alpha}^{\bar{\nu}} u(t,u_0;\alpha)| \leqslant |\hat{R}_\nu^1(t,u_0,\alpha)| + |\hat{R}_\nu^2(t,u_0,\alpha)| \leqslant K_0K_{10}|u_0|e^{\delta t} + \tfrac{1}{2}K_\nu|u_0|e^{\rho_{\bar{\nu}}\delta t} \leqslant K_\nu|u_0|e^{\rho_\nu\delta t}
$$

for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$. Finally the fact that the Taylor series of $\partial_{t,\alpha}^\nu u(t,u_0;\alpha)$ is absolutely convergent for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$ provided that $|u_0| < C_{\bar{\nu}}^{-1}e^{-\delta t}$ follows from (13) using that this is true for $\partial_\alpha^{\ell_j} u(t,u_0;\alpha)$ for all j and that $\partial^\lambda P_{\nu_0}(u;\alpha)$ is polynomial in u.

So far we have proved assertions (i) and (ii) except for the validity of the upper bound $\rho_{\bar{\nu}} \leqslant \max(|\bar{\nu}|, 1)$. Lemma 2.1 shows that this is true for $|\bar{\nu}| = 0$ because we can take $\rho_0 = 1$. The proof for $|\bar{\nu}| \geqslant 1$ follows by induction taking into account that

$$
\rho_{\bar{\nu}} := \max\left(1, \max\left(\textstyle\sum_{j=1}^q k_{j0}\rho_{\ell_j};\, (k_1,\ldots,k_q;\ell_1,\ldots,\ell_q) \in p_\star(\bar{\nu})\right)\right).
$$

The base case is also true by definition because $\rho_{\bar{\nu}} = 1$ for $|\bar{\nu}| = 1$ (recall that in this case the set $p_\star(\bar{\nu})$ is empty). The inductive step follows by noting that, due to the definition of $p(\bar{\nu},\lambda)$,

$$
\sum_{j=1}^q k_{j0}\rho_{\ell_j} \leqslant \sum_{j=1}^q k_{j0}|\ell_j| \leqslant \sum_{j=1}^q (k_{j0} + \cdots + k_{jN})|\ell_j| = \sum_{j=1}^q |k_j|(\ell_{j1} + \cdots + \ell_{jN}) = |\bar{\nu}|,
$$

where in the first inequality we use the inductive step and in the last equality take $\sum_{j=1}^q |k_j|\ell_j = \bar{\nu}$ into account.

Let us prove next the statement concerning the coefficients $h_i(t,\alpha)$ in the series (6). To this aim observe that, by assertion (ii), this series converges absolutely for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$ provided that $|u_0| < C_{\bar{\nu}}^{-1}e^{-\delta t}$. This implies that, for each fixed $t \geqslant 0$ and $\alpha \in \mathcal{C}$,

$$
h_i(t,\alpha) = \frac{1}{i!}\, \partial_{u_0}^i \partial_{t,\alpha}^\nu u(t,u_0;\alpha)\big|_{u_0=0} \quad \text{for all } i \in \mathbb{N}. \tag{14}
$$

On the other hand, thanks to assertion (i), if $|u_0| < C_{\bar{\nu}}^{-1}e^{-\delta t}$ then

$$
|\partial_{t,\alpha}^\nu u(t,u_0;\alpha)| \leqslant K_\nu|u_0|e^{\rho_{\bar{\nu}}\delta t} < K_\nu C_{\bar{\nu}}^{-1}e^{(\rho_{\bar{\nu}}-1)\delta t}.
$$

Therefore, by applying the Cauchy's estimates,

$$|h_i(t;\alpha)| \leqslant K_\nu C_{\bar\nu}^{-1} e^{(\rho_{\bar\nu}-1)\delta t} \left(C_{\bar\nu} e^{\delta t}\right)^i = K_{\bar\nu} e^{\rho_{\bar\nu}\delta t} \left(C_{\bar\nu} e^{\delta t}\right)^{i-1}$$

for all $i \geqslant 1$, $\alpha \in \mathcal{C}$ and $t \geqslant 0$. Recall in addition that, by (b) in Lemma 2.1, if $|u_0| < C_0^{-1}e^{-\delta t}$ then $u(t,u_0;\alpha) = \sum_{i=1}^{+\infty} g_i(t,\alpha)u_0^i$ converges absolutely for all $t \geqslant 0$ and $\alpha \in \mathcal{C}$. In particular, for each fixed $t \geqslant 0$ and $\alpha \in \mathcal{C}$, we can assert that $g_i(t,\alpha) = \frac{1}{i!}\,\partial_{u_0}^i u(t,u_0;\alpha)\big|_{u_0=0}$ holds for all $i \in \mathbb{N}$. Consequently

$$h_i(t,\alpha) = \frac{1}{i!}\,\partial_{u_0}^i \partial_{t,\alpha}^\nu u(t,u_0;\alpha)\big|_{u_0=0} = \frac{1}{i!}\,\partial_{t,\alpha}^\nu \partial_{u_0}^i u(t,u_0;\alpha)\big|_{u_0=0} = \partial_{t,\alpha}^\nu g_i(t,\alpha),$$

where in the first equality we use (14) and in the second one Corollary 2.2.

It only remains to be proved the upper bound for $|\partial_{u_0}^2 u(t,u_0,\alpha)|$. To this end we observe that

$$\partial_{u_0} u(t,u_0;\alpha) = \exp\left(\int_0^t \partial_u P\big(u(s,u_0;\alpha);\alpha\big)ds\right) = B(t,u_0,\alpha),$$

where we use $\partial_t\partial_{u_0}u = \partial_u P(u;\alpha)\partial_{u_0}u$ in the first equality and (8) in the second one. Therefore

$$\partial_{u_0}^2 u(t,u_0;\alpha) = B(t,u_0,\alpha)\int_0^t \partial_u^2 P\big(u(s,u_0;\alpha);\alpha\big)B(s,u_0,\alpha)ds.$$

Setting $K_{11} := \sup\left\{|\partial_u^2 P(u;\alpha)|; |u| \leqslant K_0 C_0^{-1}, \alpha \in \mathcal{C}\right\}$, (b) in Lemma 2.1 and the inequalities in (10) show that if $|u_0| < (2C_0)^{-1}e^{-\delta t}$ then

$$|\partial_{u_0}^2 u(t,u_0;\alpha)| \leqslant K_{11} e^{2K_2} e^{\alpha_1 t}\int_0^t e^{\alpha_1 s}ds = M e^{\alpha_1 t}\Omega(t,\alpha_1),$$

where we take $M = K_{11}e^{2K_2}$. This completes the proof of the result. ■

Remark 2.4. Let us mention that Theorem 2.3 corrects a mistake in the proof of [4, Propositon 23]. The authors of that paper split the proof into two intermediate claims. The second one is a particular case of assertion (i) in Theorem 2.3 (it corresponds to $|\bar\nu| = 1$ and $N = 1$) but the proof given there is not right. Indeed, they consider in page 283 the series $p(u(\xi,u_0)) = \sum_{i=1}^{+\infty} p_i(\xi)u_0^i$ but the summation index should run from $i = 0$. This may seem a typo but it has important consequences in order to bound the derivative with respect to parameters because, transferred to our proof, it yields to the factor $e^{\alpha_1 t}$ in (10). That being said, except for this bridgeable mistake in the proof of [4, Propositon 23], the main result in that paper with regard to the period of the limit cycle emerging from a saddle loop bifurcation is perfectly correct. □

At this point let us denote by $t \mapsto (x(t,p_0;\alpha), y(t,p_0;\alpha))$ the solution of the differential system given by the vector field X_α in (3) passing through $p_0 = (x_0,y_0) \in \mathbb{R}^2$. It is clear that $x(t,p_0;\alpha) = x_0e^t$. We are interested in the analytical properties of $y(t,p_0;\alpha)$ with the initial condition $p_0 = (s,1)$. This is the reason why we first studied $u = x^py^q$ and in this respect, by Lemma 2.1, we know that

$$u(t,u_0;\alpha) = \sum_{i=1}^{+\infty} g_i(t;\alpha)u_0^i = u_0 e^{\alpha_1 t}\sum_{i=0}^{+\infty} \bar g_i(t;\alpha)u_0^i,$$

where the series converge absolutely and we use that $g_i(t;\alpha) = e^{\alpha_1 t}\bar g_{i-1}(t;\alpha)$. Thus, since $x(t,p_0;\alpha) = x_0e^t$,

$$\big(y(t,p_0;\alpha)\big)^q = e^{-pt}y_0^q e^{\alpha_1 t}\sum_{i=0}^{+\infty} \bar g_i(t;\alpha)u_0^i = y_0^q e^{(\alpha_1-p)t}\left(1 + \sum_{i=1}^{+\infty} \bar g_i(t;\alpha)u_0^i\right). \tag{15}$$

Since $(1+z)^\eta=\sum_{k=0}^{+\infty}\binom{\eta}{k}z^k$ for $|z|<1$, with the aim of computing $\big(y(t,x_0,y_0;\alpha)\big)^j$ for any $j\in\mathbb{Z}$ we set

$$\psi_0^j:=1 \text{ and, for } k\in\mathbb{N},\ \psi_k^j:=\sum_{r=1}^{k}\binom{j/q}{r}\sum_{i_1+\cdots+i_r=k}\bar g_{i_1}\cdots\bar g_{i_r}. \tag{16}$$

Our next task is to prove the following result.

Lemma 2.5. *For each compact set $\mathcal{C}\subset U_\delta$ with $\delta\in(0,\frac12]$ there exist $C_0,M>0$ such that the identity*

$$\big(y(t,s,1;\alpha)\big)^j=e^{-\lambda jt}\sum_{k=0}^{+\infty}\psi_k^j(t;\alpha)s^{kp}$$

holds for all $j\in\mathbb{Z}$, $t\geqslant 0$, $\alpha\in\mathcal{C}$ and $s>0$ with $s^p\max\big(M\Omega(t,\alpha_1),4C_0e^{\delta t}\big)<2$, Moreover under these conditions the series is absolutely convergent.

Proof. Since $g_i(t;\alpha)=e^{\alpha_1 t}\bar g_{i-1}(t;\alpha)$ and $\bar g_0=1$, from (5) we get

$$\sum_{i=1}^{+\infty}\bar g_i(t;\alpha)u_0^i=\frac{u(t,u_0;\alpha)-u_0e^{\alpha_1 t}}{u_0e^{\alpha_1 t}}=\frac{\partial_{u_0}^2u(t,\xi;\alpha)u_0^2}{2u_0e^{\alpha_1 t}}\text{ for some }\xi\in[0,u_0],$$

where in the second equality we apply Taylor's theorem taking $u(t,0;\alpha)=0$ and $\partial_{u_0}u(t,0;\alpha)=e^{\alpha_1 t}$ into account. By applying Theorem 2.3, there exist $C_0,M>0$ such that if $|u_0|<(2C_0)^{-1}e^{-\delta t}$ then

$$|\partial_{u_0}^2u(t,u_0;\alpha)|\leqslant Me^{\alpha_1 t}\Omega(t,\alpha_1)\text{ for all }t\geqslant 0\text{ and }\alpha\in\mathcal{C}.$$

Hence if $|u_0|<(2C_0)^{-1}e^{-\delta t}$ then $\left|\sum_{i=1}^{+\infty}\bar g_i(t;\alpha)u_0^i\right|\leqslant\frac{M}{2}\Omega(t,\alpha_1)|u_0|$ for all $t\geqslant 0$ and $\alpha\in\mathcal{C}$. Therefore, if $|u_0|<\min\left(\frac{2}{M\Omega(t,\alpha_1)},(2C_0)^{-1}e^{-\delta t}\right)$ then $\left|\sum_{i=1}^{+\infty}\bar g_i(t;\alpha)u_0^i\right|<1$ for all $t\geqslant 0$ and $\alpha\in\mathcal{C}$. Accordingly, since $(1+z)^{j/q}=\sum_{k=0}^{+\infty}\binom{j/q}{k}z^k$ for $|z|<1$, from (15) and (16) it follows that

$$\big(y(t,x_0,y_0;\alpha)\big)^j=y_0^je^{j(\alpha_1-p)t/q}\left(1+\sum_{i=1}^{+\infty}\bar g_i(t;\alpha)u_0^i\right)^{j/q}=y_0^je^{j(\alpha_1-p)t/q}\sum_{k=0}^{+\infty}\psi_k^j(t;\alpha)u_0^k$$

for all $t\geqslant 0$ and $\alpha\in\mathcal{C}$ provided that $|u_0|<1/\max\big(\frac{M}{2}\Omega(t,\alpha_1),2C_0e^{\delta t}\big)$. Furthermore the second series converges absolutely because so it does the first one thanks to Lemma 2.1. Finally, since $u_0=x_0^py_0^q$ and $\lambda=\frac{p-\alpha_1}{q}$, the result follows taking $(x_0,y_0)=(s,1)$. ■

The following is a technical lemma that will be used in the proof of our last result in this section.

Lemma 2.6. *For each $m,n\in\mathbb{Z}_{\geq 0}$ there exist $P_{mn},Q_{mn}\in\mathbb{Z}[x,y]$ with $\deg_x P_{mn}=\deg_x Q_{mn}=m$ and $\deg_y P_{mn}=\deg_y Q_{mn}=n$ such that for any $a\in\mathbb{R}$,*

$$\partial_x^n\partial_y^m e^{axy}=a^me^{axy}P_{mn}(x,ay)\ \textit{and}\ \partial_x^n\partial_y^m x^y=x^{y-n}Q_{mn}(\ln x,y).$$

In particular, there exist $M_1,M_2>0$ such that

$$\left|\partial_x^n\partial_y^m e^{axy}\right|\leqslant M_1\max(1,|x|,|ay|)^{m+n}|a|^me^{axy}\ \textit{and}\ \left|\partial_x^n\partial_y^m x^y\right|\leqslant M_2\max(1,|\ln x|,|y|)^{m+n}x^{y-n}.$$

Proof. Note that $\partial_x^n\partial_y^m e^{axy}=\partial_x^n\big(e^{axy}(ax)^m\big)$ and $\partial_x^n\partial_y^m x^y=\partial_x^n\big(x^y(\ln x)^m\big)$. From here the proof follows by induction on n. To this end we set $P_{m0}(x,y)=Q_{m0}(x,y)=x^m$. Then the inductive step follows by taking $P_{m,n+1}=yP_{mn}+\partial_xP_{mn}$ and $Q_{m,n+1}=(y-n)Q_{mn}+\partial_xQ_{mn}$. ■

Lemma 2.7. *For each compact set $\mathcal{C}\subset U_\delta$ with $\delta\in(0,\frac{1}{2}]$, $j\in\mathbb{Z}$ and $\nu\in\mathbb{Z}_{\geqslant 0}^{N+1}$ there exist $C_{j\nu},K_\nu>0$ such that*

$$|\partial_{t,\alpha}^{\nu}\psi_k^j(t;\alpha)|\leqslant K_\nu(k+1)^{3|\nu|}(C_{j\nu})^k\max(1,t)^{|\nu|}e^{(3k+|\nu|)\delta t}$$

for all $k\in\mathbb{Z}_{\geqslant 0}$, $t\geqslant 0$ and $\alpha\in\mathcal{C}$.

