This is a preprint of: "The center problem for the class of Λ–Ω differential systems", Jaume Llibre, Rafael Orlando Ramírez, Valentín Ramírez, *Rend. Circ. Mat. Palermo (2)*, 2020.
DOI: [10.1007/s12215-020-00568-5]

The center problem for the class of $\Lambda - \Omega$ differential systems

Jaume Llibre · Rafael Ramírez ·
Valentín Ramírez

Abstract The center problem, i.e. distinguish between a focus and a center is a classical problem in the qualitative theory of planar differential equations which go back to Darboux, Poincaré and Liapunov. Here we solve the center problem for the class of planar analytic or polynomial differential systems

$$\dot{x} = -y + X = -y + \sum_{j=2}^{k} X_j, \quad \dot{y} = x + Y = x + \sum_{j=2}^{k} Y_j, \quad k \leq \infty,$$

where $X_j = X_j(x, y)$ and $Y_j = Y_j(x, y)$ are homogenous polynomials of degree $j > 1$, under the condition

$$(x^2 + y^2)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = \mu\,(xX + yY) \quad \text{with} \quad \mu \in \mathbb{R} \setminus \{0\}.$$

This work is supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and PID2019-104658GB-I00 (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.
J. Llibre is the corresponding author

J. Llibre
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.
Tel.: +34 5811303
Fax: +34 5812790
E-mail: jllibre@mat.uab.cat

R. Ramírez
Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, Avinguda dels Països Catalans 26, 43007 Tarragona, Catalonia, Spain.
E-mail: rafaelorlando.ramirez@urv.cat

V. Ramírez
Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain.
E-mail: valentin.ramirez@e-campus.uab.cat

Moreover we prove that these centers are weak centers, additionally, we provide their first integrals.

Keywords First integral · Poincaré-Liapunov first integral · analytic planar differential system · polynomial differential system · weak center

Mathematics Subject Classification (2010) 34C05 · 34C25

1 Introduction and main results

Let $\mathcal{X} = (-y + X)\dfrac{\partial}{\partial x} + (x + Y)\dfrac{\partial}{\partial y}$ be the real planar analytic or polynomial vector field associated to the real planar differential system

$$\dot{x} = -y + X = -y + \sum_{j=2}^{k} X_j, \quad \dot{y} = x + Y = x + \sum_{j=2}^{k} Y_j, \quad \text{for} \quad k \leq \infty \tag{1}$$

where $X_j = X_j(x, y)$ and $Y_j = Y_j(x, y)$ are homogenous polynomials of degree j. The *Poincaré center-focus problem* asks about conditions on the coefficients of X and Y under which all trajectories of system (1) contained in a small open neighborhood of the origin are closed, of course with exception of the origin. One of the mechanism to solve the center-focus problem is the following result due to Poincaré and Liapunov.

Theorem 1 *A planar analytic or polynomial differential system* (1) *has a center at the origin if and only if it has a first integral of the form*

$$H = \sum_{j=2}^{\infty} H_j(x, y) = \frac{1}{2}(x^2 + y^2) + \sum_{j=3}^{\infty} H_j(x, y), \tag{2}$$

where H_j are homogenous polynomials of degree j.

The first integral H is called the Poincaré-Liapunov first integral.

A center is called a *weak center* if the Poincaré-Liapunov first integral satisfies

$$H = \frac{x^2 + y^2}{2}(1 + h.o.t.) := H_2\Phi.$$

The next theorem is proved in [4].

Theorem 2 *A center of an analytic (polynomial)* (2) *differential system is a weak center if and only if* (2) *can be written as*

$$\begin{aligned} \dot{x} &= -y(1 + \Lambda(x, y)) + x\Omega(x, y), \\ \dot{y} &= x(1 + \Lambda(x, y)) + y\Omega(x, y). \end{aligned} \tag{3}$$

Differential system (3) is called $\Lambda - \Omega$ differential equation. Another well-know mechanism to solve the center-focus problem is the *Reeb criterium.*

Theorem 3 *[Reeb's criterion] (see for instance [8]). The analytic differential system* (1) *has a center at the origin if and only if there is a local nonzero analytic inverse integrating factor of the form* $V = 1 + \sum_{j=1}^{\infty} g_j(x,y)$, *called in what follows the Reeb inverse integrating factor, in a neighborhood of the origin, where* $g_j = g_j(x,y)$ *is the homogenous polynomial of degree* $j > 0$.

The following result is proved in [7].

Corollary 1 *Assuming that differential system* (1) *has a center at the origin. Then the Poincaré-Liapunov first integral* H *can be written as*

$$H = \sum_{n=1}^{\infty} (xY_n - yX_n) \left(\frac{1}{n+1} - \sum_{k=1}^{\infty} \frac{T_k}{k+n+1} \right), \tag{4}$$

where T_k *is a homogenous polynomial for* $k \geq 1$ *such that*

$$\frac{1}{V} = \frac{1}{1 + \sum_{j=2}^{\infty} g_j} = 1 - \sum_{j=1}^{\infty} T_j, \qquad T_k = g_k - \sum_{j=1}^{k-1} g_j T_{k-j},$$

for $k \geq 1$ *where* $T_1 = g_1$, *and* V *is the Reeb inverse integrating factor.*

In particular if the vector field $\mathcal{X}$ *is polynomial of degree* m *then* (4) *becomes*

$$H = \sum_{n=1}^{m} (xY_n - yX_n) \left(\frac{1}{n+1} - \sum_{k=1}^{\infty} \frac{T_k}{k+n+1} \right).$$

Moreover if the center is a weak center then

$$H = H_2 \sum_{n=1}^{\infty} 2\Lambda_{n-1}(x,y) \left(\frac{1}{n+1} - \sum_{k=1}^{\infty} \frac{T_k}{k+n+1} \right),$$

where $\Lambda_0 = 1$.

