Hypersurface model-fields of definition for smooth hypersurfaces and their twists

by

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1. Introduction. Let $X_0, \ldots, X_n$ be a homogeneous coordinate system for $\mathbb{P}^n_{\overline{k}}$, the n-dimensional projective space over $\overline{k}$. Given a smooth projective variety $\overline{V} \subset \mathbb{P}^n_{\overline{k}}$, the group of birational transformations of $\overline{V}$ onto itself is denoted by $\mathrm{Bir}(\overline{V})$, the group of automorphisms of $\overline{V}$ (that is, the group of biregular transformations of $\overline{V}$ onto itself) is $\mathrm{Aut}(\overline{V})$, and we denote by $\mathrm{Lin}(\overline{V})$ the subgroup of automorphisms of $\overline{V}$ induced by projective linear transformations in $\mathrm{Aut}(\mathbb{P}^n_{\overline{k}}) = \mathrm{PGL}_{n+1}(\overline{k})$, the general linear group of $(n+1) \times (n+1)$ projective matrices.

Now, let $\overline{V}$ be a smooth hypersurface in $\mathbb{P}^n_{\overline{k}}$, that is, an $(n-1)$-dimensional smooth projective variety identified with a hypersurface model $H_{\overline{V},d,n}$ represented by a single homogeneous polynomial equation, say $F(X_0, \ldots, X_n) = 0$ of some degree d over $\overline{k}$ without singularities (assume once and for all that $d \geq 4$). It is known that a smooth plane curve $\overline{V}$ of degree $d \geq 4$ has finitely many automorphisms, and moreover any automorphism is induced by a projective linear transformation of $\mathbb{P}^2_{\overline{k}}$, thus $\mathrm{Aut}(\overline{V}) = \mathrm{Lin}(\overline{V})$. Matsumura and Monsky (1946) showed that, for $n \geq 3$, $\mathrm{Lin}(\overline{V})$ is a finite group and moreover $\mathrm{Aut}(\overline{V}) = \mathrm{Lin}(\overline{V})$ except possibly when $(n, d) = (3, 4)$ (see [6, Theorems 1, 2]).

DEFINITION 1.1. A smooth projective variety V defined over a field k is called a *smooth L-hypersurface over k of degree d in $\mathbb{P}^n_{\overline{k}}$* and L is a *hypersurface model-field of definition for V*, where L/k is a field extension inside $\overline{k}$, if the base extension $V \otimes_k L$ is L-isomorphic to a non-singular hypersurface

[1]

model $H_{V\otimes_k L,d,n} : F_{V\otimes_k L}(X_0,\ldots,X_n) = 0$ of degree d with coefficients in L. For the special case $L = k$, V is simply called a *smooth hypersurface over k*.

Suppose that V is a smooth $\overline{k}$-hypersurface over k of degree $d \geq 4$ in $\mathbb{P}^n_{\overline{k}}$. Hence, by finiteness and linearity of $\mathrm{Aut}(V \otimes_k \overline{k})$ for $(n,d) \neq (3,4)$, we find that $\overline{V} := V \otimes_k \overline{k}$ has a linear series, which allows us to embed $\Upsilon : \overline{V} \hookrightarrow \mathbb{P}^n_{\overline{k}}$ as a smooth hypersurface.

LEMMA 1.2. *The linear system Υ for $n, d \geq 3$ such that $(n,d) \neq (3,4)$, is unique up to $\mathrm{PGL}_{n+1}(\overline{k})$-conjugation.*

Proof. Suppose that $H_{\overline{V},d,n}$ and $H'_{\overline{V},d,n}$ are two non-singular hypersurface models for $\overline{V}$ in $\mathbb{P}^n_{\overline{k}}$ of degree d. Then, both models are $\overline{k}$-isomorphic via an isomorphism, say ϕ. The same argument in the proof of [6, Theorem 2] assures that ϕ^* maps hyperplane sections on $H_{\overline{V},d,n}$ to hyperplane sections on $H'_{\overline{V},d,n}$, and so ϕ is induced by a linear transformation of $\mathbb{P}^n_{\overline{k}}$ (for $n \geq 4$ such an invariance property on hyperplane sections follows from a theorem of Grothendieck–Lefschetz or following [6] a theorem of Severi–Lefschetz–Andreotti). ∎

