BIELLIPTIC SMOOTH PLANE CURVES AND QUADRATIC POINTS

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Abstract. Let C_k be a smooth plane curve of degree $d \geq 4$ defined over a global field k of characteristic $p = 0$ or $p > (d-1)(d-2)/2$ (up to an extra condition on $\mathrm{Jac}(C_k)$). Unless the curve is bielliptic of degree 4, we observe that it always admits finitely many quadratic points. We further show that there are only finitely many quadratic extensions $k(\sqrt{D})$ when k is a number field, in which we may have more points of C_k than these over k. In particular, we have this asymptotic phenomenon valid for the Fermat's and the Klein's equations.

Second, we conjecture that there are two infinite sets $\mathcal{E}$ and $\mathcal{D}$ of isomorphism classes of smooth projective plane quartic curves over k with a prescribed automorphism group, such that all members of $\mathcal{E}$ (respectively, $\mathcal{D}$) are bielliptic and have finitely (respectively, infinitely) many quadratic points over a number field k. We verify the conjecture over $k = \mathbb{Q}$ for $G = \mathbb{Z}/6\mathbb{Z}$ and $\mathrm{GAP}(16, 13)$. The analogue of the conjecture over global fields with $p > 0$ is also considered.

1. Introduction

Let C_k be a smooth projective curve defined over a global field k of characteristic p, which is neither rational nor elliptic. The set of K-points of $C_{\overline{k}} := C_k \otimes \overline{k}$ is denoted by $C(K)$, where K/k is a field extension inside a fixed algebraic closure $\overline{k}$ of k. For a finite field extension $L/k \subseteq \overline{k}$, the set of quadratic points of $C_L = C_k \otimes L$ is denoted by $\Gamma_2(C, L)$. That is, $\Gamma_2(C, L) = \bigcup \left\{ C(K) \,:\, K/L \subseteq \overline{k} \text{ and } [K : L] \leq 2 \right\}$.

The work of Harris and Silverman over zero characteristic [19], and Schweizer over positive characteristic [37] (assuming an extra condition on the Jacobian variety $\mathrm{Jac}(C_k)$) confirms that the set $\Gamma_2(C, L)$ is infinite for some L if and only if $C_{\overline{k}}$ is hyperelliptic or bielliptic.

Suppose now that $C_{\overline{k}}$ is a smooth plane curve of degree $d \geq 4$, so it is never rational or hyperelliptic. Over characteristic $p = 0$, it is not hard to see that $C_{\overline{k}}$ is bielliptic if and only if it is a smooth quartic curve with an automorphism of order 2 (an involution). For instance, Bardelli and Del Centina [7] showed that a complex smooth plane quartic curve with an involution is the canonical model of a $\mathbb{C}$-point of the locus $\mathcal{M}_{3,\mathrm{bi}}$ whose points are the $\mathbb{C}$-isomorphism classes of smooth bielliptic curves of genus 3.

In this direction, we introduce the notation $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(G)$ for the locus of $\overline{k}$-isomorphism classes of smooth plane quartic curves whose automorphism groups are exactly isomorphic to the group G. Next, motivated by the arithmetic geometry point of view, we find interest to study the set of its k-points $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(G)(k)$ i.e. modulo k-isomorphism of smooth plane non-hypererelliptic genus 3 curves defined over k (equivalently smooth plane non singular curves over k, by 3.1) whose automorphism group in $\overline{k}$ is isomorphic to G. in particular the bielliptic ones in $\widetilde{\mathcal{M}_{3,\mathrm{bi}}^{\mathrm{Pl}}}(G) := \widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(G) \cap \mathcal{M}_{3,\mathrm{bi}}$. The research of Abramovich, Harris, and Silverman over number fields allows us to say that a k-point C of $\widetilde{\mathcal{M}_{3,\mathrm{bi}}^{\mathrm{Pl}}}(G)(k)$ has infinitely many quadratic points over k if and only if $\mathrm{Aut}(C_{\overline{k}})$ contains a (bielliptic) involution w defined over k such that the bielliptic quotient C_k/w is an elliptic curve of positive rank over k. In this case, we call C bielliptic over k. However, the arithmetic (geometric) progressions of those quadratic points must be finite [6].

Now, once a finite group G is fixed such that $\widetilde{\mathcal{M}_{3,\mathrm{bi}}^{\mathrm{Pl}}}(G) \neq \emptyset$, we partition the $\widetilde{\mathcal{M}_{3,\mathrm{bi}}^{\mathrm{Pl}}}(G)(k)$ into two disjoint subsets $\mathcal{E}$ and $\mathcal{D}$ as follows: given $C \in \widetilde{\mathcal{M}_{3,\mathrm{bi}}^{\mathrm{Pl}}}(G)(k)$, $C \in \mathcal{E}$ if and only if C has finitely many quadratic points over k, and $C \in \mathcal{D}$ otherwise. In section §4, we conjectured that $\mathcal{E}$ and $\mathcal{D}$ are of infinite cardinality. We verified the claim using the twisting theory for curves if $k = \mathbb{Q}$ and $G = \mathbb{Z}/6\mathbb{Z}$ or $\mathrm{GAP}(16, 13)$; see Theorems 4.2, 4.4, 4.7 and 4.9, also if $k = \mathbb{F}_p(T)$ and $G = \mathbb{Z}/6\mathbb{Z}$; see Theorem 4.11. It is obvious to the reader that this has

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some connection with ranks distributions for families of elliptic curves. This topic finds a lot interest and it has been investigated by so many people around. Examples including, but not limited to Hulek-Kloosterman [21] or Salgado [35].

The first two sections are just a state of the art that can be interpreted as immediate consequences of well-known results. However, they are needed to introduce and motivate the problem. In our way to do this, we observed an asymptotic phenomenon for any smooth projective plane curve of degree $d \geq 5$ over k, which is different from the asymptotic phenomenon for the Fermat's equations [15]; see Corollary 3.7.

2. The interplay between hyperelliptic (respectively, bielliptic) curves and quadratic points

Let $\overline{k}$ (respectively, k^{sep}) be a fixed algebraic (respectively, separable) closure of a field k of characteristic $p \neq 2$, once and for all. By C_k we mean a smooth projective curve defined over k such that the base extension $C_{\overline{k}} := C \otimes_k \overline{k}$ has genus $g \geq 2$ and non-trivial automorphism group $\mathrm{Aut}(C_{\overline{k}})$. The set of all k-points of C_k is denoted by $C(k)$.

An arithmetic geometer finds a lot of interest to investigate the cardinality of $C(k)$ when k is a global field; (i) k is a finite field extension of either $\mathbb{Q}$ or (ii) k is a finite extension of the quotient field of the polynomial ring over $\mathbb{F}_p$ with the variable T. In the latter case, the finite field $k \cap \overline{\mathbb{F}}_p$ is denoted by $\mathbb{F}_q$ where q is some power of p.

Over zero characteristic, we have the following result on *Mordell's Conjecture* due to Faltings [13, 14]:

Theorem 2.1 (Faltings)**.** *Given a smooth projective curve C_k as above defined over a number field k, the set $C(k)$ is always finite.*

Over positive characteristic, the *genus of C_k relative to k* is the smallest integer $g_{C,k}$ such that, for any k-divisor D of C of sufficiently large degree, $\ell(D) = \deg(D) + 1 - g_{C,k}$ where $\ell(D)$ is the dimension of the associated k-(Riemann-Roch) vector space. For instance, the (geometric) genus g defined earlier is actually the genus of $C_{\overline{k}}$ relative to $\overline{k}$.

The relative genus may change under inseparable extensions of k inside $\overline{k}$, see for example [41]. Also, the relative genus $g_{C,k}$ is an upper bound for the absolute genus g. In this sense, the curve C_k is *conservative over k* if $g = g_{C,k}$, hence it is not genus-changing under inseparable extensions between k and $\overline{k}$.

For smooth plane curves, we have:

Lemma 2.2. *Let C_k be a smooth plane curve of degree $d \geq 4$ over a global field k of characteristic $p > (d-1)(d-2)+1$. Then, C_k and all its quotients are conservative over k.*

Proof. The result follows by [41, Corollary 2], since the relative genus $g_{C,k} = (d-1)(d-2)/2 < (p-1)/2$, and also all its quotients. □

Similar to Theorem 2.1, we have the next result over positive characteristic due to Grauert [17] and Samuel [36, Theorem 4 and 5b]:

Theorem 2.3 (Grauert-Samuel)**.** *Let C_k be a smooth projective curve over a global field k of characteristic $p > 0$. If C_k is conservative, then $C(k)$ is a finite set except possibly when C_k is an isotrivial curve, that is, when $C_k \otimes_k k^{\text{sep}}$ isomorphic to a smooth projective curve over a finite field $\mathbb{F}_{q^n}$.*

Remark 2.4. *Grauert-Samuel Theorem requires C_k to be geometrically non-singular instead of being conservative. But if C_k is conservative, then it is geometrically non-singular by [17].*

This following is an example of Theorem 2.3 where an infinite number of points occurs.

