Bielliptic modular curves $X_0^*(N)$

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Abstract

Let $N \geq 1$ be a integer such that the modular curve $X_0^*(N)$ has genus ≥ 2. We prove that $X_0^*(N)$ is bielliptic exactly for 69 values of N. In particular, we obtain that $X_0^*(N)$ is bielliptic over the base field for all these values of N, except $X_0^*(160)$ that is not bielliptic over $\mathbb{Q}$ but it does over $\mathbb{Q}(\sqrt{-1})$. Moreover, we prove that the set of all quadratic points over $\mathbb{Q}$ for the modular curve $X_0^*(N)$ is infinite exactly for 100 values of N.

1 Introduction

Let X be a smooth projective curve defined over a number field K of genus g_X at least two. For a finite extension L/K, the set of points of X defined over L, $X(L)$, is finite. When we consider the set

$$\Gamma_2(X,K) := \cup_{[L:K]\leq 2} X(L)\,,$$

we know that this set is infinite if, and only if, X is hyperelliptic over K, i.e. there is an involution w defined over K such that the quotient curve has genus zero and K-rational points, or X is bielliptic over K, i.e. there exist an involution u (called bielliptic) whose quotient curve E has genus one, such that E is an elliptic curve over K and its K-rank is at least one (cf. [HS91] and [Bar18, Theorem 2.14]).

We focus our attention on the modular curves $X_0^*(N)$ defined as the quotient of the modular curve $X_0(N)$ by the group of all Atkin-Lehner involutions, which is defined over $\mathbb{Q}$. In this case, we know that $X_0^*(N)$ is hyperelliptic for sixty-four values of N (cf. [Has97]), and in [BG19] the following result is presented.

Theorem 1.1. *Let $N > 1$ be a square-free integer and assume that the genus of $X_0^*(N)$ is at least two. Then*

(i) *The modular curve $X_0^*(N)$ is bielliptic if, and only if, N is in the following table*

genus	N
2	106, 122, 129, 158, 166, 215, 390
3	178, 183, 246, 249, 258, 290, 303, 318, 430, 455, 510
4	370

In all these cases $X_0^*(N)$ is bielliptic over $\mathbb{Q}$.

ii) *The set $\Gamma_2(X_0^*(N),\mathbb{Q})$ is infinite if, and only if, N is in the set*

$\{67, 73, 85, 93, 103, 106, 107, 115, 122, 129, 133, 134, 146, 154, 158, 161, 165, 166, 167, 170, 177, 178, 183, 186, 191, 205, 206, 209, 213, 215, 221, 230, 246, 249, 255, 258, 266, 285, 286, 287, 290, 299, 303, 318, 330, 357, 370, 390, 430, 455, 510\}$.

*The first author is supported by MTM2016-75980-P and MDM-2014-0445.

†The second author is partially supported by DGI grant MTM2015-66180-R.

The goal of this paper is to complete the list of all bielliptic curves $X_0^*(N)$. The strategy and some tools used when N is square-free in [BG19] are based on the two following properties. On the one hand, all bielliptic involutions of the curve are defined over $\mathbb{Q}$. So, reductions at a prime $p \nmid N$ of bielliptic quotients can be treated in the finite field $\mathbb{F}_p$. On the other hand, all endomorphisms of the jacobian are defined over $\mathbb{Q}$ and, moreover, the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(\mathrm{Jac}(X_0^*(N))) \otimes \mathbb{Q}$ is isomorphic to the product of totally real number fields. So, the action of the involutions on these number fields can only be the product by ± 1.

When N is non square-free and $X_0^*(N)$ is bielliptic, both properties can fail. It may be that a bielliptic involution of $X_0^*(N)$ is not defined over $\mathbb{Q}$ when the genus of the curve is ≤ 5. Also, the algebra $\mathrm{End}_{\mathbb{Q}}(\mathrm{Jac}(X_0^*(N)))\otimes\mathbb{Q}$ can be non commutative and it can exist endomorphisms of the jacobian not defined over $\mathbb{Q}$. For these reasons, it is important to know the decomposition of the jacobian of $X_0(N)$ over $\mathbb{Q}$ and over $\overline{\mathbb{Q}}$, the number field where the involutions are defined and to determine the action of these involutions on the jacobian of the curve. Accordingly, the procedure used in [BG19] have to be adapted and expanded to this panorama.

The paper is organized as follows. In section 2, we present preliminary results. Some of these, Propositions 2.2, 2.4 and 2.6, will be used to decide when a non hyperelliptic curve $X_0^*(N)$, with N non square-free, is bielliptic or not. In section 3, we determine a finite set of positive integers that contains all values N such that $X_0^*(N)$ is bielliptic. Differently from the square-free case, a key point to delimit this finite set is based on the fact that if $X_0^*(N)$ is bielliptic and M is the biggest square-free integer such that $M|N$ and for each prime $p|M$ the p-valuation at N is odd, then $X_0^*(M)$ is bielliptic or its genus is ≤ 2. In sections 4, 5, 6 and 7 we determine the values N such that $X_0^*(N)$ is bielliptic with genus 2, 3, 4 and 5 respectively. Section 8 is devoted to determine the bielliptic curves $X_0^*(N)$ with genus ≥ 6. In fact, we prove that the curve $X_0^*(558)$ of genus 7 is the unique bielliptic curve $X_0^*(N)$ with genus ≥ 6. Finally, in section 9 we determine the curves such that the set $\Gamma_2(X_0^*(N), \mathbb{Q})$ is infinite.

The main result of this article is the following.

Theorem 1.2. *Let $N > 1$ be a non square-free integer. Assume that the genus of the modular curve $X_0^*(N)$ is at least 2. Then,*

(i) *The modular curve $X_0^*(N)$ is bielliptic if, and only if, N is in the following table*

genus	N
2	$88, 112, 116, 121, 153, 180, 184, 198, 204, 276, 284, 380$
3	$128, 144, 152, 164, 189, 196, 207, 234, 236, 240, 245, 248, 252, 294, 312, 315, 348, 420, 476$
4	$148, 160, 172, 200, 224, 225, 228, 242, 260, 264, 275, 280, 300, 306, 342$
5	$364, 444, 495$
7	558

The curve $X_0^*(160)$ is bielliptic over $\mathbb{Q}(\sqrt{-1})$, but not over $\mathbb{Q}$. For the remaining values of N, the curve $X_0^*(N)$ is bielliptic over $\mathbb{Q}$.

(ii) *The set $\Gamma_2(X_0^*(N), \mathbb{Q})$ is infinite if, and only if, N is in the set*

$\{88, 104, 112, 116, 117, 121, 125, 128, 135, 136, 147, 148, 152, 153, 164, 168, 171, 172, 176, 180, 184, 198, 204, 207, 224, 225, 228, 234, 236, 240, 248, 252, 260, 264, 276, 279, 280, 284, 312, 315, 342, 348, 364, 380, 420, 444, 476, 495, 558\}$.

2 Preliminary results

Let $N > 1$ be an integer. We fix, once and for all, the following notation.

(i) We denote by $B(N)$ the group of the Atkin-Lehner involutions of $X_0(N)$. If $N'|N$ and $\gcd(N', N/N') = 1$, $B(N')$ denotes the subgroup of $B(N)$ formed by the Atkin-Lehner involutions w_d such that $d|N'$.

(ii) The integer $\omega(N)$ is the number of primes dividing N. So, $|B(N)| = 2^{\omega(N)}$.

(iii) The integers g_N and g_N^* are the genus of $X_0(N)$ and $X_0^*(N)$ respectively.

(iv) We denote by New_N the set of normalized newforms in $S_2(\Gamma_0(N))$ and New_N^* is the subset of New_N formed by the newforms invariant under the action of the group of the Atkin-Lehner involutions $B(N)$.

(v) $S_2(N)^*$ is the vector space $S_2(\Gamma_0(N))^{B(N)}$, $J_0(N) = \mathrm{Jac}(X_0(N))$ and $J_0^*(N) = \mathrm{Jac}(X_0^*(N))$.

(vi) For a normalized eigenform $g \in S_2(\Gamma_0(N))$, A_g denotes the abelian variety defined over $\mathbb{Q}$ attached by Shimura to g. This abelian variety can be viewed as the optimal quotient of $J_0(N)$ such that the pullback of $\Omega^1_{A_g}$ is the vector space generated by the Galois conjugates of $g(q)\, dq/q$.

(vii) Given two abelian varieties A and B defined over the number field K, the notation $A \stackrel{K}{\sim} B$ stands for A and B are isogenous over K.

(viii) For an integer $m \geq 1$ and $f \in \mathrm{New}_N$, $a_m(f)$ is the m-th Fourier coefficient of f.

(ix) As usual, ψ denotes the Dedekind psi function.

When we deal with the bielliptic curves $X_0^*(N)$, there are two important differences depending on whether N is square-free or not. One of them concerns to the pullback of the regular differentials of the elliptic quotients and the other to the field of definition of the bielliptic involutions.

Indeed, on the one hand, we know that

$$J_0^*(N) \stackrel{\mathbb{Q}}{\sim} \prod_{i=1}^{s} A_{f_i}\,,$$

for some $f_i \in \mathrm{New}_{M_i}$ with $M_i|N$. When N is square-free, these abelian varieties A_{f_i} are pairwise non-isogenous over $\mathbb{Q}$ and, in particular, $\mathrm{End}_{\mathbb{Q}}(J_0^*(N))\otimes\mathbb{Q}$ is a commutative algebra isomorphic the product of totally real number fields. Every newform f_i provides an only normalized eigenform $g_i \in S_2(N)^*$ lying in the vector space generated by $\{f_i(q^d)\colon 1 \leq d|N/M_i\}$. More precisely,

$$g_i = \sum_{1\leq d|N/M_i} w_d(f_i) = \sum_{1\leq d|N/M_i} d\, f_i(q^d)\,,$$

where w_d is the Atkin-Lehner involution attached to d. Hence, if $\dim A_{f_i} = 1$, then the pullback of $\Omega^1_{A_{f_i}}$ in $\Omega^1_{X_0^*(N)}$ under any morphism $X_0^*(N) \twoheadrightarrow A_{f_i}$ is always the vector space generated by the differential $g_i(q)\, dq/q$. This property can fail for N non square-free.

The next lemma and Proposition 2.2 allow us to find a basis of normalized eigenforms of the old part in $S_2(N)^*$ coming from an eigenform whose level is a strict divisor of N.

By $B_d\colon S_2(\Gamma_0(D)) \to S_2(\Gamma_0(d\cdot D))$ we mean the linear map defined as $B_d(g(q)) = g(q^d)$.

Lemma 2.1. *Let M and N be positive integers such that $M|N$. Let M_1 be a positive divisor of M such that* $\gcd(M, M/M_1) = 1$ *and let d be a positive divisor of N/M such that* $\gcd(M_1\, d, N/(M_1\, d)) = 1$. *If $f \in S_2(\Gamma_0(M))$ is an eigenvector of the Atkin-Lehner involution w_{M_1} with eigenvalue $\varepsilon(f)$ and $\varepsilon \in \{-1, 1\}$, then $f + \varepsilon\, d\, B_d(f) \in S_2(\Gamma_0(N))$ is an eigenvector of the Atkin-Lehner involution $w_{M_1\, d}$ with eigenvalue $\varepsilon(f)\, \varepsilon$.*

Proof. An automorphism u on $X_0(N)$ whose action on the upper half-plane is given by a matrix $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})$ with positive determinant, sends a weight two cusp form h to $h(\gamma(z))\frac{\det \gamma}{(Cz+D)^2}$. The action on the Atkin-Lehner involution $w_{M_1\, d}$ of $X_0(N)$ is given for any matrix $\gamma = \begin{pmatrix} M_1\, d\, A & B \\ N\, C & M_1\, d\, D \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})$ with determinant $M_1\, d$. The following matrix $\begin{pmatrix} M_1\, A & B \\ N/d\, C & M_1\, d\, D \end{pmatrix}$ gives the action of the Atkin-Lehner involution w_{M_1} of $X_0(M)$. It is easy to check that $w_{M_1\, d}$ sends f to $\varepsilon(f)\, d\, B_d(f)$. Hence, $f + \varepsilon(f) \cdot \varepsilon\, w_{M_1\, d}(f)$ is an eigenvector of the Atkin-Lehner involution $w_{M_1\, d}$ with eigenvalue $\varepsilon(f)\, \varepsilon$. □

Proposition 2.2. *Assume that $N = p^k \cdot M$, where $k \geq 1$, p is a prime and M is an integer coprime to p. For $0 \leq i < k$, let $f \in S_2(\Gamma_0(p^i \cdot M))^{B(M)}$ be such that $w_{p^i}(f) = \varepsilon \cdot f$ (clearly $\varepsilon = 1$ when $i = 0$). Let $\mathcal{S}$ be the vector subspace of $S_2(\Gamma_0(p^k \cdot M))$ generated by the $k - i + 1$ linearly independent eigenforms $\{f, B_p(f), \cdots, B_p^{k-i}(f)\}$. Then,*

(i) *The following normalized eigenforms*

$$g_0 = (1 + pB_p)^{k-i} f, \cdots, g_j = (1 + pB_p)^{k-i-j}(1 - pB_p)^j f, \cdots, g_{k-i} = (1 - pB_p)^{k-i} f\,,$$

form a basis of $\mathcal{S}$.

(ii) *Every g_j is an eigenvector of w_{p^k} with eigenvalue $(-1)^j\varepsilon$.*

(iii) *The dimension s of the vector space $S_2(p^k \cdot M)^* \cap \mathcal{S}$ is*

$$s = \begin{cases} \dfrac{k - i + 1}{2} & \text{if } k - i \text{ is odd,} \\ \dfrac{k - i + 1 + \varepsilon}{2} & \text{if } k - i \text{ is even.} \end{cases}$$

Proof. To prove the statement, we fix i and proceed by induction over k. For $k = i + 1$, it is clear that the statement is true. Indeed, by Lemma 2.1, the eigenforms $g_0 = (1 + pB_p)f$ and $g_1 = (1 - pB_p)f$ are eigenvectors of w_{i+1} with eigenvalues ε and $-\varepsilon$ respectively and, moreover, $s = 1$.

