This is a preprint of: "Around some extensions of Casas-Alvero conjecture for non-polynomial functions", Anna Cima, Armengol Gasull, Francesc Mañosas, *Extracta Math.*, vol. 35(2), 221–228, 2020.
DOI: [10.17398/2605-5686.35.2.221]

Around some extensions of Casas-Alvero conjecture for non-polynomial functions

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Abstract. We show that two natural extensions of the real Casas-Alvero conjecture in the non-polynomial setting do not hold.

2010 MSC: Primary: 30C15. Secondary: 12D10; 13P15; 26C10.
Keywords: polynomial, Casas-Alvero conjecture, zeroes of functions

1 Introduction and main results

The Casas-Alvero conjecture affirms that if a complex polynomial P of degree $n>1$ shares roots with all its derivatives, $P^{(k)}, k=1,2\ldots,n-1$, then there exist two complex numbers, a and $b\neq 0$, such that $P(z)=b(z-a)^n$. Notice that, in principle, the common root between P and each $P^{(k)}$ might depend on k. Casas-Alvero arrived to this problem at the turn of this century, when he was working in his paper [1] trying to obtain an irreducibility criterion for two variable power series with complex coefficients. See [2] for an explanation of the problem in his own words.

Although several authors have got partial answers, to the best of our knowledge the conjecture remains open. For $n\leq 4$ the conjecture is a simple consequence of the wonderful Gauss-Lucas Theorem ([6]). In 2006 it was proved in [5], by using Maple, that it is true for $n\leq 8$. Afterwards in [6, 7] it was proved that it holds when n is $p^m, 2p^m, 3p^m$ or $4p^m$, for some prime number p and $m\in\mathbb{N}$. The first cases left open are those where $n=24,28$ or 30. See again [6] for a very interesting survey or [3, 8] for some recent contributions on this question.

Adding the hypotheses that P is a real polynomial and all its n roots, taking into account their multiplicities, are real, the conjecture has a real counterpart, that also remains open. It says that $P(x)=b(x-a)^n$ for some real numbers a and $b\neq 0$. For this real case, the conjecture can be proved easily for $n\leq 4$, simply by using Rolle's Theorem. This tool does not suffice for $n\geq 5$, see for instance [4] for more details, or next section.

Also in the real case, in [6] it is proved that if the condition for one of the derivatives of P is removed, then there exist polynomials satisfying the remaining $n-2$ conditions, different from $b(x-a)^n$. The construction of some of these polynomials presented in that paper is very nice and is a consequence of the Brouwer's fixed point Theorem in a suitable context.

Finally, it is known that if the conjecture holds in $\mathbb{C}$, then it is true over all fields of characteristic 0. On the other hand, it is not true over all fields of characteristic p, see again [7]. For instance, consider $P(x) = x^2(x^2+1)$ in characteristic 5 with roots $0, 0, 2$ and 3. Then $P'(x) = 2x(2x^2+1)$, $P''(x) = 12x^2 + 2 = 2(x^2+1)$ and $P'''(x) = 4x$ and all them share roots with P.

The aim of this note is to present two natural extensions of the real Casas-Alvero conjecture to smooth functions and show that none of them holds.

Question 1. *Fix $1 < n \in \mathbb{N}$. Let F be a class $\mathcal{C}^n$ real function such that $F^{(n)}(x) \neq 0$ for all $x \in \mathbb{R}$, and has n real zeroes, taking into account their multiplicities. Assume that F shares zeroes with all its derivatives, $F^{(k)}, k = 1, 2 \ldots, n-1$. Is it true that $F(x) = b(f(x))^n$ for some $0 \neq b \in \mathbb{R}$ and some f, a class $\mathcal{C}^n$ real function, that has exactly one simple zero?*

Notice that one of the hypotheses of the real Casas-Alvero conjecture can be reformulated as follows: The polynomial F shares roots with all its derivatives but one, precisely the one corresponding to its degree. Trivially, this is so, because all the derivatives of order higher than n are identically zero. The second question that we consider is:

Question 2. *Fix $1 < n \in \mathbb{N}$. Let F be a real analytic function that shares zeroes with all its derivatives but one, say $F^{(n)}$. Is it true that $F(x) = b(f(x))^n$ for some $0 \neq b \in \mathbb{R}$ and some real analytic function f, that has exactly one simple zero?*

Theorem A. (i) *The answer to the Question 1 is "yes" for $n \leq 4$ and "no" for $n = 5$.*

(ii) *The answer to the Question 2 is already "no" for $n = 2$.*

Our result reinforces the intuitive idea that Casas-Alvero conjecture is mainly a question related with the rigid structure of the polynomials.