Thus, for $(n,d) \neq (3,4)$, we can always think about $\mathrm{Aut}(\overline{V})$ as a finite subgroup of $\mathrm{PGL}_{n+1}(\overline{k})$, leaving invariant a fixed non-singular hypersurface model $H_{\overline{V},d,n} : F_{\overline{V}}(X_0,\ldots,X_n) = 0$ of degree d coming from the embedding $\Upsilon : \overline{V} \hookrightarrow \mathbb{P}^n_{\overline{k}}$ over $\overline{k}$. In other words, any other non-singular hypersurface model over $\overline{k}$ is defined by an equation of the form

$$F_{P^{-1}\overline{V}}(X_0,\ldots,X_n) := F_{\overline{V}}(P(X_0,\ldots,X_n)) = 0$$

for some $P \in \mathrm{PGL}_{n+1}(\overline{k})$.

The aim of this paper is to make a study for fields of definition of non-singular hypersurface models of a smooth $\overline{k}$-hypersurface V over k, also for its twists, by considering the embedding $\mathrm{Aut}(\overline{V}) \hookrightarrow \mathrm{PGL}_{n+1}(\overline{k})$. We note that if the smooth projective variety V, or any of its twists over k, is a smooth hypersurface over k, then we have an embedding of $\mathrm{Gal}(\overline{k}/k)$-groups for its automorphism group into $\mathrm{PGL}_{n+1}(\overline{k})$. This approach leads to two natural questions: first, given a smooth projective variety V defined over a field k and admitting a non-singular $\overline{k}$-hypersurface model, does it have a non-singular hypersurface model over k; and secondly, if the answer is yes, does every twist of V over k also have a non-singular hypersurface model over k? For both questions the answer is **No**: in general, it does not. We obtain results for the varieties for which the above questions always have an affirmative answer, and we show different examples concerning the negative general answer.

The paper is a generalization of our joint work [1] with Elisa Lorenzo García, where the same problem was addressed, but for smooth hypersurfaces in $\mathbb{P}^2$, that is, for smooth plane curve of degree $d \geq 4$.

2. Statements of the results. First, we study the minimal field L where there exists a non-singular model over L for a smooth $\overline{k}$-hypersurface V defined over k.

We show the following, which follows from [7].

THEOREM 2.1. *Let V be a smooth $\overline{k}$-hypersurface over a perfect field k of degree $d \geq 4$ in $\mathbb{P}^n_{\overline{k}}$ such that $(n, d) \neq (3, 4)$. Then V is not necessarily a smooth hypersurface over k. However, it is so in any of the following situations:*

(i) *if V has k-rational points, i.e. when $V(k) \neq \emptyset$,*
(ii) *if $\gcd(d, n + 1) = 1$,*
(iii) *if the $(n + 1)$-torsion $\mathrm{Br}(k)[n + 1]$ of the Brauer group $\mathrm{Br}(k)$ is trivial.*

In general, V has a non-singular hypersurface model over a field extension L/k of degree $[L : k] \mid n + 1$. Also, in the case of number fields k, we show in §4.1 an example of a smooth $\overline{k}$-hypersurface over k, which is not a smooth hypersurface over k, but it does over a Galois field extension of degree $n + 1$ over k.

Notation and conventions. We write $\mathrm{Gal}(L/k)$ for the Galois group of the extension L/k, and we consider left actions. The Galois cohomology sets of a $\mathrm{Gal}(L/k)$-group G when L/k is Galois are denoted by $\mathrm{H}^i(\mathrm{Gal}(L/k), G)$ with $i \in \{0, 1\}$ respectively. For the particular case $L = \overline{k}$, we use G_k instead of $\mathrm{Gal}(\overline{k}/k)$ and $\mathrm{H}^1(k, G)$ instead of $\mathrm{H}^1(\mathrm{Gal}(\overline{k}/k), G)$.

Second, we assume that V is a smooth hypersurface over k. We obtain the next theorem characterizing the twists of V, which are also smooth hypersurfaces over k. In particular, we prove the following.