Assume that the statement is right for $k - 1 > i$. The linear map

$$1 + pB_p \colon \langle f, \cdots, f(q^{k-i-1}) \rangle \longrightarrow \mathcal{S} = \langle f, \cdots, f(q^{k-i}) \rangle$$

is injective and, moreover, f is not in its image. By induction, $\{g_0, \cdots, g_{k-i-1}\}$ is a basis of the image of $1 + pB_p$ and, thus, $\{g_0, \cdots, g_{k-i-1}, f\}$ is a basis of $\mathcal{S}$. From the equality

$$2^{k-i} f = (1 + pB_p + 1 - pB_p)^{k-i} f = \sum_{j=0}^{k-i} \binom{k-i}{j} g_j\,,$$

it follows part (i) of the statement. By Lemma 2.1, each g_j is an eigenvector of the Atkin-Lehner involution w_{p^k} with eigenvalue $(-1)^j\varepsilon$. Part (iii) can be easily checked. □

For instance, consider a prime p not dividing M for which there is $f \in \mathrm{New}_{p\cdot M}$ such that $f \in S_2(\Gamma_0(p\cdot M))^{B(M)}$ and $w_p(f) = \varepsilon\, f$. In this case, f provides a vector subspace in the old part of $S_2(p^3\, M)^*$ of dimension 1 or 2 depending on whether $\epsilon = -1$ or not. Indeed, for $\varepsilon = -1$, we get the only normalized eigenform

$$g = (1 + p\, B_p)(1 - p\, B_p)(f) = f(q) - p^2 f(q^{p^2})\,.$$

For $\varepsilon = 1$, we get the two following eigenforms

$$g_1 = (1+p\, B_p)^2(f) = f(q)+2\, p\, f(q^p)+p^2 f(q^{p^2})\,, \quad g_2 = (1-p\, B_p)^2(f) = f(q)-2\, p\, f(q^p)+p^2 f(q^{p^2})\,.$$

In this particular case, if $\dim A_f = 1$, any direction in the vector space generated by $g_1(q)\, dq/q$ and $g_2(q)\, dq/q$ can be the pullback of $\Omega^1_{A_f}$ in $\Omega^1_{X_0^*(p^3\, M)}$ under a morphism $X_0^*(p^3\, M) \twoheadrightarrow A_f$.

On the other hand, when N is square-free, all automorphisms of $X_0^*(N)$ are defined over $\mathbb{Q}$, but this property can fail for N non square-free. The following results will be useful to know the number field where bielliptic involutions are defined.

Lemma 2.3 (Silverman-Harris)**.** *[HS91, Prop.2.1] Let X be a curve defined over a number field K with genus ≥ 6. If X is bielliptic, then there is an only bielliptic involution and, moreover, it is defined over K.*

Note that if $X_0^*(N)$ has a bielliptic involution u defined over a number field K, the bielliptic quotient is an elliptic curve defined over K because $X_0^*(N)$ is non empty.

Proposition 2.4. *Let A be a modular abelian variety defined over $\mathbb{Q}$ such that $A \overset{\mathbb{Q}}{\sim} \prod_{i=1}^m A_{f_i}^{n_i}$ for some $f_i \in \mathrm{New}_{N_i}$, where A_{f_i} are pairwise non-isogenous over $\mathbb{Q}$. All endomorphisms of A are defined over $\mathbb{Q}$ if, and only if, for every non trivial quadratic Dirichlet character χ, the newform $f_i \otimes \chi$ is different from any Galois conjugate of f_j for all i and j.*

Proof. For two newforms $f \in \mathrm{New}_N$ and $f' \in \mathrm{New}_{N'}$, we have that $A_f \overset{\mathbb{Q}}{\sim} A_{f'}$ if, and only if, $N = N'$ and f' is a Galois conjugate of f (see Proposition3.2 in [BGGP05]). It is well-known that the abelian variety A_f is simple over $\overline{\mathbb{Q}}$ if, and only if, f does not have any inner twist, i.e. there is not any quadratic Dirichlet character χ such that $f \otimes \chi$ is a Galois conjugated of f. This condition amounts to saying that $\mathrm{End}_{\mathbb{Q}}(A_f) = \mathrm{End}_{\overline{\mathbb{Q}}}(A_f)$.

Assume that f and f' does not have any inner twist. If A_f and $A_{f'}$ are isogenous over $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$, then there exists a Dirichlet character χ such that $A_{f'} = A_{f\otimes\chi}$ (cf. Proposition 4.2 in [GJU12]). $\square$

Remark 2.5. *If χ is the quadratic Dirichlet character attached to the quadratic number field $K = \mathbb{Q}(\sqrt{D})$, then there is an isogeny between the abelian varieties A_f and $A_{f\otimes\chi}$ defined over K.*

By using Petri's Theorem, in [BG19, Lemma 13], it is characterized the existence of a bielliptic involution of $X_0^*(N)$, when $g_N^* > 2$, $X_0^*(N)$ is not hyperelliptic and N is square-free. Next, we present Proposition 2.6 that will be useful to determine the existence of a bielliptic involution of $X_0^*(N)$ with bielliptic quotient E, when E appears in the splitting of $J_0^*(N)$ with exponent > 1 and $X_0^*(N)$ is not hyperelliptic. This result generalizes the mentioned lemma and can be applied to any curve X (modular or not).

Let X be a non-hyperelliptic curve of genus $g \geq 3$ defined over a number field K. For a fixed basis $\omega_1, \cdots, \omega_g$ of $\Omega^1_{X/K}$ and an integer $i \geq 2$, we denote by $\mathcal{L}_i$ the K-vector space formed by the homogenous polynomials $Q \in K[x_1, \cdots, x_g]$ of degree i such that $Q(\omega_1, \cdots, \omega_g) = 0$.

Proposition 2.6. *With the above notation, assume that* $\mathrm{Jac}(X) \stackrel{K}{\sim} E^m \times A$, *where* E *is an elliptic curve and* A *an abelian variety that does not have* E *as a quotient defined over* K. *Denote by* $I_{g-m} \in M_{g-m}(K)$ *the identity matrix. Take the basis* $\{\omega_i\}$ *such that* $\{\omega_1, \cdots, \omega_m\}$ *and* $\{\omega_{m+1}, \cdots, \omega_g\}$ *are bases of the pullbacks of* $\Omega^1_{E^m/K}$ *and* $\Omega^1_{A/K}$ *respectively. Then,* E *is* K*-isogenous to the Jacobian of a bielliptic quotient of* X *over* K *if, and only if, there exists a matrix* $\mathcal{A} \in \mathrm{GL}_m(K)$ *that satisfies*

$$Q((-x_1, x_2, \cdots, x_g) \cdot \mathcal{B}) \in \mathcal{L}'_i \text{ for all } Q \in \mathcal{L}_i \text{ and for all } i \geq 2\,, \tag{2.1}$$

where $\mathcal{B}$ *is the matrix* $\left(\begin{array}{c|c} \mathcal{A} & 0 \\ \hline 0 & I_{g-m} \end{array}\right) \in \mathrm{GL}_g(K)$ *and* $\mathcal{L}'_i = \{Q((x_1, x_2, \cdots, x_g) \cdot \mathcal{B}))\colon Q \in \mathcal{L}_i\}$.

Proof. Let u be a bielliptic involution of X over K such that the curve E is K-isogenous to the jacobian of the quotient curve $X/\langle u \rangle$. The linear map u^* acting on the vector space $\Omega^1_{X/K}$ is diagonalizable and has an only eigenvalue equals to 1, because $X/\langle u\rangle$ has genus one. Since E is not a quotient of A defined over K, the linear map u^* leaves the vector spaces $\Omega^1_{E^m/K}$ and $\Omega^1_{A/K}$ stable. Hence, $u(\omega_i) = -\omega_i$ for $i > m$ and there exists a matrix $\mathcal{A} \in \mathrm{GL}_m(K)$ such that the regular differentials $(\omega'_1, \omega'_2, \cdots, \omega'_m) = (\omega_1, \omega_2, \ldots, \omega_m) \cdot \mathcal{A}^{-1}$ are eigenvectors of u with eigenvalues $1, -1, \cdots, -1$ respectively. The condition 2.1 follows from the fact that the vector spaces $\mathcal{L}'_i$ are those attached to the basis $\omega'_1, \cdots, \omega'_m, \omega_{m+1}, \cdots, \omega_g$. For the converse, see the proof of [BG19, Lemma 13]. □

Remark 2.7. *We recall that if* $g = 3$, *then the condition (2.1) can be restricted to* $i = 4$ *and* $\dim \mathcal{L}_4 = 1$. *When* $g > 3$, $\dim \mathcal{L}_2 = (g-3)(g-2)/2$. *In this case, it suffices to check (2.1) only for* $i = 2, 3$ *and, in the particular case that* X *is neither a smooth quintic plane curve* ($g = 6$) *nor a trigonal curve, we can restrict the condition to* $i = 2$.

As in [BG19], for $1 \leq j \leq g$, we introduce the K-vector space

$$\mathcal{L}_{2,j} = \{Q \in \mathcal{L}_2 \colon Q(x_1, \cdots, x_{j-1}, -x_j, x_{j+1}, \cdots, x_g) \in \mathcal{L}_2\}\,.$$

By using that the polynomials in $\mathcal{L}_2$ are irreducible, in [BG19] it is proved that

$$\mathcal{L}_{2,j} = \{Q \in \mathcal{L}_2 \colon Q(x_1, \cdots, x_{j-1}, x_j, x_{j+1}, \cdots, x_g) = Q(x_1, \cdots, x_{j-1}, -x_j, x_{j+1}, \cdots, x_g)\}\,.$$

Since the spaces $\mathcal{L}'_i$ in Proposition 2.6 are the spaces $\mathcal{L}_i$ corresponding to the basis $\omega'_1, \cdots, \omega'_m$, $\omega_{m+1}, \cdots, \omega_g$ of $\Omega^1_{X/K}$, the space $\mathcal{L}'_{2,1}$ is formed by the polynomials in $\mathcal{L}'_2$ that are even in the first variable x_1.

A similar result is obtained when $g = 3$ and we replace $\mathcal{L}_2$ with $\mathcal{L}_4$.

Remark 2.8. *We have* $J_0^*(N) \stackrel{\mathbb{Q}}{\sim} \prod_{i=1}^s A_{f_i}^{n_i}$, *for some* $f_i \in \mathrm{New}_{M_i}$ *with* $M_i | N$ *and the abelian varieties* A_{f_i} *are pairwise non isogenous over* $\mathbb{Q}$. *Any* f_i *determines* n_i *normalized eigenforms* g_j *in* $S_2(N)^*$ *such that* $J_0^*(N) \stackrel{\mathbb{Q}}{\sim} \prod_{j=1}^r A_{g_j}$, *where* $r = \sum_{i=1}^s n_i$ *and* $g_1, \cdots, g_r$ *are all of these eigenforms.*

The basis of the Galois conjugates of the newforms f_i *together with the exponents* n_i *allow us to compute* $|X_0^*(\mathbb{F}_p)|$ *for all primes* $p \nmid N$, *thanks to the Eichler-Shimura congruence. The basis of the regular differentials formed by all Galois conjugates of* $g_j(q)\,dq/q$ *allows us to compute equations for* $X_0^*(N)$. *When* $\dim A_{f_i} = 1$, *we take as basis of* $\Omega^1_{A_{f_i}^{n_i}/\mathbb{Q}}$ *the regular differentials attached to the normalized eigenforms obtained from* f_i *via Proposition 2.2.*

Now, we assume that $X_0^*(N)$ has a bielliptic involution u defined over $\mathbb{Q}$. Let us denote by E the elliptic quotient $X_0^*(N)/\langle u\rangle$ and by π the non constant morphism $X_0(N)\to X_0^*(N)\to E$ of degree $2^{\omega(N)+1}$ which is defined over $\mathbb{Q}$. Let M be the conductor of E. It is well-known, that $M|N$ and there exists a morphism $\pi_M\colon X_0^*(M)\twoheadrightarrow E$ and a normalized newform $f_E\in \text{New}_M^*$ such that $\pi_M^*(\Omega^1_{E/\mathbb{Q}})=\mathbb{Q}(f_E(q)d\,q/q)$. Moreover, $\pi^*(\Omega^1_{E/\mathbb{Q}})=\mathbb{Q}(g(q)d\,q/q)$, where $g\in\ S_2(N)^*$ is an eigenform in the vector space generated by $\{f_E(q^i)\colon 1\leq i|N/M\}$. Note that for a prime $p\nmid M$, $|E(\mathbb{F}_{p^2})|\leq (p+1)^2$. More precisely, due to the congruence of Eichler-Shimura, $|E(\mathbb{F}_{p^2})|=(p+1)^2-a_p^2(f_E)$.

From [BG19, Lemmas 5 and 7], we have the following two lemmas when E is a bielliptic quotient of $X_0^*(N)$ over $\mathbb{Q}$.

Lemma 2.9. *The following inequalities hold:*

(i) *For a prime $p\nmid N$,*

$$\text{(a) } \frac{\psi(N)}{2^{\omega(N)}}\leq 12\cdot\frac{2|E(\mathbb{F}_{p^2})|-1}{p-1},\quad \text{(b) } g_N^*\leq 2\frac{|E(\mathbb{F}_{p^2})|}{p-1},\quad \text{(c) } g_N\leq 2^{\omega(N)+1}\frac{|E(\mathbb{F}_{p^2})|}{p-1}.$$

(ii) *For a prime p such that the p-adic valuation $v_p(N)=1$, then $g^*_{N/p}\leq 1$ or*

$$\frac{\psi(N/p)}{2^{\omega(N)-1}}\leq 12\frac{2\,|E(\mathbb{F}_{p^2})|-1}{p-1},\quad g^*_{N/p}\leq 2\frac{|E(\mathbb{F}_{p^2})|}{p-1},\quad g_{N/p}\leq 2^{\omega(N)}\frac{|E(\mathbb{F}_{p^2})|}{p-1},\quad \text{if } p\nmid M,$$
$$\frac{\psi(N/p)}{2^{\omega(N)-1}}\leq 12\frac{2p^2+1}{p-1},\quad g^*_{N/p}\leq 2\frac{p^2+1}{p-1},\quad g_{N/p}\leq 2^{\omega(N)}\frac{p^2+1}{p-1},\quad \text{if } p|M.$$

We recall that the condition $v_p(N)=1$ implies that the normalization of the curve $X_0^*(N)/\mathbb{F}_p$ is $X_0^*(N/p)/\mathbb{F}_p$.

Lemma 2.10. *Let E' be the elliptic curve in the $\mathbb{Q}$-isogeny class of E that is an optimal quotient of the jacobian of $X_0^*(M)$. If $M=N$, then the degree of the modular parametrization $\pi_N\colon X_0(N)\to E'$ divides $2^{\omega(N)+1}$.*

3 Selecting candidates for bielliptic curves $X_0^*(N)$

The starting point of our selection is based on the following result.

Lemma 3.1. *Let ℓ be a prime. If for an integer $k\geq 2$, the curve $X_0^*(\ell^k\cdot M)$ is bielliptic, then $X_0^*(\ell^{k-2}\cdot M)$ is hyperelliptic, bielliptic or has genus at most one.*

Proof. We can assume without loss of generality that $\gcd(M,\ell)=1$. We claim that the morphism $\pi\colon X_0(\ell^k\cdot M)\to X_0(\ell^{k-2}\cdot M)$ induced by the map $z\mapsto \ell z$ on the upper half-plane, provides a morphism from $X_0^*(\ell^k\cdot M)$ to $X_0^*(\ell^{k-2}\cdot M)$. Indeed, for $w_d\in B(M)$ one has $\pi\circ w_d=w_d\circ\pi$ and for $w_{\ell^k}\in B(\ell^k\cdot M)$ one has $\pi\circ w_{\ell^k}=w_{\ell^{k-2}}\circ\pi$. The statement follows from [HS91, Proposition 1]. □

Corollary 3.2. *Let $N>1$ be an integer such that $X_0^*(N)$ has genus ≥ 2. Let M be the biggest square-free integer such that $M|N$ and each prime $p|M$ its p-valuation at N is odd. If $X_0^*(N)$ is bielliptic, then $X_0^*(M)$ is bielliptic, hyperelliptic (equivalently $g_M^*=2$) or $g_M^*\leq 1$.*

When N is a power of a prime, we have that $X_0^*(N)=X_0^+(N)$ and from [Jeo18] we know all bielliptic curves $X_0^*(N)$.