Example 2.5. *Consider the Fermat curve $C_k : X^d + Y^d = Z^d$ over the global field $k = \mathbb{F}_p(w, v)/(w^d + v^d - 1)$ with d and p coprime. First, it is isomorphic to the Fermat curve over $\mathbb{F}_p$. Second, it has infinitely many points over k namely, $(f^n(w), f^n(v), 1)$ for each positive integer n. Those points are constructed from the point $(w, v, 1) \in C(k) \setminus C(\overline{\mathbb{F}}_p)$ via the Frobenious $f : x \mapsto x^p$.*

Motivated by these results, it became natural to investigate the infinitude of the set $\Gamma_2(C, L)$ of quadratic points of C over L.

Definition 2.6. A smooth projective curve C_k is called *hyperelliptic* (respectively, *bielliptic*) *over* k if there exists a degree two k-morphism to a projective line $\mathbb{P}^1_k$ (respectively, to an elliptic curve E_k) over k. We simply call it *hyperelliptic* (respectively, *bielliptic*) if $C_{\overline{k}}$ is hyperelliptic (respectively, bielliptic) over $\overline{k}$.

Clearly, if C_k is hyperelliptic over k and k is not a finite field, then $\Gamma_2(C, k)$ is an infinite set. Also, if C_k is bielliptic over k and E_k has infinitely many k-points, then $\Gamma_2(C, k)$ is again an infinite set.

The following result is well-known in the literature (cf. [37] for (*ii*)).

Proposition 2.7. *Let C_k be a smooth projective curve over k with $g \geq 2$, recall $p \neq 2$. Then,*

(i) C_k is hyperelliptic if and only if there exists a (hyperelliptic) involution $w \in \mathrm{Aut}(C_{\overline{k}})$ having exactly $2g+2$ fixed points. In particular, if C_k is hyperelliptic, then w is unique, defined over a finite purely inseparable extension of k, and it is called the hyperelliptic involution *of C_k.*

(ii) C_k is bielliptic if and only if there exists a (bielliptic) involution $\tilde{w} \in \mathrm{Aut}(C_{\overline{k}})$ having $2g-2$ fixed points. If C_k is bielliptic and $g \geq 6$, then $\tilde{w}$ is unique, defined over a finite purely inseparable extension of k, and it belongs to the center of $\mathrm{Aut}(C_{\overline{k}})$.

2.1. **Number fields.** Assume that k is a global field of characteristic zero, or simply a number field. Because inseparable extensions between k and $\overline{k}$ do not exist, Proposition 2.7 leads to the following well-known result (cf. [9, Lemma 2.5] and [19, Lemma 5]):

Proposition 2.8. *Let C_k be a smooth projective curve C_k defined over a number field k. It is hyperelliptic with hyperelliptic involution w such that $C/\langle w\rangle(k) \neq \emptyset$ if and only if it is hyperelliptic over k. Also, for $g \geq 6$, C_k is bielliptic if and only if it is bielliptic over k.*

The next result is known to the specialists and it follows from Abramovich-Harris in [1], or Harris-Silverman in [19]. One also can read a proof in [9, Theorem 2.14].

Theorem 2.9. *Let C_k be a smooth projective curve over a number field k with $g \geq 2$. The set $\Gamma_2(C, k)$ is an infinite set if and only if C_k is hyperelliptic over k or C_k is bielliptic over k such that it exists a degree two k-morphism form C_k to an elliptic curve E_k of positive rank over k.*

2.2. **Global fields with $p > 0$.** First, we prove:

Proposition 2.10. *Let C_k be a smooth projective curve defined over a global field k of characteristic $p > 0$, that is conservative over k. Assume also that C_k is hyperelliptic with hyperelliptic involution w defined over a finite purely inseparable extension ℓ/k in $\overline{k}$. Then, there is a (unique) degree two ℓ-morphism φ to a conic Q over ℓ. Moreover, if C_ℓ (or more generally if $C_\ell/\langle w\rangle$) has an ℓ-point, then Q is ℓ-isomorphic to $\mathbb{P}^1_\ell$.*

Proof. By assumption $C_\ell/\langle w\rangle$ is conservative genus 0 curve defined over ℓ (Lemma 2.2), so it corresponds to a conic over ℓ. Next, the covering $\pi : C_\ell \to C_\ell/\langle w\rangle$ is cyclic of order 2, hence it is Galois. Thus π is defined over a separable extension of ℓ inside $\overline{k}$. Now, fix a separable closure $\ell^{\mathrm{sep}} \subseteq \overline{k}$ of ℓ and put $\mathrm{Gal}(\ell^{\mathrm{sep}}/\ell)$ for the absolute Galois group. By the uniqueness of the hyperelliptic involution w, π and ${}^{\sigma}\pi$ only differs by an automorphism ξ_σ of $\mathbb{P}^1_{\ell^{\mathrm{sep}}}$ for any $\sigma \in \mathrm{Gal}(\ell^{\mathrm{sep}}/\ell)$. In other words, for any $\sigma \in \mathrm{Gal}(\ell^{\mathrm{sep}}/\ell)$, we obtain $\xi_\sigma \in \mathrm{PGL}_2(\ell^{\mathrm{sep}})$; the projective general linear group of 2×2 matrices over ℓ^{sep}, where ${}^{\sigma}\pi = \xi_\sigma \circ \pi$. It can be easily checked that $\xi_{\sigma\tau} = {}^{\sigma}\xi_\tau \circ \xi_\sigma$ for all $\sigma, \tau \in \mathrm{Gal}(\ell^{\mathrm{sep}}/\ell)$. Therefore, $\xi : \mathrm{Gal}(\ell^{\mathrm{sep}}/\ell) \to \mathrm{PGL}_2(\ell^{\mathrm{sep}}) : \sigma \mapsto \xi_\sigma$ defines a 1-cocycle, in particular, an element of the first Galois cohomology set $\mathrm{H}^1(\mathrm{Gal}(\ell^{\mathrm{sep}}/\ell), \mathrm{PGL}_2(\ell^{\mathrm{sep}}))$. Using the Twisting Theory for Varieties (cf. [38, III.1]), it exists a conic Q (a twist for $\mathbb{P}^1_\ell$) over ℓ and an isomorphism $\varphi_0 : Q \to C_\ell/\langle w\rangle$ given by the rule $\xi_\sigma = {}^{\sigma}\varphi_0 \circ \varphi_0^{-1}$, for all $\sigma \in \mathrm{Gal}(\ell^{\mathrm{sep}}/\ell)$. Consequently, $\varphi := \varphi_0^{-1} \circ \pi : C_\ell \to Q$ is an ℓ-morphism from C_ℓ to Q.

The rest is direct because a conic over ℓ with an ℓ-point is the trivial twist for $\mathbb{P}^1_\ell$. This obviously happens through the morphism φ if C_ℓ or $C_\ell/\langle w\rangle$ have ℓ-points. □

Corollary 2.11. *Let C_k be a smooth projective curve defined over a global field k of characteristic $p > 0$, that is conservative over k. Then, C_k is hyperelliptic if and only if there exists a finite extension L/k inside $\overline{k}$ such that $C_k \otimes_k L$ is hyperelliptic over L. In this situation, $\Gamma_2(C, L)$ is an infinite set.*

Corollary 2.12. *Let C_k be a smooth projective curve defined over a global field k of characteristic $p > 2$, that is conservative over k and such that $C_{\overline{k}}$ is not hyperelliptic. If C_k is bielliptic, then there exists a finite extension L/k inside $\overline{k}$ such that $C_k \otimes_k L$ is bielliptic over L. Consequently, $\Gamma_2(C, L')$ is an infinite set for some finite extension L'/L inside $\overline{k}$.*