A *partial integral* of the vector field $\mathcal{X}$ is a differentiable function $G : D \longrightarrow \mathbb{R}$ where D is an open subset of $\mathbb{R}^2$ satisfying

$$\mathcal{X}(G) = P\frac{\partial G}{\partial x} + Q\frac{\partial G}{\partial y} = KG,$$

with K a function of the same class than G.

We say that an analytic (polynomial) vector field $\mathcal{X}$ is *quasi–Darboux integrable* if there exist r polynomial partial integrals $g_1, \ldots, g_r \in \mathbb{R}[x,y]$ and s non-polynomial C^r with $r > 0$ partial integrals where $K_j = K_j(x,y)$ is a convenient analytic (polynomial) for $j = 1, \ldots, s$ such that the function

$$F = e^{k(x,y)/h(x,y)} g_1^{\lambda_1}(x,y) \ldots\, g_r^{\lambda_r}(x,y) f_1^{\kappa_1}(x,y) \ldots\, f_s^{\kappa_s}(x,y),$$

is a first integral, where $k = k(x,y)$, $h = h(x,y)$ are analytic (polynomial) functions, and $\lambda_1, \ldots, \lambda_r$, $\kappa_1, \ldots, \kappa_s$, are complex constants.

In [4] the following conjecture was stated.

Conjecture 1. *Any analytic (polynomial) vector field with a weak center at the origin is quasi–Darboux integrable.*

This conjecture is supported in particular by the results given in the proof of the next Theorem 5 and the following two results. Note that the first result is a local one.

Theorem 4 *Any analytic (polynomial) vector field with a weak center at the origin is locally quasi–Darboux integrable.*

Theorem 4 is proved in section 2

The weak conditions provided by Alwash and Lloyd [1] give sufficient conditions for the existence a center. More precisely, they proved:

Proposition 1 *The origin of coordinates of the polynomial differential system*

$$\dot{x} = -y + X_m, \quad \dot{y} = x + Y_m, \tag{5}$$

where X_m and Y_m are homogenous polynomial of degree m is a center if there exists $\mu \in \mathbb{R} \setminus \{0\}$ such that

$$(x^2+y^2)\left(\frac{\partial(-y+X_m)}{\partial x} + \frac{\partial(x+Y_m)}{\partial y}\right) = \mu\left(x(-y+X_m) + y(x+Y_m)\right), \tag{6}$$

and either $m = 2k$; or $m = 2k-1$ and $\mu \neq 2k$; or $m = 2k-1$, $\mu = 2k$ and $\displaystyle\int_0^{2\pi} \left(\frac{\partial X_{2k-1}}{\partial x} + \frac{\partial Y_{2k-1}}{\partial y}\right)\bigg|_{x=\cos t,\, y=\sin t} dt = 0.$

The main objective of this paper is to extend and improve the result of Proposition 1 to analytic and polynomial differential systems. Thus our two main results are the following ones.

Theorem 5 *Differential system* (1)*, satisfying the condition*

$$(x^2+y^2)\left(\frac{\partial(-y+X)}{\partial x} + \frac{\partial(x+Y)}{\partial y}\right) = \mu\left(x(-y+X) + y(x+Y)\right), \tag{7}$$

where $\mu \in \mathbb{R} \setminus \{0\}$, has a weak center at the origin if and only if:

For the analytic systems one of the following conditions holds

(i) $\mu \notin \mathbb{N} \setminus \{1\}$,
(ii) $\mu = 2$,
(iii) $\mu = 2k+1 \in \mathbb{N}$
(iV) $\mu = 2k \in \mathbb{N}\setminus\{2\}$, *and*

$$\beta_{2k-1} := \int_0^{2\pi} \left(\frac{\partial X_{2k-1}}{\partial x} + \frac{\partial Y_{2k-1}}{\partial y}\right)\bigg|_{x=\cos t,\, y=\sin t} dt = 0. \tag{8}$$

For the polynomial systems of degree m one of the following conditions holds

(i) $\mu \notin \{1, 3, \ldots, m+1\}$,
(ii) $\mu = 2$,
(iii) $\mu = 2k + 1 \in \{1, 2, \ldots, m+1\}$,
(iv) $\mu = 2k \in \{3, \ldots, m+1\}$ *and* $\beta_{2k-1} = 0$.

Moreover differential system (1) *satisfying* (7) *and having a weak center at the origin is quasi-Darboux integrable.*

Corollary 2 *Differential system* (5) *under condition* (6) *has a weak center at the origin if and only if*

(i) If $\mu \notin \{2, m+1\}$,
(ii) If $\mu = 2$,
(iii) If $\mu = 2k + 1 \in \{1, m+1\}$,
(iv) If $\mu = 2k = m + 1$ *and* $\beta_{2k-1} = 0$.

Theorem 5 and Corollary 2 are proved in section 3.

2 Preliminary results

In the proofs of our results it plays an important role the following propositions and corollaries.

The Poisson bracket of two functions f and g is defined as

$$\{f,\, g\} := \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}.$$

Proposition 2 *The next relation holds*

$$\int_0^{2\pi} \{H_2,\, \Psi\}|_{x=\cos t,\, y=\sin t}\, dt = \Psi(\cos 2\pi, \sin 2\pi) - \Psi(\cos 0, \sin 0)$$

for arbitrary C^1 function $\Psi = \Psi(x, y)$ defined in $U \subseteq \mathbb{R}^2$.