THEOREM 2.2. *Let V be a smooth hypersurface over a perfect field k identified with a fixed non-singular hypersurface model $H_{V,d,n} : F_V(X_0, \ldots, X_n) = 0$, where $F_V(X_0, \ldots, X_n) \in k[X_0, \ldots, X_n]$ with $(n, d) \neq (3, 4)$. Then there exists a natural map*

$$\Sigma : \mathrm{H}^1(k, \mathrm{Aut}(\overline{V})) \to \mathrm{H}^1(k, \mathrm{PGL}_{n+1}(\overline{k})),$$

defined by the inclusion $\mathrm{Aut}(H_{V,d,n} \otimes_k \overline{k}) \subseteq \mathrm{PGL}_{n+1}(\overline{k})$ as G_k-groups. The preimage $\Sigma^{-1}([\mathbb{P}^n_k])$ is formed by the set of all twists of V over k that are smooth hypersurfaces over k, where $[\mathbb{P}^n_k]$ denotes the class of the trivial Brauer–Severi variety of dimension n over k. Moreover, any such twist is obtained through an automorphism of $\mathbb{P}^n_{\overline{k}}$, that is, the twist is k-isomorphic to

$$H_{F_{M^{-1}V,d,n}} : F_{M^{-1}V}(X_0, \ldots, X_n) := F_V(M(X_0, \ldots, X_n)) \in k[X_0, \ldots, X_n],$$

for some $M \in \mathrm{PGL}_{n+1}(\overline{k})$.

We can reinterpret the map Σ as the map sending a twist V' over k to the Brauer–Severi variety B where it lives (cf. [7, Lemma 5]).

Then, we have assertions similar to those in Theorem 2.1.

COROLLARY 2.3. *Let V be a smooth hypersurface over a perfect field k of degree $d \geq 4$ inside $\mathbb{P}_{\overline{k}}^n$ with $(n, d) \neq (3, 4)$. The map Σ in Theorem 2.2 is trivial if $\gcd(d, n + 1) = 1$ or $\mathrm{Br}(k)[n + 1]$ is trivial. In particular, for such situations, any twist V' for V over k is also a smooth hypersurface over k.*

On the other hand, we construct in §4.4 a one-parameter family $H_{F_a, d=2p, n}$, for $a \in k$, of smooth hypersurfaces over a number field k_d where p is an odd prime integer, and we show that each member of the family has a twist over k_d that does not admit a hypersurface model over k_d.

Finally, we study the twists for a smooth hypersurface V over k such that $\mathrm{Aut}(\overline{V})$ is a cyclic group.

DEFINITION 2.4. Let $V/k : F_V(X_0, \ldots, X_n) = 0$ be a smooth hypersurface over a perfect field k, where $F_V(X_0, \ldots, X_n) \in k[X_0, \ldots, X_n]$. We call a twist V' for V over k a *diagonal twist* if there exists an $M \in \mathrm{PGL}_{n+1}(k)$ and a diagonal matrix $D \in \mathrm{PGL}_{n+1}(\overline{k})$ such that V' is k-isomorphic to

$$F_{(MD)^{-1}V}(X_0, \ldots, X_n) := F_V(MD(X_0, \ldots, X_n)) = 0,$$

where $F_{(MD)^{-1}V}(X_0, \ldots, X_n) \in k[X_0, \ldots, X_n]$.

We prove the following.

THEOREM 2.5 (Diagonal twists). *Let $V/k : F_V(X_0, \ldots, X_n) = 0$ be a smooth hypersurface over a perfect field k of degree $d \geq 4$ with $(n, d) \neq (3, 4)$. Assume that $\mathrm{Aut}(V \otimes_k \overline{k}) \hookrightarrow \mathrm{PGL}_{n+1}(\overline{k})$, given by the linear system, is a non-trivial cyclic group of order m generated by $\psi = \mathrm{diag}(1, \zeta_m^{a_1}, \ldots, \zeta_m^{a_n})$ for some $a_i \in \mathbb{N}$, where ζ_m denotes a fixed primitive mth root of unity and m is coprime to the characteristic of k. Then all the twists in $\mathrm{Twist}_k(V)$ are diagonal over k given by a non-singular polynomial equation*

$$F_{D^{-1}V}(X_0, \ldots, X_n) = 0$$

where $F_{D^{-1}V}(X_0, \ldots, X_n) \in k[X_0, \ldots, X_n]$ and D is a diagonal matrix in $\mathrm{PGL}_{n+1}(\overline{k})$. In particular, the map Σ in Theorem 2.2 is trivial.