2 Proof of Theorem A

(i) The answer to Question 1 is "yes" for $n = 2, 3, 4$ because the proof of the real Casas-Alvero conjecture for the same values of n, based on the Rolle's Theorem and given in [4], does not uses at all that P is a polynomial. Let us adapt it to our setting. Since $F^{(n)}$ does not vanish we know that F has exactly n real zeroes, taking into account their multiplicites. Moreover we know that F has to have at least a double zero, that without loss of generality can be taken as 0. Next we can do a case by case study to discard all situations except that F has only a zero and it is of multiplicity n. For the sake of brevity, we give all the details only in the most difficult case, $n = 4$.

Assume, to arrive to a contradiction, that $n = 4$, F is under the hypotheses of Question 1 and $x = 0$ is not a zero of multiplicity four. Notice that by Rolle's theorem, for $k = 1, 2, 3$, each $F^{(k)}$ has exactly $4 - k$ zeroes, taking into account their multiplicities. Moreover, the only zero of F''' must be one of the zeroes of F.

If $F''(0) = 0$ and $F'''(0) \neq 0$ then F has only another zero at $x = a$ and, without loss of generality, we can assume that $a > 0$. Applying three times Rolle's theorem we get that $F'''(b) = 0$ for some $b \in (0, a)$ which is a contradiction with the hypotheses. If $F''(0) \neq 0$ then F has two more zeroes counting multiplicities. There are three possibilities. The first one is that there is $a > 0$ such that $F(a) = F'(a) = 0$. In this case, applying two times Rolle's theorem we obtain that there exist $b, c \in (0, a)$ with $F''(b) = F''(c) = 0$ and they are the only zeroes of F''. This fact gives again a contradiction because none of them is a zero of F. The second one is that there exist $a_1, a_2 \in \mathbb{R}$ with $0 \in (a_1, a_2)$ such that $F(a_1) = F(a_2) = 0$. Also in this case, by applying two times Rolle's theorem we obtain that there exist $b, c \in (a_1, a_2)$ such that $0 \in (b, c)$ and $F''(b) = F''(c) = 0$ giving us the desired contradiction. Lastly, assume that the other two zeroes of F are a_1 and a_2, with $0 < a_1 < a_2$. By Rolle's Theorem the zeroes of F' are $0, b_1$ and b_2 and satisfy $0 < b_1 < a_1 < b_2 < a_2$. Then, since F'' has to have two zeroes, say c_1, c_2, and they satisfy $0 < c_1 < b_1 < c_2 < b_2$, the hypotheses force that $c_2 = a_1$. Hence the zero of F''' has to be between c_1 and $c_2 = a_1$, that is in particular in $(0, a_1)$, interval that contains no zero of F, arriving once more to the desired contradiction.

In short, we have proved for $n \leq 4$, that $F(x) = x^n G(x)$, for some class $\mathcal{C}^n$ function G, that does not vanish. Hence

$$F(x) = \operatorname{sign}(G(0)) \left(x \sqrt[n]{\frac{G(x)}{\operatorname{sign}(G(0))}} \right)^n = b(f(x))^n,$$

where f has only one zero, $x = 0$, that is simple, as we wanted to prove.

To find a map F for which the answer to Question 1 is "no" we consider $n = 5$ and a configuration of zeroes of F and its derivatives proposed in [4] as the simplest one, compatible with the hypotheses of the Casas-Alvero conjecture and Rolle's Theorem. Specifically, we will search for a function F, of class at least $\mathcal{C}^5$, with the five zeroes $0, 0, 1, c, d$, to be fixed, satisfying $0 < 1 < c < d$, and moreover

$$F'(0) = 0, \quad F''(1) = 0, \quad F'''(c) = 0, \quad F^{(4)}(1) = 0, \tag{1}$$

and such that $F^{(5)}$ does not vanish. Notice that $F'(0) = 0$ is not a new restriction.