Proposition 3.3 (Jeon). *Assume N is a non-square free integer which is a power of a prime, with $g_N^* \geq 2$. Then, N is bielliptic if, and only if, $N = 121 = 11^2$, or $N = 128 = 2^7$ ($g_{121}^* = 2$ and $g_{128}^* = 3$).*

For a non-square free integer $N > 1$, let M be the biggest square-free divisor of N such that for each prime $p|M$ its p-valuation at N is odd. Therefore, if $X_0^*(N)$ is bielliptic, then M is a value appearing in part (i) of Theorem 1.1 or in table 4 of the Appendix corresponding to $g_N^* \leq 2$. Moreover, we can assume that N is divisible by at least two primes. Let $\mathcal{M}$ be the set of such square-free integers M and let $\mathcal{N}$ be the set of integers N that are not a power of a prime, such that, its biggest square-free divisor such that for each prime $p|M$ its p-valuation at N is odd is in the set $\mathcal{M}$.

To discard curves $X_0^*(N)$ with $N \in \mathcal{N}$ that are not bielliptic over $\mathbb{Q}$, we will apply the same strategy as the used in [BG19], that is summarized in Proposition 3.4. Indeed, for a prime $p \nmid N$, $X_0^*(N)(\mathbb{F}_{p^k})$ denotes the set of the $\mathbb{F}_{p^k}$-points of the reduction modulo p of $X_0^*(N)$. Set

$$P_p(k) := \operatorname{mod}\,[(\sum_{d|k} \mu(k/d)|X_0^*(N)(\mathbb{F}_{p^k})|)/k, 2]$$

where $\operatorname{mod}\,[m, 2]$ denotes 0 or 1 depending on whether m is even or not, and μ is the Moebius function. Set $Q_p(2j+1) = \sum_{i\geq 0}^{j}(2i+1)P_p(2i+1)$. By [Gon17, Theorem 2.1], if $X_0^*(N)$ has an involution defined over $\mathbb{Q}$, then

$$Q_p(2j+1) \leq 2g_N^* + 2 \text{ for all } j \geq 0\,.$$

Moreover, if E is a bielliptic quotient over $\mathbb{Q}$ of $X_0^*(N)$, then

$$n(N, E, {p_E}^{k_E}) := |X_0^*(N)(\mathbb{F}_{p^k})| - 2|E(\mathbb{F}_{p^k})| \leq 0$$

for all $k > 0$.

From previous arguments and Lemmas 2.9(i)(a) and 2.10, it follows the next result.

Proposition 3.4. *Suppose that the curve $X_0^*(N)$ with $g_N^* \geq 2$ satisfies one of the three following conditions:*

(i) There exists a prime $p \nmid N$ such that $\frac{1}{2^{\omega(N)}} \cdot \frac{p-1}{12}\psi(N) + 1 > 2 \cdot (p+1)^2$,

(ii) There exist a prime $p \nmid N$ and an integer $j \geq 0$ such that $Q_p(2j+1) > 2g_N^* + 2$,

(iii) Every elliptic curve E defined over $\mathbb{Q}$ of conductor N_E which appear in the $\mathbb{Q}$-decomposition of $J_0^*(N)$ satisfies:

- *If N_E is a strict divisor or N, then there exist a prime $p_E \nmid N$ and a positive natural k_E such that $n(N, E, {p_E}^{k_E}) > 0$.*
- *If $N_E = N$, then the above condition on the existence of a prime $p_E \nmid N$ and a positive natural k_E is satisfied or the degree of the Weil strong parametrization of E does not divide $2^{\omega(N)+1}$.*

Then, the curve $X_0^*(N)$ is not bielliptic over $\mathbb{Q}$.

All objects appeared in Proposition 3.4 can be computed by Magma (and Cremona tables). We refer to the web page http://mat.uab.cat/~francesc/Magmaprogrammesxostar.html for the programmes in Magma computing the different objects involved in Proposition 3.4, and in particular the $\mathbb{Q}$-decomposition of the Jacobian of $X_0^*(N)$ for any N.

To find all bielliptic curves $X_0^*(N)$ over $\mathbb{Q}$, we proceed by determining a finite set of pairs (N, E) such that E is the $\mathbb{Q}$-isogeny class of an elliptic curve defined over $\mathbb{Q}$ and E can be a bielliptic quotient of $X_0^*(N)$ for $N \in \mathcal{N}$.

3.1 Case $g_N^* > 5$

If $g_N^* > 5$ and $X_0^*(N)$ is bielliptic, then $X_0^*(N)$ is bielliptic over $\mathbb{Q}$ (cf. Lemma 2.3).

We present the following example to show how Proposition 3.4 and Lemma 3.1 are applied. Take $M = 178 = 2 \cdot 89$ for which $X_0^*(178)$ is bielliptic and $g_M^* = 3$. We claim that for all $N = 2^a \cdot 89^b \cdot d^2$ with $a, b, d \geq 1$ and $a + b > 2$ with $\gcd(2 \cdot 89, d) = 1$, the curves $X_0^*(N)$ are not bielliptic curves. Indeed, by Lemma 3.1, it is enough to prove the assertion for $N = 2^2 \cdot 89$, $N = 89^2$, $N = 2 \cdot 89^2$, $N = 2^3 \cdot 89$ and $N = 2 \cdot 89^3$. For $N = 89^2$, $2 \cdot 89^2$,$2 \cdot 89^3$ and $2^3 \cdot 89$ one has $g_N^* > 6$ and we can discard these cases by applying part (i) of Proposition 3.4 because $\psi(N)$ satisfy the inequality with $p = 3$. For $N = 4 \cdot 89 = 356$, we have $g_N^* = 8$ and the $\mathbb{Q}$-isogeny classes of the elliptic curves over $\mathbb{Q}$ in $J_0^*(356)$ are $E89a$ and $E178a$ with Cremona notation. For $p = 3 \nmid N$, we obtain $n(356, E89a, 9) = n(356, E178a, 9) = 32 - 30 > 0$. Hence, by part (iii) of Proposition 3.4, the curve $X_0^*(356)$ is not bielliptic.

Applying part (i) of Lemma 2.9, Proposition 3.4 and Lemma 3.1, for $g_N^* \geq 6$ we can reduce the study to the following pairs (N, E) (see the detailed results for all levels and elliptic curves involved in http://mat.uab.cat/∼francesc/Tablestudy.pdf).

g_N^*	(N, E)
6	$(244, 61a), (272, 34a), (332, 83a), (332, 166a), (336, 42a), (336, 112a),$ $(564, 94a), (620, 62a), (780, 65a), (780, 130c)$
7	$(320, 32a), (324, 27a), (360, 20a), (360, 30a), (450, 15a)$ $(450, 75b), (456, 57a), (456, 76a), (456, 152a), (492, 123b),$ $(504, 21a), (504, 36a), (504, 42a), (550, 55a),$ $(550, 275a), (550, 550a), (558, 558a), (636, 53a), (660, 110b)$ $(924, 77a), (924, 462a)$
8	$(408, 102a), (468, 26b), (468, 234b), (468, 234c), (480, 20a), (480, 24a), (480, 80b),$ $(480, 160a), (540, 45a), (540, 54b), (990, 66a), (990, 99a),$ $(1020, 102a)$
9	$(560, 56a), (560, 70a), (560, 280a)(1140, 190b), (1140, 285b)$
10	$(840, 20a), (840, 140b), (840, 210d), (1050, 175b)$
11	$(672, 112c), (672, 224a)$
13	$(1260, 21a), (1260, 70a), (1260, 90b), (1260, 210d)$

Table 1

where the $\mathbb{Q}$-isogeny class of E is described with Cremona notation.

Lemma 3.5. *The curve $X_0^*(1020)$ of genus 8 is not bielliptic.*

Proof. If $X_0^*(1020)$ is bielliptic, then it is bielliptic over $\mathbb{Q}$ because $g_{1020}^* \geq 6$. Computing the values $n(1020, E, p^n)$ for all possible bielliptic quotients, we get that E should be $E102a$ (see http://mat.uab.cat/∼francesc/Tablestudy.pdf). Suppose that there is a degree two morphism defined over $\mathbb{Q}$ from $X_0^*(1020)$ to an elliptic curve in the $\mathbb{Q}$-isogeny class of $E102a$. Since $9 \nmid 1020$, the normalization of $X_0^*(1020)/\mathbb{F}_3$ is the curve $X_0^*(340)/\mathbb{F}_3$. Hence, this curve has an involution defined over $\mathbb{F}_3$ whose quotient is a curve of genus zero defined over $\mathbb{F}_3$ because $3 \mid 102$. But this is not possible because $|X_0^*(340)(\mathbb{F}_3)| = 13 > 8$. □

3.2 Case $g_N^* \leq 5$

In Table 4 of the Appendix, there are all the values of N such that $g_N^* \leq 5$. We point out that the abelian variety A_f has an elliptic quotient over $\overline{\mathbb{Q}}$ if, and only if, A_f is isogenous to the power of an elliptic curve over $\overline{\mathbb{Q}}$. When $\dim A_f > 1$ and f does not have complex multiplication (CM), i.e. $f \neq f \otimes \chi$ for all quadratic Dirichlet characters, a necessary condition to have A_f an elliptic quotient over $\overline{\mathbb{Q}}$ is $a_p(f)^2 \in \mathbb{Z}$ for all primes p (cf. [Pyl04]).

When the genus $g_N^* \leq 5$ and $N \in \mathcal{N}$, we compute the splitting of $J_0^*(N)$ over $\mathbb{Q}$. We can discard the values N such that none of the abelian varieties A_f involved in the splitting of $J_0^*(N)$ over $\mathbb{Q}$ is a power of an elliptic curve over $\overline{\mathbb{Q}}$. This occurs for $N = 117, 147, 261$ and 279. For instance, for $N = 261$, one has $J_0^*(261) \stackrel{\mathbb{Q}}{\sim} A_{f_1} \times A_{f_2}$ with $\dim A_{f_i} = 2$, where

$$\begin{aligned} f_1(q) = &\quad q + a\,q^2 + q^3 + (a-1)\,q^4 + (-2a+2)\,q^5 + a\,q^6 + (-2a-1)q^7 + \dots\,, \\ f_2(q) = &\quad\quad\quad\quad\quad\quad q - a\,q^2 + (a-1)\,q^4 - 2\,q^5 + (2a-1)\,q^7 + \cdots\,, \end{aligned}$$

and $a^2 - a - 1 = 0$. Since $a^2 \notin \mathbb{Z}$, the curve $X_0^*(261)$ does not have any elliptic quotient and, thus, it is not bielliptic.

For $N = 136, 164, 175, 196, 234, 245, 250, 294, 376$ and 495, all involutions are defined over $\mathbb{Q}$ because in all these cases in the splitting of $J_0^*(N)$ over $\mathbb{Q}$, there is only one abelian variety A_f of dimension 1 and the abelian varieties A_f with dimension > 1 do not have any elliptic quotient over $\overline{\mathbb{Q}}$.

When all abelian varieties A_f with dimension > 1 involved in the splitting of $J_0^*(N)$ are not isogenous to a power of an elliptic curve over $\overline{\mathbb{Q}}$, if $X_0^*(N)$ is bielliptic over $\overline{\mathbb{Q}}$ but not over $\mathbb{Q}$, then there exist two elliptic curves over $\mathbb{Q}$ in the splitting of $J_0^*(N)$ such that they are not isogenous over $\mathbb{Q}$ but their attached newforms are twisted by a quadratic Dirichlet character.

By the list of the values of N such that $2 \leq g_N^* \leq 5$ (see the appendix) and the above paragraph, we obtain that the only possible values of N for which $X_0^*(N)$ is a bielliptic curve when $g_N^* \leq 5$ are the following,

g_N^*	N
2	$88, 104, 112, 116, 135, 153, 168, 180, 184, 198, 204, 276, 284, 380$
3	$136, 144, 152, 162, 164, 171, 189, 196, 207, 234, 236, 240, 245, 248, 252,$ $270, 294, 312, 315, 348, 420, 476.$
4	$148, 160, 172, 176, 200, 224, 225, 228, 242, 260, 264, 275, 280, 300, 306, 308, 342, 350.$
5	$192, 208, 212, 216, 316, 364, 376, 378, 396, 414, 440, 444, 495, 572, 630.$

Table 2

When $X_0^*(N)$ is bielliptic over $\mathbb{Q}$ and $g_N^* \leq 5$, after applying Proposition 3.4, for each N we can restrict our study to the following possible bielliptic quotients E.

g_N^*	$\mathbf{N} : E$
2	**88** : $44a, 88a$; **104** : $26b, 52a$; **112** : $56a, 112a$; **116** : $58a, 58b$; **135** : $45a, 135a$; **153** : $51a, 153a$; **168** : $42a, 84b$; **180** : $30a, 90b$; **184** : $92a, 184b$; **198** : $66a, 99a$; **204** : $34a, 102a$; **276** : $138a, 138c$; **284** : $142b, 142d$; **380** : $38b, 190b$.
3	**136** : $34a$; **144** : $24a, 36a, 48a$; **152** : $38b, 152a$; **162** : $27a, 54a$; **164** : $82a$; **171** : $57a, 57c$; **189** : $21a$; **196** : $14a$; **207** : $69a$; **234** : $234c$; **236** : $118a, 118b, 118c$; **240** : $20a, 24a, 240c$; **245** : $35a$; **248**; $62a, 124b$; **252** : $21a, 42a$; **270** : $30a, 45a, 135a$; **294** : $14a$; **312** : $26b, 52a, 312b$; **315** : $21a$; **348**; $58a, 58b, 174c$; **420** : $42a, 70a, 210d$; **476** : $34a, 238b, 238d$.
4	**148** : $37a$; **160** : $20a, 160a$; **172** : $43a$; **176** : $44a$; **200**; $40a, 50b$; **224** : $56a, 224a$; **225** : $15a, 225a$; **228** : $57a$; **242** : $11a, 121b$; **260** : $65a$; **264**; $44a, 66b, 88a, 132b$; **275** : $55a$; **280** : $20a, 70a, 280a$; **300** : $15a, 50b, 150c$; **306** : $102a, 102c$; **342** : $342e$; **350** : $175b, 350c$.
5	**192** : $24a, 32a, 192a$; **216** : $36a, 54b$; **364** : $91a$; **376**; $94a$; **378** : $63a, 189a$; **396**; $66a, 99a, 198c$; **414** : $69a, 138a$; **444** : $37a, 222b$; **495** : $99a$; **630**; $21a, 210d$.