Proof Indeed,

$$\int_0^{2\pi} \{H_2, \Psi\}|_{x=\cos t,\, y=\sin t}\, dt = \int_0^{2\pi} \left(x\frac{\partial \Psi}{\partial\, y} - y\frac{\partial \Psi}{\partial\, x} \right)\Bigg|_{x=\cos t,\, y=\sin t} dt$$
$$= \int_0^{2\pi} \left(\dot{x}\frac{\partial \Psi}{\partial\, x} + \dot{y}\frac{\partial \Psi}{\partial\, y} \right)\Bigg|_{x=\cos t,\, y=\sin t} dt = \int_0^{2\pi} \frac{d}{dt}\left(\Psi(\cos t, \sin t)\right) dt$$
$$= \Psi(\cos 2\pi, \sin 2\pi) - \Psi(\cos 0, \sin 0).$$

The following corollary is due to Liapunov (see Theorem 1, page 276 of [3])

Corollary 3 *Let $U = U(x, y)$ be a homogenous polynomial of degree m. The linear partial differential equation*

$$x\frac{\partial V}{\partial y} - y\frac{\partial V}{\partial x} := \{H_2, V\} = U,$$

has a unique homogenous polynomial solution V of degree m if m is odd; and if V is a homogenous polynomial solution when m is even then any other homogenous polynomial solution is of the form $V + c(x^2+y^2)^{m/2}$ with $c \in \mathbb{R}$. Moreover, for m even these solutions exist if and only if $\int_0^{2\pi} U(x, y)|_{x=\cos t,\, y=\sin t}\, dt = 0$.

Proof (Proof of Theorem 4) Indeed, if the analytic vector field has a weak center at the origin then it admits a Poincaré-Liapunov local analytic first integral at the origin $H = H_2(1 + h.o.t) = H_2\Phi(x, y)$. From Theorem 2 we get that $\dot{H}_2 = 2H_2\Omega(x, y)$, i.e. H_2 is a partial integral with analytic cofactor $2\Omega(x, y)$. It is easy to show that the analytic function $\Phi(x, y)$ is an analytic partial integral with cofactor $-2\Omega(x, y)$. In short the theorem is proved.

In the proof of Theorem 5 and Corollary 2 we need the following two propositions which also appeared in the Ph.D. [7], and for completeness we prove them here.

Proposition 3 *Let*

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y) \tag{9}$$

be an analytic (polynomial) differential system . Then this system can be written as

$$\dot{x} = P = -\frac{\partial F}{\partial y} - y\, G, \quad \dot{y} = Q = \frac{\partial F}{\partial x} + x\, G. \tag{10}$$

where F and G are convenient analytic (polynomial) functions, if and only if

$$\int_0^{2\pi} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) |_{x=\cos t,\, y=\sin t} dt = 0. \tag{11}$$

holds.

Moreover if (11) holds then differential system (9) can be written as

$$\begin{aligned} \dot{x} &= -\frac{\partial \Phi}{\partial y} - y\frac{\tilde{\Lambda}}{\mu} + \frac{x}{\mu}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right), \\ \dot{y} &= \frac{\partial \Phi}{\partial x} + x\frac{\tilde{\Lambda}}{\mu} + \frac{y}{\mu}\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right), \end{aligned} \tag{12}$$

where $\Phi = \mu F + 2H_2G$, $\mu \in \mathbb{R}\backslash\{0\}$, and $\tilde{\Lambda}$ is a convenient function.

Proof Indeed, if (11) holds then there exist a function G such that

$$\{H_2,\, G\} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \Longleftrightarrow x\frac{\partial G}{\partial y} - y\frac{\partial G}{\partial x} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \Longleftrightarrow$$

$$\frac{\partial (Q - xG)}{\partial y} + \frac{\partial (P + yG)}{\partial x} = 0 \Longleftrightarrow P + yG = -\frac{\partial F}{\partial y}, \quad Q - xG = \frac{\partial F}{\partial x},$$

consequently (10) holds.

Assume that (10) holds then

$$\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} = \{H_2,\, G\} + \frac{\partial^2 \tilde{H}}{\partial y \partial x} - \frac{\partial^2 \tilde{H}}{\partial x \partial y},$$

and by considering that $\frac{\partial^2 \tilde{H}}{\partial y \partial x} = \frac{\partial^2 \tilde{H}}{\partial x \partial y}$ we get that

$$\{H_2,\, G\} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}.$$

Hence in view of the relation $\int_0^{2\pi} \{H_2, G\}\,|_{x=\cos t,\, y=\sin t} dt = 0$ we obtain that (11) holds. In short the proposition is proved.

Now we prove formula (12) for differential system (9) under the condition (11).

If (11) holds then from (10) we get that

$$\{H_2,\, G\} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}, \qquad \{F,\, H_2\} = xP + yQ.$$

So

$$\{\mu\, F + 2H_2 G,\, H_2\} = (x^2 + y^2)\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) - \mu\,(xP + yQ)\,.$$

Thus

$$x\left(\mu\, P - x\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) + \frac{\partial \Phi}{\partial y}\right) + y\left(\mu\, Q - y\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) - \frac{\partial \Phi}{\partial x}\right) = 0.$$

Consequently, by introducing the function $\Phi = \mu\, F + 2H_2 G$, we get that

$$\mu\, P - x\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) - \frac{\partial \Phi}{\partial y} = -\tilde{\Lambda}\, y,$$

$$\mu\, Q - y\left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) - \frac{\partial \Phi}{\partial x} = \tilde{\Lambda}\, x,$$

Thus formula (12) is proved. In short the proposition is proved.