We start assuming that $F^{(5)}(x) = r - \sin(x)$, for some $r > 1$ to be determined. By imposing that conditions (1) hold, together with $F(0) = 0$, we get that

$$F(x) = \int_0^x \int_0^u \int_1^w \int_c^z \int_1^y \big(r - \sin(t)\big)\, \mathrm{d}t\, \mathrm{d}y\, \mathrm{d}z\, \mathrm{d}w\, \mathrm{d}u.$$

Some straightforward computations give that

$$\begin{aligned} F(x) =& \frac{r}{120}x^5 - \frac{r + \cos(1)}{12}x^4 + \frac{2rc - 2\sin(c) + 2\cos(1)c - rc^2}{12}x^3 \\ &+ \frac{6\sin(c) + 2r + 9\cos(1) - 6rc + 3rc^2 - 6\cos(1)c}{12}x^2 - 1 + \cos(x). \end{aligned}$$

Imposing now that $F(1) = 0$ we obtain that

$$r = \frac{5\big(8\cos(1)c - 41\cos(1) - 8\sin(c) + 24\big)}{4(5c^2 - 10c + 4)} = R(c).$$

Next we have to impose that $F(c) = 0$. By replacing the above expression of r in F we obtain that

$$F(c) = \frac{G(c)}{96(5c^2 - 10c + 4)},$$

where

$$G(c) = -\,c^2\left(12\,c^4-369\,c^3+1437\,c^2-1708\,c+532\right)\cos(1)-8\,c^2\,(c-1)\,(c-2)^2\sin(c)$$
$$+\left(480\,c^2-960\,c+384\right)\cos(c)-24\,\left(c-1\right)\left(9\,c^4-36\,c^3+24\,c^2+24\,c-16\right).$$

A carefully study shows that G has exactly one real zero $c_1 \in (17/10, 19/10) = I$, with $c_1 \approx 1.79343096$. To prove its existence it suffices to show that

$$G\left(\frac{17}{10}\right) = -\frac{99211099}{500000}\cos(1)-\frac{18207}{12500}\sin\left(\frac{17}{10}\right)+\frac{696}{5}\cos\left(\frac{17}{10}\right)+\frac{1583211}{12500} > 0,$$
$$G\left(\frac{19}{10}\right) = -\frac{180110481}{500000}\cos(1)-\frac{3249}{12500}\sin\left(\frac{19}{10}\right)+\frac{1464}{5}\cos\left(\frac{19}{10}\right)+\frac{3616677}{12500} < 0.$$

By using Taylor's formula we know that for any $c>0$, $S^-(c) < \sin(c) < S^+(c)$ and $C^-(c) < \cos(c) < C^+(c)$ where

$$S^{\pm}(c) = c-\frac{c^3}{3!}+\frac{c^5}{5!}-\frac{c^7}{7!}+\frac{c^9}{9!}\pm\frac{c^{11}}{11!} \quad \text{and} \quad C^{\pm}(c) = 1-\frac{c^2}{2!}+\frac{c^4}{4!}-\frac{c^6}{6!}+\frac{c^8}{8!}\pm\frac{c^{10}}{10!}.$$

Hence we can replace the values of the trigonometric functions in G by rational numbers to have upper or lower bounds of this function evaluated at 1, 17/10 or 19/10. For instance,

$$0.5403023 \approx \frac{1960649}{3628800} = C^-(1) < \cos(1) < C^+(1) = \frac{280093}{518400} \approx 0.5403028.$$

We obtain that

$$\begin{aligned} G\left(\frac{17}{10}\right) &> -\frac{99211099}{500000}\,C^+(1)-\frac{18207}{12500}\,S^+\left(\frac{17}{10}\right)+\frac{696}{5}\,C^-\left(\frac{17}{10}\right)+\frac{1583211}{12500} \\ &= \frac{34446000995619698569 69}{4989600000000000000000} > 0 \end{aligned}$$

and

$$\begin{aligned} G\left(\frac{19}{10}\right) &< -\frac{180110481}{500000}\,C^-(1)-\frac{3249}{12500}\,S^-\left(\frac{19}{10}\right)+\frac{1464}{5}\,C^+\left(\frac{19}{10}\right)+\frac{3616677}{12500} \\ &= -\frac{1689627895469649855823}{1663200000000000000000} < 0. \end{aligned}$$