Table 3

By using results due to Hasegawa, we can prove that some values of N in table 2 corresponds to bielliptic curves $X_0^*(N)$ over $\mathbb{Q}$. First, we recall that if $4|N$ (resp. $9|N$) the map $z \mapsto z+1/2$ (resp. $z \mapsto z+1/3$) on the upper-half plane provides an automorphism S_2 (resp. S_3) of $X_0(N)$. The automorphism S_2 is an involution defined over $\mathbb{Q}$ and S_3 has order 3, defined over $\mathbb{Q}(\sqrt{-3})$ and $\sigma(S_3) = S_3^{-1}$ for the non trivial conjugation σ of $\mathbb{Q}(\sqrt{-3})$.

Proposition 3.6 (Proposition 7, Has97)**.** *Assume that $g_N^* \geq 2$ and $N = 4N'$ with N' odd. Then, the three following quotients curves of $X_0(N)$ are isomorphic over $\mathbb{Q}$:*

$$X_0^*(N) \cong X_0(N)/\langle S_2 \text{ and } B(N')\rangle \cong X_0(2N')/\langle B(N')\rangle .$$

Corollary 3.7. *Assume that $g_N^* \geq 2$ and $N = 4N'$ with N' odd. If $g_{2N'}^* = 1$, then $X_0^*(N)$ is a bielliptic curve over $\mathbb{Q}$.*

Proof. Consider the degree two morphism of $X_0(2N')/\langle w_d \in B(N')\rangle \to X_0^*(2N')$ given by the quotient of the Atkin-Lehner involution w_2. □

Corollary 3.8. *The curve $X_0^*(N)$ is a bielliptic curve over $\mathbb{Q}$ for the following values of N:*

$g_{N/2}^*$	g_N^*	N
1	2	116, 180, 204, 276, 284, 380.
1	3	164, 196, 236, 252, 348, 420, 476.
1	4	148, 172, 228, 260, 300.
1	5	364, 444.

In order to study involutions of $X_0^*(N)$, Hasegawa obtained the following result.

Proposition 3.9 (Corollary, p.371, Has97)**.** *Suppose $8|N$ (resp. $9||N$). Let 2^ν be the maximal power of 2 dividing N. Then, $V_2 := S_2 w_{2^\nu} S_2$ (resp. $V_3 := S_3 w_9 S_3^2$) defines an involution on $X_0^*(N)$.*

It is clear that V_2 is defined over $\mathbb{Q}$. Also, V_3 is defined over $\mathbb{Q}$, because if σ is the non trivial conjugation of $\mathbb{Q}(\sqrt{-3})$, then $\sigma(V_3) = V_3^{-1} = V_3$.

Hasegawa determined the genus of the quotients curves by these involutions and, in particular, those which are 1.

Corollary 3.10 (Table 3, Has97)**.** *For the following N, $X_0^*(N)$ is bielliptic over $\mathbb{Q}$ and some bielliptic involutions are also listed,*

g_N^*	N	Bielliptic involution
3	144	V_2, V_3, V_2V_3
	152	V_2
	234	V_3
	240	V_3
	312	V_2
4	264	V_2
	280	V_2
	306	V_3
	342	V_3

Corollary 3.11. *The hyperelliptic curves $X_0^*(207)$ and $X_0^*(315)$ of genus 3 are bielliptic over $\mathbb{Q}$.*

Proof. By Table 3 in [Has97], we know that the curves $X_0^*(207)/\langle V_3\rangle$ and $X_0^*(315)/\langle V_3\rangle$ have genus two. By [Acc94], if X is a genus 3 curve which has a map to a genus 2 curve then X is hyperelliptic and bielliptic. Since the hyperelliptic involution w is defined over $\mathbb{Q}$, the involution $w\cdot V_3$ is bielliptic and defined over $\mathbb{Q}$. □

Remark 3.12. *The levels N for which we know that $X_0^*(N)$ is bielliptic, by the work of Hasegawa, and $g_N^*>2$ will be studied again when we compute equations for these curves. In this way, we will be able to precise the bielliptic quotients and their ranks and, as a consequence, to determine the infinitude of the set $\Gamma_2(X_0^*(N),\mathbb{Q})$.*

Remark 3.13. *Given a curve X over a number field of genus g_X, its gonality γ, i.e. the lowest degree map to the projective line, satisfies the inequality*

$$2\le\gamma\le\left\lfloor\frac{g_X+3}{2}\right\rfloor.$$

Therefore if $X_0^*(N)$ is bielliptic and non-hyperelliptic with $g_N^*=3$ or 4, then it has gonality three and is a trigonal curve. Hasegawa and Shimura listed all trigonal modular curves $X_0^*(N)$ in [HS00]. This result (jointly with [Has97]) allows us to present the complete list of the values N such that $3\le g_N^*\le 4$ in Appendix A. Also, it is useful when we find equations for the candidate non hyperelliptic curves $X_0^*(N)$ such that $g_N^*>4$ (see Remark 2.7).

4 Case $g_N^*=2$

Proposition 4.1. *Let N be a non square-free integer. The curve $X_0^*(N)$ has genus two and is bielliptic if, and only if, N lies in the set*

$$\{88,112,116,121,153,180,184,198,204,276,284,380\}.$$

In all theses cases, the curves are bielliptic over $\mathbb{Q}$.

Proof. In all possible cases in Table 2, the splitting of the Jacobian of these curves over $\mathbb{Q}$ is the product of two elliptic curves. For $N=121$, the curve $X_0^*(N)$ is bielliptic over $\mathbb{Q}$ (see [Gon16] or [Jeo18]). For $N\neq 121$, the Jacobian has all its endomorphisms defined over $\mathbb{Q}$ and, in particular, the bielliptic involutions are defined over $\mathbb{Q}$. When $N=116,180,204,276,284,380$, we know that $X_0^*(N)$ is bielliptic (cf. Corollary 3.8). For the remaining cases to be considered, $J_0^*(N)\stackrel{\mathbb{Q}}{\sim}A_{f_1}\times A_{f_2}$, where

N	f_1	f_2
88	$f_1\in\mathrm{New}_{88}^*$	$f_2\in\mathrm{New}_{44}^{B(11)}$, $w_4(f_2)=-f_2$
104	$f_1\in\mathrm{New}_{52}^{B(13)}$, $w_4(f_1)=-f_1$	$f_2\in\mathrm{New}_{26}^{B(13)}$, $w_2(f_2)=-f_2$
112	$f_1\in\mathrm{New}_{112}^*$	$f_2\in\mathrm{New}_{56}^{B(7)}$, $w_8(f_2)=-f_2$
135	$f_1\in\mathrm{New}_{135}^*$	$f_2\in\mathrm{New}_{45}^{B(5)}$, $w_9(f_2)=-f_2$
153	$f_1\in\mathrm{New}_{153}^*$	$f_2\in\mathrm{New}_{51}^{B(17)}$, $w_3(f_2)=-f_2$
168	$f_1\in\mathrm{New}_{84}^{B(21)}$, $w_4(f_1)=-f_1$	$f_2\in\mathrm{New}_{42}^{B(21)}$, $w_2(f_2)=-f_2$
184	$f_1\in\mathrm{New}_{184}^*$	$f_2\in\mathrm{New}_{92}^{B(23)}$, $w_4(f_2)=-f_2$
198	$f_1\in\mathrm{New}_{99}^*$	$f_2\in\mathrm{New}_{66}^{B(22)}$, $w_3(f_2)=-f_2$

Taking g_1 and g_2 as the following eigenforms

N	g_1	g_2
88	$f_1(q)$	$f_2(q)-2f_2(q^2)$
104	$f_1(q)-2f_1(q^2)$	$f_2(q)-4f_2(q^4)$
112	$f_1(q)$	$f_2(q)-2f_2(q^2)$
135	$f_1(q)$	$f_2(q)-3f_2(q^3)$
153	$f_1(q)$	$f_2(q)-3f_2(q^3)$
168	$f_1(q)-2f_1(q^2)$	$f_2(q)-4f_2(q^4)$
184	$f_1(q)$	$f_2(q)-2f_2(q^3)$
198	$f_1(q)+2f_1(q^2)$	$f_2(q)-3f_2(q^3)$,

$g_1(q)\,d\,q/q$ and $g_2(q)\,d\,q/q$ is a basis of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ formed by the pullbacks of regular differentials of the elliptic quotients. The functions

$$X=\frac{g_1}{g_2}\,,\quad Y=\frac{\mathrm{d}\,X}{g_2}q\,,$$

satisfy the following equations

$$\begin{aligned}
N=88:&\quad & Y^2 &= 1+X^2+3X^4-X^6\,,\\
N=104:&\quad & 81\,Y^2 &= 40+48X+105X^2+244X^3-12X^4+336X^5-32X^6\,,\\
N=112:&\quad & Y^2 &= 1+X^4+2X^6\,,\\
N=135:&\quad & 9Y^2 &= -4+24X-3X^2+32X^3+12X^4+12X^5+8X^6\,,\\
N=153:&\quad & Y^2 &= 3+7X^2+9X^4-3X^6\,,\\
N=168:&\quad & 81Y^2 &= 112+192X+105X^2+280X^3-48X^4+120X^5-32X^6\,,\\
N=184:&\quad & 2Y^2 &= 1-X^2+7X^4+X^6\,,\\
N=198:&\quad & Y^2 &= (3+X^2)(-2+3X^2+3X^4)\,.
\end{aligned}$$

Only the curves whose equations are of the form $Y^2=P(X^2)$ for a polynomial P of degree 3 are bielliptic over $\mathbb{Q}$ (cf. [BG19, Remark 10]). Since for $N=104,135,168$ the newforms involved in the splitting of $J_0^*(N)$ are not twisted and without complex multiplication, all endomorphisms of $J_0^*(N)$ are defined over $\mathbb{Q}$ and $X_0^*(N)$ is not bielliptic over $\overline{\mathbb{Q}}$. □

5 Case $g_N^*=3$

Proposition 5.1. *Let N be a non square-free integer. The curve $X_0^*(N)$ has genus 3 and is bielliptic if, and only if, N lies in the set*

$$\{128,144,152,164,189,196,207,234,236,240,245,248,252,294,312,315,348,420,476\}\,.$$

In all these cases, the curves are bielliptic over $\mathbb{Q}$.

Proof. In Table 2, we only have hyperelliptic curves for $N=136,171,207,252,315$ and, in all cases, the jacobians have all their endomorphisms defined over $\mathbb{Q}$. Their equations can be found in [Has97]:

$$\begin{array}{lll} N=136: & Y^2= & X(1+X)(-2+3X+X^2)(-4+2X+5X^2+4X^3+X^4)\,,\\ N=171: & Y^2= & (3+3X+X^2)(67+241X+324X^2+209X^3+72X^4+13X^5+X^6)\,,\\ N=207: & Y^2= & 1-6X+11X^2-12X^3+9X^4-12X^5+11X^6-6X^7+X^8\,,\\ N=252: & Y^2= & 21-42X+73X^2-74X^3+64X^4-38X^5+17X^6-6X^7+X^8\,,\\ N=315: & Y^2= & (1+X+3X^2+X^3+X^4)(1+5X+3X^2+5X^3+X^4)\,. \end{array}$$

It can be checked that the automorphisms group for $N=136,171$ has order 2 and, thus, the curves are not bielliptic. For $N=207,252,315$, the automorphism group is always isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, generated by the hyperelliptic involution and the involutions $(X,Y)\mapsto(1/X,Y/X^4)$, $(X,Y)\mapsto((X+1)/(X-1),4Y/(X-1)^4)$ and $(X,Y)\mapsto(1/X,Y/X^4)$ respectively . Hence, $X_0^*(207)$, $X_0^*(252)$ and $X_0^*(315)$ are bielliptic. More precisely, their elliptic quotients for such bielliptic involutions are respectively

N	equation	E
207	$V^2= U^4-6U^3+7U^2+6U-11\,,$ with $U=\frac{X^2+1}{X}\,,\quad V=\frac{Y}{X^2}$	$E69a\,,$
252	$V^2= U^4-6U^3+7U^2-6U+1\,,$ with $U=\frac{X^2+1}{X-1}\,,\quad V=\frac{Y}{(X-1)^2}$	$E21a\,,$
315	$V^2= U^4+6U^3+7U^2+6U+1\,,$ with $U=\frac{X^2+1}{X}\,,\quad V=\frac{Y}{X^2}$	$E21a\,.$

We know that $X_0^*(128)$ is bielliptic over $\mathbb{Q}$ (cf. [Jeo18]). For the remaining 18 cases to be considered, the splitting of the jacobian over $\mathbb{Q}$ is as follows:

$$\begin{array}{lllll}
J_0^*(144)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E24a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E36a\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E48a\,,\\
J_0^*(152)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E38b\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E76a\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E152a\,,\\
J_0^*(162)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E27a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E54a\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E162a\,,\\
J_0^*(164)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^2 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E82a\,, & f_2\in\mathrm{New}_{82}^{B(41)}\,, & \dim A_{f_2}=2\,,\\
J_0^*(189)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E21a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E63a\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E189a\,,\\
J_0^*(196)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^2 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E14a\,, & f_2\in\mathrm{New}_{98}^{B(49)}\,, & \dim A_{f_2}=2\,,\\
J_0^*(234)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^2 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E234c\,, & f_2\in\mathrm{New}_{39}^{B(13)}\,, & \dim A_{f_2}=2\,,\\
J_0^*(236)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E118a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E118b\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E118c\,,\\
J_0^*(240)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E20a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E24a\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E240c\,,\\
J_0^*(245)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^2 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E35a\,, & A_{f_2}\in\mathrm{New}_{245}^*\,, & \dim A_{f_2}=2\,,\\
J_0^*(248)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E62a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E124b\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E248a\,,\\
J_0^*(270)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E30a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E45a\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E135a\,.\\
J_0^*(294)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^2 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E14a\,, & f_2\in\mathrm{New}_{147}^*\,, & \dim A_{f_2}=2\,,\\
J_0^*(312)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E26b\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E52a\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E312b\,.\\
J_0^*(348)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E58a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E58b\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E174c\,.\\
J_0^*(420)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E42a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E70a\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E210d\,.\\
J_0^*(476)\stackrel{\mathbb{Q}}{\sim}\prod_{i=1}^3 A_{f_i}\,, & A_{f_1}\stackrel{\mathbb{Q}}{\sim}E34a\,, & A_{f_2}\stackrel{\mathbb{Q}}{\sim}E238b\,, & A_{f_3}\stackrel{\mathbb{Q}}{\sim}E238d\,.
\end{array}$$

The existence of twists only occurs for $N=144$:

$$f_3=f_1\otimes\chi_{-1}\,,\quad f_2=f_2\otimes\chi_{-3}\,,$$

where χ_D denotes the quadratic Dirichlet character attached to the quadratic field $\mathbb{Q}(\sqrt{D})$. The splitting of $J_0^*(144)$ over $\mathbb{Q}$ and over $\mathbb{Q}(\sqrt{-3})$ agree. This fact implies that if A_{f_2} is a bielliptic quotient of $X_0^*(144)$, then the bielliptic involution is defined over $\mathbb{Q}$. The splitting over $\mathbb{Q}(\sqrt{-1})$ is $J_0^*(144) \overset{\mathbb{Q}(\sqrt{-1})}{\sim} A_{f_1}^2 \times A_{f_2}$. Hence, it may be that A_{f_1} is the quotient of the curve $X_0^*(144)$ by an involution defined over $\mathbb{Q}(\sqrt{-1})$ but not over $\mathbb{Q}$.