Proposition 4 *The analytic differential system* (2) *satisfying condition* (7) *can be written as the* $\Lambda - \Omega$ *differential system*

$$\begin{aligned} \dot{x} &= -y(1+\Lambda) + \frac{x}{\mu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = -y(1+\Lambda) + x\Omega(x,y), \\ \dot{y} &= x(1+\Lambda) + \frac{y}{\mu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = x(1+\Lambda) + y\Omega(x,y). \end{aligned} \tag{13}$$

Where $\Lambda = \Lambda(x,y)$ *is a solution of the first order partial differential equation*

$$\{H_2, \Lambda\} = (\mu - 2)\Omega - x\frac{\partial \Omega}{\partial x} - y\frac{\partial \Omega}{\partial y}, \tag{14}$$

which in polar coordinates $x = r\cos\theta,\ y = r\sin\theta$ *becomes*

$$\frac{\partial}{\partial \theta}\left(\tilde{\Lambda}\right) = \frac{\partial}{\partial r}\left(\frac{\tilde{\Omega}}{r^{\mu-2}}\right)$$

where $\tilde{\Lambda} = \Lambda(r\cos\theta, r\sin\theta)$ *and* $\tilde{\Omega} = \Omega(r\cos\theta, r\sin\theta)$.

Proof Indeed, from (7) it follows that

$$x\left(\mu X - x\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\right) + y\left(\mu Y - y\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)\right) = 0.$$

Hence

$$\mu X - x\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = -\mu\Lambda, \qquad \mu Y - y\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = \mu\Lambda,$$

where $\Lambda = \Lambda(x,y)$ is a convenient function. Thus we obtain that

$$X = -y\Lambda + x\Omega, \qquad Y = x\Lambda + y\Omega,$$

where

$$\Omega = \frac{1}{\mu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right), \tag{15}$$

Thus we easily obtain (13). From these relations we get that

$$\begin{aligned} \frac{\partial X}{\partial x} &= -y\frac{\partial \Lambda}{\partial x} + \Omega + x\frac{\partial \Omega}{\partial x}, \\ \frac{\partial Y}{\partial y} &= x\frac{\partial \Lambda}{\partial y} + \Omega + y\frac{\partial \Omega}{\partial y}. \end{aligned}$$

Consequently the formula (14) it follows. Finally we observe that if

$$X = -\frac{\partial \Phi}{\partial y} - y\Lambda + x\Omega, \qquad Y = \frac{\partial \Phi}{\partial x} + x\Lambda + y\Omega, \qquad \text{with} \quad \Omega = \frac{1}{\mu}\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right)$$

where Ω is given by formula (15), and Λ is a solution of equation (14). Thus the proposition is proved.

A center O of system (1) is a *uniform isochronous center* if the equality $x\dot{y}-y\dot{x}=\kappa(x^2+y^2)$ holds for a nonzero constant κ; or equivalently in polar coordinates (r,θ) such that $x=r\cos\theta$, $y=r\sin\theta$, we have that $\dot{\theta}=\kappa$. Clearly that from Theorem 2 it follows that uniform isochronous centers are weak center.

Example 1 For differential system (5) under the condition (6) we get that differential system (13) and condition (14) becomes

$$\begin{aligned} \dot{x} &= -y(1+\mathit{\Lambda}_{m-1})+x\mathit{\Omega}_{m-1},\\ \dot{y} &= x(1+\mathit{\Lambda}_{m-1})+y\mathit{\Omega}_{m-1}, \\ \{H_2,\, \mathit{\Lambda}_{m-1}\} &= (\mu-m-1)\mathit{\Omega}_{m-1}, \end{aligned} \tag{16}$$

respectively. Consequently if $\mu-m-1\neq 0$, then system (16) can be written as

$$\dot{x}=-y(1+\mathit{\Lambda}_{m-1})+x\frac{\{H_2,\,\mathit{\Lambda}_{m-1}\}}{\mu-m-1},\qquad \dot{y}=x(1+\mathit{\Lambda}_{m-1})+y\frac{\{H_2,\,\mathit{\Lambda}_{m-1}\}}{\mu-m-1},$$

and if $\mu-m-1=0$, then

$$\dot{x}=-y(1+\mathit{\Lambda}_{m-1}(H_2))+x\mathit{\Omega}_{m-1},\qquad \dot{y}=x(1+\mathit{\Lambda}_{m-1}(H_2))+y\mathit{\Omega}_{m-1}. \tag{17}$$

Since the differential system (17) in polar coordinates writes

$$\dot{r}=r\mathit{\Omega}_{m-1}(r\cos\theta,r\sin\theta),\qquad \dot{\theta}=1+\mathit{\Lambda}(1/2),$$

we get that the weak center in this case is a uniform center.

3 Proof of Theorem 5 and Corollary 2

Proof of Theorem 5. Multiplying (7) by $(x^2+y^2)^{(\mu-2)/2}$ we get

$$V\left(\frac{\partial}{\partial y}(-y+X)+\frac{\partial}{\partial y}(x+Y)\right)=(-y+X)\frac{\partial V}{\partial x}+(x+Y)\frac{\partial V}{\partial y},$$

where $V=(x^2+y^2)^{\mu/2}$, or equivalently

$$\frac{\partial}{\partial x}\left(\frac{-y+X}{(x^2+y^2)^{\mu/2}}\right)+\frac{\partial}{\partial y}\left(\frac{x+Y}{(x^2+y^2)^{\mu/2}}\right)=0.$$

Hence there exists a function $\tilde{F}$ such that

$$\frac{-y+X}{(x^2+y^2)^{\mu/2}}=-\frac{\partial\tilde{F}}{\partial y},\qquad \frac{x+Y}{(x^2+y^2)^{\mu/2}}=\frac{\partial\tilde{F}}{\partial x}. \tag{18}$$

Thus

$$-y+X=-(x^2+y^2)^{\mu/2}\frac{\partial\tilde{F}}{\partial y},\quad x+Y=(x^2+y^2)^{\mu/2}\frac{\partial\tilde{F}}{\partial x}. \tag{19}$$