Proof. Note first that, on account of the definition in (16),

$$\partial^\nu\psi_k^j=\sum_{r=1}^{k}\binom{j/q}{r}\sum_{i_1+\cdots+i_r=k}\partial^\nu(\bar g_{i_1}\cdots\bar g_{i_r}),$$

where, due to $g_i(t;\alpha)=e^{\alpha_1 t}\bar g_{i-1}(t;\alpha)$ and applying Theorem B.2,

$$\partial^\nu(\bar g_{i_1}\cdots\bar g_{i_r})=\sum_{\ell_0+\ldots+\ell_r=\nu}a_{\ell_0,\ldots,\ell_r}\partial^{\ell_0}(e^{-\alpha_1 rt})\partial^{\ell_1}g_{i_1+1}\cdots\partial^{\ell_r}g_{i_r+1}.$$

We remark that this summation is multidimensional with $\ell_0,\ldots,\ell_r\in\mathbb{Z}_{\geqslant 0}^{N+1}$ and $a_{\ell_0,\ldots,\ell_r}=\binom{\nu}{\ell_0,\ldots,\ell_r}$ are the generalized multinomial coefficients (cf. Remark B.3). Setting $\ell_0=(\ell_{00},\ldots,\ell_{0N})$ then, by Lemma 2.6, $\partial^{\ell_0}(e^{-\alpha_1 rt})=\partial_t^{\ell_{00}}\partial_{\alpha_1}^{\ell_{01}}(e^{-\alpha_1 rt})=(-r)^{\ell_{01}}e^{-\alpha_1 rt}P_{\ell_{01}\ell_{00}}(t,-\alpha_1 r)$ if $\ell_{02}=\ldots=\ell_{0N}=0$ and zero otherwise. In addition $|\partial^{\ell_0}(e^{-\alpha_1 rt})|\leqslant M_{\ell_0}\max(1,|t|,|r\alpha_1|)^{|\ell_0|}r^{\ell_{01}}e^{-\alpha_1 rt}$. On the other hand, by Theorem 2.3,

$$|\partial_{t,\alpha}^{\ell}g_{i+1}(t;\alpha)|\leqslant K_\ell e^{\rho_\ell\delta t}(C_\ell e^{\delta t})^i\text{ for all }i\in\mathbb{N},\ \alpha\in\mathcal{C}\text{ and }t\geqslant 0.$$

Thus, if we set $\hat M_\nu:=\max(M_\ell;\ell\leqslant\nu)$, $\hat C_\nu:=\max(C_\ell;\ell\leqslant\nu)$ and $\hat K_\nu:=\max(K_\ell;\ell\leqslant\nu)$, then

$$|\partial^\nu\left(\bar g_{i_1}\cdots\bar g_{i_r}\right)(t;\alpha)|\leqslant\sum_{\ell_0+\ldots+\ell_r=\nu}a_{\ell_0,\ldots,\ell_r}\hat M_\nu(\hat K_\nu)^r\max(1,|t|)^{|\nu|}r^{2|\nu|}e^{(r+\rho_{\ell_1}+\ldots+\rho_{\ell_r})\delta t}(\hat C_\nu e^{\delta t})^{i_1+\ldots+i_r}.$$

Here we use $|\alpha_1|\leqslant\delta<1$, $\ell_0\leqslant\nu$ and $r\geqslant 1$, which implies

$$\max(1,|t|,|r\alpha_1|)^{|\ell_0|}r^{\ell_{01}}\leqslant\max(1,|t|,|r|)^{|\nu|}r^{\nu_1}\leqslant\max(1,|t|)^{|\nu|}r^{2|\nu|}.$$

Hence, since $\rho_\nu\leqslant\max(|\nu|,1)\leqslant 1+|\nu|$ thanks to Theorem 2.3 and, on the other hand, $|\ell_0|+\ldots+|\ell_r|=|\nu|$ and $r\leqslant k=i_1+\cdots+i_r$, we obtain

$$|\partial^\nu\left(\bar g_{i_1}\cdots\bar g_{i_r}\right)(t;\alpha)|\leqslant\hat M_\nu(\hat K_\nu\hat C_\nu)^k k^{2|\nu|}\max(1,t)^{|\nu|}e^{(3k+|\nu|)\delta t}\sum_{\ell_0+\ldots+\ell_r=\nu}a_{\ell_0,\ldots,\ell_r}.$$

Thus, since $\sum_{\ell_0+\ldots+\ell_r=\nu}a_{\ell_0,\ldots,\ell_r}=(r+1)^{|\nu|}\leqslant(k+1)^{|\nu|}$ thanks to Remark B.3, we get

$$|\partial^\nu\left(\bar g_{i_1}\cdots\bar g_{i_r}\right)(t;\alpha)|\leqslant\hat M_\nu(\hat K_\nu\hat C_\nu)^k(k+1)^{3|\nu|}\max(1,t)^{|\nu|}e^{(3k+|\nu|)\delta t}.$$

Accordingly, since $|\binom{j/q}{r}|\leqslant\max(|j/q|,1)^r\leqslant\max(|j|,1)^k$ for all $j\in\mathbb{Z}$,

$$\begin{aligned}\left|\partial^\nu\psi_k^j(t;\alpha)\right|&\leqslant\hat M_\nu(\hat K_\nu\hat C_\nu)^k(k+1)^{3|\nu|}\max(1,t)^{|\nu|}e^{(3k+|\nu|)\delta t}\sum_{r=1}^{k}\max(|j|,1)^k\sum_{i_1+\cdots+i_r=k}1\\&=\hat M_\nu p(k)\left(\max(|j|,1)\hat K_\nu\hat C_\nu\right)^k(k+1)^{3|\nu|}\max(1,t)^{|\nu|}e^{(3k+|\nu|)\delta t},\end{aligned}$$

where $p(k)$ is the number of partitions of k and it is easy to see that $p(k)\leqslant\binom{2k-1}{k}\leqslant 2^{2k-1}\leqslant 4^k$. Hence, setting $C_{j\nu}=4\max(|j|,1)\hat K_\nu\hat C_\nu$ and $K_\nu=\hat M_\nu$, the result follows. ■

3 Dulac map

This section is entirely devoted to prove Theorem A, that will follow almost immediately from Theorem 3.3. In the proof of this result, and the forthcoming Proposition 4.2, we will use the following lemma together with this easy observation:

Remark 3.1. The function $\phi(s) = s^\alpha(-\ln s)^m$ is monotonous increasing on the interval $(0, \frac{1}{e})$ provided that $\alpha > m \geqslant 0$ because $\partial_s\phi(s) = -s^{\alpha-1}(-\ln(s))^{m-1}(m + \alpha \ln s)$. □

Lemma 3.2. *For every $\rho \in (0,1)$ and $n \in \mathbb{Z}_+$ there exists $A > 0$ such that $\sum_{k\geqslant K} k^n r^k \leqslant AK^n r^K$ for all $K \in \mathbb{N}$ and $0 \leqslant r \leqslant \rho$.*

Proof. Setting $c_\ell := \binom{n}{\ell}\sum_{i=0}^{+\infty} i^{n-\ell}\rho^i$ and $A := \sum_{\ell=0}^{n} c_\ell$ we obtain

$$\sum_{k=K}^{+\infty} k^n r^k = \sum_{i=0}^{+\infty}(i+K)^n r^{i+K} = r^K \sum_{\ell=0}^{n}\binom{n}{\ell}K^\ell \sum_{i=0}^{+\infty} i^{n-\ell}r^i \leqslant r^K \sum_{\ell=0}^{n} c_\ell K^\ell \leqslant AK^n r^K,$$

where in the last inequality we take $K \geqslant 1$ into account. ■

Theorem 3.3. *Consider the family of vector fields $\{X_\alpha\}_{\alpha\in U_\delta}$ defined in (3) and let $D(\,\cdot\,;\alpha)$ be the Dulac map of X_α between the transversal sections $\{y = 1\}$ and $\{x = 1\}$. Then the following holds:*

(*a*) *For each compact set $\mathcal{C} \subset U_\delta$ with $\delta \in (0, \frac{1}{2}]$ there exists $s_0 > 0$ such that*

$$D(s;\alpha) = \sum_{k=0}^{+\infty} \psi_k^1(-\ln s;\alpha)s^{kp+\lambda}, \text{ for all } s \in (0, s_0) \text{ and } \alpha \in \mathcal{C},$$

and the series is absolutely convergent. Moreover, for each $K \in \mathbb{N}$ there exists $\Delta(z, w;\alpha) \in \mathbb{Q}[z, w, \alpha]$ with $\deg_{z,w}(\Delta) < K$ and $\Delta(0, 0;\alpha) = 1$ such that

$$\sum_{k=0}^{K-1} \psi_k^1(-\ln s;\alpha)s^{kp+\lambda} = s^\lambda\Delta(s^p, s^p\omega;\alpha), \text{ where } \omega = \omega(s;\alpha_1).$$

(*b*) *Finally, for each $L \in \mathbb{R}$ there exists $K_L \in \mathbb{Z}_{\geq 0}$ such that*

$$\sum_{k=K_L}^{+\infty} \psi_k^1(-\ln s;\alpha)s^{kp+\lambda} \in \mathcal{F}_L^\infty(U_0). \tag{17}$$

Proof. The solution $x(t, x_0, y_0;\alpha) = x_0e^t$ of X_α with initial condition $(x_0, y_0) = (s, 1)$ intersects the transversal section $\{x = 1\}$ at $t = -\ln s$. Hence the Dulac map is given by $D(s;\alpha) = y(t, s, 1;\alpha)|_{t=-\ln s}$. On account of this, the first assertion in (*a*) will follow by applying Lemma 2.5 once we show that we can take $s_0 > 0$ small enough such that

$$s^p \max\left(M\Omega(t,\alpha_1), 4C_0e^{\delta t}\right)\big|_{t=-\ln s} < 2 \text{ for all } s \in (0, s_0).$$

In this respect observe that, by applying (*b*) in Lemma A.4, $s^p\Omega(-\ln s, \alpha_1) = s^p\omega(s;\alpha_1)$ tends to 0 as $s \to 0^+$ uniformly in $\alpha_1 \in (-\delta, \delta)$, and this is also true for $s^{p-\delta}$ because $p - \delta \geqslant p - \frac{1}{2} > 0$. Consequently it is clear that there exists $s_0 > 0$ small enough such that the above inequality holds and so the first assertion is true. With regard to second one, from (*a*) in Lemma 2.1 and (16) it follows that $\psi_k^1(-\ln s;\alpha) = \eta_k(\omega;\alpha)$ where $\eta_k \in \mathbb{Q}[\omega, \alpha]$ with $\deg_\omega(\eta_k) \leqslant k$. Then it is clear that, for each $k = 0, 1, \ldots, K-1$, there exists a homogeneous polynomial $\hat{\eta}_k \in \mathbb{Q}[z,w,\alpha]$ with $\deg_w(\hat{\eta}_k) \leqslant k$ such that we can write $\psi_k^1(-\ln s;\alpha)s^{kp} = \hat{\eta}_k(s^p, s^p\omega;\alpha)$. Since $\hat{\eta}_0 \equiv 1$, this shows the validity of the second assertion in (a).

In order to prove (b) we claim that for each $\nu \in \mathbb{Z}_{\geqslant 0}^{N+1}$ there exists $s_0 > 0$ small enough such that the series $\sum_{k\geqslant 0} \partial_{s,\alpha}^{\nu}(\psi_k^1(-\ln s;\alpha)s^{pk+\lambda})$ converges uniformly on $(0,s_0)\times\mathcal{C}$, where $\mathcal{C}$ is any compact set in U_δ that we hereafter. By the Weierstrass M-test, to this end it suffices to show that there exists a sequence of positive numbers $\{M_k\}_{k\in\mathbb{N}}$ with $\sum_{k\geqslant 1} M_k < \infty$ such that, for some $k_\nu \in \mathbb{N}$ large enough,

$$|\partial_{s,\alpha}^{\nu}(\psi_k^1(-\ln s;\alpha)s^{kp+\lambda})| \leqslant M_k, \text{ for all } k \geqslant k_\nu,\ s\in(0,s_0) \text{ and } \alpha\in\mathcal{C}.$$

By applying Theorem B.2 we have that

$$\partial^{\nu}(\psi_k^1(-\ln s;\alpha)s^{kp+\lambda}) = \sum_{\ell_1+\ell_2=\nu} a_{\ell_1\ell_2}\partial^{\ell_1}(\psi_k^1(-\ln s;\alpha))\partial^{\ell_2}(s^{pk+\lambda}) \tag{18}$$

with $a_{\ell_1\ell_2} = \binom{\nu}{\ell_1,\ell_2}$. Setting $\hat{\ell}_1 = (0,\ell_{11},\ldots,\ell_{1N})$ it turns out that, for each fixed s and α,

$$|\partial_{s,\alpha}^{\ell_1}(\psi_k^1(-\ln s;\alpha))| = |\partial_s^{\ell_{10}}(\partial_\alpha^{\hat{\ell}_1}\psi_k^1)(-\ln s;\alpha)| \leqslant C_{\ell_{10}}s^{-\ell_{10}} \max_{j\in\{0,\ldots,\ell_{10}\}} |(\partial^{(j,\ell_{11},\ldots,\ell_{1N})}\psi_k^1)(-\ln s,\alpha)|,$$

where $C_{\ell_{10}} > 0$ depends only on ℓ_{10}. The above inequality is clear in case that $\ell_{10} = 0$, whereas for $\ell_{10} \geqslant 1$ it follows easily by applying the one-dimensional Faa di Bruno formula

$$\partial_s^n(f(g(s))) = \sum_{j=1}^n (\partial_s^j f)(g(s)) \sum_{p(n,j)} n! \prod_{i=1}^n \frac{(\partial^i g(s))^{k_i}}{(k_i!)(i!)^{k_i}}$$

taking $n = \ell_{10}$, $f = \partial_\alpha^{\hat{\ell}_1}\psi_k^1$ and $g = -\ln s$ and noting that, in doing so, $\partial^i g(s) = (-1)^i(i-1)!s^{-i}$ and $\sum_{i=1}^n ik_i = n$. Thus by applying Lemma 2.7 we deduce that, for all $s\in(0,1/e)$ and α inside a compact subset $\mathcal{C}$ of U_δ with $\delta\in(0,\frac{1}{2}]$,

$$|\partial_{s,\alpha}^{\ell_1}(\psi_k^1(-\ln s;\alpha))| \leqslant \hat{K}_{\ell_1}(k+1)^{3|\ell_1|}(\hat{C}_{\ell_1})^k(-\ln s)^{|\ell_1|}s^{-(3k+|\ell_1|)\delta-\ell_{10}}, \tag{19}$$

where, following the notation in that result,

$$\hat{K}_{\ell_1} = C_{\ell_{10}}\max(K_{(j,\ell_{11},\ldots,\ell_{1N})}; j=0,\ldots,\ell_{10}) \text{ and } \hat{C}_{\ell_1} = \max(C_{1,(j,\ell_{11},\ldots,\ell_{1N})}; j=0,\ldots,\ell_{10})$$

and we use that $\max(1,-\ln s) = -\ln s$ for $s\in(0,1/e)$. In addition, since $\lambda = \frac{p-\alpha_1}{q}$, Lemma 2.6 shows that

$$\begin{aligned}|\partial^{\ell_2}(s^{pk+\lambda})| = |\partial_s^{\ell_{20}}\partial_{\alpha_1}^{\ell_{21}}(s^{pk+\lambda})| &\leqslant M_2\max(-\ln s, pk+\lambda)^{|\ell_2|}s^{pk+\lambda-\ell_{20}}q^{-\ell_{21}}\\ &\leqslant C_{\ell_2}(k+1)^{|\ell_2|}(-\ln s)^{|\ell_2|}s^{pk+\lambda-\ell_{20}},\end{aligned}$$

because $pk+\lambda \leqslant p(k+1)+1 \leqslant 2p(k+1)$ due to $p,q\geqslant 1$, $|\alpha_1|\leqslant\delta<1$ and we set $C_{\ell_2} = (2p)^{|\ell_2|}q^{-\ell_{21}}M_2$. Here we also use that $\max(x,y)\leqslant xy$ when $x,y\geqslant 1$. Using this inequality and the one in (19), from (18) we obtain

$$|\partial^{\nu}(\psi_k^1(-\ln s;\alpha)s^{kp+\lambda})| \leqslant \bar{K}_\nu(\bar{C}_\nu)^k(k+1)^{3|\nu|}(-\ln s)^{|\nu|}s^{(p-3\delta)k+\lambda-|\nu|\delta-\nu_0}, \tag{20}$$

where we set $\bar{C}_\nu := \max(\hat{C}_{\ell_1};\ell_1\leqslant\nu)$ and, on account of $\sum_{\ell_1+\ell_2=\nu} a_{\ell_1\ell_2} = 2^{|\nu|}$,

$$\bar{K}_\nu := 2^{|\nu|}\max(\hat{K}_{\ell_1}C_{\ell_2};\ell_1+\ell_2=\nu).$$

Let us remark that the above estimate holds for all $s\in(0,1/e)$ and $\alpha\in\mathcal{C}\subset U_\delta$ with $\delta\in(0,\frac{1}{2}]$. As a matter of fact, at this point we shrink it so that $\delta\in(0,\frac{1}{4})$, which in particular implies $p-3\delta\geqslant\frac{1}{4}$. Consequently, using also the fact that $\lambda>0$, from (20) we get

$$|\partial^{\nu}(\psi_k^1(-\ln s;\alpha)s^{kp+\lambda})| \leqslant \bar{K}_\nu(\bar{C}_\nu)^k(k+1)^{3|\nu|}(-\ln s)^{|\nu|}s^{(k-|\nu|)/4-\nu_0} =: m_k(s). \tag{21}$$

On account of Remark 3.1 it easily follows that a sufficient condition for $s \mapsto m_k(s)$ to be monotonous increasing on $(0,1/e)$ is that

$$k > 9|\nu| =: k_\nu.$$

Note on the other hand that

$$m_k(s) = \bar{K}_\nu(\bar{C}_\nu s^{1/4})^k (k+1)^{3|\nu|}(-\ln s)^{|\nu|} s^{-|\nu|/4-\nu_0}.$$

Thus if we take $s_0 := \min\left(\frac{1}{e}, (2\bar{C}_\nu)^{-4}\right)$ then the series with general term $M_k := m_k(s_0)$ is summable and, additionally, from (21) and the monotonicity of $m_k(s)$ on $(0,1/e)$,

$$|\partial^\nu(\psi_k^1(-\ln s;\alpha)s^{kp+\lambda})| \leqslant m_k(s) < M_k \text{ for all } s \in (0,s_0) \text{ and } k \geqslant k_\nu.$$

This proves the validity of the claim and consequently, by applying Lemma B.4 recursively, if $s \in (0,s_0)$ and $\alpha \in U_\delta$ then

$$\partial_{s,\alpha}^\nu\left(\sum_{k=K_L}^{+\infty} \psi_k^1(-\ln s;\alpha)s^{pk+\lambda}\right) = \sum_{k=K_L}^{+\infty} \partial_{s,\alpha}^\nu\left(\psi_k^1(-\ln s;\alpha)s^{pk+\lambda}\right) \tag{22}$$

for all $\nu \in \mathbb{Z}_{\geq 0}^{N+1}$ and $K_L \in \mathbb{N}$. (We stress that the above identity is valid regardless of $K_L \geqslant k_\nu$ and this is crucial in what follows because k_ν depends on ν.)