Proof. Suppose that C_k is bielliptic, and consider a bielliptic involution $\tilde{w} \in \mathrm{Aut}(C_{\overline{k}})$ as in Proposition 2.7. Since $g \geq 2$ and $\mathrm{Aut}(C_{\overline{k}})$ is finite, we have only finitely many possibilities for the Galois group $\mathrm{Gal}(\overline{k}/k)$-action on $\tilde{w}$. So $\tilde{w}$ must be defined over a finite field extension L_0/k inside $\overline{k}$. Because C_k is also conservative over k, then by making a finite extension L/L_0 with $L \subseteq \overline{k}$, we get a degree two L-morphism from $C_k \otimes_k L$ to a genus one curve (by Lemma 2.2) that has L-points, thus to an elliptic curve E over L, and hence C_k is bielliptic over L. Finally, it suffices to apply base extension to some $L \subseteq L' \subseteq \overline{k}$ of finite index so that $E \otimes_L L'$ has positive rank. Consequently, $\Gamma_2(C, L')$ is infinite and we conclude. □

Furthermore, inspired by the case of number fields (Theorem 2.9 above), we have [37, Theorem 5.1]:

Theorem 2.13 (Schweizer)**.** *Let C_k be a conservative smooth projective curve over a finite field extension k of $\mathbb{F}_q(T)$; i.e. k is a global field of characteristic $p > 2$. Under the conditions: $g \geq 3$, $C(k) \neq \emptyset$, C_k is not hyperelliptic over k, and that the Jacobian variety $\mathrm{Jac}(C_k)$ over $\overline{k}$ has no non-zero homomorphic images defined over $\overline{\mathbb{F}}_q$, the set $\Gamma_2(C, k)$ is infinite only if $\mathrm{Jac}(C_k)$ over k contains an elliptic curve E_k of positive rank. Moreover, there is a degree two morphism from C_k to E_k.*

Remark 2.14. *The condition that an abelian variety A over k has no non-zero homomorphic images defined over $\overline{\mathbb{F}_q}$ means that A has no isogeny factor defined over $\overline{\mathbb{F}}_q$. The isogeny, however, can be defined over $\overline{k}$. So what is excluded the existence of an abelian variety B over $\overline{F}_q$ such that over $\overline{k}$ there exists an isogeny from a isogeny factor of the base change of A onto B.*

As an example, the conditions on $\mathrm{Jac}(C_k)$ imply that E_k is not isotrivial (that is, its j-invariant does not belong to $\overline{\mathbb{F}}_q$) and that any factor of the Jacobian is a Jacobian of an isotrivial curve.

The curve $C_k : Z^4 + X^4 + Y^4 - 3X^2Y^2 = 0$ is an example for which the theorem does not apply. It appears for its Jacobian an elliptic curve with j-invariant 1728, hence isotrivial. See the proof of Lemma 4.6

Remark 2.15. *We observe that if $g \geq 6$ and $p > 2$, then the degree two morphism from C_k to E_k described in Theorem 2.13 is defined over k, and C_k is a bielliptic curve over k. Effectively, if $g \geq 6$, then the same argument of* [19, Lemma 5] *shows the result provided that Castellnuovo's inequality holds over the function field extensions of k which are involved (all are of degree 2 in Theorem 2.13).*

Castellnouvo's inequality is true when k is perfect, see [40, Theorem 3.11.3]. *However, following the proof of* [40, Theorem 3.11.3] *under the assumption that C_k is conservative, Castellnouovo inequality remains true if the characteristic p is big enough to guarantee that inseparable extensions do not appear in the extensions of the fields involved in the inequality. Consequently, under the hypothesis of Theorem 2.13 together with the assumptions $g \geq 6$ and $p > 2$, we get C_k is bielliptic over k.*

3. Bielliptic smooth plane curves and their quadratic points

Let C/k be a smooth projective curve over a field k. Suppose that $L/k \subseteq \overline{k}$ is a field extension such that $C_L := C \otimes_k L$ is a non-singular plane curve $F(X, Y, Z) = 0$ over L. In such a case we call L a *plane model field of definition for C* and C is a *smooth (k, L)-plane curve.* We simply call C a *smooth plane curve over k* if $L = k$ is plane model field of definition for C.

We investigated (minimal) plane models fields of definition for smooth (k, k^{sep})-plane curves in [5, §2]. For instance, we showed:

Theorem 3.1 (Badr-Bars-García). *Given a smooth (k, k^{sep})-plane curve C of degree $d \geq 4$, it does not necessarily have a non-singular plane model defined over the field k. However, it does in any of the following cases; if d is coprime with 3, if $C(k) \neq \emptyset$, or if the 3-torsion $\text{Br}(k)[3]$ of the Brauer group $\text{Br}(k)$ is trivial. In general, C is a smooth (k, L)-plane curve of degree d for some cubic Galois field extension L/k.*

When $\overline{k}$ is a plane model field of definition for C, then $C_{\overline{k}}$ has a unique g^2_d-linear series modulo $\text{PGL}_3(\overline{k})$-conjugation, see [22, Lemma 11.28]. This allows us to embed $C_{\overline{k}}$ into $\mathbb{P}^2_{\overline{k}}$ as a non-singular plane model over $\overline{k}$ of some degree d, in particular, the genus $g := g = (d-1)(d-2)/2$. Moreover, we have:

Proposition 3.2. *A smooth $(k, \overline{k})$-plane curve C of degree $d \geq 5$ is neither hyperelliptic nor bielliptic. Also, C is never hyperelliptic when $d = 4$.*

Proof. It is a basic fact in algebraic geometry that a smooth $(k, \overline{k})$-plane curve of degree $d \geq 4$ is non-hyperelliptic. On the other hand, the (geometric) gonality of a smooth projective curve C is defined to be the minimum degree of a $\overline{k}$-morphism from $C_{\overline{k}}$ to the projective line $\mathbb{P}^1_{\overline{k}}$. For the special case of smooth $(k, \overline{k})$-plane curves C of degree $d \geq 4$, the gonality equals to $d - 1$ (cf. Namba [33] for zero characteristic and Homma [23] for positive characteristic, also one can read the assertion in [45, page 341]). Consequently, the gonality of C is at least 5 when $d \geq 6$ and C can not be bielliptic. For $d = 5$ and $k = \mathbb{C}$, see [2, Chapter V, Exercises A-6 and A-7], but here we will introduce a general argument: By assumption, $C_{\overline{k}}$ admits a bielliptic involution, that is, an automorphism $\tilde{w}$ of order 2 fixing $2g - 2 = 10$ points on $C_{\overline{k}}$. By [22, Lemma 11.28], we can take $\tilde{w} : X \mapsto X, Y \mapsto Y, Z \mapsto -Z$ up to $\text{PGL}_3(\overline{k})$-conjugation. In particular, $C_{\overline{k}}$ is defined by an equation of the form $Z^4 L_{1,Z} + Z^2 L_{2,Z} + L_{5,Z} = 0$, where $L_{i,Z}$ denotes a homogenous degree i binary form in X, Y. This means that $\tilde{w}$ fixes exactly $d + 1 = 6 < 10$ points on $C_{\overline{k}}$, which contradicts the fact that C is bielliptic. □

3.1. **Smooth plane quartic curves, which are bielliptic.** Over the complex field $\mathbb{C}$, Bardelli and Del Centina [7] showed that any non-singular plane quartic curve $F(X, Y, Z) = 0$ over $\mathbb{C}$ that admits an automorphism of order 2 is the canonical model of a bielliptic curve of genus 3. Moreover, they characterized a distinguished (canonical) equation for any $(\mathbb{C}, \mathbb{C})$-plane quartic curve C, once a bielliptic structure on C is fixed. As a consequence, a nice configuration was obtained for the 84 proper second order Weierstrass point of the Klein quartic curve $K : X^3Y + Y^3Z + Z^3X$. See [7, Section 2] for further details.

We remark that the Klein quartic K is the unique $\mathbb{C}$-point of the locus $\mathcal{M}_{3,\text{bi}}(\widetilde{\text{PSL}}(2, 7))$, which is 0-dimensional. However, this locus has several $\mathbb{Q}$-points that are not isomorphic over $\mathbb{Q}$, but isomorphic over $\mathbb{C}$ to the Klein quartic curve K above. In particular, those $\mathbb{Q}$-points come up with different arithmetic properties. Since the paper is closer to arithmetics, we will be following the approach developed by Lercier-Ritzenthaler-Rovetta-Sijsling in [27].

Let $\mathcal{M}_g$ be the (coarse) moduli space, representing $\overline{k}$-isomorphism classes of smooth curves of genus g. For a finite non-trivial group G, consider the locus $\widetilde{\mathcal{M}^{\text{Pl}}_g(G)} \subset \mathcal{M}_g$ consisting of all $\overline{k}$-isomorphism classes of smooth plane curves $[C]$ with automorphism group $\text{PGL}_g(\overline{k})$-conjugated to $\varrho(G)$ for some injective representation $\varrho : G \hookrightarrow \text{PGL}_g(\overline{k})$.