REMARK 2.6. Let V/k be a smooth hypersurface over a perfect field k of characteristic $p \geq 0$, and identify it with a non-singular hypersurface model $H_{V,d,n} : F_V(X_0, \ldots, X_n) = 0$ over k with $d \geq 4$ and $(n, d) \neq (3, 4)$. Suppose also that $\mathrm{Aut}(H_{V,d,n} \otimes_k \overline{k}) \subseteq \mathrm{PGL}_{n+1}(\overline{k})$ is a cyclic group of order n, generated by a matrix ψ whose conjugacy class in $\mathrm{PGL}_{n+1}(k)$ does not contain elements of diagonal shapes. Then the twists of V over k whose image under Σ is trivial (i.e., the ones that are smooth hypersurfaces over k) are expected not to be represented by diagonal twists. For example, for $n = 2$

in [1], we provide a smooth plane curve with a cyclic non-diagonal automorphism group in the above sense, where not all of its twists are diagonal. Constructing such examples in higher-dimensional $\mathbb{P}^n$ requires knowledge of the structure of automorphism groups and also the twisting theory for smooth hypersurfaces living there.

3. Brauer–Severi varieties and central simple algebras. Let U be a quasi-projective variety defined over a perfect field k. A *twist* for U is a variety U' defined over k that is isomorphic over $\overline{k}$ to U, but not necessarily over k. A twist U' is called *trivial* if U and U' are k-isomorphic. The set of all twists of U modulo k-isomorphisms is denoted by $\mathrm{Twist}_k(U)$. It is well known that the set $\mathrm{Twist}_k(U)$ is in one-to-one correspondence with the first Galois cohomology set $\mathrm{H}^1(k, \mathrm{Aut}(U \otimes_k \overline{k}))$ given by $[U'] \mapsto \xi : \tau \mapsto \xi_\tau := \phi \circ {}^\tau\phi^{-1}$ for $\tau \in G_k$, where $\phi : U' \otimes_k \overline{k} \to U \otimes_k \overline{k}$ is a fixed $\overline{k}$-isomorphism (see [9, Chap. III]).

Brauer–Severi varieties. A *Brauer–Severi variety* $\mathcal{B}$ of dimension n over a perfect field k is a twist of the n-dimensional projective space $\mathbb{P}^n_k$ over k. A field extension L/k is said to be a *splitting field* of $\mathcal{B}$ if $\mathcal{B} \otimes_k L \simeq \mathbb{P}^n_L$, and we say that *$L/k$ splits $\mathcal{B}$* or that *$\mathcal{B}$ splits over L*.

In particular, we obtain

COROLLARY 3.1. *The set* $\mathrm{Twist}_k(\mathbb{P}^n_k)$ *is in bijection with* $\mathrm{H}^1(k, \mathrm{Aut}(\mathbb{P}^n_{\overline{k}}))$ $= \mathrm{H}^1(k, \mathrm{PGL}_{n+1}(\overline{k}))$.

Moreover, we have

THEOREM 3.2 (Severi, Châtelet, Lichtenbaum). *Let $\mathcal{B}$ be a Brauer–Severi variety of dimension n over a perfect field k. Then there exists a field extension L/k of degree $[L : k] \mid n + 1$ such that L splits $\mathcal{B}$. Moreover, $\mathcal{B}$ splits over k if it has k-rational points or contains a hypersurface of degree relatively prime with $n + 1$.*

Proof. The result is due to Severi (cf. [8, X, §6, Exercise 1] in the original French edition), Châtelet (cf. his PhD thesis [2]), and Lichtenbaum (cf. [4, Theorem 5.4.10]). One can also read the proof of [7, Theorem 5]. ∎

We deduce from [7, Lemma 4] the following.

THEOREM 3.3 (Roé–Xarles). *Let V be a smooth projective variety over a perfect field k. Suppose that, for some fixed $n \geq 2$, there is a unique (modulo automorphisms) n-dimensional linear series over $\overline{k}$ invariant under the G_k-action, giving a morphism $h : V \otimes_k \overline{k} \to \mathbb{P}^n_{\overline{k}}$. Then there exists a Brauer–Severi variety $\mathcal{B}$ of dimension n defined over k, together with a k-morphism $g : V \hookrightarrow \mathcal{B}$, such that $g \otimes_k \overline{k} : V \otimes_k \overline{k} \to \mathbb{P}^n_{\overline{k}}$ is equal to h.*

Central simple algebras. A *central simple algebra* over a field k is a finite-dimensional associative algebra over k which is simple, i.e. it contains no non-trivial (two-sided) ideal and the multiplication operation is not uniformly zero, and for which the center is exactly k.

A field extension L/k is said to be a *splitting field* of a central simple algebra $\mathcal{A}$ over k if $\mathcal{A} \otimes_k L \simeq M_n(L)$ for some n, and we say that L/k *splits* $\mathcal{A}$.