We are now in position to finish the proof. We will show that (17) holds taking $K_L := \max(0, \lceil 4L \rceil + 4)$. To this end, recall Definition 1.1, we fix any $\nu \in \mathbb{Z}_{\geq 0}^{N+1}$ and $\alpha^\star = (0,\alpha_2,\ldots,\alpha_N) \in U_0 = \{0\} \times \mathbb{R}^{N-1}$, and we take a relatively compact neighbourhood V of $\alpha^\star$ contained in U_δ with $\delta = \min(\frac{1}{4}, \frac{1}{|\nu|})$. Then, from (22) and using the upper bound in (20), for each $s \in (0,s_0)$ and $\alpha \in V$ we have

$$\begin{aligned}
\left|\partial_{s,\alpha}^\nu\left(\sum_{k=K_L}^{+\infty} \psi_k^1(-\ln s;\alpha)s^{pk+\lambda}\right)\right| &\leqslant \sum_{k=K_L}^{+\infty} \left|\partial_{s,\alpha}^\nu(\psi_k^1(-\ln s;\alpha)s^{pk+\lambda})\right| \\
&\leqslant \bar{K}_\nu(-\ln s)^{|\nu|} s^{\lambda-|\nu|\delta-\nu_0} \sum_{k=K_L}^{+\infty} (k+1)^{3|\nu|}(\bar{C}_\nu s^{p-3\delta})^k \\
&\leqslant \bar{K}_\nu M_\nu s^{-|\nu|\delta-\nu_0} A(K_L+1)^{3|\nu|}(\bar{C}_\nu)^{K_L} s^{(p-3\delta)K_L} \\
&\leqslant \bar{K}_\nu M_\nu A(K_L+1)^{3|\nu|}(\bar{C}_\nu)^{K_L} s^{\frac{1}{4}K_L-1-\nu_0} \leqslant C s^{L-\nu_0}.
\end{aligned}$$

In the third inequality above we apply Lemma 3.2 and set $M_\nu := \sup\{s^\lambda(-\ln s)^{|\nu|};\ s \in (0,s_0), |\alpha_1| \leqslant \delta\}$. Next, in the fourth inequality, we take $\delta = \min(\frac{1}{4}, \frac{1}{|\nu|})$ into account. Finally in the last inequality we set $C := \bar{K}_\nu M_\nu A(K_L+1)^{3|\nu|}(\bar{C}_\nu)^{K_L}$ and use that $K_L \geqslant 4(L+1)$. This completes the proof of the result. ■

Proof of Theorem A. By Theorem 3.3, for each compact set $\mathcal{C} \subset U_\delta$ with $\delta \in (0,\frac{1}{2}]$ there exists $s_0 > 0$ such that

$$D(s;\alpha) = \sum_{k=0}^{+\infty} \psi_k^1(-\ln s;\alpha)s^{kp+\lambda} \text{ for all } s \in (0,s_0) \text{ and } \alpha \in \mathcal{C}.$$

In addition, for each $L \in \mathbb{R}$ there exists $K_L \in \mathbb{Z}_{\geq 0}$ such that

$$\mathscr{D}_L(s;\alpha) := \sum_{k=K_L}^{+\infty} \psi_k^1(-\ln s;\alpha)s^{kp+\lambda} \in \mathcal{F}_L^\infty(U_0).$$

If $K_L = 0$ then the result follows taking $\Delta \equiv 0$. If, on the contrary, $K_L \in \mathbb{N}$ then by Theorem 3.3 we know that there exists $\hat{\Delta}(z,w;\alpha) \in \mathbb{Q}[z,w,\alpha]$ with $\hat{\Delta}(0,0;\alpha) = 1$ such that

$$\sum_{k=0}^{K_L-1} \psi_k^1(-\ln s;\alpha)s^{kp+\lambda} = s^\lambda \hat{\Delta}(s^p, s^p\omega;\alpha),$$

where $\omega=\omega(s;\alpha_1)$. By gathering the homogenous part of $\hat{\Delta}$ of i-th degree, for $i=0,1,\ldots,\hat{d}:=\deg_{(z,w)}\hat{\Delta}$, it turns out that we can write $s^\lambda\hat{\Delta}(s^p,s^p\omega)=\sum_{i=0}^{\hat{d}}s^{\lambda+pi}p_i(\omega;\alpha)$ where $p_i(w;\alpha)\in\mathbb{Q}[w,\alpha]$ with $\deg_w p_i\leqslant i$. Then, due to $\lambda=\frac{p-\alpha_1}{q}$ and by (d) in Lemma A.4, note that $s^{\lambda+pi}p_i(\omega;\alpha)\in\mathcal{F}_L^\infty(U_0)$ provided that $i>\frac{L}{p}-\frac{1}{q}$. Consequently if $L\geqslant\frac{p}{q}$ then there exists a unique polynomial $\Delta(z,w;\alpha)\in\mathbb{Q}[z,w,\alpha]$ with $\Delta(0,0;\alpha)=1$ and $\deg_{(z,w)}\Delta\leqslant\lfloor\frac{L}{p}-\frac{1}{q}\rfloor=:d$, such that

$$\Delta(z,w;\alpha)=\sum_{i=0}^{d}s^{pi}p_i(\omega;\alpha)\text{ and }\sum_{i=d+1}^{\hat{d}}s^{\lambda+pi}p_i(\omega;\alpha)\in\mathcal{F}_L^\infty(U_0),$$

where, in case that $\hat{d}\leqslant d$, the second summation is void and we set $p_i\equiv 0$ for $i>\hat{d}$. Hence the result follows taking Δ and $\sum_{k=d+1}^{+\infty}\psi_k^1(-\ln s;\alpha)s^{kp+\lambda}$ instead of $\hat{\Delta}$ and $\mathscr{D}_L$ respectively. Observe on the other hand that if $L<\frac{p}{q}$ then $s^\lambda\hat{\Delta}(s^p,s^p\omega)=\sum_{i=0}^{d}s^{\lambda+pi}p_i(\omega;\alpha)\in\mathcal{F}_L^\infty(U_0)$ and so in this case the result follows taking $\Delta\equiv 0$ instead of $\hat{\Delta}$. This concludes the proof of the result since the uniqueness of the polynomial Δ in the statement follows from the fact that $s^{\lambda+pi}\omega^\ell\notin\mathcal{F}_L^\infty(U_0)$ if $i\leqslant d$. ∎

4 Dulac time

In this section we will prove Theorem B. To this aim, for the sake of convenience, we begin by introducing

$$T_{ijk}(s;\alpha):=\int_0^{-\ln s}e^{(i-\lambda j)t}\psi_k^j(t;\alpha)dt,\text{ for }i,j\in\mathbb{Z}\text{ and }k\in\mathbb{N},\tag{23}$$

and in its regard we prove the following result.

Lemma 4.1. *For each $i,j\in\mathbb{Z}$ and $\delta\in(0,\frac{1}{2}]$ there exists $k_0\in\mathbb{Z}_{\geq 0}$ such that for all $\nu\in\mathbb{Z}_{\geq 0}^{N+1}$ and compact set $\mathcal{C}\subset U_\delta$ there exist $C_\nu,K_\nu>0$ so that the upper bound*

$$|\partial^\nu T_{ijk}(s;\alpha)|\leqslant K_\nu(k+1)^{3|\nu|}(C_\nu)^k(-\ln s)^{|\nu|}s^{\lambda j-i-\nu_0-(3k+|\nu|)\delta}$$

holds for all $k\geqslant k_0$, $s\in(0,1/e)$ and $\alpha\in\mathcal{C}$.

Proof. The result follows by applying Lemma 2.7 to the given compact set $\mathcal{C}\subset U_\delta$ and $\nu\in\mathbb{Z}_{\geq 0}^{N+1}$. Denote $\nu=(\nu_0,\nu_1,\ldots,\nu_N)$ and suppose first that $\nu_0=0$. In this case if $s\in(0,1/e)$ and $\alpha\in\mathcal{C}$ then

$$\begin{aligned}|\partial^\nu(T_{ijk}(s;\alpha)|&\leqslant\sum_{\ell_1+\ell_2=\nu}\binom{\nu}{\ell_1,\ell_2}\int_0^{-\ln s}\left|\partial_\alpha^{\ell_1}(e^{(i-\lambda j)t})\partial_\alpha^{\ell_2}\psi_k^j(t;\alpha)\right|dt\\&\leqslant 2^{|\nu|}\hat{K}_\nu(k+1)^{3|\nu|}(C_\nu)^k\int_0^{-\ln s}e^{(i-\lambda j)t}(jt/q)^{\ell_{11}}\max(1,t)^{|\ell_2|}e^{(3k+|\ell_2|)\delta t}dt\\&\leqslant K_\nu(k+1)^{3|\nu|}(C_\nu)^k(-\ln s)^{|\nu|}\int_0^{-\ln s}e^{(i-\lambda j+(3k+|\nu|)\delta)t}dt\\&\leqslant K_\nu(k+1)^{3|\nu|}(C_\nu)^k(-\ln s)^{|\nu|}s^{\lambda j-i-(3k+|\nu|)\delta},\end{aligned}$$

where in the first inequality we apply Theorem B.2, in the second one Lemma 2.7 and Remark B.3, in the third one we set $K_\nu:=2^{|\nu|}(j/q)^{\nu_1}\hat{K}_\nu$ and we use that $\max(1,t)\leqslant-\ln s$ for all $t\in(0,-\ln s)$ due to $s\in(0,1/e)$, and in the last one we take

$$k\geqslant k_0:=\max\left(0,\left\lceil\tfrac{1}{3\delta}\left(1-i+\tfrac{p+\delta}{q}|j|\right)\right\rceil\right)\tag{24}$$

in order that $i-\lambda j+(3k+|\nu|)\delta)\geqslant 1$ holds for all $\alpha\in U_\delta$ and $k\geqslant k_0$. Here we use that $\lambda\in\left(\frac{p-\delta}{q},\frac{p+\delta}{q}\right)$ due to $|\alpha_1|<\delta$ and $\lambda=\frac{p-\alpha_1}{q}$. We stress, and this is crucial, that k_0 is independent from ν and $\mathcal{C}$. This proves the result for $\nu_0=0$. Let us consider next the case $\nu_0\geqslant 1$ and to this end we denote $\nu':=(\nu_0-1,\nu_1,\ldots,\nu_N)$. Thus, from (23) and Theorem B.2,

$$\partial^\nu T_{ijk}(s;\alpha)=-s^{-1}\partial^{\nu'}\left(s^{\lambda j-i}\psi_k^j(-\ln s;\alpha)\right)=-s^{-1}\sum_{\ell_1+\ell_2=\nu'}\binom{\nu'}{\ell_1,\ell_2}\partial^{\ell_1}(s^{\lambda j-i})\partial^{\ell_2}(\psi_k^j(-\ln s;\alpha)).$$

Then the application of Lemma 2.6 and Lemma 2.7 show respectively

$$\left|\partial^{\ell_1}(s^{\lambda j-i})\right|\leqslant M_{\ell_1}(-\ln s)^{|\ell_1|}\max(1,|\lambda j-i|)^{|\ell_1|}s^{\lambda j-i-\ell_{10}}(|j|/q)^{\ell_{10}}$$

and, since $\max(1,-\ln s)=-\ln s$ due to $s\in(0,1/e)$,

$$\left|\partial^{\ell_2}(\psi_k^j(-\ln s;\alpha))\right|\leqslant\hat{K}_{\ell_2}(k+1)^{3|\ell_2|}(\hat{C}_{j\ell_2})^k(-\ln s)^{|\ell_2|}s^{-(3k+|\ell_2|)\delta}.$$

Setting $\bar{K}_\nu:=\sup\left\{M_{\ell_1}\hat{K}_{\ell_2}\max(1,|\lambda j-i|)^{|\ell_1|}(|j|/q)^{\ell_{10}};\alpha\in\mathcal{C},\ell_1+\ell_2=\nu'\right\}$ and $C_\nu:=\max(\hat{C}_{j\ell_2};\ell_2\leqslant\nu')$, we can assert that if $s\in(0,1/e)$ and $\alpha\in\mathcal{C}$ then

$$|\partial^\nu T_{ijk}(s;\alpha)|\leqslant 2^{|\nu|-1}\bar{K}_\nu(k+1)^{3|\nu|}(C_\nu)^k(-\ln s)^{|\nu|}s^{\lambda j-i-\nu_0-(3k+|\nu|)\delta}.$$

Here we also take $\ell_1+\ell_2=\nu'=\nu-(1,0,\ldots,0)$ and Remark B.3 into account. Consequently, setting $k_0:=0$ and $K_\nu:=2^{|\nu|-1}\bar{K}_\nu$, the result follows in case that $\nu_0\geqslant 1$. ■

Recall at this point, see (2), that Theorem B concerns with the Dulac time associated to

$$Y_{\alpha,\beta}=\frac{1}{\beta_0x^my^n+u^\ell\sum_{i=1}^M\beta_iu^{i-1}}X_\alpha,$$

where $m,n,\ell\in\mathbb{Z}$ and $u=x^py^q$ with $p,q\in\mathbb{N}$. For this reason, as an intermediate step, we next consider the Dulac time $T_{ij}(\,\cdot\,;\alpha)$ of $\frac{1}{x^iy^j}X_\alpha$ for any $i,j\in\mathbb{Z}$. In its regard the next statement explains the convenience of introducing T_{ijk}, see (23).