Lercier-Ritzenthaler-Rovetta-Sijsling in [27, §2] introduced the notions of complete, and representative families for loci of $\mathcal{M}_g$ when k is any field of characteristic $p = 0$ or $p > 2g + 1$. In the case of plane curves, a family $\mathcal{C}$ of smooth plane curves over k is said to be *complete over k for $\widetilde{\mathcal{M}^{\text{Pl}}_g(G)}$* if, for any algebraic extension k'/k inside $\overline{k}$ and any k'-point C in the $\widetilde{\mathcal{M}^{\text{Pl}}_g(G)}(k')$, there exists a $\overline{k}$-isomorphic non-singular plane model for C defined over k' in the family $\mathcal{C}$. If such a model in the family is always unique, then the family $\mathcal{C}$ is *representative over k for $\widetilde{\mathcal{M}^{\text{Pl}}_g(G)}$*. A family $\mathcal{C}$ is *geometrically complete* (respectively, *geometrically representative*) if $\mathcal{C} \otimes_k \overline{k}$ is complete (respectively, representative) over $\overline{k}$.

The following result is a reformulation of [7, Corollary 2.2 and Thoerem 2.5] in terms of the notions developed in [27].

Theorem 3.3 (Bielliptic quartic curves). *Let k be a field of characteristic $p = 0$ or $p > 7$. A smooth $(k, \overline{k})$-plane quartic curve C is bielliptic if and only if $C \otimes_k \overline{k}$ is isomorphic to a non-singular plane model of the form $Z^4 + Z^2 L_{2,Z} + L_{4,Z} = 0$, where $L_{i,Z}$ is a homogenous binary form in $\overline{k}[X, Y]$ of degree i.*

Table 1 below gives a geometrically complete family for each locus of $\mathcal{M}_3$ of smooth plane bielliptic quartic curves over $\overline{k}$.

Table 1. Bielliptic geometrically complete classification

$\mathrm{Aut}(C_{\overline{k}})$	*Families*	*Restrictions*
$\mathbb{Z}/2\mathbb{Z}$	$Z^4+Z^2L_{2,Z}(X,Y)+L_{4,Z}(X,Y)$	$L_{2,Z}(X,Y)\neq 0$, *not below*
$\mathbb{Z}/2\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}$	$Z^4+Z^2(bY^2+cX^2)+(X^4+Y^4+aX^2Y^2)$	$a\neq\pm b\neq c\neq\pm a$
$\mathbb{Z}/6\mathbb{Z}$	$Z^4+aZ^2Y^2+(X^3Y+Y^4)$	$a\neq 0$
$\mathbf{S_3}$	$(X^3+Y^3)Z+X^2Y^2+aXYZ^2+bZ^4$	$a\neq b,\ ab\neq 0$
D_4	$Z^4+bXYZ^2+(X^4+Y^4+aX^2Y^2)$	$b\neq 0,\pm\frac{2a}{\sqrt{1-a}}$
$\mathrm{GAP}(16,13)$	$Z^4+(X^4+Y^4+aX^2Y^2)$	$\pm a\neq 0,2,6,2\sqrt{-3}$
S_4	$Z^4+aZ^2(Y^2+X^2)+(X^4+Y^4+aX^2Y^2)$	$a\neq 0,\ \frac{-1\pm\sqrt{-7}}{2}$
$\mathrm{GAP}(48,33)$	$Z^4+(X^4+Y^4+(4\zeta_3+2)X^2Y^2)$	–
$\mathrm{GAP}(96,64)$	$Z^4+(X^4+Y^4)$	–
$\mathbf{PSL_2}(\mathbb{F}_7)$	$X^3Y+Y^3Z+Z^3X$	–

The algebraic restrictions for the parameters over $\overline{k}$, in the last column, are taken so that the defining equation is non-singular and has no bigger automorphism group. For example, the term "not below" means to assume more restrictions for no larger automorphism group to occur.

Here ζ_3 denotes a fixed primitive 3rd root of unity. The automorphism groups are given in GAP *library* [16] *notations.*

Remark 3.4. *The two highlighted cases in Table 1 are not in the prescribed form $Z^4+Z^2L_{2,Z}+L_{4,Z}=0$, that is,* $\mathrm{diag}(1,1,-1)$ *is not an explicit bielliptic involution. However, they do up to $\mathrm{PGL}_3(\overline{k})$-conjugation. The way to see this is elementary; any 3×3 projective linear transformation of order two of the plane fixes (point-wise) a line $\mathcal{L}\subset\mathbb{P}^2_{\overline{k}}$, called its axis. So, we may ask for a change of variables ϕ such that the transformed $\mathcal{L}$ becomes the projective line $\mathcal{L}':\ Z=0$. For example, for $G=\mathrm{S}_3$, the change of variables*

$$\begin{pmatrix}1&-1&1\\1&-1&-1\\0&2&0\end{pmatrix}$$

does the job. It transforms the axis $\mathcal{L}:X-Y=0$ of involution $X\leftrightarrow Y, Z\mapsto Z$ to $\mathcal{L}':\ Z=0$. In particular, we obtain the $\overline{k}$-equivalent family

$$Z^4-2Z^2\left(X^2-8XY+(2a+7)Y^2\right)+\left(X^4+2(2a-3)(XY)^2-8(a-1)XY^3+(4a+16b-3)Y^4\right).$$

Proof of Theorem 3.3. The assertion that non-singular plane models of the form $Z^4+Z^2L_{2,Z}+L_{4,Z}=0$ characterizes smooth $(k,\overline{k})$-plane bielliptic quartic curves up to $\mathrm{PGL}_3(\overline{k})$-equivalence can be found in [7].

The stratification by automorphism groups follows from the work of Henn [20] (see also [8]). The geometrically complete families are issued from [27, 28]. □

Remark 3.5. *Any locus of the shape $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}(G)}$ corresponds to a unique representation $\varrho:G\hookrightarrow\mathrm{PGL}_3(\overline{k})$, provided that it is non-empty. The uniqueness of ϱ up to conjugation in $\mathrm{PGL}_3(\overline{k})$ is not guaranteed for $d>4$, since it might happen that two smooth $(k,\overline{k})$-plane curves C,C' of degree $d>4$ have isomorphic non-conjugated automorphism groups. In particular, C and C' are not $\overline{k}$-isomorphic. See* [3, 4] *for more details and examples.*

Theorem 3.6. *Let C be a smooth plane curve of degree $d\geq 4$ over a number field k. Unless C is a smooth plane bielliptic quartic curve, the set $\Gamma_2(C,L)$ has finite cardinality for any finite field extension $L/k\subseteq\overline{k}$.*

Proof. We know by Proposition 3.2 that C is never hyperelliptic. Moreover, by Proposition 3.2 and Theorem 3.3, it is not bielliptic if (i) $d\geq 5$ or (ii) $d=4$ such that $\mathrm{Aut}(C_{\overline{k}})\cong 1,\mathbb{Z}/3\mathbb{Z}$, or $\mathbb{Z}/9\mathbb{Z}$. In those situations, we guarantee that $\Gamma_2(C,L)$ is finite due to Theorems 2.13, and 2.9. Otherwise, $\Gamma_2(C,L)$ would be an infinite set for some finite extension $L/k \subseteq \overline{k}$ by Theorem 2.9. □

The next asymptotic phenomenon is probably known to specialists, but we present a quite simple proof:

Corollary 3.7. *Let C be a smooth plane curve of degree $d \geq 5$ over a number field k. Then, there are only finitely many quadratic field extensions $k'/k \subseteq \overline{k}$ such that $C(k') \neq C(k)$. In particular, for the Fermat curve $C: X^d + Y^d - Z^d = 0$ or the Klein curve $C: X^{d-1}Y + Y^{d-1}Z + Z^{d-1}X = 0$ of degree $d \geq 5$ over $\mathbb{Q}$, we have $C(\mathbb{Q}) = C(\mathbb{Q}(\sqrt{D}))$ for all but finitely many square-free integers D. This means that only finitely many quadratic extensions $\mathbb{Q}(\sqrt{D})$ satisfy $\#C(\mathbb{Q}(\sqrt{D})) \neq \#C(\mathbb{Q})$.*

Proof. By definition, $C(k) \subseteq C(k') \subseteq \Gamma(C,k)$ for any quadratic field extension $k \subset k' \subseteq \overline{k}$. By Theorem 3.6, $\Gamma_2(C,k)$ is a finite set. So $C(k) \subsetneq C(k')$ occurs for for finitely many k'.