EXAMPLE 3.4 (Cyclic algebras). Let L/k be a cyclic extension of degree $n+1$ with $\mathrm{Gal}(L/k) = \langle \sigma \rangle$. Then an element of $\mathrm{H}^1(\mathrm{Gal}(L/k), \mathrm{PGL}_{n+1}(L))$ represented by a 1-cocycle $f : \mathrm{Gal}(L/k) \to \mathrm{PGL}_{n+1}(L)$ is completely determined by the value of $f(\sigma)$, which is subject to

$$f(\sigma) \cdot {}^{\sigma}(f(\sigma)) \cdot {}^{\sigma^2}(f(\sigma)) \cdot \ldots \cdot {}^{\sigma^n}(f(\sigma)) = 1.$$

For instance, let $a \in k^*$ and consider the matrices

$$C_a := \begin{pmatrix} 0 & 0 & \ldots & 0 & a \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}, \quad D_a := \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & & & \\ 0 & 0 & \ldots & \ldots & 0 & 1 \\ a & 0 & \ldots & \ldots & 0 & 0 \end{pmatrix}.$$

Define a 1-cocycle f by setting $f(\sigma) = C_a \bmod L^*$. Hence

$$f(\sigma) \cdot {}^{\sigma}(f(\sigma)) \cdot {}^{\sigma^2}(f(\sigma)) \cdot \ldots \cdot {}^{\sigma^n}(f(\sigma)) = C_a^{n+1} \bmod L^* = I \bmod L^*,$$

where I is the identity matrix. According to [11, Theorem 5.4], we can associate to this 1-cocycle a central simple algebra $\mathcal{A}$ over k of dimension $(n+1)^2$ that splits over L, by considering the set of matrices $M \in M_{n+1}(L)$ satisfying $C_a \, {}^{\sigma}M \, C_a^{-1} = M$. One finds that $I, C_a, \ldots, C_a^n \in \mathcal{A}$ as well as

$$S_b := \mathrm{diag}(b, \sigma(b), \ldots, \sigma^n(b)) \quad \text{for } b \in L.$$

Therefore, $\bigoplus S_b C_a^i$ is a k-subalgebra of the correct dimension $(n+1)^2$, and corresponds to the algebra $\mathcal{A}$ defined by the 1-cocycle f. This kind of k-algebras is also known as the *cyclic algebra* (χ, a) associated to $a \in k^*$ and the character $\chi : \mathrm{Gal}(L/k) \xrightarrow{\cong} \mathbb{Z}/(n+1)\mathbb{Z}$ defined by $\chi(\sigma) = -1 \bmod (n+1)$.

In the above computations, we may replace C_a with D_a and we get symmetrically the cyclic algebra (χ, a) associated to $a \in k^*$ and the character $\chi : \mathrm{Gal}(L/k) \xrightarrow{\cong} \mathbb{Z}/(n+1)\mathbb{Z}$ sending σ to $1 \bmod (n+1)$, since $C_a S_b = {}^{\sigma^{-1}}S_b \, C_a$ is changed to $D_a S_b = {}^{\sigma}S_b \, D_a$.

For complete details, we refer to [11, Example 5.5].

THEOREM 3.5 (J. H. Maclagan-Wedderburn, R. Brauer). *Given a central simple algebra $\mathcal{A}$ over k, there exists a unique (up to isomorphism) division algebra $\mathcal{D}$ with center k and a positive integer n such that $\mathcal{A}$ is isomorphic to $M_n(\mathcal{D})$. Consequently, the dimension $\dim_k(\mathcal{A})$ of $\mathcal{A}$ over k is always a square.*

Theorem 3.5 gives a strict relation between central simple algebras and division algebras, and suggests the introduction of the following equivalence relation: Two central simple algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ over the same field k are equivalent if there are positive integers m, n such that $M_m(\mathcal{A}_1) \simeq M_n(\mathcal{A}_2)$. Equivalently, $\mathcal{A}_1$ and $\mathcal{A}_2$ are equivalent if $\mathcal{A}_1$ and $\mathcal{A}_2$ are (up to isomorphism) matrix algebras over a division algebra.

DEFINITION 3.6. The set of all Brauer equivalence classes of central simple algebras over k equipped with the tensor product of k-algebras is an abelian group (cf. [4, Proposition 2.4.8]), known as the *Brauer group* of k and denoted by $\mathrm{Br}(k)$. The *period* of a central simple algebra over k is defined to be its order as an element of the Brauer group. The *m-torsion* $\mathrm{Br}(k)[m]$ of the Brauer group $\mathrm{Br}(k)$ is the set of all elements of $\mathrm{Br}(k)$ of order dividing m.