Proposition 4.2. *For each compact set $\mathcal{C}\subset U_\delta$ with $\delta\in(0,\frac{1}{4}]$ there exists $s_0>0$ such that the Dulac time $T_{ij}(\,\cdot\,;\alpha)$ of the vector field $\frac{1}{x^iy^j}X_\alpha$, where $i,j\in\mathbb{Z}$, writes as*

$$T_{ij}(s;\alpha)=\sum_{k=0}^{+\infty}s^{i+pk}T_{ijk}(s;\alpha)\text{ for all }s\in(0,s_0)\text{ and }\alpha\in\mathcal{C}\tag{25}$$

and the series is absolutely convergent. Moreover, for each $L\in\mathbb{R}$ there exists $K_L\in\mathbb{Z}_{\geq0}$ such that

$$\sum_{k=K_L}^{+\infty}s^{i+pk}T_{ijk}(s;\alpha)\in\mathcal{F}_L^\infty(U_0).\tag{26}$$

Proof. Let $t\mapsto\big(x(t,p_0;\alpha),y(t,p_0;\alpha)\big)$ be the solution of X_α passing through $p_0\in\mathbb{R}^2$ at $t=0$. Note that if $p_0=(s,1)$ with $s>0$ then $x(t,p_0;\alpha)=se^t$ intersects the transversal section $\{x=1\}$ at $t=-\ln s$. Thus the time $T_{ij}(s;\alpha)$ that spends the solution of $\frac{1}{x^iy^j}X_\alpha$ starting at $(s,1)$ with $s>0$ to reach the transversal section $\{x=1\}$ is given by

$$T_{ij}(s;\alpha)=\int_0^{-\ln s}(x(t,s,1;\alpha))^i(y(t,s,1;\alpha))^jdt=\int_0^{-\ln s}s^ie^{(i-\lambda j)t}\sum_{k=0}^{+\infty}\psi_k^j(t;\alpha)s^{kp}dt,$$

where in the second equality we apply Lemma 2.5. In this respect observe that, due to $\partial_t\Omega(t,\alpha_1)=e^{\alpha_1 t}>0$, for all $t\in(0,-\ln s)$ we have

$$s^p\max\left(\tfrac{M}{2}\Omega(t,\alpha_1),2C_0e^{\delta t}\right)<s^p\max\left(\tfrac{M}{2}\Omega(t,\alpha_1),2C_0e^{\delta t}\right)\big|_{t=-\ln s}=s^p\max\left(\tfrac{M}{2}\omega(s;\alpha_1),2C_0s^{-\delta}\right)<1,$$

provided that $s>0$ is small enough because $\lim_{s\to 0^+}s^{p-\delta}=0$ and, by (b) in Lemma A.4, $s^p\omega(s;\alpha_1)$ tends to zero as $s\to 0^+$ uniformly on U_δ. Consequently, recall the definition in (23), the first assertion in the statement will follow by applying Lemma B.5 once we show that for each compact set $\mathcal{C}\subset U_\delta$ with $\delta\in(0,\frac14]$ there exists $s_0>0$ such that

$$\sum_{k=0}^{+\infty}\int_0^{-\ln s}s^{i+kp}e^{(i-\lambda j)t}\left|\psi_k^j(t;\alpha)\right|dt<+\infty \text{ for all } s\in(0,s_0) \text{ and } \alpha\in\mathcal{C}. \tag{27}$$

With this aim let us note that, by applying Lemma 2.7 with $|\nu|=0$,

$$\begin{aligned}\int_0^{-\ln s}s^{i+kp}e^{(i-\lambda j)t}\left|\psi_k^j(t;\alpha)\right|dt &\leqslant K_0(C_{j0})^k s^{i+kp}\int_0^{-\ln s}e^{(i-\lambda j+3k\delta)t}dt\\ &=K_0(C_{j0})^k s^{i+kp}\frac{s^{-(i-\lambda j+3k\delta)}-1}{i-\lambda j+3k\delta}\leqslant K_0(C_{j0})^k s^{k(p-\frac34)+\lambda j},\end{aligned}$$

where in the last inequality we use that $p-3\delta\geqslant\frac14$, due to $\delta\in(0,\frac14)$, and we take k large enough so that $i-\lambda j+3k\delta\geqslant 1$. Thus the above upper bound readily shows the validity of (27) taking $s_0=(C_{j0})^{-1/(p-\frac34)}$ because it guarantees that $C_{j0}s^{p-\frac34}<1$ for all $s\in(0,s_0)$.

With regard to the last assertion in the statement let us first note that, by applying Theorem B.2,

$$\partial^\nu\left(s^{i+pk}T_{ijk}(s;\alpha)\right)=\sum_{\ell_1+\ell_2=\nu}\binom{\nu}{\ell_1,\ell_2}\partial^{\ell_1}(s^{i+pk})\partial^{\ell_2}(T_{ijk}(s;\alpha)).$$

Accordingly, by Lemma 4.1, there exists $k_0\in\mathbb{Z}_{\geqslant 0}$ such that, for all $\nu\in\mathbb{Z}_{\geqslant 0}^{N+1}$ and compact set $\mathcal{C}\subset U_\delta$,

$$|\partial^\nu(s^{i+pk}T_{ijk}(s;\alpha))|\leqslant\sum_{\ell_1+\ell_2=\nu}\binom{\nu}{\ell_1,\ell_2}K_{\ell_2}|i+kp|^{\ell_{10}}(k+1)^{3|\ell_2|}(C_{\ell_2})^k(-\ln s)^{|\ell_2|}s^{\lambda j+pk-\ell_{10}-\ell_{20}-(3k+|\nu|)\delta}$$

provided that $k\geqslant k_0$, $s\in(0,\frac1e)$ and $\alpha\in\mathcal{C}$. Since $|i+pk|\leqslant(k+1)(|i|+p)$, setting $\hat{C}_\nu=\max(C_{\ell_2};\ell_2\leqslant\nu)$ and $\hat{K}_\nu:=2^{|\nu|}\max(K_{\ell_2}(|i|+p)^{\ell_{10}};\ell_1+\ell_2=\nu)$, we can assert that if $k\geqslant k_0$, $s\in(0,1/e)$ and $\alpha\in\mathcal{C}$ then

$$\begin{aligned}|\partial^\nu(s^{i+pk}T_{ijk}(s;\alpha))|&\leqslant\hat{K}_\nu(k+1)^{4|\nu|}(\hat{C}_\nu)^k(-\ln s)^{|\nu|}s^{\lambda j+(p-3\delta)k-\nu_0-|\nu|\delta}\\&\leqslant\hat{K}_\nu(k+1)^{4|\nu|}(\hat{C}_\nu)^k(-\ln s)^{|\nu|}s^{\gamma+(k-|\nu|)/4-\nu_0}=:m_k(s),\end{aligned}\tag{28}$$

where in the first inequality we also take $\ell_1+\ell_2=\nu$ and Remark B.3 into account, and in the second one we use that $\delta\in(0,\frac14)$, $p\geqslant 1$ and $\lambda j\geqslant-\frac{p+\delta}{q}|j|=:\gamma$. (Let us remark, it will be important later on when we use the previous inequalities, that k_0 is independent from ν and $\mathcal{C}$.) On account of Remark 3.1, a sufficient condition for $s\mapsto m_k(s)$ to be monotonous increasing on $(0,1/e)$ is that $k>9|\nu|+4(\nu_0-\gamma)$, and for this reason we set

$$\bar{k}_\nu:=\max\left(\lceil 5|\nu|+4(\nu_0-\gamma)\rceil,k_0\right).$$

Note on the other hand that, due to

$$m_k(s)=\hat{K}_\nu(\hat{C}_\nu s^{1/4})^k(k+1)^{4|\nu|}(-\ln s)^{|\nu|}s^{\gamma-|\nu|/4-\nu_0},$$

if we set $s_0 := \min(1/e, (2\hat{C}_\nu)^{-4})$ then the series with general term $M_k := m_k(s_0)$ is summable and, moreover, thanks to the monotonicity of $m_k(s)$ on $(0, 1/e)$,

$$|\partial^\nu (s^{i+pk} T_{ijk}(s;\alpha))| \leqslant m_k(s) \leqslant M_k \text{ for all } s \in (0, s_0),\ \alpha \in \mathcal{C} \text{ and } k \geqslant \bar{k}_\nu.$$

Hence, thanks to the Weierstrass M-test, for each $\nu \in \mathbb{Z}_{\geq 0}^{N+1}$ the series $\sum_{k=0}^{\infty} \partial^\nu (s^{i+pk} T_{ijk}(s;\alpha))$ converges uniformly for $s \in (0, s_0)$ and $\alpha \in \mathcal{C}$. Consequently, by applying recursively Lemma B.4 starting from (25), we have that for each compact set $\mathcal{C} \subset U_\delta$ and $\nu \in \mathbb{Z}_{\geq 0}^{N+1}$ there exists $s_0 > 0$ small enough such that if $s \in (0, s_0)$ and $\alpha \in \mathcal{C}$ then

$$\partial^\nu T_{ij}(s;\alpha) = \partial^\nu \left(\sum_{k=0}^{+\infty} s^{i+pk} T_{ijk}(s;\alpha) \right) = \sum_{k=0}^{+\infty} \partial^\nu \left(s^{i+pk} T_{ijk}(s;\alpha) \right). \tag{29}$$

We are now in position to finish the proof. Indeed, we claim that (26) holds taking

$$K_L := \max\left(k_0, \left\lceil 4L + \tfrac{4p+1}{q} |j| \right\rceil + 8 \right).$$

(Recall that k_0 is the nonnegative integer given by Lemma 4.1, see (24), which is relevant for our purpose because it guarantees the upper bound (28) for $k \geqslant k_0$.) We point out that K_L is independent from ν and $\mathcal{C}$. In order to show (26), recall Definition 1.1, we fix any $\nu \in \mathbb{Z}_{\geq 0}^{N+1}$ and $\alpha^\star = (0, \alpha_2, \ldots, \alpha_N) \in U_0 = \{0\} \times \mathbb{R}^{N-1}$, and we take a relatively compact neighbourhood V of $\alpha^\star$ contained in U_δ with $\delta = \min(\frac{1}{4}, \frac{1}{|\nu|})$. Then, from (29) and using the upper bound in (28), for each $s \in (0, s_0)$ and $\alpha \in V$ we have

$$\begin{aligned}
\left| \partial^\nu_{s,\alpha} \left(\sum_{k=K_L}^{+\infty} (s^{i+pk} T_{ijk}(s;\alpha) \right) \right| &\leqslant \sum_{k=K_L}^{+\infty} \left| \partial^\nu_{s,\alpha} (s^{i+pk} T_{ijk}(s;\alpha)) \right| \\
&\leqslant \hat{K}_\nu (-\ln s)^{|\nu|} s^{\lambda j - |\nu|\delta - \nu_0} \sum_{k=K_L}^{+\infty} (k+1)^{4|\nu|} (\hat{C}_\nu s^{p-3\delta})^k \\
&\leqslant \hat{K}_\nu \hat{M}_\nu s^{\lambda j - |\nu|\delta - \nu_0 - 1} A (K_L+1)^{4|\nu|} (\hat{C}_\nu)^{K_L} s^{(p-3\delta)K_L} \\
&\leqslant \hat{K}_\nu \hat{M}_\nu A (K_L+1)^{4|\nu|} (\hat{C}_\nu)^{K_L} s^{\lambda j + \frac{1}{4} K_L - 2 - \nu_0} \leqslant C s^{L - \nu_0}.
\end{aligned}$$

In the third inequality above we apply Lemma 3.2 and set $\hat{M}_\nu := \sup\{ s(-\ln s)^{|\nu|};\ s \in (0, s_0) \}$. Next, in the fourth inequality, we take $\delta = \min(\frac{1}{4}, \frac{1}{|\nu|})$ and $p \geqslant 1$ into account. Finally in the last inequality we use the definition of K_L, which implies $\lambda j + \frac{1}{4} K_L - 2 \geqslant L$ due to $\lambda < \frac{p+\delta}{q}$, and we set $C := \hat{K}_\nu \hat{M}_\nu A (K_L+1)^{4|\nu|} (\hat{C}_\nu)^{K_L}$. This completes the proof of the result. ∎

Finally, and this will be the last ingredient for the proof of Theorem B, we next study the finite truncation of the series given in (25). We will show that it can be written in terms of polynomials in s^p and $s^p \omega$.

Lemma 4.3. *Consider $i, j \in \mathbb{Z}$ and $K \in \mathbb{N}$ and define*

$$T_{ij}^K(s;\alpha) := \sum_{k=0}^{K-1} s^{i+pk} T_{ijk}(s;\alpha).$$

Then, setting $\omega = \omega(s;\alpha_1)$, the following holds:

(a) *If $iq - jp \neq 0$ then there exists $\tau_{ij}^K(z, w; \alpha) \in \mathbb{Q}(\alpha_1)[z, w, \alpha_2, \ldots, \alpha_N]$, with $\deg_{z,w}(\tau_{ij}^K) < K$ and not having poles along $\alpha_1 = 0$, such that $T_{ij}^K(s) = s^{\lambda j} \tau_{ij}^K(s^p, s^p \omega; \alpha) - s^i \tau_{ij}^K(s^p, 0; \alpha)$.*

(b) *If $(i, j) = \nu(p, q)$ with $\nu \in \mathbb{N}$ then there exists $\varrho_{ij}^K(z, w; \alpha) \in \mathbb{Q}[z, w, \alpha]$, with $\deg_{z,w}(\varrho_{ij}^K) < K + \nu$ and $\varrho_{ij}^K(z, 0; \alpha) = 0$, such that $T_{ij}^K(s) = \varrho_{ij}^K(s^p, s^p \omega; \alpha)$.*

Proof. By applying (a) in Lemma 2.1, from the definition in (16) we get the existence of a polynomial $R_k^j(z;\alpha) \in \mathbb{Q}[z,\alpha]$ with $\deg_z(R_k^j) \leqslant k$ such that

$$\psi_k^j(t;\alpha) = R_k^j(\Omega(t;\alpha_1);\alpha), \text{ where } \Omega(t;\alpha) = \tfrac{e^{\alpha_1 t}-1}{\alpha_1}.$$

Accordingly, from the definition in (23) and by performing the coordinate change $w = \Omega(t;\alpha_1)$, we get

$$T_{ijk}(s;\alpha) = \int_0^{-\ln s} e^{(i-\lambda j)t} R_k^j\left(\Omega(t;\alpha_1);\alpha\right) dt = \int_0^{\omega(s;\alpha_1)} (1+\alpha_1 w)^{\frac{i-\lambda j}{\alpha_1}-1} R_k^j(w;\alpha) dw, \tag{30}$$

where we use that $\Omega(-\ln s;\alpha_1) = \omega(s;\alpha_1)$ by definition. If $i-\lambda j|_{\alpha_1=0} \neq 0$, which is equivalent to $pj-qi \neq 0$, after integrating by parts k times we obtain

$$T_{ijk}(s;\alpha) = \frac{(1+\alpha_1 w)^{\frac{i-\lambda j}{\alpha_1}}}{i-\lambda j}\left(R_k^j(w;\alpha) - \frac{\partial_w R_k^j(w;\alpha)(1+\alpha_1 w)}{i-\lambda j+\alpha_1}\right.$$
$$\left. + \frac{\partial_w^2 R_k^j(w;\alpha)(1+\alpha_1 w)^2}{(i-\lambda j+\alpha_1)(i-\lambda j+2\alpha_1)} + \cdots + \frac{(-1)^k \partial_w^k R_k^j(w;\alpha)(1+\alpha_1 w)^k}{(i-\lambda j+\alpha_1)\cdots(i-\lambda j+k\alpha_1)}\right)\Bigg|_0^{\omega(s;\alpha_1)}.$$

It is clear then that there exists a polynomial $\tau_{ijk}(w;\alpha) \in \mathbb{Q}(\alpha_1)[w,\alpha_2,\ldots,\alpha_N]$, not having poles along $\alpha_1 = 0$ and with $\deg_w(\tau_{ijk}) \leqslant k$, such that we can write

$$s^{i+kp} T_{ijk}(s;\alpha) = s^{i+kp}\left((1+\alpha_1\omega)^{\frac{i-\lambda j}{\alpha_1}} \tau_{ijk}(\omega;\alpha) - \tau_{ijk}(0;\alpha)\right) = s^{\lambda j+kp}\tau_{ijk}(\omega;\alpha) - \tau_{ijk}(0;\alpha)s^{i+kp},$$

where we set $\omega = \omega(s;\alpha_1)$ for shortness and in the second equality we use that $1+\alpha_1\omega = s^{-\alpha_1}$. On account of this there exists $\hat{\tau}_{ijk}(z,w;\alpha) \in \mathbb{Q}(\alpha_1)[z,w,\alpha_2,\ldots,\alpha_N]$, which is homogenous of degree k in z and w, such that

$$s^{i+kp} T_{ijk}(s;\alpha) = s^{\lambda j}\hat{\tau}_{ijk}\left(s^p, s^p\omega;\alpha\right) - s^i \hat{\tau}_{ijk}(s^p,0;\alpha).$$

In view of this it is clear that the assertion in (a) follows taking $\tau_{ij}^K := \sum_{k=0}^{K-1} \hat{\tau}_{ijk}$. With regard to the one in (b) we note that, since $(p,q) = 1$, the equality $pj - qi = 0$ holds if and only if there exists $\nu \in \mathbb{Z}$ such that $(i,j) = \nu(p,q)$. In this case, from (30), we deduce that

$$T_{ijk}(s;\alpha) = \int_0^{\omega(s;\alpha_1)} (1+\alpha_1 w)^{\nu-1} R_k^j(w;\alpha) dw.$$

If $\nu \in \mathbb{N}$ then $T_{ijk}(s;\alpha) = \varrho_{ijk}(\omega;\alpha) - \varrho_{ijk}(0;\alpha)$, where $\varrho_{ijk}(z;\alpha) \in \mathbb{Q}[z,\alpha]$ with $\deg_z(\varrho_{ijk}) \leqslant k+\nu$. Hence there exists $\hat{\varrho}_{ijk}(z,w;\alpha) \in \mathbb{Q}[z,w,\alpha]$, homogeneous of degree $k+\nu$ in z and w, such that

$$s^{i+kp} T_{ijk}(s;\alpha) = s^{p(\nu+k)} T_{ijk}(s;\alpha) = \hat{\varrho}_{ijk}\left(s^p, s^p\omega;\alpha\right) - \hat{\varrho}_{ijk}(s^p,0;\alpha).$$

Since $T_{ij}^K(s;\alpha) = \sum_{k=0}^{K-1} s^{i+pk} T_{ijk}(s;\alpha)$, this shows that (b) follows taking

$$\varrho_{ij}^K(z,w;\alpha) := \sum_{k=0}^{K-1} \left(\varrho_{ijk}(z,w;\alpha) - \varrho_{ijk}(z,0;\alpha)\right),$$

which concludes the proof of the result. ■

We are now in position to prove our second main result.