From this relation we easily obtain

$$\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}=\mu(x^2+y^2)^{(\mu-2)/2}\{\tilde{F},H_2\}.$$

Hence in view of Proposition 2 we get that

$$\begin{aligned}\int_0^{2\pi}\left(\frac{\partial X}{\partial x}+\frac{\partial Y}{\partial y}\right)\Big|_{x=\cos t,\,y=\sin t}dt=\\ \mu\left(\tilde{F}(\cos(2\pi),\,\sin(2\pi))-\tilde{F}(\cos(0),\,\sin(0))\right).\end{aligned}\tag{20}$$

On the other hand, from (19) and in view of Proposition 4 it follows

$$\begin{aligned}d\tilde{F}&=\left(\frac{x+Y}{(x^2+y^2)^{\mu/2}}\right)dx+\left(\frac{y-X}{(x^2+y^2)^{\mu/2}}\right)dy\\ &=\frac{(x(1+\Lambda(x,y))+y\Omega(x,y))}{(x^2+y^2)^{\mu/2}}dx+\frac{(y(1+\Lambda(x,y))-x\Omega(x,y))}{(x^2+y^2)^{\mu/2}}dy.\end{aligned}\tag{21}$$

From the condition $\dfrac{\partial^2\tilde{F}}{\partial y\partial x}=\dfrac{\partial^2\tilde{F}}{\partial x\partial y}$ we get that the function Λ must be satisfies the equation (14).

Clearly from (21) we obtain

$$d\tilde{F}=\frac{(1+\Lambda(x,y))(xdx+ydy)}{(x^2+y^2)^{\mu/2}}+\frac{\Omega(x,y)(y\,dx-x\,dy)}{(x^2+y^2)^{\mu/2}}.\tag{22}$$

First we study the analytic case. Thus we assume that $\Lambda(x,y)=\displaystyle\sum_{j=1}^{\infty}\Lambda_j(x,y)$ and $\Omega(x,y)=\displaystyle\sum_{j=1}^{\infty}\Omega_j(x,y)$, where $\Lambda_j(x,y)$ and $\Omega_j(x,y)$ are homogeneous polynomials of degree j. Equation (22) in polar coordinates $x=r\cos\theta,\ y=r\sin\theta$ becomes

$$df=\frac{1+\Lambda(r\cos\theta,r\sin\theta)}{r^{\mu-1}}dr-\frac{\Omega(r\cos\theta,r\sin\theta\,)}{r^{\mu-2}}d\theta,$$

where $f:=f(r,\theta)=\tilde{F}(r\cos\theta,r\sin\theta)$ and

$$\Lambda_j(r\cos\theta,r\sin\theta\,)=r^j\alpha_j(\theta),\ \ \Omega_j(r\cos\theta,r\sin\theta\,)=r^j\tau_j(\theta),\ \text{for}\quad j\in\mathbb{N},$$

or equivalently

$$df=\left(r^{1-\mu}+\sum_{j=1}^{\infty}r^{j+1-\mu}\alpha_j(\theta)\right)dr-\left(\sum_{j=1}^{\infty}r^{j+2-\mu}\tau_j(\theta)\right)d\theta.\tag{23}$$

Hence

$$\frac{\partial f}{\partial r}=r^{1-\mu}+\sum_{j=1}^{\infty}r^{j+1-\mu}\alpha_j(\theta),\quad \frac{\partial f}{\partial\theta}=-\sum_{j=1}^{\infty}r^{j+2-\mu}\tau_j(\theta),$$

From the compatibility conditions $\dfrac{\partial^2 f}{\partial r \partial \theta} = \dfrac{\partial^2 f}{\partial \theta \partial r}$ we get that

$$\frac{\partial}{\partial \theta}\left(\alpha_j(\theta)\right) = \alpha_j'(\theta) = -(j+2-\mu)\tau_j(\theta), \tag{24}$$

or equivalently $\dfrac{\partial}{\partial \theta}\left(r^j \alpha_j(\theta)\right) = -(j+2-\mu) r^j \tau_j(\theta)$, which in cartesian coordinates becomes

$$\begin{aligned} \{H_2, \Lambda_j\} &= -(j+2-\mu)\Omega_j \\ &= (\mu-2)\Omega_j - x\frac{\partial \Omega_j}{\partial x} - y\frac{\partial \Omega_j}{\partial y}, \end{aligned}$$

Assuming that $\mu \notin \mathbb{N}\setminus\{1\}$ then in (24) $j+2-\mu \neq 0$. Consequently from (26) we obtain that

$$\begin{aligned} df &= \left(r^{1-\mu} + \sum_{j=1}^{\infty} r^{j+1-\mu}\alpha_j(\theta)\right) dr + \left(\sum_{j=1}^{\infty} \frac{r^{j+2-\mu}}{j+2-\mu}\alpha_j'(\theta)\right) d\theta \\ &= d\left(\frac{r^{2-\mu}}{2-\mu} + \sum_{j=1}^{\infty} \frac{r^{j+2-\mu}}{j+2-\mu}\alpha_j(\theta)\right). \end{aligned}$$

Therefore

$$f = \frac{r^{2-\mu}}{2-\mu} + \sum_{j=1}^{\infty} \frac{r^{j+2-\mu}}{j+2-\mu}\alpha_j(\theta),$$

is a first integral, which in cartesian coordinates becomes

$$\tilde{F}(x,y) = \frac{\dfrac{1}{2-\mu} + \displaystyle\sum_{j=1}^{\infty} \frac{\Lambda_j(x,y)}{j+2-\mu}}{(x^2+y^2)^{(\mu-2)/2}}.$$

Therefore

$$H = \frac{1}{\tilde{F}(x,y)^{2/(\mu-2)}} = \frac{2H_2}{\left(\dfrac{1}{2-\mu} + \displaystyle\sum_{j=1}^{\infty} \frac{\Lambda_j(x,y)}{j+2-\mu}\right)^{2/(\mu-2)}}, \tag{25}$$

is an analytic first integral defined in a neighborhood of the origin, i.e. it is a Poincaré-Liapunov first integral. From the expression of this first integral we obtain that the origin is a weak center. Clearly that H_2 and $\dfrac{1}{2-\mu} + \displaystyle\sum_{j=1}^{\infty} \frac{\Lambda_j(x,y)}{j+2-\mu}$ are partial integrals with analytic cofactor 2Ω and $(\mu-2)\Omega$ respectively.