□

4. Quadratic points on smooth plane curves fixing the base field

Let C be a smooth $(k,\overline{k})$-plane quartic curve as in Theorem 3.3. Because the degree $d=4$ is coprime with 3, we deduce by Theorem 3.1 that k is not only a field of definition for C, but also a plane-model field of definition.

We find some interest to conjecture the following over a number field k.

Conjecture 4.1. *Fix a non-empty loci $\widetilde{\mathcal{M}_{3,biell}}(G)$, and denote its k-points by $\widetilde{\mathcal{M}_{3,biell}}(G)(k)$. Then, there exists an infinite subset $\mathcal{E}$ (respectively, $\mathcal{D}$) of $\subseteq \widetilde{\mathcal{M}_{3,biell}}(G)(k)$ such that each element C of $\mathcal{E}$ (respectively, $\mathcal{D}$) corresponds to a k-isomorphism class of smooth plane quartic curves over k with $\Gamma_2(C,k)$ finite (respectively, infinite).*

We refer to section §4.3 for a version of this conjecture over positive characteristic.

By the work of Lercier-Ritzenthaler-Rovetta-Sijsling in [27, §2], we have a parametrization of the loci $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(G)$ in terms of complete and representative families over $\mathbb{Q}$, except when $G = \mathbb{Z}/2\mathbb{Z}$ (a representative family over $\mathbb{R}$ does not exist). On the other hand, the work of García in [28] and [29] detailed the study of the twists of smooth plane quartic curves over $\mathbb{Q}$. These results help us to support the previous conjecture in two different situations over $k = \mathbb{Q}$. The main idea is to start with a family of smooth plane quartic curves whose defining equation over $\mathbb{Q}$ involves many parameters. This in turns allows us to construct subfamilies of infinite cardinalities of non $\mathbb{Q}$-isomorphic smooth plane quartic curves with the same automorphism group G, up to group isomorphism. The members of these families are mapped to concrete elliptic curves of rank zero (respectively, positive rank) over $\mathbb{Q}$.

Notations. We use ζ_n for a fixed primitive n-th root of unity inside $\overline{k}$ provided that the characteristic of k is coprime with n.

A projective linear transformation $A = (a_{i,j})_{1 \leq i,j \leq 3}$ of the projective plane $\mathbb{P}^2_{\overline{k}}$ is often written as

$$[a_{1,1}X + a_{1,2}Y + a_{1,3}Z : a_{2,1}X + a_{2,2}Y + a_{2,3}Z : a_{3,1}X + a_{3,2}Y + a_{3,3}Z],$$

where $\{X,Y,Z\}$ are the homogenous coordinates of $\mathbb{P}^2_{\overline{k}}$.

4.1. **The conjecture 4.1 is true for $\widetilde{\mathcal{M}_{3,biell}}(\mathbb{Z}/6\mathbb{Z}) = \widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(\mathbb{Z}/6\mathbb{Z})$.** Let k be a field of characteristic $p=0$ or $p>7$. The one parameter family $\mathcal{C}_a$ defined by

$$\mathcal{C}_a : aZ^4 + Y^2(Y^2 + aZ^2) + X^3Y = 0,$$

where $a \neq 0, 4$, is a representative family over k for $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}}(\mathbb{Z}/6\mathbb{Z})$, see [27, Theorem 3.3]. Thus, any smooth plane quartic curve C over k with automorphism group isomorphic to $\mathbb{Z}/6\mathbb{Z}$ (up to $\overline{k}$-isomorphism) has a non-singular plane model in $\mathcal{C}_a$ for an unique $a \in k$. That is, there exists a finite extension $L/k \subseteq \overline{k}$ such that $C \otimes_k L$ is L-isomorphic to an unique non-singular polynomial equation of the form $aZ^4 + Y^2(Y^2 + aZ^2) + X^3Y = 0$ for some $a \in k$. In particular, $\Gamma_2(C,L) = \Gamma_2(aZ^4 + Y^2(Y^2 + aZ^2) + X^3Y = 0, L)$. We also note that the above family is $\overline{k}$-isomorphic to the geometrically complete family described in Theorem 3.3 through a diagonal change of variables [27]. So we can assume that $\tilde{w} = \mathrm{diag}(1, 1, -1)$ is the unique (bielliptic) involution for any smooth curve in the family $\mathcal{C}_a$.

Theorem 4.2. *Consider a smooth quartic plane bielliptic curve C_a of the form $aZ^4+Y^2(Y^2+aZ^2)+X^3Y = 0$ for some $a \in \mathbb{Q}\setminus\{0, 4\}$. The quotient $C_a/\langle\tilde{w}\rangle$ is an elliptic curve of positive rank over $\mathbb{Q}$. In particular, the family $\mathcal{C}_a$ for $a \in \mathbb{Q}\setminus\{0, 4\}$ contains infinitely many smooth plane quartic bielliptic curves C over $\mathbb{Q}$ with automorphism group $\mathbb{Z}/6\mathbb{Z}$, and such that $\Gamma_2(C, \mathbb{Q})$ is infinite.*

Proof. We work affine with $az^4+az^2+x^3+1 = 0$ by taking $Y = 1$, in particular, $C_a/\langle\tilde{w}\rangle : az^2+az+x^3+1 = 0$. It is straightforward to see that $C_a/\langle\tilde{w}\rangle$ is $\mathbb{Q}$-isomorphic to the elliptic curve $E/\mathbb{Q} : z^2 = x^3 - a^3(1-a/4)$ whose j-invariant equals to zero. One can use MAGMA to check that the point $P_a := (x, z) = (a, a^2/2)$ on $E/\mathbb{Q}$ has order $m > 10$ and $m \neq 12$. So it is a non-torsion point on $E/\mathbb{Q}$ (cf. [31, Theroem 7']). Thus $\mathrm{rank}_{\mathbb{Q}}(C_a/\langle\tilde{w}\rangle) \geq 1$ for any $a \in \mathbb{Q}\setminus\{0, 4\}$, and $\Gamma_2(C_a, \mathbb{Q})$ is an infinite set by Theorem 2.9.

Finally, the family $\mathcal{C}_a$ is representative for $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}(\mathbb{Z}/6\mathbb{Z})}$ over $\mathbb{Q}$. This means that two curves C_a and $C_{a'}$ in the family with $a \neq a' \in \mathbb{Q}$ can not be $\overline{\mathbb{Q}}$-isomorphic. Thus, we have infinitely many smooth plane quartic curves each of which admits infinitely many quadratic points over $\mathbb{Q}$. □

Now, if we aim to parameterize the set of k-isomorphism classes of smooth plane quartic curves over k with automorphism group isomorphic to $\mathbb{Z}/6\mathbb{Z}$, then we may use [28, Proposition 3.2.8] (or [29, Proposition 5.3]) to deduce that the family

$$\mathcal{C}_{A,n,m} : Am^2Z^4 + mY^2Z^2 + nX^3Y + Y^4 = 0,$$

where $(A, n, m) \in k^* \times k^*/k^{*^3} \times k^*/k^{*^2}$ is such a parametrization[1]. More precisely, any smooth plane quartic curve over k with automorphism group $\mathbb{Z}/6\mathbb{Z}$ is k-isomorphic to a non-singular plane model in the family $\mathcal{C}_{A,n,m}$ for some triple $(A, n, m) \in k^* \times k^*/k^{*^3} \times k^*/k^{*^2}$. In particular, it is bielliptic with unique bielliptic involution $\tilde{w} = \mathrm{diag}(1, 1, -1)$. Moreover, two such curves are $\overline{k}$-isomorphic if and only if they have the same parameter $A \in k^*$. Therefore, it suffices to assume $\Gamma_2(\mathcal{C}_{A,n,m}, k)$ for $(A, n, m) \in k^* \times k^*/k^{*^3} \times k^*/k^{*^2}$ in order to investigate the infinitude of quadratic points over k of smooth plane quartic curves inside $\widetilde{\mathcal{M}_3^{\mathrm{Pl}}(\mathbb{Z}/6\mathbb{Z})}$ over k.