We take a basis $\omega_1, \omega_2, \omega_3$ of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ obtained as the union of the bases corresponding to each A_{f_i} and following the order exhibited in the splitting of its Jacobian. A basis of $\Omega^1_{A_{f_i}}$ formed by eigenforms is obtained by applying Proposition 2.2 to all Galois conjugate of f_i. If $\dim A_{f_i} > 1$, we replace this with a basis of the $\mathbb{Z}$-module $\Omega^1_{A_{f_i}/\mathbb{Q}} \cap \mathbb{Z}[[q]]$. We compute a nonzero homogenous polynomial $Q \in \mathbb{Q}[x, y, z]$ of degree 4 such that $Q(\omega_1, \omega_2, \omega_3) = 0$. We obtain

N	Q
144	$x^4 - 4x^2y^2 + 3y^4 + 2x^2z^2 - 6y^2z^2 + 4z^4$
152	$8x^4 + 16x^3y + 3x^2y^2 - 32xy^3 + 5y^4 - 27x^2z^2 - 108xyz^2 + 54y^2z^2 + 81z^4$
162	$x^4 + 2x^3y - 3x^2y^2 + 2xy^3 - 2y^4 + 9xy^2z - 9xyz^2 + 9y^2z^2 - 6xz^3 + 6yz^3 - 9z^4$
164	$x^4 - 4x^2y^2 - 4x^2yz + 32y^3z - 2x^2z^2 + 4y^2z^2 - 12yz^3 + z^4$
189	$9x^4 - 18x^2y^2 + 9y^4 - 12x^2yz - 4y^3z - 24x^2z^2 + 24y^2z^2 + 16z^4$
196	$x^4 - 12x^2y^2 + 32y^4 - 4x^2yz + 32y^3z - 2x^2z^2 + 12y^2z^2 - 12yz^3 + z^4$
234	$3x^4 - 13x^2y^2 + 12y^4 + 19x^2yz - 12y^3z - x^2z^2 + y^2z^2 + yz^3 - 2z^4$
236	$81x^4 + 54x^2y^2 + 73y^4 - 108x^2yz + 92y^3z - 108x^2z^2 - 84y^2z^2 + 32yz^3 - 32z^4$
240	$x^4 - y^4 + 2x^2z^2 + 2y^2z^2 - 4z^4$
245	$x^4 + 2x^2y^2 - 3y^4 - 8x^2yz + 8y^3z - 8x^2z^2 - 8y^2z^2 + 16yz^3 + 16z^4$
248	$16x^4 + 32x^3y + 6x^2y^2 + 44xy^3 - 17y^4 - 54x^2z^2 + 108xyz^2 - 54y^2z^2 - 81z^4$
270	$9x^4 + 36x^3y - 42x^2y^2 + 4xy^3 - 7y^4 + 72x^3z + 72xy^2z - 16y^3z + 24x^2z^2 - 48xyz^2$ $+24y^2z^2 - 64xz^3 + 64yz^3 - 128z^4$
294	$x^4 - y^4 + 6x^2yz + 2y^3z - 10x^2z^2 + 2y^2z^2 - 24yz^3 + 24z^4$
312	$32x^4 + 64x^3y + 30x^2y^2 - 56xy^3 + 11y^4 - 126x^2z^2 - 288xyz^2 + 90y^2z^2 + 243z^4$
348	$27x^4 - 30x^2y^2 - 13y^4 - 84x^2yz + 4y^3z + 60x^2z^2 + 36y^2z^2 - 32yz^3 + 32z^4$
420	$5x^4 + 12x^3y - 4x^2y^2 - 8xy^3 + 3y^4 + 2x^2z^2 - 12xyz^2 - 6y^2z^2 + 8z^4$
476	$3x^4 + 6x^2y^2 + 8y^4 + 4x^3z - 4xy^2z - 4x^2z^2 - 18y^2z^2 + 8xz^3 - 3z^4$

Let ω be the regular differential attached to an elliptic quotient E of one of these curves $X_0^*(N)$. An equivalent condition for E to be a bielliptic quotient is that the polynomial Q must be even in the variable corresponding to ω. Only the curves corresponding to $N = 162, 175, 270$ are not bielliptic. The $\mathbb{Q}$-isogeny classes of all bielliptic involutions for the remaining cases are

N	E	N	E	N	E
144	$24a, 36a, 48a$	234	$234a$	294	$14a$
152	$152a$	236	$118a$	312	$312b$
164	$82a$	240	$20a, 24a, 240c$	348	$58a$
189	$21a$	245	$35a$	420	$210d$
196	$14a$	248	$248a$	476	$238b$

The curve $X_0^*(144)$ is bielliptic over $\mathbb{Q}$ and has 3 bielliptic involutions. Applying Proposition 2.6, it can be checked that this curve only has these 3 bielliptic involutions. □

6 Case $g_N^* = 4$

Proposition 6.1. *Let N be a non square-free integer. The curve $X_0^*(N)$ has genus 4 and is bielliptic if, and only if, $N \in \{148, 160, 172, 200, 224, 225, 228, 242, 260, 264, 275, 280, 300, 306, 342\}$.*

For $N=160$, the curve is bielliptic over $\mathbb{Q}(\sqrt{-1})$ but not over $\mathbb{Q}$. For the remaining cases $X_0^*(N)$ is bielliptic over $\mathbb{Q}$.

Proof. For $N \neq 160, 200, 225, 242, 275$ in Table 2, the absence of twists between the newforms involved in the splitting of $J_0^*(N)$ over $\mathbb{Q}$ implies that all involutions of the curve are defined over $\mathbb{Q}$. By using Table 3, we can discard $N=308$.

Only for $N=176$, the curve $X_0^*(N)$ is hyperelliptic. In [Has97], it is determined the following equation:

$$Y^2 = X(X^3-4\,X+4)(X^3-2\,X^2)(X^3+2\,X-2)\,.$$

For instance, by using Magma, it can be checked that the automorphism group has order 2 and, thus, the curve is not bielliptic.

For the remaining 16 cases, the splitting of the Jacobian over $\mathbb{Q}$ is as follows:

$$\begin{array}{llllll}
J_0^*(148) \stackrel{\mathbb{Q}}{\sim} & A_{f_1}^2 \times A_{f_2}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E37a\,, & f_2 \in \text{New}_{74}^{B(37)}\,, & \dim A_{f_2}=2\,, & \\
J_0^*(160) \stackrel{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E20a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E80b\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E160a\,, & \\
J_0^*(172) \stackrel{\mathbb{Q}}{\sim} & A_{f_1}^2 \times A_{f_2}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E43a\,, & f_2 \in \text{New}_{86}^{B(43)}\,, & \dim A_{f_2}=2\,, & \\
J_0^*(200) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E20a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E40a\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E50a\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E100a\,, \\
J_0^*(224) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E56a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E112a\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E112c\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E224a\,, \\
J_0^*(225) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E15a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E75b\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E75c\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E225a\,, \\
J_0^*(228) \stackrel{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E57a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E38b\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E114c\,, & \\
J_0^*(242) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E11a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E121b\,, & A_{f_3} \in \text{New}_{242}^*\,, & \dim A_{f_3}=2\,, \\
J_0^*(260) \stackrel{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E65a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E26a\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E130c\,, & \\
J_0^*(264) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E44a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E66b\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E88a\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E132b\,, \\
J_0^*(275) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E55a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E275a\,, & A_{f_3} \in \text{New}_{275}^*\,, & \dim A_{f_3}=2\,, \\
J_0^*(280) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E20a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E70a\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E140b\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E280a\,, \\
J_0^*(300) \stackrel{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E15a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E50b\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E150c\,, & \\
J_0^*(306) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E51a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E102a\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E102c\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E153c\,, \\
J_0^*(342) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E57a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E57b\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E57c\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E342e\,, \\
J_0^*(350) \stackrel{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E175b\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E350c\,, & A_{f_3} \in \text{New}_{35}^{B(7)}\,, & \dim A_{f_3}=2\,,
\end{array}$$

For the values $N \in \{160, 200, 225, 242, 275\}$, we have the following twists:

160	200	225	242	275
$f_2 = f_1 \otimes \chi_{-1}$	$f_4 = f_1 \otimes \chi_5$	$f_3 = f_1 \otimes \chi_5$	$f_2 = f_2 \otimes \chi_{-11}$	$f_2 = f_1 \otimes \chi_5\,.$

The splitting over $\overline{\mathbb{Q}}$ of $J_0^*(N)$ is as follows:

$$J_0^*(160) \stackrel{\mathbb{Q}(\sqrt{-1})}{\sim} A_{f_1}^3 \times A_{f_3}\,, \quad J_0^*(200) \stackrel{\mathbb{Q}(\sqrt{5})}{\sim} A_{f_1}^2 \times A_{f_2} \times A_{f_3}\,,\; J_0^*(225) \stackrel{\mathbb{Q}(\sqrt{5})}{\sim} A_{f_1}^2 \times A_{f_2} \times A_{f_4}\,,$$

$$J_0^*(242) \stackrel{\mathbb{Q}(\sqrt{-11})}{\sim} A_{f_1} \times A_{f_2} \times A_{f_3}\,, \quad J_0^*(275) \stackrel{\mathbb{Q}(\sqrt{5})}{\sim} A_{f_1}^2 \times A_{f_3}\,.$$

For $N=242$, A_{f_2} is an elliptic curve with CM by $\mathbb{Q}(\sqrt{-11})$ and $\text{End}(J_0^*(242)) \otimes \mathbb{Q}$ is isomorphic to $\mathbb{Q} \times \mathbb{Q}(\sqrt{-11}) \times \mathbb{Q}$. So, an involution of $X_0^*(242)$ has to be defined over $\mathbb{Q}$. For the remaining four values of N, all involutions are defined over $\mathbb{Q}(\sqrt{D})$.

We take a basis $\omega_1, \omega_2, \omega_3, \omega_4$ of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ following the order exhibited in the splitting of its Jacobian. We know that $\dim \mathcal{L}_2 = 1$. It turns out that all these curves are trigonal (cf. Remark 3.13). Hence, the curves are determined by $\mathcal{L}_2$ and $\mathcal{L}_3$, whose dimension is 5. So, the curve is determined by a nonzero polynomial $Q_2 \in \mathcal{L}_2$ and any polynomial $Q_3 \in \mathcal{L}_3$ that is not a multiple of Q_2.

Now, we focus our attention to bielliptic curves over $\mathbb{Q}$. First, we only consider the values of $N \neq 148, 160, 172, 228, 260, 300$. For each of these values, the endomorphism algebra $\mathrm{End}_{\mathbb{Q}}(J_0^*(N)) \otimes \mathbb{Q}$ is isomorphic to the product of totally real number fields and we can apply the same procedure as in [BG19]. We compute a nonzero polynomial $Q_2 \in \mathcal{L}_2$:

N	Q_2
200	$-10t^2 + 15tx + 9x^2 - 18y^2 - 10tz - 6xz + 20z^2$
224	$-4t^2 + 2x^2 + xy - y^2 + 3xz + yz - 2z^2$
225	$-9t^2 + 3x^2 + 5y^2 + z^2$
242	$-36t^2 + 9x^2 + 24ty - 5y^2 + 12tz + 4yz - 8z^2$
264	$-112t^2 - 51tx + 63x^2 + 80ty - 30xy - 175y^2 + 225z^2$
275	$-8t^2 + 3x^2 + 5y^2 + 4tz - 20z^2$
280	$-75t^2 + 12x^2 + 20xy + 25y^2 + xz + 30yz - 13z^2$
306	$27t^2 + x^2 - 3xy - 9y^2 - 17xz - 15yz + 16z^2$
342	$-216t^2 + 117x^2 + 72xy - 56y^2 + 90xz - 104yz + 97z^2$
350	$-171t^2 + 150tx + 85x^2 - 204ty - 20xy + 160y^2 + 468tz - 420xz + 240yz - 432z^2$

We can discard the value $N = 350$ because for every elliptic differential ω_i, one has

$$Q_2(\cdots, x_i, \cdots) \neq Q_2(\cdots, -x_i, \cdots)\,.$$

For the remaining values, we compute a polynomial $Q_3 \in \mathcal{L}_3$ that is not a multiple of Q_2:

N	Q_3
200	$-5t^3 - 27tx^2 + 54ty^2 - 54xy^2 + 99txz + 27x^2z - 162y^2z - 30tz^2 - 18xz^2 + 116z^3$
224	$4x^3 - 3xy^2 + 2y^3 + 4x^2z - 2xyz - 3y^2z + xz^2 - 3z^3$
225	$-9x^2y + 5y^3 + 4z^3$
242	$144t^3 - 36tx^2 - 24x^2y - 44ty^2 + 12y^3 - 3x^2z - 32tyz - 9y^2z + 40tz^2 + 24yz^2$
264	$20t^2x - 45tx^2 - 72t^2y + 104txy + 18x^2y + 180ty^2 + 20xy^2 - 255tz^2 + 180xz^2 - 150yz^2$
275	$-12t^3 + 12tx^2 - 45x^2y + 20ty^2 + 25y^3 + 51x^2z + 85y^2z + 48tz^2$
280	$351t^2x - 156xy^2 + 130y^3 - 276t^2z - 12x^2z + 58xyz - 64y^2z + 77xz^2 - 108yz^2$
306	$-32t^2x - 96t^2y + 18xy^2 + 33y^3 + 50t^2z + 22x^2z + 74xyz + 33y^2z - 52xz^2 - 83yz^2 + 33z^3$
342	$-11349t^2x - 5626t^2y - 1067xy^2 + 970y^3 + 18421t^2z - 468x^2z - 1064xyz + 2067y^2z$ $-4628xz^2 + 2744yz^2$

Hence, for all these values $N \neq 225$, the curve $X_0^*(N)$ is bielliptic over $\mathbb{Q}$ with a unique bielliptic involution, because there is only an elliptic differential ω_i such that

$$\begin{aligned} Q_2(\cdots, \omega_i, \cdots) - Q_2(\cdots, -\omega_i, \cdots) &= 0\,, \\ \frac{Q_3(\cdots, \omega_i, \cdots) - Q_3(\cdots, -\omega_i, \cdots)}{x_i} &\in \mathcal{L}_2\,. \end{aligned} \tag{6.1}$$

The $\mathbb{Q}$-isogeny classes of their bielliptic quotients are $E40a$, $E224a$, $E11a$, $E88a$, $E55a$, $E280a$, $E153c$ and $E342e$ respectively. For $N = 225$ there are two bielliptic involutions corresponding to the elliptic differentials ω_1 and ω_4 and their bielliptic quotients are $E15a$ and $E225a$.