Proof of Theorem B. Recall that the family of vector fields under consideration is given by

$$Y_{\alpha,\beta} = \frac{1}{\beta_0 x^m y^n + \sum_{i=\ell}^{M+\ell-1} \beta_{i+1-\ell}\, u^i} X_\alpha,$$

where $\ell \in \mathbb{Z}$ is defined in (1), $u = x^p y^q$, $(p,q) = 1$ and

$$X_\alpha = x\partial_x + \tfrac{1}{q}\left(-p + \textstyle\sum_{i=0}^{N-1} \alpha_{i+1} u^i\right) y\partial_y.$$

Let us denote the solution of X_α passing through $p_0 \in \mathbb{R}^2$ at $t=0$ by $t \mapsto \big(x(t,p_0;\alpha), y(t,p_0;\alpha)\big)$. Then, if $p_0 = (s,1)$ with $s>0$, $x(t,p_0;\alpha) = se^t$ intersects the transversal section $\{x=1\}$ at $t = -\ln s$. Consequently the time $T(s;\alpha,\beta)$ that spends the solution of $Y_{\alpha,\beta}$ starting at $(s,1)$ with $s>0$ to reach the transversal section $\{x=1\}$ is given by

$$\begin{aligned} T(s;\alpha,\beta) &= \int_0^{-\ln s} \left(\beta_0 x^m y^n + \sum_{i=\ell}^{M+\ell-1} \beta_{i+1-\ell}(x^p y^q)^i\right)\Bigg|_{\{x=x(t,s,1;\alpha),y=y(t,s,1;\alpha)\}} dt \\ &= \beta_0 T_{mn}(s;\alpha) + \sum_{i=\ell}^{M+\ell-1} \beta_{i+1-\ell} T_{ip,iq}(s;\alpha), \end{aligned}$$

where $T_{ij}(\,\cdot\,;\alpha)$ is the Dulac time of $\frac{1}{x^i y^j} X_\alpha$, which is precisely our concern in Proposition 4.2 and Lemma 4.3. It is clear then that, by applying Proposition 4.2, for each compact set $C \subset U_\delta$ with $\delta \in (0,\frac{1}{4}]$ there exists $s_0 > 0$ such that

$$T(s;\alpha,\beta) = \sum_{k=0}^{+\infty} s^{kp}\left(\beta_0 s^m T_{mnk}(s;\alpha) + \sum_{i=\ell}^{M+\ell-1} \beta_{i+1-\ell} s^{ip} T_{ip,iq,k}(s;\alpha)\right)$$

for all $s \in (0,s_0)$ and $\alpha \in \mathcal{C}$ and the series is absolutely convergent. Furthermore we can assert that, for the given $L \in \mathbb{R}$, there exists $K_L \in \mathbb{Z}_{\geq 0}$ such that

$$\mathcal{T}_L(s;\alpha,\beta) := \sum_{k=K_L}^{+\infty} s^{kp}\left(\beta_0 s^m T_{mnk}(s;\alpha) + \sum_{i=\ell}^{M+\ell-1} \beta_{i+1-\ell} s^{ip} T_{ip,iq,k}(s;\alpha)\right) \in \mathcal{F}_L^\infty(U_0 \times \mathbb{R}^{M+1}),$$

where $U_0 \times \mathbb{R}^{M+1}$ stands for the set $\{(\alpha,\beta) \in \mathbb{R}^{M+N+1}; \alpha_1 = 0\}$. This assertion follows by taking (26) into account and applying, in this order, (c), (b) (g) and (e) in Lemma A.3.

On the other hand, by Lemma 4.3, there exist $\tau_0(z,w;\alpha) \in \mathbb{Q}(\alpha_1)[z,w,\alpha_2,\ldots,\alpha_N]$ without poles along $\alpha_1 = 0$ and $\varrho_i(z,w;\alpha) \in \mathbb{Q}[z,w,\alpha]$ with $\varrho_i(z,0;\alpha) = 0$, $i = 0,1,\ldots,M$, such that such that setting

$$\mathscr{L}_0(s;\alpha) := \begin{cases} s^{\lambda n}\tau_0(s^p, s^p\omega;\alpha) - s^m \tau_0(s^p,0;\alpha) & \text{if } mq - np \neq 0, \\ s^{\kappa p}\varrho_0(s^p, s^p\omega;\alpha) & \text{if } mq-np = 0 \text{ and } (m,n) \neq (0,0), \\ -\ln s & \text{if } (m,n) = (0,0), \end{cases}$$

$$\mathscr{L}_1(s;\alpha) := \begin{cases} s^{\ell p}\varrho_1(s^p, s^p\omega;\alpha) & \text{if } \ell > 0, \\ -\ln s & \text{if } \ell = 0, \end{cases}$$

and

$$\mathscr{L}_i(s;\alpha) := s^{(\ell+i-1)p}\varrho_i(s^p, s^p\omega;\alpha), \text{ for } i = 2,3,\ldots,M,$$

then

$$
\begin{aligned}
T^L(s;\alpha,\beta) :=& \sum_{k=0}^{K_L-1} s^{kp}\left(\beta_0 s^m T_{mnk}(s;\alpha) + \sum_{i=\ell}^{M+\ell-1} \beta_{i+1-\ell} s^{ip} T_{ip,iq,k}(s;\alpha)\right)\\
=& \beta_0 \mathscr{L}_0(s;\alpha) + \beta_1 \mathscr{L}_1(s;\alpha) + \sum_{i=2}^{M} \beta_i \mathscr{L}_i(s;\alpha).
\end{aligned} \tag{31}
$$

With regard to the cases considered in the definition of $\mathscr{L}_0$, let us note that if $mq-np=0$ then, due to $(p,q)=1$, there exists $\eta\in\mathbb{Z}$ such that $(m,n)=\eta(p,q)$. Thus, by assumption, $\eta=\kappa:=\lceil\max\left(\frac{m}{p},\frac{n}{q}\right)\rceil\geqslant 0$ and hence, on account of Definition 1, $\ell=\eta+1>0$. (In particular, if $mq-np=0$ and $(m,n)\neq(0,0)$ then $\eta=\kappa\in\mathbb{N}$, and so the assertion with respect to $\mathscr{L}_0$ follows by (b) in Lemma 4.3.) If, on the contrary, $mq-np\neq 0$ then, by Definition 1 again, $\ell=\kappa\geqslant 0$. Note also that if $(i,j)=(0,0)$ then $T_{ij}(s;\alpha)=-\ln s$, which yields to the subcases $(m,n)=(0,0)$ and $\ell=0$ in $\mathscr{L}_0$ and $\mathscr{L}_1$, respectively. In this respect, $\mathscr{L}_0(s;\alpha)=-\ln s$ in case that $(m,n)=(0,0)$, which implies $\ell\geqslant 1$, and then, $\mathscr{L}_1(s;\alpha)\neq-\ln s$. On the other hand, $\mathscr{L}_1(s;\alpha)=-\ln s$ in case that $\ell=0$, which implies $mp-nq\neq 0$ due to (1) and the assumption $\kappa\geqslant 0$. Accordingly, in this case, $\mathscr{L}_0(s;\alpha)\neq-\ln s$.

Taking the previous considerations into account, the assertions with respect to T^L follow from (31). This concludes the proof of the result. ■

A Results about the class $\mathcal{F}_L^K(W)$

The present section is devoted to show a number of general properties about the class $\mathcal{F}_L^K(W)$. We first prove that any $g(s;\mu)\in\mathcal{F}_L^K(W)$ extends to a finitely smooth function (on s and the parameter μ) along $s=0$. (This applies in particular to the remainder $\mathscr{D}_L$ in Theorem A, as well as to $\mathcal{T}_L$ in Theorem B.) On the contrary, we will provide an example showing that a function $g(s;\mu)$ verifying the estimates in (4) but only with respect to the s derivative (i.e., with $\nu_1=\ldots=\nu_N=0$) may not have an extension along $s=0$ which is $\mathscr{C}^L$ on s and the parameter μ (see Example A.2).

Lemma A.1. *Let U be an open set of $\mathbb{R}^N$, $K\in\mathbb{Z}_{\geq 0}$ and $g(s;\mu)\in\mathscr{C}^K_{s>0}(U)$ such that, for some $W\subset U$ and $L\in\mathbb{R}$, $g(s;\mu)\in\mathcal{F}_L^K(W)$. If $L>K$ then g extends to a $\mathscr{C}^K$-function $\hat{g}$, defined in some open neighbourhood of $\{0\}\times W$ in $\mathbb{R}^{N+1}$, and satisfying $\partial^\nu\hat{g}(0;\mu)=0$ for all $\mu\in W$ and $\nu\in\mathbb{Z}^{N+1}_{\geq 0}$ with $|\nu|\leqslant K$.*

Proof. Due to $g(s;\mu)\in\mathscr{C}^K_{s>0}(U)$, by definition there exists an open neighbourhood V of $\{0\}\times U$ in $\mathbb{R}^{N+1}$ such that $(s,\mu)\mapsto g(s;\mu)$ is $\mathscr{C}^K$ on $V_+:=V\cap\big((0,+\infty)\times U\big)$. Then the function

$$
\hat{g}(s;\mu) := \begin{cases} g(|s|;\mu) & \text{if } s\neq 0 \text{ and } (|s|,\mu)\in V_+,\\ 0 & \text{if } s=0 \text{ and } \mu\in U,\end{cases}
$$

is well defined on $\big\{(s,\mu)\in\mathbb{R}^{N+1};(|s|,\mu)\in V_+\big\}\cup\big(\{0\}\times U\big)$, which is an open neighbourhood of $\{0\}\times U$ in $\mathbb{R}^{N+1}$. Moreover, for $\nu=(\nu_0,\nu_1,\ldots,\nu_N)\in\mathbb{Z}^{N+1}_{\geq 0}$ with $|\nu|\leqslant K$, it is easy to show (by induction on ν_0) that

$$
\partial^\nu\hat{g}(s;\mu)=\operatorname{sgn}(s)^{\nu_0}\partial^\nu g(|s|;\mu) \text{ for } s\neq 0 \text{ with } (|s|,\mu)\in V_+. \tag{32}
$$

Next we fix any $\hat{\mu}\in W$. Then, due to $g(s;\mu)\in\mathcal{F}_L^K(\hat{\mu})$, by definition there exist $s_0,\varepsilon,C>0$ such that, for each $\nu\in\mathbb{Z}^{N+1}_{\geq 0}$ with $|\nu|\leqslant K$,

$$
|\partial^\nu g(s;\mu)|\leqslant Cs^{L-\nu_0} \text{ for all } s\in(0,s_0) \text{ and } \|\mu-\hat{\mu}\|<\varepsilon. \tag{33}
$$

We claim that $\hat{g}(s;\mu)$ is of class $\mathscr{C}^K$ at $(0,\hat{\mu})$ and that $\partial^\nu\hat{g}(0;\hat{\mu})=0$ for all $\nu\in\mathbb{Z}^{N+1}_{\geq 0}$ with $|\nu|\leqslant K$. Since $\hat{\mu}$ is arbitrary and, on account of (32), $\hat{g}$ is $\mathscr{C}^K$ on $\big\{(s,\mu)\in\mathbb{R}^{N+1};(|s|,\mu)\in V_+\big\}$, the result will follow once we prove the claim. To prove it we will show by induction on ν_0 that if $|\nu| \leqslant K$ then $|\partial^\nu \hat{g}(s;\mu)| \leqslant C|s|^{L-\nu_0}$ for all (s,μ) with $s \in (-s_0, s_0)$ and $\|\mu - \hat{\mu}\| < \varepsilon$. (This will imply that $\partial^\nu \hat{g}$ is continuous and vanishes at any $(0,\mu)$ with $\|\mu - \hat{\mu}\| < \varepsilon$.) Denote $\bar{\nu} = (\nu_1, \nu_2, \ldots, \nu_N) \in \mathbb{Z}^N_{\geq 0}$ for shortness so that $\nu = (\nu_0, \bar{\nu})$. The base case $\nu_0 = 0$ is clear because, taking (32) and $\hat{g}(0;\mu) = 0$ into account,

$$\partial^{(0;\bar{\nu})}\hat{g}(s;\mu) = \begin{cases} \partial^{(0;\bar{\nu})} g(|s|;\mu) & \text{if } s \neq 0, \\ 0 & \text{if } s = 0, \end{cases}$$

that has absolute value smaller than $C|s|^L$ if $\|\mu - \hat{\mu}\| < \varepsilon$ and $s \in (-s_0, s_0)$ thanks to (33) and $\partial^{(0;\bar{\nu})}\hat{g}(0;\mu) = 0$. Let us take next any $\nu_0 \geqslant 1$ and show the inductive step. Then, by using (32) and that $\partial^{(\nu_0 - 1, \bar{\nu})}\hat{g}(0;\mu) = 0$ due to the induction hypothesis, we get

$$\partial^\nu \hat{g}(s;\mu) = \begin{cases} \operatorname{sgn}(s)^{\nu_0} \partial^\nu g(|s|;\mu) & \text{if } s \neq 0, \\ \lim\limits_{z \to 0} \frac{\partial^{(\nu_0 - 1, \bar{\nu})}\hat{g}(z,\mu)}{z} & \text{if } s = 0. \end{cases}$$

Therefore $|\partial^\nu \hat{g}(s;\mu)| = |\operatorname{sgn}(s)^{\nu_0} \partial^\nu g(|s|;\mu)| \leqslant C|s|^{L-\nu_0}$ in case that $0 < |s| < s_0$, thanks to (33), whereas $\partial^\nu \hat{g}(0;\mu) = 0$ because the induction hypothesis implies $\left| \frac{\partial^{(\nu_0 - 1, \bar{\nu})}\hat{g}(z,\mu)}{z} \right| \leqslant \frac{C|z|^{L-\nu_0+1}}{|z|} = C|z|^{L-\nu_0}$, which tends to zero as $z \to 0$ due to $L > K \geqslant |\nu| \geqslant \nu_0$. Accordingly $|\partial^\nu \hat{g}(s;\mu)| \leqslant C|s|^{L-\nu_0}$ for all (s,μ) with $s \in (-s_0, s_0)$ and $\|\mu - \hat{\mu}\| < \varepsilon$, and this proves the induction step. Consequently the claim is true and the result follows. ∎

Example A.2. With regard to the previous result we now exhibit a $\mathscr{C}^\infty$ function $g(s;\mu)$ on $(0, +\infty) \times \mathbb{R}$ verifying $|\partial^i_s g(s;\mu)| \leqslant C s^{L-i}$ for all $s > 0$, $\mu \in \mathbb{R}$ and $i = 0, 1, \ldots, L$, but such that $\partial_\mu g(s;\mu)$ does not have a continuous extension along $s = 0$.