Lemma 4.3. *For an arbitrary but a fixed $(A, n) \in k^* \times k^*/k^{*^3}$, the set $\Gamma_2(\mathcal{C}_{A,n,m}, k)$ being finite or infinite is independent of the choice of m.*

Proof. One finds that $\mathcal{C}_{A,n,m}/\langle\tilde{w}\rangle$ is k-isomorphic to the elliptic curve $\mathcal{D}_{A,n}/k : z^2 = x^3 + \frac{n^2A^2(1-4A)}{4}$ over k. Indeed, one works affine $Y = 1$ to easily obtain that $\mathcal{C}_{A,n,m}/\langle\tilde{w}\rangle$ is k-isomorphic to the elliptic curve $Az^2 + z + nx^3 + 1 = 0$ over k, in particular, its defining equation and its rank is independent from m. To reach the Weierstrass form $\mathcal{D}_{A,n}$, we follow the usual way (cf. [39]); first, by the change of variables $z \mapsto z - \frac{1}{2A}$ and $x \mapsto -x$, we get $z^2 = (n/A)x^3 + \frac{1-4A}{4A^2}$, After by $z \mapsto (1/nA^2)z$ and $x \mapsto (1/An)x$. □

Theorem 4.4. *There are infinitely many smooth plane quartic curves C over $\mathbb{Q}$ in the family $\mathcal{C}_{A,n,m}$, with $(A, n, m) \in \mathbb{Q}^* \times \mathbb{Q}^*/\mathbb{Q}^{*3} \times \mathbb{Q}^*/\mathbb{Q}^{*2}$, such that $\Gamma_2(C, \mathbb{Q})$ is a finite set.*

Proof. For example, consider the subfamily $\mathcal{C}_{A(t),n(t),m} : A(t)m^2Z^4 + mY^2Z^2 + n(t)X^3Y + Y^4 = 0$ where $A(t) := (108t^2 + 1)/4$ and $n(t) := 4/t(108t^2 + 1)$, for $t = a/b \in \mathbb{Q}^*$ with a, b are odd coprime integers. In this situation, $\mathcal{C}_{A(t),n(t),m}/\langle\tilde{w}\rangle$ is always $\mathbb{Q}$-isomorphic to $\mathcal{D}_{A(t),n(t)} : z^2 = x^3 - 27$ of rank 0 over $\mathbb{Q}$. Second, two curves in the family associated to the triples $(A(t), n(t), m)$ and $(A(t'), n(t'), m')$ with $A(t) \neq A(t')$ are not $\overline{\mathbb{Q}}$-isomorphic. Accordingly, we have infinitely many smooth plane quartic curves with infinitely many quadratic points over $\mathbb{Q}$, which was to be shown. □

Remark 4.5. *We can prove Theorems 4.2 and 4.4 without using the twisting theory of curves. For instance, following Henn's table (cf. [8, Theorem 29]), smooth plane curves in the family $\mathcal{C}_{a,b} : y^4 + ay^2 = x^3 + b$ have automorphism group equals to $\mathbb{Z}/6\mathbb{Z}$ and their $\overline{\mathbb{Q}}$-isomorphism classes are uniquely determined by the ratio b/a^2. Imposing $a^2+4b = 8$ yields infinitely many non-isomorhic smooth plane curves in the family $\mathcal{C}_{a,b}$ corresponding to*

[1] It remains to determine the algebraic restrictions on the parameters to ensure non-singularity and no larger automorphism group. For example, $A \neq 1/4$, since we get the singular points $(0 : \pm\sqrt{m/2} : 1)$ on $\mathcal{C}_{1/4,n,m}$, otherwise.

the elliptic curve $y^2 = x^3 + 2$ of rank one. Similarly, imposing $a^2 + 4b = 4$ yields infinitely many non-isomorphic smooth plane curves corresponding to the elliptic curve $y^2 = x^3 + 1$ of rank zero.

4.2. **The conjecture 4.1 is true for $\mathcal{M}_{3,biell}(\widetilde{\mathrm{GAP}(16,13)})$.** The family defined by

$$\mathcal{C}_a : Z^4 + (X^4 + Y^4 + aX^2Y^2) = 0,$$

where $\pm a \neq 0, 2, 6, 2\sqrt{-3}$, is a geometrically complete family for the locus $\mathcal{M}_3^{\mathrm{Pl}}(\widetilde{\mathrm{GAP}(16,13)})$ over $\overline{\mathbb{Q}}$. In this case, there are exactly seven (bielliptic) involutions, namely $\iota_1 := \mathrm{diag}(1,1,-1)$, $\iota_2 := \mathrm{diag}(-1,1,1)$, $\iota_3 := \mathrm{diag}(1,-1,1)$, $\iota_4 := [Y : X : Z]$, $\iota_5 := [Y : X : -Z]$, $\iota_6 := [Y : -X : \zeta_4 Z]$ and $\iota_7 := [Y : -X : -\zeta_4 Z]$.

Lemma 4.6. *For $a \in \mathbb{Q} \setminus \{0, \pm 2, \pm 6\}$, suppose that $\mathcal{C}_a/\langle \iota_i \rangle$ has a $\mathbb{Q}$-point. Then, $\mathcal{C}_a/\langle \iota_i \rangle$ is $\mathbb{Q}$-isomorphic to an elliptic curve $E/\mathbb{Q}$ of one of the forms:*

Involution	E	j*-invariant*
ι_1	$z^2 = x^3 - ax^2 - 4x + 4a$	$(16a^6 + 576a^4 + 6912a^2 + 27648)/(a^4 - 8a^2 + 16)$
ι_2, ι_3	$z^2 = x^3 + (a^2 - 4)x$	1728
$\iota_4, \iota_5, \iota_6, \iota_7$	$z^2 = x^3 + (1 - a^2/4)x$	1728

Proof. First, one can use MAGMA to verify that $\mathcal{C}_a/\langle \iota_i \rangle$ is a smooth curve of genus 1 over $\mathbb{Q}$. Second, by assumption, $\mathcal{C}_a/\langle \iota_i \rangle$ has rational points. So it is $\mathbb{Q}$-isomorphic to its Jacobian variety, which is an elliptic curve over $\mathbb{Q}$ of one the prescribed forms. Third, the condition that $\mathcal{C}_a/\langle \iota_i \rangle$ has rational points is verified in many cases. For example, we can see that $\mathcal{C}_a/\langle \iota_1 \rangle$ is defined inside $\mathbb{P}^3_{\mathbb{Q}}$ by the two quadrics $-X_2X_3 + X_4^2 = 0$ and $X_1^2 + X_2^2 + X_3^2 + aX_2X_3 = 0$ over $\mathbb{Q}$. Hence, if we impose $-(a+2) \in \mathbb{Q}^{*^2} \setminus \{4\}$, then $(\sqrt{-(a+2)} : 1 : 1 : 1)$ is an obvious $\mathbb{Q}$-point. □

Consider the family $\mathcal{C}'_A : AX^4 + Y^4 + Z^4 + X^2Y^2 = 0$, where $\pm A \neq \{1/4, 1/36, 1/12\}$. This is a representative family over $\mathbb{Q}$ (cf. [28, pages 36-37]), in particular, any smooth plane quartic curve over $\mathbb{Q}$ with automorphism group isomorphic to GAP(16, 13) is isomorphic (not necessarily over $\mathbb{Q}$) to a smooth plane curve in the family $\mathcal{C}'_A$ for a unique $A \in \mathbb{Q}^* \setminus \{\pm 1/4, \pm 1/36, \pm 1/12\}$. On the other hand, the transformed seven bielliptic involutions of any smooth plane quartic curve in the family $\mathcal{C}'_A$ over $\mathbb{Q}$ are $\iota'_i := P^{-1}\iota_i P$, for $i = 1, \ldots, 7$, where $P := \mathrm{diag}(\frac{1}{\sqrt[4]{A}}, 1, 1)$. If we restrict $A \in \mathbb{Q}^* \setminus \mathbb{Q}^{*^4}$, then $\iota'_1 = \iota_1$, $\iota'_2 = \iota_2$, $\iota'_3 = \iota_3$ are the only bielliptic involutions defined over $\mathbb{Q}$.