For the cases $N \in \{148, 160, 172, 228, 260, 300, 308\}$, $J_0^*(N) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \times A$, where A_{f_1} is an elliptic curve. The polynomials Q_2 are

N	Q_2
148	$-8t^2 - 30tx + 25x^2 + 6ty + 5xy + 2y^2 - 72tz - 10xz + 2yz - 112z^2$
160	$-48t^2 + 8tx + 3x^2 - 8ty + 6xy - y^2 + 36xz + 12yz - 8z^2$
172	$-32t^2 - 20tx + 25x^2 + 4ty + 20xy + 3y^2 + 272tz - 40xz + 8yz - 128z^2$
228	$512t^2 - 80tx + 75x^2 + 16ty - 120xy - 3y^2 + 80tz - 100xz + 20yz - 400z^2$
260	$68t^2 - 40tx + 25x^2 + 50xy + 25y^2 - 136tz + 40xz - 32z^2$
300	$170t^2 - 60tx + 45x^2 - 18xy - 9y^2 + 80tz - 48xz - 160z^2$

The splitting of A over $\mathbb{Q}$ does not contain any bielliptic quotient of $X_0^*(N)$ over $\mathbb{Q}$. By Proposition 2.6, in the $\mathbb{Q}$-isogeny class of A_{f_1} there is a bielliptic quotient if, and only if, there exists a matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ such that the polynomials

$$R_2(x,y,z,t) := Q_2(a_1x + a_2y, b_1x + b_2y, z, t), \quad R_3(x,y,z,t) := Q_3(a_1x + a_2y, b_1x + b_2y, y, z, t)$$

satisfy the conditions

$$\begin{aligned} R_2(x,y,z,t) - R_2(-x,y,z,t) &= 0\,, \\ \frac{R_3(x,y,z,t) - R_3(-x,y,z,t)}{x} &= \lambda\, R_2(x,y,z,t)\,, \end{aligned} \tag{6.2}$$

for some $\lambda \in \mathbb{Q}$. For instance, we can assume $a_1 \in \{0,1\}$.

It can be checked that the first condition in (6.2) does not happen for $N = 160$. For the remaining 5 cases, we obtain the following polynomial $Q_3 \in \mathcal{L}_3$:

N	Q_3
148	$344t^2x - 318tx^2 + 35x^3 - 72t^2y + 28x^2y - 18ty^2 + xy^2 + 2304t^2z + 216txz$
	$-26x^2z - 72tyz - 80xyz - 6y^2z + 768tz^2 + 16xz^2 - 48yz^2$
172	$-272t^2x + 125x^3 + 112t^2y + 84txy - 4ty^2 - 37xy^2 - 8y^3 - 640t^2z + 712txz$
	$-250x^2z - 632tyz - 32xyz - 38y^2z - 480tz^2 + 112xz^2 + 208yz^2$
228	$-3248t^2x + 7725tx^2 + 1350x^3 - 4880t^2y + 4056txy + 555ty^2 + 1242xy^2 + 2352t^2z$
	$-4700txz - 3000x^2z + 76tyz - 600xyz - 528y^2z - 4080tz^2 - 800xz^2 + 4480yz^2$
260	$-9916t^2x + 2245tx^2 + 1700x^3 + 3810txy + 1905ty^2 - 5440t^2z + 7672txz$
	$-2720x^2z - 1360xyz - 680y^2z + 2240tz^2 + 544xz^2$
300	$4850t^2x - 1725tx^2 + 1530x^3 - 330txy - 165ty^2 - 1140t^2z + 1160txz$
	$-1020x^2z + 816xyz + 408y^2z + 240tz^2 - 4624xz^2\,.$

Taking the following matrix $\mathcal{A}$

	148, 172, 228	260, 300
$\mathcal{A}$	$\begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$,

the reader can check that the conditions (6.2) are satisfied. The bielliptic $\mathbb{Q}$-isogeny class of the bielliptic quotients are $E37a$, $E43a$, $E57a$, $E65a$ and $E15a$ respectively.

Finally, we claim that $X_0^*(160)$ is bielliptic over $\overline{\mathbb{Q}}$. For the basis chosen to compute Q_2, we obtain as $Q_3(x,y,z,t)$ the following polynomial:

$$-80t^3+13tx^2+2x^3+10txy+ty^2-2xy^2+60txz+6x^2z+20tyz-4xyz-2y^2z-24tz^2-4xz^2+4yz^2\,.$$

Again by Proposition 2.6, it suffices to find a matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in \mathrm{GL}_3(\mathbb{Q}(\sqrt{-1}))$ such that the polynomials

$$\begin{aligned} R_2(x,y,z,t) &= Q_2(a_1x+a_2y+a_3z, b_1x+b_2y+b_3z, c_1x+c_2y+c_3z, t)\,, \\ R_3(x,y,z,t) &= Q_3(a_1x+a_2y+a_3z, b_1x+b_2y+b_3z, c_1x+c_2y+c_3z, t)\,, \end{aligned}$$

satisfy

$$R_2(x,y,z,t) = R_2(-x,y,z,t)\,, \quad R_3(-x,y,z,t) \in \mathcal{L}'_3\,. \tag{6.3}$$

The second condition in (6.3) amounts to saying

$$R_3(x,y,z,t) - R_3(-x,y,z,t) = \lambda\, x\, R_2(x,y,z,t)\,, \lambda \in \mathbb{Q}(\sqrt{-1})\,. \tag{6.4}$$

As usual, set $i = \sqrt{-1}$. Taking the matrix

$$\mathcal{A} = \begin{pmatrix} i & i & 1 \\ 1 & -3 & 0 \\ 0 & -4\,i & 1 \end{pmatrix},$$

we get

$$\begin{aligned} R_2(x,y,z,t) = {} & -48t^2-(16-48i)x^2+32ty-24y^2+32i\,tz-48i\,yz+(8-48i)z^2\,, \\ R_3(x,y,z,t) = {} & -80t^3-(48-80i)tx^2-(32-32i)x^2y-8ty^2-16y^3-(32+32i)x^2z \\ & -16ityz-48iy^2z-(40+80i)tz^2+(16-32i)yz^2+(32-16i)z^3\,, \end{aligned}$$

which satisfy the conditions (6.4). More precisely, put

$$(\omega'_1, \omega'_2, \omega'_3) = (\omega_1, \omega_2, \omega_3)\cdot \mathcal{A}^{-1}\,.$$

The linear map $u : (\omega'_1, \omega'_2, \omega'_3, \omega_4) \mapsto (\omega'_1, -\omega'_2, -\omega'_3, -\omega_4)$ acts on the basis ω_i as follows

$$u\colon (\omega_1,\omega_2,\omega_3,\omega_4) \mapsto (-\frac{1}{4}\omega_1 + \frac{1}{4}\omega_2 + i\,\omega_3, \frac{3}{4}\omega_1 - \frac{3}{4}\omega_2 + i\,\omega_3, -\frac{3}{4}i\,\omega_1 - \frac{1}{4}i\,\omega_2, -\omega_4)\,,$$

which defines a bielliptic involution on $X_0^*(160)$. □

Remark 6.2. *When $X_0^*(N)$ is bielliptic and $J_0^*(N)$ has endomorphisms not defined over $\mathbb{Q}$, we do not determine all bielliptic involutions in the group $\mathrm{Aut}_{\overline{\mathbb{Q}}}(X_0^*(N))$. For instance, the curve $X_0^*(275)$ has an only bielliptic involution defined over $\mathbb{Q}$:*

$$(\omega_1,\omega_2,\omega_3,\omega_4) \mapsto (\omega_1, -\omega_2, -\omega_3, -\omega_4)\,.$$

Nevertheless, it has another involution defined over $\mathbb{Q}(\sqrt{5})$:

$$(\omega_1,\omega_2,\omega_3,\omega_4) \mapsto (-\frac{1}{2}\omega_1 - \frac{\sqrt{5}}{2}\omega_2, 3\frac{\sqrt{5}}{10}\omega_1 - \frac{1}{2}\omega_2, -\omega_3, -\omega_4)\,.$$

7 Case $g_N^* = 5$

Proposition 7.1. *Let N be a non square-free integer. The curve $X_0^*(N)$ has genus 5 and it is bielliptic if, and only if, $N = 364, 444, 495$. In all these cases, the curve is bielliptic over $\mathbb{Q}$.*

Proof. For $N \neq 192, 208, 216, 378$ in Table 2, the absence of twists between the newforms involved in the splitting of $J_0^*(N)$ implies that all involutions of the curve are defined over $\mathbb{Q}$. By Table 3, we can exclude $N = 212, 316, 440, 572$ because for these values none of the pairs (N, E) is bielliptic over $\mathbb{Q}$.

The splitting of the Jacobian over $\mathbb{Q}$ is as follows:

$$
\begin{array}{lllllll}
J_0^*(192) & \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E24a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E32a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E96b\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E192a\,,\\
J_0^*(208) & \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E26b\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E52a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E104a\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E208b\,,\\
J_0^*(216) & \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^5 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E36a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E54b\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E72a\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E108a\,,\\
 & & & & & & A_{f_5} \overset{\mathbb{Q}}{\sim} E216a\,,\\
J_0^*(364) & \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E91a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E26b\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E182a\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E182d\,,\\
J_0^*(376) & \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E94a\,, & f_2 \in \mathrm{New}_{188}^{B(47)}\,, & \dim A_{f_2} = 2\,, & f_3 \in \mathrm{New}_{376}^*\,,\\
 & & & & & & \dim A_{f_3} = 2\,,\\
J_0^*(378) & \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^5 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} 21a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E63a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E126b\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E189a\,,\\
 & & & & & & A_{f_5} \overset{\mathbb{Q}}{\sim} E378d\,,\\
J_0^*(396) & \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E99a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E66a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E66b\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E198c\,,\\
J_0^*(414) & \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E69a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E138a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E138b\,, & A_{f_4} \in \mathrm{New}_{207}^*\,,\\
 & & & & & & \dim A_{f_4} = 2\,,\\
 & & & & & & A_{f_5} \overset{\mathbb{Q}}{\sim} E440a\,,\\
J_0^*(444) & \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E37a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} 222b\,, & f_3\, \mathrm{New}_{74}^{B(37)}\,, & \dim A_{f3} = 2\,,\\
J_0^*(495) & \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^3 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E99a\,, & f_2 \in \mathrm{New}_{165}^*\,, & \dim A_{f_2} = 2\,, & f_3 \in \mathrm{New}_{165}^{B(55)}\,,\\
 & & & & & & \dim A_{f_3} = 2\,,\\
J_0^*(630) & \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^4 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E21a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E30a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E210d\,, & A_{f_4} \in \mathrm{New}_{315}^*\,,\\
 & & & & & & \dim A_{f_4} = 2\,,
\end{array}
$$

For $N \in \{192, 208, 216, 378\}$, we have the following twists:

192	208	216	378
$f_4 = f_3 \otimes \chi_{-2}$	$f_4 = f_3 \otimes \chi_{-1}$	$f_1 = f_1 \otimes \chi_{-3}$	$f_2 = f_1 \otimes \chi_{-3}\,.$

The splitting over $\overline{\mathbb{Q}}$ of the jacobians of these curves is as follows:

$$J_0^*(192) \overset{\mathbb{Q}(\sqrt{-2})}{\sim} A_{f_1}^2 \times A_{f_2} \times A_{f_3}^2\,, \quad J_0^*(208) \overset{\mathbb{Q}(\sqrt{-1})}{\sim} A_{f_1}^2 \times A_{f_2} \times A_{f_3}^2\,,$$

$$J_0^*(216) \overset{\mathbb{Q}(\sqrt{-3})}{\sim} \prod_{i=1}^{5} A_{f_i}\,, \quad J_0^*(378) \overset{\mathbb{Q}(\sqrt{-3})}{\sim} A_{f_1}^2 \prod_{i=3}^{5} A_{f_i}\,.$$

First, we find which curves $X_0^*(N)$ are bielliptic curves over $\mathbb{Q}$. For the values of N different from $192, 208, 216, 364, 396$ and 444, the endomorphism algebra of the Jacobian of $X_0^*(N)$ is isomorphic to the product of totally real number fields and we can apply the same procedure as in [BG19]. As before, we take a basis $\omega_1, \omega_2, \omega_3, \omega_4, \omega_5$ of $\Omega^1_{X_0^*(N)/\mathbb{Q}}$ following the order exhibited in the splitting of its Jacobian.

We know that $\dim \mathcal{L}_2 = 3$. For all these values, the curve $X_0^*(N)$ is neither trigonal (cf. [HS00]) nor a smooth plane quintic. Hence, if ω_j is the regular differential of an elliptic curve E, the pair $(X_0^*(N), E)$ is bielliptic over $\mathbb{Q}$ if, and only if, $\dim \mathcal{L}_{2,j} = 3$.

N	$\dim \mathcal{L}_{2,j}$	N	$\dim \mathcal{L}_{2,j}$
216	$\dim \mathcal{L}_{2,1} = \dim \mathcal{L}_{2,2} = 1$	414	$\dim \mathcal{L}_{2,1} = \dim \mathcal{L}_{2,2} = 0$
376	$\dim \mathcal{L}_{2,1} = 0$	495	$\dim \mathcal{L}_{2,1} = 3$
378	$\dim \mathcal{L}_{2,2} = \dim \mathcal{L}_{2,4} = 0$	630	$\dim \mathcal{L}_{2,1} = \dim \mathcal{L}_{2,3} = 0$

Hence, for $N \in \{216, 376, 378, 414, 630\}$, the curve $X_0^*(N)$ is not bielliptic over $\mathbb{Q}$. The curve $X_0^*(495)$ is bielliptic and the $\mathbb{Q}$-isogeny class of its bielliptic quotient is $E99a$.