Let us begin by taking a $\mathscr{C}^\infty$ bump function $\varphi\colon \mathbb{R} \longrightarrow [0, +\infty)$ defined by $\varphi(x) = \exp(-x^2/(x^2-1)^2)$ if $|x| \leqslant 1$ and zero otherwise. Let us fix besides any $\alpha \in (0,1)$ and define $\beta = \frac{1+\alpha}{2}$. Then, for each $k \in \mathbb{Z}_{\geq 0}$, define $E_k := \{(s,\mu) \in \mathbb{R}^2;\ p_k(s,\mu) \leqslant 1\}$ where

$$p_k(s,\mu) := \left(\frac{2(s - \beta\alpha^k)}{\alpha^k(1-\beta)} \right)^2 + \left(\frac{\mu}{\alpha^{(L+1)k}} \right)^2.$$

The sets E_k, $k \in \mathbb{Z}_{\geq 0}$, are pairwise disjoint and, furthermore, every $(s,\mu) \neq (0,0)$ has an open neighbourhood that intersects at most one E_k. This shows that

$$g(s;\mu) := \sum_{k=0}^{+\infty} \alpha^{Lk} \varphi(p_k(s,\mu))$$

is a well defined $\mathscr{C}^\infty$ function on $\mathbb{R}^2 \setminus \{(0,0)\}$. For the same reason we can commute derivation and summation and then, by applying Theorem B.1,

$$\partial^n_s g(s;\mu) = \sum_{k=0}^{+\infty} \alpha^{Lk} \sum_{j=1}^{n} \varphi^{(j)}\big(p_k(s;\mu)\big) \sum_{r_1,\ldots,r_n} n! \prod_{i=1}^{n} \frac{\big(\partial^i_s p_k(s,\mu)\big)^{r_i}}{r_i!(i!)^{r_i}}, \text{ for all } (s,\mu) \neq (0,0),$$

where the third summation is subject to the coupling conditions $\sum_{i=1}^n r_i = j$ and $\sum_{i=1}^n i r_i = n$. Observe that $\partial^i_s p_k(s,\mu) = 2\left(\frac{2(s-\beta\alpha^k)}{\alpha^k(1-\beta)}\right)^{2-i} \left(\frac{2}{\alpha^k(1-\beta)}\right)^i$ for $i = 1, 2$ and zero for $i \geqslant 3$. Thus, if $(s,\mu) \in E_k$ then $|\partial^i_s p_k(s,\mu)| \leqslant 2\left(\frac{2}{\alpha^k(1-\beta)}\right)^i$ for all $i \in \mathbb{N}$. Consequently, if $(s,\mu) \in E_{k_0}$ and $n \in \mathbb{N}$ then we get

$$|\partial^n_s g(s,\mu)| \leqslant C' \alpha^{Lk_0} \prod_{i=1}^{n} (\alpha^{-k_0})^{ir_i} = C' \alpha^{k_0(L-n)} \leqslant C \alpha^{(k_0+1)(L-n)} \leqslant C s^{L-n},$$

where C' is a positive constant (depending on n, α and $\|\varphi^{(j)}\|$, $j=1,2,\ldots,n$), $C:=C'\alpha^{n-L}$ and we use use that $s\in[\alpha^{k_0+1},\alpha^{k_0}]$. The same inequality is valid for $n=0$ since $|g(s;\mu)|\leqslant\alpha^{Lk_0}=\alpha^{-L}\alpha^{L(k_0+1)}\leqslant\alpha^{-L}s^L$. Accordingly g verifies the desired bounds with respect to the s derivatives.

The sequence of points $(s_i,\mu_i):=(\beta\alpha^i,2^{-1/2}\alpha^{Li})\in E_i$ tends to $(0,0)$ as $i\to\infty$ and, on the other hand, an easy computations gives

$$|\partial_\mu g(s_i;\mu_i)|=\alpha^{Li}|\partial_\mu\varphi(p_i(s_i,\mu_i))|=\alpha^{-i}|\varphi'(1/2)|,$$

which tends to $+\infty$ as $i\to\infty$. This shows that $\partial_\mu g(s;\mu)$ does not have a continuous extension at $(0,0)$. □

Next result gathers some general properties with regard to operations between functions in $\mathcal{F}_L^K(W)$ with $K\in\mathbb{Z}_{\geq 0}\cup\{\infty\}$ and $L\in\mathbb{R}$.

Lemma A.3. *Let U and U' be open sets of $\mathbb{R}^N$ and $\mathbb{R}^{N'}$ respectively and consider $W\subset U$ and $W'\subset U'$. Then the following holds:*

(a) $\mathcal{F}_L^K(W)\subset\mathcal{F}_L^K(\hat{W})$ *for any* $\hat{W}\subset W$ *and* $\bigcap_n\mathcal{F}_L^K(W_n)=\mathcal{F}_L^K\left(\bigcup_n W_n\right)$.

(b) $\mathcal{F}_L^K(W)\subset\mathcal{F}_L^K(W\times W')$.

(c) $\mathscr{C}^K(U)\subset\mathscr{C}^K_{s=0}(U)\subset\mathcal{F}_0^K(W)$.

(d) If $K\geqslant K'$ *and* $L\geqslant L'$ *then* $\mathcal{F}_L^K(W)\subset\mathcal{F}_{L'}^{K'}(W)$.

(e) $\mathcal{F}_L^K(W)$ *is closed under addition.*

(f) If $f\in\mathcal{F}_L^K(W)$ *and* $\nu\in\mathbb{Z}_{\geq 0}^{N+1}$ *with* $|\nu|\leqslant K$ *then* $\partial^\nu f\in\mathcal{F}_{L-\nu_0}^{K-|\nu|}(W)$.

(g) $\mathcal{F}_L^K(W)\cdot\mathcal{F}_{L'}^K(W)\subset\mathcal{F}_{L+L'}^K(W)$.

(h) Assume that $\phi\colon U'\longrightarrow U$ *is a* $\mathscr{C}^K$ *function with* $\phi(W')\subset W$ *and let us take* $g\in\mathcal{F}_{L'}^K(W')$ *with* $L'>0$ *and verifying* $g(s;\eta)>0$ *for all* $\eta\in W'$ *and* $s>0$ *small enough. Consider also any* $f\in\mathcal{F}_L^K(W)$*. Then* $h(s;\eta):=f(g(s;\eta);\phi(\eta))$ *is a well-defined function that belongs to* $\mathcal{F}_{LL'}^K(W')$.

Proof. Let us begin by showing (g) since the previous assertions are straightforward. Take $f(s;\mu)\in\mathcal{F}_L^K(W)$ and $g(s;\mu)\in\mathcal{F}_{L'}^K(W)$ and fix $\hat{\mu}\in W$ and $\hat{\nu}\in\mathbb{Z}_{\geq 0}^{N+1}$ with $|\hat{\nu}|\leqslant K$. Then, by definition, it follows that there exist a neighbourhood V of $\hat{\mu}$ and $C,s_0>0$ such that $|\partial^\nu f(s;\mu)|\leqslant Cs^{L-\nu_0}$ and $|\partial^\nu g(s;\mu)|\leqslant Cs^{L'-\nu_0}$ for all $\mu\in V$, $s\in(0,s_0)$ and $\nu\in\mathbb{Z}_{\geq 0}^{N+1}$ with $|\nu|\leqslant|\hat{\nu}|$. Thus, by applying Leibniz's rule (see Theorem B.2), if $\mu\in V$ and $s\in(0,s_0)$ then

$$\left|\partial^{\hat{\nu}}\big(f(s;\mu)g(s;\mu)\big)\right|\leqslant\sum_{\nu_1+\nu_2=\hat{\nu}}\binom{\hat{\nu}}{\nu_1,\nu_2}|\partial^{\nu_1}f(s;\mu)|\,|\partial^{\nu_2}g(s;\mu)|\leqslant\hat{C}s^{L+L'-\hat{\nu}_0},$$

where we use that $\nu_{10}+\nu_{20}=\hat{\nu}_0$ and set $\hat{C}:=C^2\sum_{\nu_1+\nu_2=\hat{\nu}}\binom{\hat{\nu}}{\nu_1,\nu_2}=C^2 2^{|\hat{\nu}|}$. Thus $fg\in\mathcal{F}_{L+L'}^K(W)$.

Let us turn next to show the assertion in (h). To this end fix any $\hat{\nu}\in\mathbb{Z}_{\geq 0}^{N'+1}$ and $\hat{\eta}\in U'\subset\mathbb{R}^{N'}$. Then, by definition, it follows that there exist a neighbourhood V' of $\hat{\eta}$ and $C',s_1>0$ such that $|\partial^\nu g(s;\eta)|\leqslant C's^{L'-\nu_0}$ for all $\eta\in V'$, $s\in(0,s_1)$ and $\nu\in\mathbb{Z}_{\geq 0}^{N'+1}$ with $|\nu|\leqslant|\hat{\nu}|$. On the other hand, there exist a neighbourhood V of $\hat{\mu}:=\phi(\hat{\eta})\in U\subset\mathbb{R}^N$ and $C,s_2>0$ such that $|\partial^\nu f(s;\mu)|\leqslant Cs^{L-\nu_0}$ for all $\mu\in V$, $s\in(0,s_2)$ and $\nu\in\mathbb{Z}_{\geq 0}^{N+1}$ with $|\nu|\leqslant|\hat{\nu}|$. Consider now a relatively compact neighbourhood V'' of $\hat{\eta}$ with $V''\subset V'$ and $\phi(V'')\subset V$. Then, on account of $L'>0$, there exists $s_3\in(0,s_1)$ such that $g(s;\eta)\in(0,s_2)$ for all $s\in(0,s_3)$

and $\eta \in V''$. The application of Faà di Bruno formula (see Theorem B.1) to compute the derivative of $h(s;\eta) = f(g(s;\eta);\phi(\eta))$ yields

$$\partial^{\hat{\nu}} h(s;\eta) = \sum_{1 \leqslant |\lambda| \leqslant |\hat{\nu}|} \partial^{\lambda} f(u;\mu)\big|_{\{u=g(s;\eta),\mu=\phi(\eta)\}} \sum_{p(\hat{\nu},\lambda)} (\hat{\nu}!) \prod_{i=1}^{q} C_{k_i\ell_i} (\partial^{\ell_i} g(s,\eta))^{k_{i0}} \prod_{j=1}^{N} (\partial^{\ell_i}\phi_i(\eta))^{k_{ij}}.$$

Here we set $C_{k_i\ell_i} := \frac{1}{k_i!(\ell_i!)^{|k_i|}}$ and $q := -1 + \prod_{i=0}^{N'}(\hat{\nu}_i + 1)$ for shortness. Note that the vectors $\lambda, k_i \in \mathbb{Z}_{\geq 0}^{N+1}$ and $\ell_i \in \mathbb{Z}_{\geq 0}^{N'+1}$ are subject to the coupling conditions $\sum_{i=1}^{q} k_i = \lambda$ and $\sum_{i=1}^{q} |k_i|\ell_i = \hat{\nu}$. So if we define $C_{k\ell}^1 := \prod_{i=1}^{q} C_{k_i\ell_i}$ and $C_{k\ell}^2 := \sup\left\{\prod_{i=1}^{q}\prod_{j=1}^{N} |\partial^{\ell_i}\phi_i(\eta)|^{k_{ij}}; \eta \in V''\right\}$ and we take any $s \in (0,s_3)$ and $\eta \in V''$,

$$\begin{aligned} |\partial^{\hat{\nu}} h(s;\eta)| &\leqslant \sum_{1 \leqslant |\lambda| \leqslant |\hat{\nu}|} Cg(s;\eta)^{L-\lambda_0} \sum_{p(\hat{\nu},\lambda)} (\hat{\nu}!) C_{k\ell}^1 C_{k\ell}^2 \prod_{i=1}^{q} (C' s^{L'-\ell_{i0}})^{k_{i0}} \\ &= \sum_{1 \leqslant |\lambda| \leqslant |\hat{\nu}|} Cg(s;\eta)^{L-\lambda_0} \sum_{p(\hat{\nu},\lambda)} C_{k\ell}^3 s^{\sum_{i=1}^{q}(L'-\ell_{i0})k_{i0}} \\ &\leqslant \sum_{1 \leqslant |\lambda| \leqslant |\hat{\nu}|} C(C' s^{L'})^{L-\lambda_0} \sum_{p(\hat{\nu},\lambda)} C_{k\ell}^3 s^{L'\lambda_0 - \hat{\nu}_0} \end{aligned}$$

where we set $C_{k\ell}^3 := (\hat{\nu}!) C_{k\ell}^1 C_{k\ell}^2 \prod_{i=1}^{q} (C')^{k_{i0}} = (\hat{\nu}!) C_{k\ell}^1 C_{k\ell}^2 (C')^{\lambda_0}$ and we use that $\sum_{i=1}^{q} k_{i0}\ell_{i0} \leqslant \hat{\nu}_0$. Consequently, setting $\hat{C} := \sum_{1 \leqslant |\lambda| \leqslant |\hat{\nu}|} C(C')^{L-\lambda_0} \sum_{p(\hat{\nu},\lambda)} C_{k\ell}^3$, this shows that $|\partial^{\hat{\nu}} h(s;\eta)| \leqslant \hat{C} s^{LL'-\hat{\nu}_0}$ for all $s \in (0,s_3)$ and $\eta \in V''$, which proves the validity of (h). This completes the proof of the result. ■

Next result gathers some interesting properties of the Ecalle-Roussarie compensator that will be used in this (and a subsequent) paper. In the statement we use the notation $x^+ := \max(x,0)$ and $x^- := \max(-x,0)$ for, respectively, the positive and negative part of a given $x \in \mathbb{R}$. Note in particular that then $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

Lemma A.4. *The following assertions hold:*

(*a*) *For each compact set $I \subset \mathbb{R}$ and $\nu \in \mathbb{Z}_{\geq 0}^2$ there exists a constant $C > 0$ such that*

$$|\partial^{\nu}\omega(s;\alpha)| \leqslant Cs^{-\alpha^+ - \nu_0} |\ln s|^{|\nu|+1} \textit{for all } \alpha \in I \textit{ and } s \in (0,1/e).$$

Moreover $\lim_{s\to 0^+} \frac{1}{\omega(s;\alpha)} = \alpha^-$ uniformly on $\alpha \in \mathbb{R}$ so that, in particular, $\lim_{(s,\alpha)\to(0^+,0)} \frac{1}{\omega(s;\alpha)} = 0$.

(*b*) *For each $\varepsilon > 0$, $(s,\alpha) \mapsto \omega(s;\alpha)$ belongs to $\mathcal{F}_{-\varepsilon}^{\infty}(\{\alpha < \varepsilon\})$ and $(s;\alpha) \mapsto \frac{1}{\omega(s;\alpha)}$ belongs to $\mathcal{F}_{-\varepsilon}^{\infty}(\mathbb{R})$.*

(*c*) *For each $L \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, $(s,\alpha,\beta) \mapsto s^{\beta}\omega^{\ell}(s;\alpha)$ belongs to $\mathcal{F}_L^{\infty}(\{(\alpha,\beta) \in \mathbb{R}^2\,;\, \beta > L + \ell^+\alpha^+\})$.*

(*d*) *If $p(z;\mu) \in \mathscr{C}^K(U)[z,z^{-1}]$, where U is some open set of $\mathbb{R}^N$, then the function $(s,\alpha,\beta,\mu) \mapsto s^{\beta}p(\omega(s;\alpha);\mu)$ belongs to $\mathcal{F}_L^K(\{(\alpha,\beta,\mu) \in \mathbb{R}^2 \times U\,;\, \alpha = 0, \beta > L\})$.*

Proof. For the sake of convenience we prove first the assertion (*c*) for $\ell = 0$. To this end we apply Lemma 2.6, which shows that for each $i,j \in \mathbb{Z}_{\geq 0}$ there exists $M > 0$ so that, for every $s \in (0,1/e)$,

$$|\partial_s^i \partial_\beta^j s^{\beta}| \leqslant Ms^{\beta-i} \max(|\ln s|, |\beta|)^{i+j} = Ms^{L-i}s^{\beta-L} \max(|\ln s|, |\beta|)^{i+j}. \tag{34}$$

Let us fix $\hat{\beta} \in \mathbb{R}$ with $\hat{\beta} > L$ and take a compact neighborhood I of $\hat{\beta}$ such that $\beta - L > 0$ for all $\beta \in I$. Thus $C := M\sup\left\{s^{\beta-L}\max(|\ln s|,|\beta|)^{i+j}; \beta \in I, s \in (0,1/\varepsilon)\right\}$ is finite and so, from (34), $|\partial_s^i\partial_\beta^j s^{\beta}| \leqslant Cs^{L-i}$ for all

$s \in (0, 1/e)$ and $\beta \in I$. Hence s^β belongs to $\mathcal{F}_L^\infty(\{\beta > L\})$, which is a subset of $\mathcal{F}_L^\infty(\{(\alpha, \beta) \in \mathbb{R}^2; \beta > L\})$ by (b) in Lemma A.3.