By the work of García in [28, Chp. 3], we know that any *diagonal twist* of C_A, for a fixed A, in the family $\mathcal{C}'_A$ is $\mathbb{Q}$-isomorphic to

$$C_{A,m,q} : mAX^4 + q^2mY^4 + Z^4 + qmX^2Y^2 = 0,$$

for some $A, m, q \in \mathbb{Q}^*$. Two of the twists $\{A, m, q\}$ and $\{A', m', q'\}$ are $\mathbb{Q}$-isomorphic if $A = A'$, $m \equiv m' \mod \mathbb{Q}^{*^4}$ and $q \equiv q' \mod \mathbb{Q}^{*^2}$. First, we consider smooth curves of the form $C_{A,m,q}$ with $(A, m, q) \in \mathbb{Q}^* \times \mathbb{Q}^*/\mathbb{Q}^{*^4} \times \mathbb{Q}^*/\mathbb{Q}^{*^2}$. Next, the quotient curve $C_{A,m,q}/\langle \iota'_3 \rangle$ is a genus one curve that is $\mathbb{Q}$-isomorphic to $y^2 + (1/m)z^4 - (1/4 - A) = 0$ independently from the parameter $q \in \mathbb{Q}^*/\mathbb{Q}^{*^2}$. Assuming that $1/4 - A$ is a square implies that $C_{A,m,q}/\langle \iota'_3 \rangle$ has $\mathbb{Q}$-points, and its Weierstrass form in the variables u, ν reduces to

$$\nu^2 = u^3 + ((1 - 4A)/m)\, u.$$

Theorem 4.7. *Consider a smooth bielliptic quartic plane curve in the family $\mathcal{C}_{A,m,q} : mAX^4 + q^2mY^4 + Z^4 + qmX^2Y^2 = 0$, for some $q \in \mathbb{Q}^* \setminus \mathbb{Q}^{*^2}$ and $A, m \in \mathbb{Q}^* \setminus \mathbb{Q}^{*^4}$, such that $1/4 - A$ is a square. Then, the quotient $\mathcal{C}_{A,m,q}/\langle \tilde{w} \rangle$, where $\tilde{w} = \mathrm{diag}(1, -1, 1)$, is an elliptic curve of positive rank over $\mathbb{Q}$. In particular, $\mathcal{C}_{A,m,q}$ gives an infinite family of smooth plane quartic curves C over $\mathbb{Q}$ whose automorphism group is isomorphic to* GAP(16, 13)*, and $\Gamma_2(C, \mathbb{Q})$ is an infinite set.*

Proof. We have seen above that $\mathcal{C}_{A,m,q}/\langle \tilde{w} \rangle$ is $\mathbb{Q}$-isomorphic to $\mathcal{D}_{A,m} : \nu^2 = u^3 + Du$ with $D := (1 - 4A)/m$ independently from the parameter $q \in \mathbb{Q}^*/\mathbb{Q}^{*^2}$. It suffices now to specialize A, m so that $\mathrm{rank}_{\mathbb{Q}}(\mathcal{D}_{A,m})$ is positive, so $\Gamma_2(C, \mathbb{Q})$ is infinite. For example, if we take $D := (1 - 4A)/m = p$, where p is a prime integer < 1000 and congruent to 5 modulo 8, then the rank is always 1 in accordance with the conjecture of Selmer and Mordell (see [10]). If we take $D := (1 - 4A)/m = -p$, where p is a Fermat or a Mersenne prime, then the ranks

0, 1 and 2 were found (see [26]). If we take $D := (1-4A)/m = -n$, where n is related to the positive integer solutions of the diophantine equation $n = \alpha^4 + \beta^4 = \gamma^4 + \mu^4$, then the rank is at least 3 (see [24]). If we take $D := (1-4A)/m = -pq$, where p and q are two different odd primes, then, up to an extra condition, a family of rank 4 was found (see [30]), etc... Finally, by [28, Proposition 3.2.9], two twists $C_{A,q,m}$ and $C_{A,q',m'}$, with $(A, m, q), (A, m', q') \in \mathbb{Q}^* \times \mathbb{Q}^*/{\mathbb{Q}^*}^2 \times \mathbb{Q}^*/{\mathbb{Q}^*}^4$ are $\mathbb{Q}$-isomorphic only if $m = m'$ in $\mathbb{Q}^*/{\mathbb{Q}^*}^4$. Consequently, for any fixed A, we can run m and q as before to obtain infinitely many non-$\mathbb{Q}$-isomorphic smooth plane quartic curves with infinitely many quadratic points over $\mathbb{Q}$. $\square$

Now, we aim to construct an infinite family of smooth quartic curves over $\mathbb{Q}$ inside $\mathcal{M}_3^{\mathrm{Pl}}(\widetilde{\mathrm{GAP}(16,13)})$, such that the quotient by any bielliptic involution may only produces elliptic curves of rank 0 over $\mathbb{Q}$. In particular, the set quadratic points over $\mathbb{Q}$ is finite. To do this, we turn our attention to *non-diagonal* twists of C_A, for a fixed $A \in \mathbb{Q}^*$. Those twists are parameterized by

$$\mathcal{C}_{\underline{a},\underline{b},m,q,A} : 2\underline{a}X^4 + 8\underline{b}mX^3Y + 12\underline{a}mX^2Y^2 + 8\underline{b}m^2XY^3 + 2\underline{a}m^2Y^4 + q(X^2 - mY^2)^2 + Z^4 = 0$$

where $m \in \mathbb{Q}^*$, $\underline{a}, \underline{b}, q \in \mathbb{Q}$ satisfy $\underline{a}^2 - \underline{b}^2 m = q^4 A$ (see [28, Proposition 3.2.9], or alternatively see [29, Proposition 5.5]). Two such twists $\{\underline{a}, \underline{b}, m\}$ and $\{\underline{a}', \underline{b}', m'\}$ for C_A are equivalent if and only if $m \equiv m' \bmod {\mathbb{Q}^*}^2$ and that there exist $c, d \in \mathbb{Q}$ such that $\underline{a} + \underline{b}\sqrt{m} = (c + d\sqrt{m})^4(\underline{a}' + \underline{b}'\sqrt{m})$.

The transformed seven bielliptic involutions of any smooth plane quartic curve $C_{\underline{a},\underline{b},m,q,A}$ in the family $\mathcal{C}_{\underline{a},\underline{b},m,q,A}$ over $\mathbb{Q}$ are $\iota_i'' := P'^{-1}\iota_i' P'$, for $i = 1, \ldots, 7$, where

$$P' := \begin{pmatrix} \sqrt[4]{\underline{a} + \underline{b}\sqrt{m}} & \sqrt{m}\sqrt[4]{\underline{a} + \underline{b}\sqrt{m}} & 0 \\ \sqrt[4]{\underline{a} - \underline{b}\sqrt{m}} & -\sqrt{m}\sqrt[4]{\underline{a} - \underline{b}\sqrt{m}} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

One can check that $\mathrm{diag}(1, 1, -1)$ is the unique bielliptic involution defined over $\mathbb{Q}$ for $C_{\underline{a},\underline{b},m,q,A}$, when $A \notin {\mathbb{Q}^*}^4$ and $m \notin {\mathbb{Q}^*}^2$. Thus, the only way to obtain degree two $\mathbb{Q}$-morphisms to elliptic curves over $\mathbb{Q}$ is to quotient by $\tilde{w} := \mathrm{diag}(1, 1, -1)$. Suppose that the quotient family $\mathcal{C}_{\underline{a},\underline{b},m,q,A}/\langle\tilde{w}\rangle$ has $\mathbb{Q}$-points, which is verified when $q = -2\underline{a}$ through the point $(0 : 1 : 0)$, for example. Thus $\mathcal{C}_{\underline{a},\underline{b},m,q=-2\underline{a},A}/\langle\tilde{w}\rangle$ is $\mathbb{Q}$-isomorphic to its Jacobian variety given by:

$$\mathcal{E}_{m,q} : y^2 = x^3 + 8qmx^2 + 16q^2m^3x$$

when $\underline{a} = -\underline{b}$. In this case, $A = \frac{(1-m)}{4q^2}$ and we always impose $A \notin {\mathbb{Q}^*}^4$.