By Table 3, we know that the curve $X_0^*(208)$ is not bielliptic over $\mathbb{Q}$. For the values of $N \in \{192, 364, 396, 444\}$, A_{f_1} is a bielliptic quotient of $X_0^*(N)$ over $\mathbb{Q}$ if, and only if, there is a matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ such that for all $Q_2 \in \mathcal{L}_2$ the polynomials

$$R_2(x_1, x_2, x_3, x_4, x_5) := \quad Q_2(a_1x_1 + a_2x_2, b_1x_1 + b_2x_2, x_3, x_4, x_5)\,,$$

satisfy the condition

$$R_2(x_1, x_2, x_3, x_4, x_5) = R_2(-x_1, x_2, x_3, x_4, x_5)\,. \tag{7.1}$$

One can check that, for $N = 192, 396$, there is not any matrix $\mathcal{A}$ satisfying the condition (7.1). For $N = 364, 444$, the condition (7.1) is satisfied for the same matrix $\mathcal{A} = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}$. By Table 3, there are no more bielliptic quotients over $\mathbb{Q}$ because one has

N	$\dim \mathcal{L}_{2,j}$	N	$\dim \mathcal{L}_{2,j}$	N	$\dim \mathcal{L}_{2,j}$
192	$\dim \mathcal{L}_{2,3} = \dim \mathcal{L}_{2,5} = 1$	396	$\dim \mathcal{L}_{2,3} = 2, \dim \mathcal{L}_{2,5} = 1$	444	$\dim \mathcal{L}_{2,3} = 0$

To determine bielliptic curves over $\overline{\mathbb{Q}}$, we only have to consider the cases $N = 192, 208, 216$ and 378. The case $N = 216$ can be discarded. Indeed, the splitting of $J_0^*(216)$ over $\mathbb{Q}(\sqrt{-3})$ and over $\mathbb{Q}$ are the same and we get $\dim \mathcal{L}_{2,1} = 1$. For the remaining cases, $f_i = f_j \otimes \chi_D$ with $i < j$. One can check that there is no matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\sqrt{D})$ such that, for all $Q_2 \in \mathcal{L}_2$, one has that

$$Q_2(x_1, \cdots, x_{i-1}, a_1x_i + a_2x_j, x_{i+1}, \cdots, x_{j-1}, b_1x_i + b_2x_j, x_{j+1}, \cdots, x_5)$$

is an even function in the variable x_i . $\square$

8 Case $g_N^* \geq 6$

For $g^* \geq 6$, all bielliptic involutions are defined over $\mathbb{Q}$. For the possible values of N to be considered, the curve $X_0^*(N)$ is not trigonal.

Proposition 8.1. *The curve $X_0^*(558)$ of genus 7 is the only bielliptic curve $X_0^*(N)$ of genus ≥ 6.*

We split the proof according to the genus, following the values N in Table 1, except for the value appearing in Lemma 3.5.

8.1 $g_N^* = 6$

Proof. In this case, $\dim \mathcal{L}_2 = 6$. The splitting of the Jacobian over $\mathbb{Q}$ for the seven cases to be considered is as follows:

$$\begin{array}{lllll}
J_0^*(244) \stackrel{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E61a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E122a\,, & f_3 \in \mathrm{New}_{122}^{B(61)}\,, & \dim A_{f_3} = 3,\,, \\
J_0^*(272) \stackrel{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E34a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E136b\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E272a\,, & f_4 \in \mathrm{New}_{68}^{B(17)}\,, \\
 & & & & \dim A_{f_4} = 2\,, \\
J_0^*(332) \stackrel{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^3 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E83a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E166a\,, & f_3 \in \mathrm{New}_{166}^{B(83)}\,, & \dim A_{f_3} = 3\,, \\
J_0^*(336) \stackrel{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^5 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E42a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E24a\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E36a\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E84b\,, \\
 & & & & A_{f_5} \stackrel{\mathbb{Q}}{\sim} E112a\,, \\
J_0^*(564) \stackrel{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E141d\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E94a\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E282a\,, & f_4 \in \mathrm{New}_{282}^{B(282)}\,, \\
 & & & & \dim A_{f_4} = 2\,, \\
J_0^*(620) \stackrel{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^4 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E155c\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E62a\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E310a\,, & f_4 \in \mathrm{New}_{310}^{B(310)}\,, \\
 & & & & \dim A_{f_4} = 2\,, \\
J_0^*(780) \stackrel{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^5 A_{f_i}\,, & A_{f_1} \stackrel{\mathbb{Q}}{\sim} E65a\,, & A_{f_2} \stackrel{\mathbb{Q}}{\sim} E26b\,, & A_{f_3} \stackrel{\mathbb{Q}}{\sim} E130c\,, & A_{f_4} \stackrel{\mathbb{Q}}{\sim} E390a\,, \\
 & & & & A_{f_5} \stackrel{\mathbb{Q}}{\sim} E390e\,,
\end{array}$$

After computing, we obtain

$N = 332$	$N = 336$	$N = 564$	$N = 620$	$N = 780$
$\dim \mathcal{L}_{2,3} = 3$	$\dim \mathcal{L}_{2,6} = 3$	$\dim \mathcal{L}_{2,3} = 2$	$\dim \mathcal{L}_{2,3} = 2$	$\dim \mathcal{L}_{2,4} = 1$

Hence, the $\mathbb{Q}$-isogeny class of an elliptic curve A_{f_i} with $i > 1$ can not be a bielliptic quotient of $X_0^*(N)$. In all cases, there is no matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ satisfying

$$Q_2(a_1x_1 + a_2x_2, b_1x_1 + b_2x_2, x_3, \cdots, x_6) = Q_2(-a_1x_1 + a_2x_2, -b_1x_1 + b_2x_2, x_3, \cdots, x_6)$$

for all $Q_2 \in \mathcal{L}_2$. So, the $\mathbb{Q}$-isogeny class of the elliptic curve A_{f_1} can not be a bielliptic quotient of $X_0^*(N)$. Hence, all these curves are not bielliptic. □

8.2 $g_N^* = 7$

Proof. Now $\dim \mathcal{L}_2 = 10$. The splitting of the Jacobian over $\mathbb{Q}$ for the 12 cases to be considered is as follows:

$$J_0^*(320) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^6 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E20a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E32a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E80b\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E160a\,, \quad A_{f_5} \overset{\mathbb{Q}}{\sim} E160b \quad A_{f_6} \overset{\mathbb{Q}}{\sim} E320b$$

$$J_0^*(324) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^6 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E27a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E54a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E54b\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E162a\,, \quad A_{f_5} \overset{\mathbb{Q}}{\sim} E162b \quad A_{f_6} \overset{\mathbb{Q}}{\sim} E162d$$

$$J_0^*(360) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 A_{f_2}^2 \prod_{i=3}^5 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E20a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E30a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E36a\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E90b\,, \quad A_{f_5} \overset{\mathbb{Q}}{\sim} E120b$$

$$J_0^*(450) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^7 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E15a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E30a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E75b\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E75c\,, \quad A_{f_5} \overset{\mathbb{Q}}{\sim} E39a\,, \quad A_{f_6} \overset{\mathbb{Q}}{\sim} E225a\,, \quad A_{f_7} \overset{\mathbb{Q}}{\sim} E450f\,,$$

$$J_0^*(456) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^6 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E57a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E38b\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E76a\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E114c\,, \quad A_{f_5} \overset{\mathbb{Q}}{\sim} E152a\,, \quad A_{f_6} \overset{\mathbb{Q}}{\sim} E228a\,,$$

$$J_0^*(492) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^5 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E123b\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E82a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E246d\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E246g\,, \quad f_5 \in \mathrm{New}_{82}^{B(41)}\,, \quad \dim A_{f_5} = 2\,,$$

$$J_0^*(504) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^6 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E21a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E36a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E42a\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E84b\,, \quad A_{f_5} \overset{\mathbb{Q}}{\sim} E168d\,, \quad A_{f_6} \overset{\mathbb{Q}}{\sim} E504a$$

$$J_0^*(550) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^5 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E55a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E275a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} 550a\,, \quad f_4 \in \mathrm{New}_{110}^{B(22)}\,, \quad \dim A_{f_4} = 2\,, \quad f_5 \in \mathrm{New}_{275}^*\,, \quad \dim A_{f_5} = 2\,,$$

$$J_0^*(558) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^4 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} 186c\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E558a\,, \quad f_3 \in \mathrm{New}_{93}^*\,, \quad \dim A_{f_3} = 2\,, \quad f_4 \in \mathrm{New}_{93}^{B(31)}\,, \quad \dim A_{f_4} = 3\,,$$

$$J_0^*(636) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^6 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E53a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E106a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E106b\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E106c\,, \quad A_{f_5} \overset{\mathbb{Q}}{\sim} E318a\,, \quad A_{f_6} \overset{\mathbb{Q}}{\sim} E318c\,,$$

$$J_0^*(660) \overset{\mathbb{Q}}{\sim} \prod_{i=1}^3 A_{f_i} A_{f_4}^2\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E66c\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E110b\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E330d\,, \quad f_4 \in \mathrm{New}_{165}^*\,, \quad \dim A_{f_4} = 2\,,$$

$$J_0^*(924) \overset{\mathbb{Q}}{\sim} A_{f_1}^2 \prod_{i=2}^6 A_{f_i}\,, \quad A_{f_1} \overset{\mathbb{Q}}{\sim} E77a\,, \quad A_{f_2} \overset{\mathbb{Q}}{\sim} E42a\,, \quad A_{f_3} \overset{\mathbb{Q}}{\sim} E66b\,, \quad A_{f_4} \overset{\mathbb{Q}}{\sim} E154a\,, \quad A_{f_5} \overset{\mathbb{Q}}{\sim} E154b \quad A_{f_6} \overset{\mathbb{Q}}{\sim} E462a$$

We get

$N = 320$	$N = 450$	$N = 456$
$\dim \mathcal{L}_{2,3} = 3$	$\dim \mathcal{L}_{2,1} = 3\,, \dim \mathcal{L}_{2,3} = 4$	$\dim \mathcal{L}_{2,4} = 4\,, \dim \mathcal{L}_{2,6} = 6$

$N = 492$	$N = 504$
$\dim \mathcal{L}_{2,3} = \dim \mathcal{L}_{2,4} = 4$	$\dim \mathcal{L}_{2,3} = 6\,, \dim \mathcal{L}_{2,4} = 4$

$N = 550$	$N = 558$	$N = 660$	$N = 924$
$\dim \mathcal{L}_{2,1} = \dim \mathcal{L}_{2,2} = \dim \mathcal{L}_{2,3} = 4$	$\dim \mathcal{L}_{2,2} = 10$	$\dim \mathcal{L}_{2,2} = 4$	$\dim \mathcal{L}_{2,7} = 4$

.

Moreover, there is no matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ satisfying

$$Q_2(a_1x_1 + a_2x_2, b_1x_1 + b_2x_2, x_3, x_4, x_5, x_6, x_7) = Q_2(-a_1x_1 + a_2x_2, -b_1x_1 + b_2x_2, x_3, x_4, x_5, x_6, x_7)$$

for all $Q_2 \in \mathcal{L}_2$ when $N = 320, 324, 360, 456, 492, 504, 636, 924$ or

$$Q_2(x_1, x_2, a_1x_3 + a_2x_4, b_1x_3 + b_2x_4, x_5, x_6, x_7) = Q_2(x_1, x_2, -a_1x_3 + a_2x_4, -b_1x_3 + b_2x_4, x_5, x_6, x_7)$$

for all $Q_2 \in \mathcal{L}_2$ when $N = 360$. Thus, the $\mathbb{Q}$-isogeny class of the elliptic curve A_{f_1} (resp. A_{f_3}) can not be a bielliptic quotient of $X_0^*(N)$ for $N = 320, 324, 360, 456, 492, 504, 636, 924$ (resp. $N = 360$). Hence, the curve $X_0^*(558)$ is the only bielliptic curve of genus 7. □

8.3 Case $g_N^* = 8$

Proof. In this case $\dim \mathcal{L}_2 = 15$. The splitting of the Jacobian over $\mathbb{Q}$ for the five cases to be considered is as follows:

$$\begin{array}{llllll}
J_0^*(408) \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^5 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} 102a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E34a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E204a\,, & f_4 \in \mathrm{New}_{68}^{B(17)}\,, \\
 & & & \dim A_{f_4} = 2 & f_5 \in \mathrm{New}_{408}^* & \dim A_{f_5} = 2 \\
J_0^*(468) \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^3 A_{f_i} A_{f_4}^2\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E26b\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E234b\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E234c\,, & f_4 \in \mathrm{New}_{39}^{B(13)}\,, \\
 & & & & & \dim A_{f_4} = 2\,, \\
J_0^*(480) \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^7 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E20a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E24a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E80b\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E160a\,, \\
 & & & A_{f_5} \overset{\mathbb{Q}}{\sim} E240b\,, & A_{f_6} \overset{\mathbb{Q}}{\sim} E240c\,, & A_{f_7} \overset{\mathbb{Q}}{\sim} E480a\,, \\
J_0^*(540) \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^5 A_{f_i} A_{f_6}^2\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E45a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E30a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E54b\,, & f_4 \overset{\mathbb{Q}}{\sim} E90b\,, \\
 & & & & A_{f_5} \overset{\mathbb{Q}}{\sim} E270b\,, & A_{f_6} \overset{\mathbb{Q}}{\sim} E135a\,, \\
J_0^*(990) \overset{\mathbb{Q}}{\sim} & \prod_{i=1}^6 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E30a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E66a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E99a\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E990a\,, \\
 & & f_5 \in \mathrm{New}_{165}^*\,, & \dim A_{f_5} = 2 & f_6 \in \mathrm{New}_{165}^{B(55)}\,, & \dim A_{f_6} = 2\,,
\end{array}$$

$N = 468$	$N = 480$	$N = 540$	$N = 990$
$\dim \mathcal{L}_{2,3} = \dim \mathcal{L}_{2,4} = 8\,,$	$\dim \mathcal{L}_{2,3} = \dim \mathcal{L}_{2,4} = \dim \mathcal{L}_{2,5} = 8$	$\dim \mathcal{L}_{2,4} = 10$	$\dim \mathcal{L}_{2,2} = 8\,, \dim \mathcal{L}_{2,3} = 10$

There is no matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q})$ satisfying

$$Q_2(a_1x_1+a_2x_2, b_1x_1+b_2x_2, x_3, x_4, x_5, x_6, x_7, x_8) = Q_2(-a_1x_1+a_2x_2, -b_1x_1+b_2x_2, x_3, x_4, x_5, x_6, x_7, x_8)$$

for all $Q_2 \in \mathcal{L}_2$ when $N = 408, 468, 480, 540$. □

8.4 Case $g_N^* = 9$

Proof. In this case $\dim \mathcal{L}_2 = 21$. The splitting of the Jacobian over $\mathbb{Q}$ for the values to be considered are as follows:

$$\begin{array}{llllll}
J_0^*(560) \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \prod_{i=2}^7 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E70a\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E20a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E56a\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E112a\,, \\
 & & A_{f_5} \overset{\mathbb{Q}}{\sim} E140b\,, & A_{f_6} \overset{\mathbb{Q}}{\sim} E280a\,, & f_7 \in \mathrm{New}_{280}^{B(35)}\,, & \dim A_{f_7} = 2\,, \\
J_0^*(1140) \overset{\mathbb{Q}}{\sim} & A_{f_1}^2 \cdot A_{f_2}^2 \prod_{i=3}^7 A_{f_i}\,, & A_{f_1} \overset{\mathbb{Q}}{\sim} E285b\,, & A_{f_2} \overset{\mathbb{Q}}{\sim} E57a\,, & A_{f_3} \overset{\mathbb{Q}}{\sim} E38b\,, & A_{f_4} \overset{\mathbb{Q}}{\sim} E114c\,, \\
 & & & A_{f_5} \overset{\mathbb{Q}}{\sim} E190b\,, & A_{f_6} \overset{\mathbb{Q}}{\sim} E570a\,, & f_7 \overset{\mathbb{Q}}{\sim} E570g\,,
\end{array}$$

For $N = 560$, $\dim \mathcal{L}_{2,4} = \dim \mathcal{L}_{2,7} = 13$, and for $N = 1140$, $\dim \mathcal{L}_{2,7} = 14$. Moreover for $N = 560$ and 1140 and there is no matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{Q})$ satisfying

$$Q_2(a_1x_1 + a_2x_2, b_1x_1 + b_2x_2, x_3, \cdots, x_9) = Q_2(-a_1x_1 + a_2x_2, -b_1x_1 + b_2x_2, x_3, \cdots, x_9)$$

for all $Q_2 \in \mathcal{L}_2$. □

8.5 Case $g_N^* = 10$

Proof. Now, $\dim \mathcal{L}_2 = 28$. We have to consider two cases $X_0^*(840)$ and $X_0^*(1050)$.