We show next the validity of the inequality in (a). Take $\nu = (\nu_0, \nu_1) \in \mathbb{Z}_{\geq 0}^2$ and a compact set I of $\mathbb{R}$ and let us consider first the case $\nu_0 > 0$. Then, if $\alpha \in I$ and $s \in (0, 1/e)$,

$$\begin{aligned}|\partial^\nu \omega(s;\alpha)| &= |\partial^{(\nu_0-1,\nu_1)} s^{-\alpha-1}| \leqslant M s^{-\alpha-\nu_0} \max(|\ln s|, |\alpha+1|)^{|\nu|-1} \\ &\leqslant C s^{-\alpha-\nu_0} |\ln s|^{|\nu|-1} \leqslant C s^{-\alpha^+-\nu_0} |\ln s|^{|\nu|+1},\end{aligned}$$

where the first inequality follows from (34) taking $i = \nu_0 - 1$, $j = \nu_1$ and $\beta = -\alpha - 1$, and the second one setting $C := M \max(1, \sup\{|\alpha+1|; \alpha \in I\})^{|\nu|-1}$ and using previously that $\max(a, b) \leqslant a \max(1, b)$ for any $a \geqslant 1$ and $b \geqslant 0$. In order to prove the same inequality for $\nu_0 = 0$ note that $\omega(s;\alpha) = F(\alpha \ln s) \ln s$ with $F(x) := \frac{e^{-x}-1}{x}$ and so, in this case, $\partial^\nu \omega(s;\alpha) = \partial_\alpha^{\nu_1}(F(\alpha \ln s) \ln s) = (\ln s)^{\nu_1+1} F^{(\nu_1)}(\alpha \ln s)$. We claim that

$$|F^{(n)}(x)| \leqslant e^{x^-} \text{ for all } x \in \mathbb{R} \text{ and } n \in \mathbb{Z}_{\geq 0}.$$

In this respect observe that, due to $x^-|_{x=\alpha \ln s} = \max(-\alpha \ln s, 0) = -\ln s \max(\alpha, 0) = \ln(s^{-\alpha^+})$, the claim will imply $|\partial^\nu \omega(s;\alpha)| \leqslant s^{-\alpha^+} |\ln s|^{\nu_1+1} = s^{-\alpha^+-\nu_0} |\ln s|^{|\nu|+1}$ for all $s \in (0, 1/e)$ and $\alpha \in \mathbb{R}$ and, consequently, the validity of the inequality in (a) for $\nu_0 = 0$ as well. To prove the claim we note that F is an entire function which, differentiating term by term its Taylor's series at $x = 0$, verifies

$$F^{(n)}(x) = -\sum_{r=n}^{+\infty} \frac{(-1)^n (-x)^{r-n}}{(r-n)!(r+1)} = (-1)^{n+1} \sum_{k=0}^{+\infty} \frac{(-x)^k}{k!(k+n+1)} \text{ for all } x \in \mathbb{R}.$$

Hence, on account of $\frac{1}{k+n+1} \leqslant 1$, we get $|F^{(n)}(x)| \leqslant e^{|x|}$ for all $x \in \mathbb{R}$. In its turn this implies the claim for $x \leqslant 0$ because, in this case, $x^- = |x|$. The proof of the claim for $x \geqslant 0$ is a little more involved. We must show that $\left|\partial_x^n \left(\frac{e^{-x}-1}{-x}\right)\right| \leqslant 1$ for all $x \geqslant 0$, and it is clear that this will follow once we prove that

$$0 < \partial_x^n \left(\frac{e^x - 1}{x}\right) \leqslant 1 \text{ for all } x \leqslant 0. \tag{35}$$

To prove these two inequalities we first check by induction on $n \in \mathbb{Z}_{\geq 0}$ that

$$\partial_x^n \left(\frac{e^x - 1}{x}\right) = e^x n! \sum_{k=0}^{+\infty} \frac{(-x)^k}{(k+n+1)!},$$

which is valid for all $x \in \mathbb{R}$ because $x \mapsto \frac{e^x-1}{x}$ is an entire function. Hence, for any $n \in \mathbb{Z}_{\geq 0}$, we can assert that $\partial_x^n \left(\frac{e^x-1}{x}\right) > 0$ for all $x \leqslant 0$. In particular this implies $\partial_x^{n-1} \left(\frac{e^x-1}{x}\right) \leqslant \partial_x^{n-1} \left(\frac{e^x-1}{x}\right)\big|_{x=0} = \frac{1}{n} \leqslant 1$ for all $x \leqslant 0$ and $n \in \mathbb{N}$. Thus both inequalities in (35) are true and so the claim follows for $x \geqslant 0$ as well.

Let us prove now that $\lim_{s \to 0^+} \frac{1}{\omega(s;\alpha)} = \alpha^-$ uniformly on $\alpha \in \mathbb{R}$. By distinguishing the cases $\alpha < 0$, $\alpha = 0$ and $\alpha > 0$, one can check that $\frac{1}{\omega(s;\alpha)} - \alpha^- = \frac{1}{\omega(s;|\alpha|)}$, which is strictly positive in case that $s \in (0, 1)$ due to $\omega(s;\alpha) = \int_s^1 x^{-\alpha-1} dx$. Accordingly, for each given $\varepsilon > 0$ we must find $s_0 \in (0, 1)$ small enough such that if $s \in (0, s_0)$ then

$$\left|\frac{1}{\omega(s;\alpha)} - \alpha^-\right| = \frac{1}{\omega(s;|\alpha|)} < \varepsilon \text{ for all } \alpha \in \mathbb{R}. \tag{36}$$

If $\alpha \neq 0$ then $\frac{1}{\omega(s;|\alpha|)} = \frac{|\alpha|}{s^{-|\alpha|}-1}$. So, in this case, the above inequality holds if and only if $s < (1 + |\alpha|/\varepsilon)^{-1/|\alpha|}$. In this regard note that, for every $\varepsilon > 0$ and $\alpha \in \mathbb{R}$,

$$e^{-\frac{1}{\varepsilon}} = \lim_{\alpha \to 0} \left(1 + \frac{|\alpha|}{\varepsilon}\right)^{-\frac{1}{|\alpha|}} \leqslant \left(1 + \frac{|\alpha|}{\varepsilon}\right)^{-\frac{1}{|\alpha|}}$$

because the function $x \mapsto \left(1+\frac{x}{\varepsilon}\right)^{-1/x}$ is increasing on $(0,+\infty)$ for every $\varepsilon>0$. Hence this shows that, for $\alpha \neq 0$, the inequality in (36) follows taking $s_0 = e^{-1/\varepsilon}$. This is also true for $\alpha = 0$ because in this case the inequality in (36) simply writes as $-\frac{1}{\ln s} < \varepsilon$. Thus $\lim_{s\to 0^+} \frac{1}{\omega(s;\alpha)} = \alpha^-$ uniformly on $\alpha \in \mathbb{R}$, as desired.

We turn next to the proof of the two assertions in (b). To show the first one we consider the given $\varepsilon>0$ and any $\hat\alpha < \varepsilon$, and we take a compact neighbourhood I of $\hat\alpha$ such that $\alpha<\varepsilon$ for all $\alpha \in I$. Then, by applying (a), for each $\nu \in \mathbb{Z}^2_{\geqslant 0}$ there exists $C>0$ such that

$$|\partial^\nu \omega(s;\alpha)| \leqslant C s^{-\varepsilon-\nu_0} s^{\varepsilon-\alpha^+} |\ln s|^{|\nu|+1} \text{ for all } \alpha \in I \text{ and } s \in (0,1/e).$$

Thus, since $\alpha^+ < \varepsilon$ if and only if $\alpha<\varepsilon$, taking $\hat C := C \sup\{s^{\varepsilon-\alpha^+}|\ln s|^{|\nu|+1}; s \in (0,1/e), \alpha \in I\}$, from the previous estimate we get $|\partial^\nu\omega(s;\alpha)| \leqslant \hat C s^{-\varepsilon-\nu_0}$ for all $s\in(0,1/e)$ and $\alpha \in I$. This proves that $\omega(s;\alpha) \in \mathcal{F}^\infty_{-\varepsilon}(\{\alpha<\varepsilon\})$, as desired. Let us prove next that $\frac{1}{\omega(s;\alpha)} \in \mathcal{F}^\infty_{-\varepsilon}(\mathbb{R})$ for all $\varepsilon>0$. So consider any $\hat\alpha \in \mathbb{R}$ and take a compact neighbourhood I of $\hat\alpha$. Theorem B.1 shows that, for any $\nu \in \mathbb{Z}^2_{\geqslant 0}$ with $|\nu| \geqslant 1$,

$$\partial^\nu\left(\frac{1}{\omega(s;\alpha)}\right) = \sum_{n=1}^{|\nu|}(-1)^n n!(\omega(s;\alpha))^{-1-n}\sum_{p(\nu,n)}\nu!\prod_{i=1}^{q}C_{k_i\ell_i}(\partial^{\ell_i}\omega(s;\alpha))^{k_i},$$

with $C_{k_i\ell_i} = \frac{1}{k_i!(\ell_i)^{|k_i|}}$ and $q=(\nu_0+1)(\nu_1+1)-1$, and where the second summation is multidimensional and subject to the coupling conditions $\sum_{i=1}^q k_i = n$ and $\sum_{i=1}^q \ell_i k_i = \nu$. On account of this and the inequality in (a) there exists $C'>0$ such that $\prod_{i=1}^q |\partial^{\ell_i}\omega(s;\alpha)|^{k_i} \leqslant C' s^{-n\alpha^+-\nu_0}|\ln s|^{n+|\nu|}$ for all $\alpha \in I$ and $s\in(0,1/\varepsilon)$. Consequently, taking $s^{-\alpha^+} = \max(1,s^{-\alpha}) = \max(1,1+\alpha\omega(s;\alpha))$ also into account, we can assert that there exist suitable positive constants C'' and C such that if $s \in (0,1/e)$ and $\alpha\in I$ then

$$\left|\partial^\nu\left(\frac{1}{\omega(s;\alpha)}\right)\right| \leqslant C'' s^{-\varepsilon-\nu_0}\sum_{n=1}^{|\nu|}\frac{\max(1,1+\alpha\omega(s;\alpha))^n}{\omega(s;\alpha)^{n+1}} s^\varepsilon |\ln s|^{n+|\nu|} \leqslant C s^{-\varepsilon-\nu_0},$$

where in the second inequality we also use that, by applying (a), $\lim_{s\to 0^+}\frac{1}{\omega(s;\alpha)} = \alpha^-$ uniformly on $\alpha \in \mathbb{R}$. Observe that, by the same reason, $\lim_{s\to0^+}\frac{s^\varepsilon}{\omega(s;\alpha)} = 0$ uniformly for $\alpha \in I$, which implies that the above inequality holds for $|\nu|=0$ as well. This proves that the function $\frac{1}{\omega(s;\alpha)}$ belongs to $\mathcal{F}^\infty_{-\varepsilon}(\mathbb{R})$ for any $\varepsilon>0$.

With regard to the assertion in (c) recall that the case $\ell=0$ is already proved. Here, for the sake of shortness in the exposition, we shall use the Heaviside step function $H(\ell)$, which is defined by $H(\ell)=0$ if $\ell<0$ and $H(\ell)=1$ if $\ell>0$. By applying (b) together with Lemma A.3, and distinguishing the cases $\ell<0$ and $\ell>0$, it can be easily checked that

$$\omega^\ell(s;\alpha) \in \mathcal{F}^\infty_{-|\ell|\varepsilon}\big(\{\alpha\in\mathbb{R}; H(\ell)\alpha<\varepsilon\}\big) \subset \mathcal{F}^\infty_{-|\ell|\varepsilon}\big(\{(\alpha,\beta)\in\mathbb{R}^2; H(\ell)\alpha<\varepsilon, \beta>L\}\big).$$

Similarly, but applying (c) with $\ell=0$, we get

$$s^\beta \in \mathcal{F}^\infty_L\big(\{\beta>L\}\big) \subset \mathcal{F}^\infty_L\big(\{(\alpha,\beta)\in\mathbb{R}^2; H(\ell)\alpha<\varepsilon,\beta>L\}\big).$$

Consequently, by (g) in Lemma A.3,

$$s^\beta\omega^\ell(s;\alpha) \in \mathcal{F}^\infty_{L-|\ell|\varepsilon}\big(\{(\alpha,\beta)\in\mathbb{R}^2; H(\ell)\alpha<\varepsilon,\beta>L\}\big) \text{ for all } L\in\mathbb{R} \text{ and } \varepsilon>0.$$

Hence $s^\beta\omega^\ell(s;\alpha) \in \mathcal{F}^\infty_L\big(\{(\alpha,\beta)\in\mathbb{R}^2; H(\ell)\alpha<\varepsilon,\beta>L+|\ell|\varepsilon\}\big)$ for all $L\in\mathbb{R}$ and $\varepsilon>0$. Thus, by (a) in Lemma A.3, the function $(s,\alpha,\beta)\mapsto s^\beta\omega^\ell(s;\alpha)$ belongs to

$$\begin{aligned}\bigcap_{\varepsilon>0}\mathcal{F}^\infty_L\big(\{(\alpha,\beta)\in\mathbb{R}^2; H(\ell)\alpha<\varepsilon,\beta>L+|\ell|\varepsilon\}\big) &= \mathcal{F}^\infty_L\left(\bigcup_{\varepsilon>0}\Big\{(\alpha,\beta)\in\mathbb{R}^2; H(\ell)\alpha<\varepsilon,\beta>L+|\ell|\varepsilon\Big\}\right)\\ &= \mathcal{F}^\infty_L\big(\{(\alpha,\beta)\in\mathbb{R}^2;\beta>L+\ell^+\alpha^+\}\big),\end{aligned}$$

where once again the second equality follows by distinguishing the cases $\ell > 0$ and $\ell < 0$. This proves assertion (c) for $\ell \neq 0$. Finally assertion (d) follows by applying (c) in the present result and, in this order, (c), (b), (g) and (e) in Lemma A.3. This concludes the proof of the result. ■

Next we introduce the set of functions $\mathcal{I}_K(W)$ that we previously used in [8, 9, 11, 12] to describe the properties of the remainder $\mathcal{T}_L$ of the Dulac time. In this respect let us quote that Mourtada uses essentially the same definition in his study of the cyclicity of the hyperbolic polycycles (see for instance [13]). This set of functions is not used in the present paper and our aim is only to relate it with the set $\mathcal{F}_K^L(W)$ for completeness and reader's convenience.

Definition A.5. Consider $K \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$ and an open subset U of $\mathbb{R}^N$. Let $\mathscr{D} := s\partial_s$ be the *Euler operator* and consider some $\hat{\mu} \in U$. We say that $\psi(s;\mu) \in \mathscr{C}^K_{s>0}(U)$ belongs to the class $\mathcal{I}_K(\hat{\mu})$ if for each $k = 0, 1, \ldots, K$ there exists a neighbourhood V of $\hat{\mu}$ such that

$$\lim_{s\to 0^+} \mathscr{D}^k \psi(s;\mu) = 0 \text{ uniformly on } \mu \in V.$$

If W is a (not necesarily open) subset of U then we define $\mathcal{I}_K(W) = \bigcap_{\hat{\mu}\in W} \mathcal{I}_K(\hat{\mu})$. □

The following result shows that the remainder $\mathscr{D}_L$ in Theorem A and $\mathcal{T}_L$ in Theorem B can be written in terms of the class $\mathcal{I}_k(W)$, which is more suitable in order to perform the derivation-division algorithm.