Lemma 4.8. *The two parameter family $\mathcal{E}_{m,q} : y^2 = x^3 + 8qmx^2 + 16q^2m^3x$, for $q \in \mathbb{Q}$ and $m \in \mathbb{Q}^* \setminus {\mathbb{Q}^*}^2$ such that $\frac{1-m}{4q^2} \notin {\mathbb{Q}^*}^4$, contains infinitely many elliptic curves $E_{m,q}$ of rank 0 over $\mathbb{Q}$. More precisely, for any fixed $m \in \mathbb{Z} \setminus \mathbb{Z}^2$ with $m \notin \{1 - h^2 \,|\, h \in \mathbb{Z}\}$, there exist infinitely many $q \in \mathbb{Q}^* \bmod {\mathbb{Q}^*}^2$ with $m\,q$ square-free integer, where $\mathrm{rank}_{\mathbb{Q}}(E_{m,q}) = 0$.*

Proof. Set $m, m\,q \in \mathbb{Z}$ such that $m \notin \mathbb{Z}^2$ and mq is square-free. Then, the elliptic curve $E_{m,q}$ is $\mathbb{Q}$-isomorphic to the quadratic twist $E_D : Dy^2 = x^3 + 8x^2 + 16mx$ for $E : y^2 = x^3 + 8x^2 + 16mx$ over $\mathbb{Q}$, where $D = mq$. Indeed, to obtain the equation of $E_{m,q}$ we multiply the defining equation for E_D by D^3 and next we apply the change of variables $y \mapsto (1/D^2)y$ and $x \mapsto (1/D)x$. We know that $E/\mathbb{Q}$ is modular by the work of so many people; mainly C. Breuil, B. Conrad, F. Diamond, R. Taylor and A. Wiles (cf. [11, 12, 42, 44]). Therefore, we deduce from M. R. Murty and V. K. Murty [32, Chapter 6, Theorem 1.1] (cf. the first line of the abstract in [34]) that, for a fixed m, there are infinitely many square-free integers $D = mq$ such that E_D has rank 0 over $\mathbb{Q}$, where each D is congruent to 1 mod $4\mathfrak{n}$, where $\mathfrak{n}$ is the conductor of the elliptic curve $E/\mathbb{Q}$. In particular, we get infinitely many elliptic curves $E_{m,q}$ of rank 0 over $\mathbb{Q}$, corresponding to infinitely many such q's mod ${\mathbb{Q}^*}^2$. The condition that $m \notin \mathbb{Z}^2$ with $m \notin \{1 - h^2 \,|\, h \in \mathbb{Z}\}$ is to ensure that A is not a fourth power. $\square$

Theorem 4.9. *The two-parameters family*

$$\mathcal{C}_{m,q} : -qX^4 + 4qmX^3Y - 6qmX^2Y^2 + 4qm^2XY^3 - qm^2Y^4 + q(X^2 - mY^2)^2 + Z^4 = 0,$$

for $q \in \mathbb{Q}^* \setminus {\mathbb{Q}^*}^2$ and $m \in \mathbb{Q}^*/{\mathbb{Q}^*}^2$ such that $A = \frac{1-m}{4q^2} \notin {\mathbb{Q}^*}^4$, contains infinitely many, non $\mathbb{Q}$-isomorphic, smooth plane curves C over $\mathbb{Q}$ that have only finitely many quadratic points over $\mathbb{Q}$.

Proof. If we specialize $q = -2\underline{a} = 2\underline{b}$ in the family $\mathcal{C}_{\underline{a},\underline{b},m,q,A}$ mentioned above, then we get the subfamily $\mathcal{C}_{m,q}$. In particular, any smooth plane curve in $\mathcal{C}_{m,q}$ is bielliptic and has only one bielliptic involution $\tilde{w}$ over $\mathbb{Q}$. Furthermore, by Lemma 4.8, for any $m \in \mathbb{Z} \setminus \mathbb{Z}^2$ with $m \notin \{1 - h^2 \,|\, h \in \mathbb{Z}\}$, there exists an infinite number of $q \in \mathbb{Q}^* \operatorname{mod} \mathbb{Q}^{*^2}$ with mq a square-free integer, such that $\mathcal{C}_{m,q}/\langle\tilde{w}\rangle$ is an elliptic curve of rank zero over $\mathbb{Q}$. Therefore, the number of quadratic points over $\mathbb{Q}$ is finite by Theorem 2.9. Moreover, if any two curves $C_{m,q}$ and $C_{m,q'}$, which give rank 0, are $\mathbb{Q}$-isomorphic, then $C_{(1-m)/4q^2} : \frac{1-m}{4q^2}X^4 + Y^4 + Z^4 + X^2Y^2 = 0$ and $C_{(1-m)/4q'^2} : \frac{1-m}{4q'^2}X^4 + Y^4 + Z^4 + X^2Y^2 = 0$ are $\overline{\mathbb{Q}}$-isomorphic (recall that $C_{m,q}$ and $C_{m,q'}$ are twists for $C_{(1-m)/4q^2}$ and $C_{(1-m)/4q'^2}$, respectively). But this contradicts the fact that the family $\mathcal{C}'_A : AX^4 + Y^4 + Z^4 + X^2Y^2 = 0$ is a representative family over $\mathbb{Q}$ for the locus $\mathcal{M}_3^{\text{Pl}}(\widetilde{\text{GAP}(16, 13)})$. ☐

4.3. **A version of Conjecture 4.1 over the global field $\mathbb{F}_p(T)$.** It is to be noted that formulating the conjecture 4.1 over global fields of positive characteristic has more complications than this over zero characteristic. For instance, a similar statement to Theorem 2.9 is not valid for the setting, also Theorem 2.13 does not apply when isotrivial elliptic curves appear in the picture.

In this setting, we reformulate the conjecture in terms of the behavior of bielliptic quotients instead of quadratic points as follows:

Conjecture 4.10. *Fix a prime $p > 7$ and consider a non-empty loci $\widetilde{\mathcal{M}_3^{\text{Pl}}(G)}$, where all of its $\overline{\mathbb{F}_p(T)}$-points are bielliptic curves. Then, there is an infinite set $\mathcal{D}$ (respectively, $\mathcal{E}$) in $\widetilde{\mathcal{M}_3^{\text{Pl}}(G)}(\mathbb{F}_q(T))$, of $\mathbb{F}_p(T)$-isomorphism classes of smooth plane quartic curves over $\mathbb{F}_p(T)$, such that for each $d \in \mathcal{D}$ one of their bielliptic quotients is an elliptic curve of positive $\mathbb{F}_p(T)$-rank (respectively, for each $e \in \mathcal{E}$ all bielliptic quotients have zero rank over $\mathbb{F}_p(T)$).*

Theorem 4.11. *Conjecture 4.10 is true for $\mathcal{M}_{3,biell}\widetilde{(\mathbb{Z}/6\mathbb{Z})}$.*

Proof. Since $G = \mathbb{Z}/6\mathbb{Z}$ contains only one bielliptic involution, then any $\mathbb{F}_p(T)$-point of $\mathcal{M}_3^{\text{Pl}}\widetilde{(\mathbb{Z}/6\mathbb{Z})}$ has a unique bielliptic quotient. Let $E_{a,b}$ be the elliptic curve $y^2 = x^3 + b$, which is $\mathbb{F}_p(T)$-isomorphic to $y^2 = x^4 + (b + \frac{a^2}{4})$. Let $C_{a,b}$ be the double cover $y^4 + ay^2 = x^3 + b$. For generic a and b in $\mathbb{F}_p(T)$, $C_{a,b}$ has automorphism group $\mathbb{Z}/6\mathbb{Z}$, and its $\overline{\mathbb{F}_p(T)}$-isomorphism class is uniquely determined by b/a^2. In particular, for (a, b), $(a', b') \in \mathbb{F}_p(T)^2$, we say that $(a, b) \sim (a', b')$ if and only if $b/a^2 = b'/(a')^2$.

First, taking $(a, b) \in \mathbb{F}_p(T)^2/\sim$ such that $4b + a^2 = 4(t^2 - t^3)$ gives infinitely many $C_{a,b}$ covering the isotrivial elliptic curve $E : y^2 = x^3 + t^2 - t^3$ with $(t, t) \in E(\mathbb{F}_p(T))$. By [18, Proposition 2.4], $E(\mathbb{F}_p(t))_{\text{tors}} = \{0\}$, so $\text{rank}_{\mathbb{F}_p(T)}(E) \geq 1$ and we get an inifinte set $\mathcal{D}$ as described in Conjecture 4.10.

Second, taking $(a, b) \in \mathbb{F}_p(T)^2/\sim$ such that $4b + a^2 = 4t$ reduces the problem to checking the $\mathbb{F}_p(t)$-rank of the elliptic curve $E : y^2 = x^3 + t$ that has a bad additive reduction at t and $1/t$. By the degree formula (cf. [43, Theorem 9.3]), the L-function $L(E, s)$ has degree zero as polynomial in p^{-s}. Because the Birch-Swinnerton-Dyer conjecture over elliptic curves over $\mathbb{F}_p(T)$ is known for isotrivial elliptic curve by Tate [43, Theorem 12.2], we can conclude that E has rank zero. This constructs an infinite set $\mathcal{E}$ as required.

☐

Remark 4.12. *Rewriting the argument of Lemma 4.6 for $\mathcal{M}_3^{\text{Pl}}(\widetilde{\text{GAP}(16, 13)})$ over positive characteristic yields $E : y^2 = x^3 + Dx$ with $D \in \mathbb{F}_p(T)$ of positive rank (by taking $D = -t^2/(t+1)^3$, for example). This allows us to get a $\mathcal{D}$ inside $\mathcal{M}_3^{\text{Pl}}(\widetilde{\text{GAP}(16, 13)})(\mathbb{F}_p(T))$.*

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