Now assume that $\mu \in \mathbb{N}\backslash\{2\}$. Let $j = \mu - 2$, then $\alpha_{\mu-2}(\theta) = \alpha_{\mu-2}$ =constant. From (26) and in view of (24) we obtain that

$$
\begin{aligned}
df =& \left(r^{1-\mu} + \frac{\alpha_{\mu-2}}{r} + \sum_{\substack{j=1 \\ j\neq\mu-2}}^{\infty} r^{j+1-\mu}\alpha_j(\theta) \right) dr \\
& - \left(\tau_{\mu-2}(\theta) + \sum_{\substack{j=1 \\ j\neq\mu-2}}^{\infty} r^{j+2-\mu}\tau_j(\theta) \right) d\theta \qquad (26) \\
=& \, d\left(\frac{r^{2-\mu}}{2-\mu} + \int \tau_{\mu-2}(\theta)d\theta + \alpha_{\mu-2} \log r + \sum_{j=1}^{\infty} \frac{r^{j+2-\mu}}{j+2-\mu}\alpha_j(\theta) \right).
\end{aligned}
$$

Thus

$$
f = \frac{r^{2-\mu}}{2-\mu} + \int \tau_{\mu-2}(\theta)d\theta + \alpha_{\mu-2} \log r + \sum_{j=1}^{\infty} \frac{r^{j+2-\mu}}{j+2-\mu}\alpha_j(\theta),
$$

or equivalently the function f is

$$
\begin{aligned}
& \frac{1}{r^{\mu-2}} \left(\frac{1}{2-\mu} + \alpha_{\mu-2} r^{\mu-2} \log r + \int r^{\mu-2}\tau_{\mu-2}(\theta)d\theta + \sum_{\substack{j=1 \\ j\neq\mu-2}}^{\infty} \frac{r^j}{j+2-\mu}\alpha_j(\theta) \right) \\
&= \frac{1}{r^{\mu-2}} \Big(\frac{1}{2-\mu} + \alpha_{\mu-2} r^{\mu-2} \log r + \int \varOmega_{\mu-1}(r\cos\theta, r\sin\theta)d\theta \\
&+ \sum_{\substack{j=1 \\ j\neq\mu-2}}^{\infty} \frac{\varLambda_j(r\cos\theta, r\sin\theta)}{j+2-\mu} \Big).
\end{aligned}
$$

We have that

$$
\int \varOmega_{\mu-2}(r\cos\theta, r\sin\theta)d\theta \Big|_{r=\sqrt{x^2+y^2},\ \theta=\arctan(y/x)} = \beta_{\mu-1} \arctan\frac{y}{x} + \varphi,
$$

where $\varphi := \varphi(x, y)$ is a polynomial of degree $\mu - 1$ and $\beta_{\mu-1}$ is a constant such that

$$
\begin{aligned}
\beta_{\mu-1} &= \int_0^{2\pi} \varOmega_{\mu-2}(\cos t, \sin t)dt = \int_0^{2\pi} \frac{1}{\mu} \left(\frac{\partial X_{\mu-1}}{\partial x} + \frac{\partial Y_{\mu-1}}{\partial y} \right) |_{x=\cos t,\, y=\sin t} dt \\
&= \begin{cases} 0 & \text{if } \mu = 2k+1, \\ \dfrac{1}{\mu}\displaystyle\int_0^{2\pi} \left(\frac{\partial X_{2k-1}}{\partial x} + \frac{\partial Y_{2k-1}}{\partial y} \right) |_{x=\cos t,\, y=\sin t} dt & \text{if } \mu = 2k, \end{cases}
\end{aligned}
$$

This first integral in cartesian coordinates becomes

$$\tilde{F}(x,y)=\frac{1}{(2H)^{(\mu-2)/2}}\Big(\frac{1}{2-\mu}+\varphi(x,y)+\beta_{\mu-1}\arctan\frac{y}{x}$$

$$+\frac{\Lambda_{\mu-2}(2H_2)}{2}\log H_2+\sum_{\substack{j=1\\ j\neq\mu-2}}^{\infty}\frac{\Lambda_j(r\cos\theta,r\sin\theta)}{j+2-\mu}\Big),$$

Therefore from (20) we obtain that

$$\tilde{F}(\cos(2\pi),\sin(2\pi))-\tilde{F}(\cos(0),\sin(0))=\frac{\beta_{\mu-1}\arctan(\tan(2\pi))}{2^{\mu/2-1}}.$$

Thus in order to obtain a center at the origin from (20) we need that $\beta_{\mu-1}=0$.