For $N = 840$, the Jacobian splits over $\mathbb{Q}$ as $A_{f_1}^2 \prod_{i=2}^{9} A_{f_i}$, where

$A_{f_1} \overset{\mathbb{Q}}{\sim} E210d\,,\ A_{f_2} \overset{\mathbb{Q}}{\sim} E20a\,,\ A_{f_3} \overset{\mathbb{Q}}{\sim} E42a\,,\ A_{f_4} \overset{\mathbb{Q}}{\sim} E70a\,,\ A_{f_5} \overset{\mathbb{Q}}{\sim} E84b\,,\ A_{f_6} \overset{\mathbb{Q}}{\sim} E140b\,,$ $A_{f_7} \overset{\mathbb{Q}}{\sim} E280a\,, A_{f_8} \overset{\mathbb{Q}}{\sim} E420a$ and $A_{f_9} \overset{\mathbb{Q}}{\sim} E840a$.

We get $\dim \mathcal{L}_{2,3} = \dim \mathcal{L}_{2,7} = 19$ and there is no matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{Q})$ satisfying

$$Q_2(a_1x_1 + a_2x_2, b_1x_1 + b_2x_2, x_3, \cdots, x_{10}) = Q_2(-a_1x_1 + a_2x_2, -b_1x_1 + b_2x_2, x_3, \cdots, x_{10})$$

for all $Q_2 \in \mathcal{L}_2$.

For $N = 1050$, the Jacobian splits over $\mathbb{Q}$ as $\prod_{i=1}^{6} A_{f_i} \times A_{f_7} \times A_{f_8}$, where

$A_{f_1} \overset{\mathbb{Q}}{\sim} E15a\,,\ A_{f_2} \overset{\mathbb{Q}}{\sim} E175b\,,\ A_{f_3} \overset{\mathbb{Q}}{\sim} E210d\,,\ A_{f_4} \overset{\mathbb{Q}}{\sim} E35a\,,\ A_{f_5} \overset{\mathbb{Q}}{\sim} E525a\,,\ A_{f_6} \overset{\mathbb{Q}}{\sim} E1050a\,,$ $f_7 \in \mathrm{New}_{35}^{B(7)}\,, \dim A_{f_7} = 2\,, f_8 \in \mathrm{New}_{525}^*\,, \dim A_{f_8} = 2$.

We get $\dim \mathcal{L}_{2,2} = 19$. □

8.6 Case $g_N^* = 11$

Proof. Now, $\dim \mathcal{L}_2 = 36$. We have to consider only the case $X_0^*(672)$.

For $N = 672$, the Jacobian splits over $\mathbb{Q}$ as $A_{f_1}^2 A_{f_2}^2 \prod_{i=3}^{9} A_{f_i}$, where

$A_{f_1} \overset{\mathbb{Q}}{\sim} E84b\,,\ A_{f_2} \overset{\mathbb{Q}}{\sim} E42a\,,\ A_{f_3} \overset{\mathbb{Q}}{\sim} E24a\,,\ A_{f_4} \overset{\mathbb{Q}}{\sim} E56a\,,\ A_{f_5} \overset{\mathbb{Q}}{\sim} E112a\,,\ A_{f_6} \overset{\mathbb{Q}}{\sim} E112c\,,$ $A_{f_7} \overset{\mathbb{Q}}{\sim} E224a\,, A_{f_8} \overset{\mathbb{Q}}{\sim} E336a$ and $A_{f_9} \overset{\mathbb{Q}}{\sim} E672a$.

We get $\dim \mathcal{L}_{2,7} = \dim \mathcal{L}_{2,8} = 26$. □

8.7 Case $g_N^* = 13$

Proof. Now, $\dim \mathcal{L}_2 = 55$. We have to consider only the case $X_0^*(1260)$.

For $N = 1260$, the Jacobian splits over $\mathbb{Q}$ as $A_{f_1}^2 A_{f_2}^2 (\prod_{i=3}^{7} A_{f_i}) \cdot A_{f_8}^2$, where

$A_{f_1} \overset{\mathbb{Q}}{\sim} E21a\,,\ A_{f_2} \overset{\mathbb{Q}}{\sim} E70a\,,\ A_{f_3} \overset{\mathbb{Q}}{\sim} E30a\,,\ A_{f_4} \overset{\mathbb{Q}}{\sim} E42a\,,\ A_{f_5} \overset{\mathbb{Q}}{\sim} E90b\,,\ A_{f_6} \overset{\mathbb{Q}}{\sim} E210d\,,$ $A_{f_7} \overset{\mathbb{Q}}{\sim} E630g\,, A_{f_8} \in \mathrm{New}_{315}^{B(315)}$ and $\dim A_{f_8} = 2$.

We are interested for $21a, 70a, 90b$ and $210d$. We get $\dim \mathcal{L}_{2,7} = \dim \mathcal{L}_{2,8} = 43$, thus the pair $(1260, 90b)$,$(1260, 210d)$ are not bielliptic. For the remaining $(1260, 21a)$ and $(1260, 70a)$ they are not bielliptic because there is no matrix $\mathcal{A} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$ in $\mathrm{GL}_2(\mathbb{Q})$ satisfying

$$Q_2(a_1x_1 + a_2x_2, b_1x_1 + b_2x_2, x_3, \cdots, x_{13}) = Q_2(-a_1x_1 + a_2x_2, -b_1x_1 + b_2x_2, x_3, \cdots, x_{13})$$

for all $Q_2 \in \mathcal{L}_2$, or respectively

$$Q_2(x_1, x_2, a_1x_3+a_2x_4, b_1x_3+b_2x_4, x_5, \cdots, x_{13}) = Q_2(x_1, x_2, -a_1x_3+a_2x_4, -b_1x_3+b_2x_4, x_5, \cdots, x_{13})$$

□

Hence, part (i) of Theorem 1.2 is proved .

8.8 A remark for levels $4M$ with M odd

After the paper was completed, Andreas Schweizer mentioned us the following argument, which helps in some levels to obtain that the modular curve $X_0^*(N)$ is not bielliptic.

Lemma 8.2 (Schweizer)**.** *Let M be odd and let the genus of $X_0^*(4M)$ be at least 6. If $X_0^*(4M)$ is bielliptic, then the genus of $X_0^*(2M)$ must be 1 or 2.*

Proof. Assume that $X_0^*(4M)$ is bielliptic and $g_{4M}^* \geq 6$. By Proposition 3.6, $X_0^*(4M)$ is isomorphic over $\mathbb{Q}$ to the curve $X := X_0(2M)/\langle B(M)\rangle$. It is well-known that the Fricke involution W_{2M} always has fixed points. By the unramified covering criterion, [JKS17, Lemma 3.10], applied to the Galois covering $X \to X/W_{2M} = X_0^*(2M)$, we get that $X_0^*(2M)$ must be hyperelliptic or of genus ≤ 1. But $X_0^*(2M)$ cannot have genus 0, because then $X_0^*(4M)$ would also be hyperelliptic, which is not possible for genus bigger or equal than 6 (cf. [Has97, Theorem B]). Since $2M$ is congruent to 2 mod 4, again applying [Has97, Theorem B] we get $g_{2M}^* < 3$. □

Corollary 8.3 (Schweizer)**.** *Let M be odd and let the genus of $X_0^*(4M)$ be at least 8. Then, $X_0^*(4M)$ is bielliptic if and only if $X_0^*(2M)$ has genus 1.*

Proof. The condition $g_{2M}^* > 1$ contradicts the Castelnuovo Inequality. □

These results could be applied, without the use of Petri's theorem and Proposition 2.6, for the followings levels $N = 324, 468, 492, 540, 564, 620, 636, 660, 924, 1140, 1260$.

9 Quadratic Points

In this section, we determine the values N such that the set $\Gamma_2(X_0^*(N), \mathbb{Q})$ is infinite. The values N square-free are in part (ii) of Theorem 1.1. Assume that N is not square-free. All hyperelliptic curves $X_0^*(N)$ are over $\mathbb{Q}$ and, thus, have infinite number of points over $\mathbb{Q}$. These values can be found in [Has97]. More precisely, for $g_N^* = 2$ the values N are in Table 4 of the Appendix and for $g_N^* \geq 3$ one has

g_N^*	3	4	5
N	$136, 171, 207, 252, 315$	176	279

When $X_0^*(N)$ is bielliptic over $\mathbb{Q}$ and not hyperelliptic, we select the values N such that $X_0^*(N)$ have at least a bielliptic quotient with positive rank. We get the following cases.

g_N^*	(N, E)
3	$(128, E128a), (152, E152a), (164, E82a), (234, E234c), (236, E118a), (240, E240c),$ $(248, E248a), (312, E312b), (348, E58a), (420, E210d), (476, E238b)$
4	$(148, E37a), (172, E43a), (224, E224a), (228, E57a), (225, E225a), (260, E65a),$ $(264, E88a), (280, E280a), (342, E342e).$
5	$(364, E91a), (444, E37a), (495, E99a).$
7	$(558, E558a).$

This concludes the proof of part (ii) of Theorem 1.2.

10 Appendix

Here, we present the list of curves $X_0^*(N)$ with $g_N^* \leq 5$. We developed a programme in Magma to obtain the genus of $X_0^*(N)$ for any N. It can be found in

http://mat.uab.cat/∼francesc/Computegenusquotientsx0N.pdf

In particular, we can list all the modular curves $X_0^*(N)$ of genus ≤ 5. For genus 0 and 1, in [GL98] are listed the cases with N square-free. For genus 2, the values N can be found in Hasegawa [Has95]. For genus 3 or 4, the curve $X_0^*(N)$ is trigonal or hyperelliptic (see Remark 3.13), then the table follows from [HS00] and [Has97]. When $g_N^* = 5$, we know that the gonality is ≤ 4. We prove that if $X_0^*(N)$ has gonality ≤ 4, then $N \leq 121337$. This is made following an argument of Ogg in [Ogg74], that in [HS00, Lemma 1,Proposition 2] is used to find an upper bound for trigonal curves $X_0^*(N)$. Applying the program we can complete the list. In fact, with these tools we could obtain the complete list of curves $X_0^*(N)$ with a fixed genus.

We obtain the following list, where for completeness of the paper, we mark by bold the square-free integers and with italic the integers that are a non trivial power of a prime.

$g_N^* = 0$	**2**, **3**, **4**, **5**, **6**,**7**,**8**,*9*, **10**,**11**, 12, **13**, **14**, **15**, *16*, **17**, 18, **19**, 20, **21**, **22**, **23**, 24, *25*, **26**, *27*, 28, **29**, **30**, **31**, *32*, **33**, **34**, **35**, 36, **38**, **39**, **41**, **42**, 44, 45, **46**, **47**, *49*, 50, **51**, 54, **55**, 56, **59**, 60, **62**, **66**, **69**, **70**, **71**, **78**, **87**, 92, **94**, **95**, **105**, **110**, **119**
$g_N^* = 1$	**37**, 40, **43**, 48, 52, **53**, **57**, **58**, **61**, 63, *64*, **65**, 68, 72, **74**, 75, 76, **77**, **79**, 80,*81*, **82**, **83**, 84, **86**, **89**, 90,**91**, 96, 98, 99, 100, **101**, **102**, 108, **111**, **114**, **118**, 120, **123**, 124, 126, **130**, **131**, 132, **138**, 140, **141**, **142**, **143**, **145**, 150, **155**, 156, **159**, **174**, **182**, 188, **190**, **195**, **210**, 220, **222**, **231**, **238**
$g_N^* = 2$	**67**, **73**, **85**, 88, **93**, **103**, 104, **106**, **107**, 112, **115**, 116, 117, *121*, **122**, *125*, **129**, **133**, **134**, 135, **146**, 147, 153, **154**, **158**, **161**, **165**, **166**, **167**, 168, **170**, **177**, 180, 184, **186**, **191**, 198, 204, **205**, **206**, **209**, **213**, **215**, **221**, **230**, **255**, **266**, 276, 284, **285**, **286**, **287**, **299**, **330**, **357**, 380, **390**.
$g_N^* = 3$	**97**, **109**, **113**, **127**, *128*, 136, **139**, 144, **149**, **151**, 152, 162, 164, *169*, 171, 175, **178**, **179**, **183**, **185**, **187**, 189, **194**, 196, **203**, 207, **217**, 234, 236, **239**, 240, 245, **246**, 248, **249**, 252, **258**, 270, **282**, **290**, 294, **295**, **303**, **310**, 312, 315, **318**, **329**, 348, 420, **429**, **430**, **455**, **462**, 476, **510**.
$g_N^* = 4$	**137**, 148, 160, 172, **173**, 176, **199**, 200, **201**, **202**, **214**, **219**, 224, 225, 228, 242, **247**, **251**, **254**, **259**, 260, 261, **262**, 264, **267**, **273**, 275, 280, 300, **305**, 306, 308, **311**, **319**, **321**, **322**, **334**, **335**, **341**, 342, **345**, 350, **354**, **355**, **366**, **370**, **374**, **385**, **395**, **399**, **426**, **434**, **483**, **546**, **570**.
$g_N^* = 5$	**157**, **181**, 192, 208, 212, 216, **218**, **226**, **227**, **235**, **237**, 250, **253**, **263**, **278**, 279, **302**, 316, **323**, **339**, 364, **371**, 376, **377**, 378, **382**, **391**, **393**, 396, **402**, **406**, **407**, **410**, **413**, 414, **418**, **435**, **438**, 440, **442**, 444, **465**, **494**, 495, **551**, **555**, 572, **574**, **595**, 630, **645**, **663**, **714**, **770**, **798**, **910**.

Table 4

Acknowledgements

We thank to Andreas Schweizer for his comments and suggestions, and specially those concerning morphisms between modular curves with different level and the remark quoted in Subsection 8.8 for levels $4N$.

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