Lemma A.6. *Let U be an open set of $\mathbb{R}^N$, $W \subset U$, $L \in \mathbb{R}$, $K \in \mathbb{Z}_{\geqslant 0} \cup \{+\infty\}$ and $\varepsilon > 0$. Then the inclusion $\mathcal{F}^K_{L+\varepsilon}(W) \subset s^L \mathcal{I}_K(W)$ holds.*

Proof. Clearly it suffices to show that $\mathcal{F}^K_{L+\varepsilon}(\hat{\mu}) \subset s^L \mathcal{I}_K(\hat{\mu})$ for any $\hat{\mu} \in W$ because then, by definition,

$$\mathcal{F}^K_{L+\varepsilon}(W) = \bigcap_{\hat{\mu}\in W} \mathcal{F}^K_{L+\varepsilon}(\hat{\mu}) \subset \bigcap_{\hat{\mu}\in W} s^L \mathcal{I}_K(\hat{\mu}) \subset s^L \bigcap_{\hat{\mu}\in W} \mathcal{I}_K(\hat{\mu}) = s^L \mathcal{I}_K(W).$$

So fix $\hat{\mu} \in W$ and let us show that $\mathcal{F}^K_{L+\varepsilon}(\hat{\mu}) \subset s^L \mathcal{I}_K(\hat{\mu})$. To this end we note that one can easily verify by induction that for all $k \in \mathbb{Z}_{\geqslant 0}$ there exist $\eta_{ik} \in \mathbb{Z}_{\geqslant 0}$, $i = 0, 1, \ldots, k$, such that the identity

$$\mathscr{D}^k g(s;\mu) = \sum_{i=0}^{k} \eta_{ik} s^i \partial_s^i g(s;\mu)$$

holds for any $\mathscr{C}^k$-function g. On the other hand, if $\psi \in \mathcal{F}^K_{L+\varepsilon}(\hat{\mu})$ then for each $i = 0, 1, \ldots, K$ there exist a neighbourhood V_i of $\hat{\mu}$ and $C_i, s_i > 0$ such that $|\partial_s^i \psi(s;\mu)| \leqslant C_i s^{L+\varepsilon-i}$ for all $s \in (0, s_i)$ and $\mu \in V_i$. Thus, setting $\bar{V}_k := \cap_{i=0}^k V_i$, $\hat{s}_k := \min(s_i; i = 0, \ldots, k)$ and $\hat{C}_k := \sum_{i=0}^k \eta_{ik} C_i$, by applying the above identity we get that if $k = 0, 1, \ldots, K$ then

$$|\mathscr{D}^k \psi(s;\mu)| \leqslant \sum_{i=0}^{k} \eta_{ik} s^i |\partial_s^i \psi(s;\mu)| \leqslant \left(\sum_{i=0}^{k} \eta_{ik} C_i\right) s^{L+\varepsilon} = \hat{C}_k s^{L+\varepsilon} \text{ for all } s \in (0, \hat{s}_k) \text{ and } \mu \in \bar{V}_k.$$

Taking this into account, since

$$\mathscr{D}^i \left(s^{-L}\psi(s;\mu)\right) = \sum_{k=0}^{i} \binom{i}{k} \mathscr{D}^{i-k}(s^{-L}) \mathscr{D}^k \psi(s;\mu) = \sum_{k=0}^{i} \binom{i}{k} (-L)^{i-k} s^{-L} \mathscr{D}^k \psi(s;\mu),$$

we can assert that

$$\left|\mathscr{D}^i \left(s^{-L}\psi(s;\mu)\right)\right| \leqslant \sum_{k=0}^{i} \binom{i}{k} |L|^{i-k} \hat{C}_k s^{\varepsilon} = \tilde{C}_i s^{\varepsilon} \text{ for all } s \in (0, \tilde{s}_i) \text{ and } \mu \in \tilde{V}_i,$$

where $\tilde{V}_i := \cap_{k=0}^{i} \bar{V}_k$, $\tilde{s}_i := \min(\hat{s}_k; k=0,\dots,i)$ and $\tilde{C}_i := \sum_{k=0}^{i} \binom{i}{k} |L|^{i-k} \hat{C}_k$. It is clear that the above upper bound implies that $\lim_{s\to 0^+} \mathscr{D}^i \left(s^{-L}\psi(s;\mu)\right) = 0$ uniformly on $\mu \in \tilde{V}_i$ for $i = 0,1,\dots,K$, which implies $s^{-L}\psi(s;\mu) \in \mathcal{I}_K(\hat{\mu})$, as desired. This proves the validity of the result. ■

Corollary A.7. *For each $\ell \in \mathbb{Z}$, $(s,\alpha,\beta) \mapsto s^{\beta}\omega^{\ell}(s;\alpha)$ belongs to $s^L\mathcal{I}_{\infty}\big(\{(\alpha,\beta)\in\mathbb{R}^2; \beta > L + \ell^+\alpha^+\}\big)$ for all $L \in \mathbb{R}$.*

Proof. The result follows by noting that

$$s^{\beta}\omega^{\ell}(s;\alpha) \in \bigcap_{\varepsilon>0} \mathcal{F}^{\infty}_{L+\varepsilon}\big(\{(\alpha,\beta)\in\mathbb{R}^2; \beta > L+\varepsilon+\ell^+\alpha^+\}\big) \subset \bigcap_{\varepsilon>0} s^L\mathcal{I}_{\infty}\big(\{(\alpha,\beta)\in\mathbb{R}^2; \beta > L+\varepsilon+\ell^+\alpha^+\}\big)$$

$$= s^L\mathcal{I}_{\infty}\left(\bigcup_{\varepsilon>0}\left\{(\alpha,\beta)\in\mathbb{R}^2; \beta > L+\varepsilon+\ell^+\alpha^+\right\}\right) = s^L\mathcal{I}_{\infty}\big(\{(\alpha,\beta)\in\mathbb{R}^2; \beta > L+\ell^+\alpha^+\}\big),$$

where we apply firstly Lemma A.4 and secondly Lemma A.6. ■

B Differentiation formulas and integration of series

In this section, for reader's convenience, we state some specific results from analysis and calculus that we use all along. To begin with, since we use several times the multivariate Faa di Bruno formula to calculate the derivative of a composition of functions, we provide its explicit expression according to [3, Theorem 2.1]. To this end some notation is needed. If $\boldsymbol{\nu} = (\nu_1,\dots,\nu_d) \in \mathbb{Z}^d_{\geq 0}$ and $\mathbf{x} = (x_1,\dots,x_d) \in \mathbb{R}^d$ then we define

$$|\boldsymbol{\nu}| = \sum_{i=1}^{d} \nu_i, \quad \boldsymbol{\nu}! = \prod_{i=1}^{d}(\nu_i!), \quad \partial^{\boldsymbol{\nu}}_{\mathbf{x}} = \frac{\partial^{|\boldsymbol{\nu}|}}{\partial^{\nu_1}_{x_1}\cdots\partial^{\nu_d}_{x_d}} \text{ and } \mathbf{x}^{\boldsymbol{\nu}} = \prod_{i=1}^{d} x_i^{\nu_i}.$$

Moreover, if $\boldsymbol{\ell} = (\ell_1,\dots,\ell_d) \in \mathbb{Z}^d_{\geq 0}$, we write $\boldsymbol{\ell} \leqslant \boldsymbol{\nu}$ provided $\ell_i \leqslant \nu_i$ for $i = 1,\dots,d$. Let $f(y_1,\dots,y_m)$ and $g^{(1)}(x_1,\dots,x_d),\dots,g^{(m)}(x_1,\dots,x_d)$ be real-valued functions and set

$$h(x_1,\dots,x_d) = f\left(g^{(1)}(x_1,\dots,x_d),\dots,g^{(m)}(x_1,\dots,x_d)\right).$$

Theorem B.1 (Multivariate Faa di Bruno formula)**.** *Let $\boldsymbol{\nu} = (\nu_1,\dots,\nu_d) \in \mathbb{Z}^d_{\geq 0}$ with $|\boldsymbol{\nu}| > 0$ and $\mathbf{x^0} \in \mathbb{R}^d$ be given. Suppose that all the partial derivatives $\partial^{\boldsymbol{\ell}}_{\mathbf{x}}$ with $\boldsymbol{\ell} \leqslant \boldsymbol{\nu}$ of $g^1,\dots,g^m$ exist and are continuous in a neighbourhood of $\mathbf{x^0}$. Assume moreover that all the partial derivatives $\partial^{\boldsymbol{\lambda}}_{\mathbf{y}} f(\mathbf{y})$, with $\boldsymbol{\lambda} \in \mathbb{Z}^m_{\geq 0}$ and $|\boldsymbol{\lambda}| \leqslant |\boldsymbol{\nu}|$, exist and are continuous in a neighbourhood of $\big(g^1(\mathbf{x^0}),\dots,g^m(\mathbf{x^0})\big) \in \mathbb{R}^m$. Then $\partial^{\boldsymbol{\nu}}_{\mathbf{x}} h(\mathbf{x})$ exits in a neighbourhood of $\mathbf{x^0}$ and it is given by*

$$h_{\boldsymbol{\nu}}(\mathbf{x}) = \sum_{1\leqslant|\boldsymbol{\lambda}|\leqslant|\boldsymbol{\nu}|} f_{\boldsymbol{\lambda}}(g(\mathbf{x})) \sum_{p(\boldsymbol{\nu},\boldsymbol{\lambda})} (\boldsymbol{\nu}!) \prod_{i=1}^{q} \frac{(g_{\boldsymbol{\ell}_i}(\mathbf{x}))^{\mathbf{k}_i}}{(\mathbf{k}_i!)(\boldsymbol{\ell}_i!)^{|\mathbf{k}_i|}},$$

where

$$p(\boldsymbol{\nu},\boldsymbol{\lambda}) = \left\{(\mathbf{k}_1,\dots,\mathbf{k}_q;\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_q) : \sum_{i=1}^{q}\mathbf{k}_i = \boldsymbol{\lambda} \text{ and } \sum_{i=1}^{q}|\mathbf{k}_i|\boldsymbol{\ell}_i = \boldsymbol{\nu}\right\}. \tag{37}$$

In the statement $\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_q \in \mathbb{Z}^d_{\geq 0}$ is a complete listing of all vectors $\boldsymbol{\ell} \leqslant \boldsymbol{\nu}$ with $|\boldsymbol{\ell}| > 0$, $\mathbf{k}_1,\dots,\mathbf{k}_q \in \mathbb{Z}^m_{\geq 0}$ and $q = -1+\prod_{i=1}^{d}(\nu_i+1)$. We also set $h_{\boldsymbol{\nu}}(\mathbf{x}) = \partial^{\boldsymbol{\nu}}_{\mathbf{x}} h(\mathbf{x})$, $f_{\boldsymbol{\lambda}}(\mathbf{y}) = \partial^{\boldsymbol{\lambda}}_{\mathbf{y}} f(\mathbf{y})$ and $g_{\boldsymbol{\ell}}(\mathbf{x}) = \Big(g^{(1)}_{\boldsymbol{\ell}}(\mathbf{x}),\dots,g^{(m)}_{\boldsymbol{\ell}}(\mathbf{x})\Big)$ where $g^{(i)}_{\boldsymbol{\ell}}(\mathbf{x}) = \partial^{\boldsymbol{\ell}}_{\mathbf{x}} g^{(i)}(\mathbf{x})$.

We will also appeal to the following Leibniz formula for the partial derivatives of a product of functions (see for instance [2, Theorem C, p. 132]).

Theorem B.2. *If $f_1,\dots,f_r \in \mathscr{C}^\infty(U)$ for some open subset U of $\mathbb{R}^d$ and $\boldsymbol{\nu} \in \mathbb{Z}^d_{\geq 0}$ then*

$$\partial^{\boldsymbol{\nu}} \prod_{i=1}^r f_i = \sum_{\boldsymbol{\ell}_1+\ldots+\boldsymbol{\ell}_r=\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_r} \prod_{i=1}^r \partial^{\boldsymbol{\ell}_i} f_i,$$

where $\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_r \in \mathbb{Z}^d_{\geq 0}$ and $\binom{\boldsymbol{\nu}}{\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_r} := \frac{\boldsymbol{\nu}!}{\boldsymbol{\ell}_1!\cdots\boldsymbol{\ell}_r!} = \prod_{i=1}^d \frac{\nu_i!}{\ell_{1i}!\cdots\ell_{ri}!}$.

Remark B.3. The *generalized multinomial coefficients* $\binom{\boldsymbol{\nu}}{\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_r}$ satisfy

$$r^{|\boldsymbol{\nu}|} = \prod_{i=1}^d \left(\sum_{j=1}^r 1\right)^{\nu_i} = \prod_{i=1}^d \left(\sum_{\boldsymbol{\ell}_1+\ldots+\boldsymbol{\ell}_r=\boldsymbol{\nu}} \frac{\nu_i!}{\ell_{1i}!\cdots\ell_{ri}!}\right) = \sum_{\boldsymbol{\ell}_1+\ldots+\boldsymbol{\ell}_r=\boldsymbol{\nu}} \prod_{i=1}^d \frac{\nu_i!}{\ell_{1i}!\cdots\ell_{ri}!} = \sum_{\boldsymbol{\ell}_1+\ldots+\boldsymbol{\ell}_r=\boldsymbol{\nu}} \binom{\boldsymbol{\nu}}{\boldsymbol{\ell}_1,\dots,\boldsymbol{\ell}_r}$$

thanks to the multinomial identity (see [2, Theorem B, p 28])

$$\left(\sum_{i=1}^m x_i\right)^n = \sum \frac{n!}{a_1!\cdots a_m!} x_1^{a_1}\cdots x_m^{a_m},$$

where the summation takes place over all $(a_1,\dots,a_m) \in \mathbb{Z}^m_{\geq 0}$ such that $a_1+\ldots+a_m = n$. □

The following result is also well-known (see [17, Theorem 7.17] for instance).

Lemma B.4. *Suppose that $\{f_n\}$ is a sequence of functions, differentiable on $[a,b]$ and such that $\{f_n(x_0)\}$ converges for some point $x_0 \in [a,b]$. If $\{f_n'\}$ converges uniformly on $[a,b]$, then $\{f_n\}$ converges uniformly on $[a,b]$ to a function f such that*

$$f'(x) = \lim_{n\to\infty} f_n'(x) \text{ for all } x \in [a,b].$$

Lemma B.5. *Let E be a measurable set of $\mathbb{R}$ and consider a sequence of measurable functions $\{f_n\}_{n\in\mathbb{N}}$. If $\sum_{n\geqslant 1}\int_E |f_n(x)|dx < +\infty$ then $\int_E \sum_{n\geqslant 1} f_n(x)dx = \sum_{n\geqslant 1}\int_E f_n(x)dx$.*

Proof. The problem is to show that

$$\lim_{k\to+\infty} \int_E \psi_k(x)dx = \int_E \lim_{k\to+\infty} \psi_k(x)dx, \text{ where } \psi_k(x) := \sum_{n=1}^k f_n(x) \text{ for each } k \in \mathbb{N},$$

and this follows by the Lebesgue's dominated convergence theorem (see [17, Theorem 11.32]) because

$$|\psi_k(x)| \leqslant \sum_{n=1}^k |f_n(x)| \leqslant \sum_{n=1}^{+\infty} |f_n(x)| =: \Psi(x) \text{ for all } k \in \mathbb{N}$$

and, on the other hand, $\int_E \Psi(x)dx < +\infty$ by hypothesis. In this regard let us remark that, due to $|f_n| \geqslant 0$ for all $n \in \mathbb{N}$, the equality $\sum_{n\geqslant 1}\int_E |f_n(x)|dx = \int_E \sum_{n\geqslant 1}|f_n(x)|dx$ holds (see [17, Theorem 11.30]). ■

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