From (24) we get that $\alpha_{\mu-2}$ is a constant, i.e.

$$\sum_{j+k=\mu-2}a_{k\,j}\cos^k\theta\sin^j\theta=\alpha_{\mu-2}=\text{constant},$$

Hence $\Lambda_{\mu-2}=r^{\mu-2}\alpha_{\mu-2}$ thus

$$\Lambda_{\mu-2}=(2H)^{(\mu-2)/2}\alpha_{\mu-2}=\begin{cases}0 & \text{if } \mu\neq 2k,\\ (2H)^{k-1}\alpha_{2k-2} & \text{if } \mu=2k.\end{cases}$$

Hence

$$\tilde{H}=\left(\frac{1}{\tilde{F}(x,y)}\right)^{2/(\mu-2)}\Bigg|_{\beta_{\mu-1}=0}$$

$$=\frac{2H_2}{\left(\dfrac{1}{2-\mu}+\displaystyle\sum_{\substack{j=1\\ j\neq\mu-2}}^{\infty}\frac{\Lambda_j}{j+2-\mu}+\frac{\Lambda_{\mu-2}(H_2)}{2}\log(2H_2)+\varphi(x,y)\right)^{2/(\mu-2)}}.$$

Thus if $\mu\neq 2k$

$$H=\frac{H_2}{\left(\dfrac{1}{2-\mu}+\displaystyle\sum_{\substack{j=1\\ j\neq\mu-2}}^{\infty}\frac{\Lambda_j}{j+2-\mu}++\varphi(x,y)\right)^{2/(\mu-2)}},\tag{27}$$

and if $\mu=2k$ then

$$\tilde{H}=\frac{H_2}{\left(\dfrac{1}{2-2k}+\displaystyle\sum_{\substack{j=1\\ j\neq 2k-2}}^{\infty}\frac{\Lambda_j}{j+2-2k}+\frac{(2H_2)^{k-1}}{2}\log H_2+\varphi(x,y)\right)^{1/(k-1)}}.$$

When $\mu = 2k > 2$ the first integral $\tilde{H}$ is such that $\lim_{(x,y)\longrightarrow (0,0)} \tilde{H} = 0$ and it is not analytic. So we have a center, by considering that is a center of a $\Lambda - \Omega$ equation we obtain that the origin is a weak center. Evidently that H_2 and $\dfrac{1}{\mu-2} + \dfrac{(2H_2)^{k-1}}{2}\log H_2 + \varphi(x,y) + \sum_{\substack{j=1\\ j\neq 2k-2}}^{\infty} \dfrac{\Lambda_j}{j+2-\mu}$ are partial integrals with analytic cofactor 2Ω and $(\mu-2)\Omega$.

If $\mu \neq 2k$ then H is a first integral an analytic at the origin. Thus the origin is a weak center. Clearly that H_2 and $\dfrac{1}{\mu-2} + \sum_{\substack{j=1\\ j\neq \mu-2}}^{\infty} \dfrac{\Lambda_j}{j+2-\mu} + \varphi(x,y)$ are partial integrals with analytic cofactor 2Ω and $(\mu-2)\Omega$.

Finally we consider the case $\mu = 2$. Then from (26) and (24) we have that

$$df(r,\theta) = \left(r^{-1} + \sum_{j=1}^{\infty} r^{j-1}\alpha_j(\theta) \right) dr + \left(\sum_{j=1}^{\infty} \frac{r^j}{j}\alpha_j'(\theta) \right) d\theta$$

$$= d\left(\log r + \sum_{j=1}^{\infty} \frac{r^j}{j}\alpha_j \right) = d\left(\log r + \sum_{j=1}^{\infty} \frac{\Lambda_j}{j} \right).$$

Therefore the first integral $\tilde{F}(x,y)$ becomes

$$\tilde{F}(x,y) = \frac{1}{2}\log 2H_2 + \sum_{j=1}^{\infty} \frac{\Lambda_j}{j} = \frac{1}{2}\log\left(2H_2 \exp\left(2\sum_{j=1}^{\infty} \frac{\Lambda_j}{j} \right) \right).$$

The Poincaré-Liapunov first integral in this case is

$$H = H_2 \exp\left(2\sum_{j=1}^{\infty} \frac{\Lambda_j}{j} \right).$$

So if $\mu = 2$ then the origin is a weak center, and Theorem 5 is proved for the analytic case.

The condition in order that a polynomial differential system of degree m satisfying condition (7) has a center at the origin are obtained directly from conditions proved in the analytic case.

Polynomial differential system of degree m satisfying condition (7) has a weak center at the origin if and only if (8) holds. We provide the expressions of the first integrals in the case of polynomial differential systems.

If $\mu \notin \{3,4,\ldots,m+1\}$, then we get the analytic first integral in the neighborhood of the origin from (25)

$$H = \frac{H_2}{\left(\dfrac{1}{2-\mu} + \displaystyle\sum_{j=1}^{m-1} \dfrac{\Lambda_j}{j+2-\mu} \right)^{2/(\mu-2)}}.$$

If $\mu \neq 2k \in \{3,4,\ldots,m+1\}$, then from (27) we get that the Poincaré-Liapunov first integral is

$$H=\frac{H_2}{\left(\dfrac{1}{2-\mu}+\varphi(x,y)+\displaystyle\sum_{\substack{j=1\\ j\neq\mu-2}}^{m-1}\frac{\Lambda_j}{j+2-\mu}+\right)^{2/(\mu-2)}},$$

where $\varphi(x,y)$ is a polynomial such that

$$\varphi(x,y)=\left(\int \Omega_{\mu-2}(r\cos\theta, r\sin\theta)d\theta\Big|_{r=\sqrt{x^2}+y^2,\ \theta=\arctan(y/x)}\right)\Bigg|_{\beta_{\mu-1}=0}.$$

and if $\mu=2k\in\{3,4,\ldots,m+1\}$, then

$$\tilde{H}=\frac{H_2}{\left(-\dfrac{1}{2-2k}+\varphi(x,y)+\displaystyle\sum_{\substack{j=1\\ j\neq 2k-2}}^{m-1}\frac{\Lambda_j}{j+2-2k}+\frac{(2H_2)^{k-1}}{2}\log H_2\right)^{1/(k-1)}}.$$

The first integral $\tilde{H}$ is such that $\lim\limits_{(x,y)\longrightarrow(0,0)}\tilde{H}=0$ and is not analytic. So we have a center, by considering that is a center of the $\Lambda-\Omega$ equation we obtain that the origin is a weak center.