To show the uniqueness of the zero in I, we will prove that G is strictly decreasing in this interval. It holds that

$$G'(c) = T(c)\cos(1)+U(c)\sin(c)+V(c)\cos(c)+W(c),$$

with

$$U(c) = -\,8\,\left(5\,c^2-10\,c+4\right)\left(c^2-2\,c+12\right), \qquad V(c) = -8\,(c-1)\left(c^4-4\,c^3+4\,c^2-120\right),$$
$$T(c) = -\,c\left(72\,c^4-1845\,c^3+5748\,c^2-5124\,c+1064\right), \quad W(c) = -120(9c^4-36c^3+36c^2-8).$$

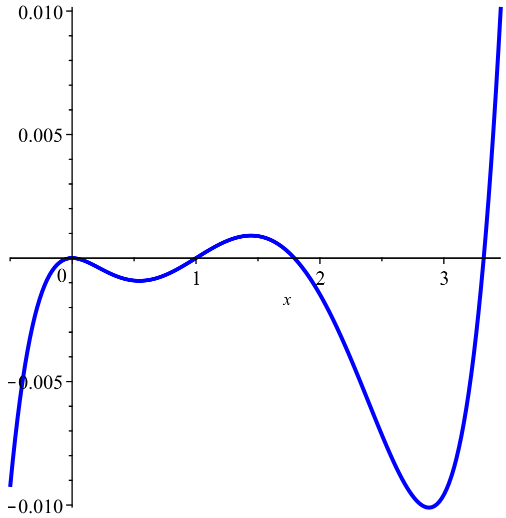

Figure 1: Plot of a map F for which the answer to Question 1 for $n = 5$ is "no".

By computing the Sturm sequences of T, U and V we can prove that $T(c) < 0$, $U(c) < 0$ and $V(c) > 0$ for all $c \in I$. Hence, for $c \in I$,

$$G'(c) < T(c)C^-(c) + U(c)S^-(c) + V(c)C^+(c) + W(c) = Q(c),$$

where

$$\begin{aligned} Q(c) =& \frac{72469}{64800}\,c - \frac{669211}{43200}\,c^2 + \frac{18852329}{302400}\,c^3 - \frac{8854991}{80640}\,c^4 + \frac{4732471}{50400}\,c^5 - \frac{532}{15}\,c^6 + \frac{8}{7}\,c^7 + \frac{191}{70}\,c^8 \\ &- \frac{797}{1890}\,c^9 - \frac{34}{405}\,c^{10} + \frac{1651}{103950}\,c^{11} + \frac{3533}{2494800}\,c^{12} - \frac{193}{623700}\,c^{13} + \frac{1}{142560}\,c^{14} - \frac{1}{831600}\,c^{15}. \end{aligned}$$

The Sturm sequence of Q shows that it has no zeroes in I. Moreover, it is negative in this interval, and as a consequence, G' is also negative, as we wanted to prove.

We fix $c = c_1$. Then, $r = R(c_1)$ and F is also totally fixed. Moreover, by using the same techniques we get that $r = R(c_1) > R(19/10) > 1$ and as a consequence $F^{(5)}$ does not vanish. In fact, $r = R(c_1) \approx 1.04591089$. Finally, F has one more real zero $d \in (33/10, 34/10)$. In fact, $d \approx 3.32178369$. This F gives our desired example, see Figure 1.

(ii) Consider $F(x) = 4x^2 + \pi^2(\cos(x) - 1)$ that has a double zero at 0 and also vanishes at $\pm\pi/2$. Moreover, $F'(x) = 8x - \pi^2 \sin(x)$ vanishes at $x = 0$, $F''(x) = 8 - \pi^2 \cos(x)$ has no common zeroes with F and, for any $k > 1$, $|F^{(2k)}(x)| = \pi^2|\cos(x)|$ vanishes at $x = \pi/2$ and $|F^{(2k-1)}(x)| = \pi^2|\sin(x)|$ vanishes at $x = 0$.

A similar example for $n = 3$ is $F(x) = 4x^3 - 6\pi x^2 + \pi^3(1 - \cos(x))$, that vanishes at $0, \pi$ (double zeroes) and $\pi/2$.

Acknowledgements

The authors are supported by Ministerio de Ciencia, Innovación y Universidades of the Spanish Government through grants MTM2016-77278-P (MINECO/AEI/FEDER, UE, first and second authors) and (MTM2017-86795-C3-1-P third author). The three authors are also supported by the grant 2017-SGR-1617 from AGAUR, Generalitat de Catalunya.

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