If $\mu=2$ the Poincaré-Liapunov first integral in this case is

$$H=H_2\exp\left(2\sum_{j=1}^{m-1}\frac{\Lambda_j}{j}\right),$$

then the origin is a weak center, and Theorem 5 is proved for polynomial case. Thus Theorem 5 is proved.

Proof of Corollary 2. It is a simple consequence of Theorem 5 for the polynomial case the Poincaré-Liapunov first integral is

$$H=\frac{1}{2\left(\tilde{F}(x,y)\right)^{2/(m-1)}}=\frac{H_2}{\left(\dfrac{1}{m-1}+\dfrac{\Lambda_{m-1}}{m+1-\mu}\right)^{2/(m-1)}}.$$

If $\mu\notin\{2,m+1\}$. Clearly that H_2 and $\dfrac{1}{m-1}+\dfrac{\Lambda_{m-1}}{m+1-\mu}$ are partial integrals with polynomial cofactor $2\Omega_{m-1}$ and $(m-1)\Omega_{m-1}$ respectively.

If $\mu=m+1\neq 2k$ then the Poincaré-Liapunov first integral is

$$H=\left(\frac{1}{\tilde{F}(x,y)}\right)^{2/(m-1)}\Bigg|_{\beta_{m-1}=0}=\frac{2H_2}{\left(\dfrac{1}{m-1}+\varphi(x,y)\right)^{1/(m-1)}},$$

where $\varphi(x,y)$ is a polynomial of degree $m-1$ and such that

$$\varphi(x,y) = \left(\int \Omega_{m-1}(r\cos\theta, r\sin\theta)d\theta \Big|_{r=\sqrt{x^2+y^2},\ \theta=\arctan(y/x)} \right)\Big|_{\beta_{m-1}=0},$$

and $\beta_{m-1} = \displaystyle\int_0^{2\pi} \left(\frac{\partial X_{2k-1}}{\partial x} + \frac{\partial Y_{2k-1}}{\partial y} \right)\Big|_{x=\cos t,\, y=\sin t} dt = 0.$

If $\mu = m+1 = 2k$ then the first integral is

$$\tilde{H} = \frac{H_2}{\left(-\dfrac{1}{2-2k} + \dfrac{(2H_2)^{k-1}}{2} \log H_2 + \varphi(x,y) \right)^{1/(k-1)}}.$$

The first integral $\tilde{H}$ is such that $\displaystyle\lim_{(x,y)\longrightarrow(0,0)} \tilde{H} = 0$ and it is not analytic. So we have a center, by considering that is a center of the $\Lambda - \Omega$ equation we obtain that the origin is a weak center.

If $\mu = 2$ then we get the following Poincaré-Liapunov first integral

$$H = H_2 \exp\left(2\frac{\Lambda_{m-1}}{m-1} \right).$$

In short the corollary is proved.

4 Example

In [5,6] we state the following conjecture

Conjecture 2. *The polynomial differential system of degree* m

$$\begin{aligned} \dot{x} &= -y(1+\kappa\,(a_1\,y - a_2\,x)) + x(a_1x + a_2y + \Omega_{m-1}) = -y + X, \\ \dot{y} &= x(1+\kappa\,(a_1\,y - a_2\,x)) + y(a_1x + a_2y + \Omega_{m-1}) = x + Y, \end{aligned} \tag{28}$$

where $(\kappa+m-2)(a_1^2+a_2^2) \neq 0$, *and* $\Omega_{m-1} = \Omega_{m-1}(x,y)$ *is a homogenous polynomial of degree* $m-1$, *has a weak center at the origin if and only if system* (28) *after a linear change of variables* $(x,y) \longrightarrow (X,Y)$ *is invariant under the transformations* $(X,Y,t) \longrightarrow (-X,Y,-t)$.

Theorem 6 *Conjecture* 2 *holds for* $m = 2,3,4,5,6$.

The center problem for the case when $(\kappa+m-2)(a_1^2+a_2^2) = 0$, we solve in the next proposition.

Proposition 5 *polynomial differential system of degree* m (28) *satisfies the condition* (6) *if* $(\kappa+m-2)(a_1^2+a_2^2) = 0$.

Proof After some computations we get that

$$
\begin{aligned}
xX + yY &= (x^2+y^2)(a_1x + a_2y + \Omega_{m-1}),\\
\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} &= (\kappa+3)(a_1x+a_2y) + x\frac{\partial \Omega_{m-1}}{\partial x} + y\frac{\partial \Omega_{m-1}}{\partial y} + 2\Omega_{m-1}\Big|_{\kappa=m-2}\\
&= (m+1)(a_1x + a_2y + \Omega_{m-1}),
\end{aligned}
$$

hence we get that

$$
(x^2+y^2)\left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y}\right) = (m+1)\,(xX+yY)\,,
$$

which coincide with (6) with $\mu = m+1$. In this case we get that $\beta_{m-1} = \int_0^{2\pi} \Omega_{m-1}(\cos t,\, \sin t)dt$.

To solution of the center problem for differential system (28) under the condition $(\kappa + m - 2)(a_1^2 + a_2^2) = 0$ is a simple consequence of Theorem 5 with $\mu = m+1$. The case when $a_1 = a_2 = 0$ was study in [2].

5 Declarations

5.1 Funding

This work is supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER) and PID2019-104658GB-I00 (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

5.2 Conflicts of interest/Competing interests

This paper has no conflicts of interest.

5.3 Availability of data and material

Not applicable.

5.4 Code availability

Not applicable.

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