Recall that each Brauer equivalence class contains a unique (up to isomorphism) division algebra. Define the *index* of a central simple algebra to be the degree of the division algebra D that is Brauer equivalent to it, i.e. the square root of the dimension of D over k.

In particular we have

COROLLARY 3.7 (cf. [3]). *The period of a central simple algebra over k divides its index, and hence is finite.*

Interplay. In the literature we find several approaches to the connection between central simple algebras and Brauer–Severi varieties; first, the connection between quaternion algebras and plane conics observed by E. Witt in [13]. In its general form, we mention for example the most elementary one promoted by J.-P. Serre in his books [8, 9]. The main observation is that central simple algebras of dimension $(n + 1)^2$ over a perfect field k as well as n-dimensional Brauer–Severi varieties over k can both be described by classes in one and the same cohomology set $\mathrm{H}^1(k, \mathrm{PGL}_{n+1}(\overline{k}))$.

4. Proofs of the results

4.1. A smooth $\overline{k}$-hypersurface without a non-singular hypersurface model over k. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$, and let $k \subset \overline{\mathbb{Q}}$ be a

number field. Suppose that

$$f(t) = t^{n+1} + \lambda_n t^n + \cdots + \lambda_1 t + (-1)^{n+1}\lambda_0 = \prod_{i=0}^{n}(t - a_i) \in k[t]$$

is an irreducible polynomial of degree $n + 1 \geq 3$ over $k[t]$ whose splitting field k_f over k satisfies $\mathrm{Gal}(k_f/k) \cong \mathbb{Z}/(n+1)\mathbb{Z}$. Such a polynomial exists since the inverse Galois problem for the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ is solvable (recall that any solvable group is realizable as a Galois group over a number field by the contribution of Shafarevich in 1954). Fix a generator

$$\sigma : a_0 \to a_1 \to a_2 \to \cdots \to a_{n-1} \to a_n \to a_0$$

for the Galois group $\mathrm{Gal}(k_f/k)$. Take a positive integer d not relatively prime to $n + 1$ such that the following holds: there exists $\beta \in k^*$ which is not a norm in k_f and $\lambda_0^{n+1}\beta^d = \prod_{i=0}^{n}\sigma^i(\alpha)$, equivalently $\beta^d = N_{k_f/k}(\alpha/\lambda_0)$, for some $\alpha \in k_f^* \setminus k^*$. Next, we define a smooth hypersurface over k_f by the equation

$$H_{f,\alpha,d,n} : \lambda_0 X_0^d + \sum_{i=1}^{n}\left(\frac{1}{\lambda_0^{i-1}}\prod_{j=0}^{i-1}\sigma^j(\alpha)\right)X_i^d = 0$$

of degree $d \geq 4$ such that $(n, d) \neq (3, 4)$. The matrix

$$\phi_\sigma := \begin{pmatrix} 0 & 0 & \ldots & 0 & \beta \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & & & & \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix}$$

defines an isomorphism $\phi_\sigma = C_\beta : {}^\sigma H_{f,\alpha,d,n} \to H_{f,\alpha,d}$, which satisfies Weil's condition of descent [12] ($\phi_{\sigma^{n+1}} = \phi_\sigma^{n+1} = 1$). We therefore find that the variety is defined over k, and that there exists an isomorphism $\varphi_0 : V_k \to H_{f,\alpha,d,n}$ where V_k is a rational model such that $\psi_\sigma = \phi_\sigma^{-1} = \varphi_0 \circ {}^\sigma\varphi_0^{-1} \in \mathrm{PGL}_{n+1}(k)$. The formula $\psi_\tau := \varphi_0 \circ {}^\tau\varphi_0^{-1}$ defines an element of $\mathrm{H}^1(\mathrm{Gal}(k_f/k), \mathrm{PGL}_{n+1}(k_f))$, corresponding to a cyclic algebra which is non-trivial because β is not a norm of an element of k_f (cf. [5, §2.1]). Consequently, φ_0 is not given by an element of $\mathrm{PGL}_{n+1}(k_f)$, or even $\mathrm{PGL}_{n+1}(\overline{k}_f)$, since its cohomology class after applying the inflation map is not trivial. Therefore, the variety V_k is not a smooth hypersurface over k (otherwise, V_k is identified via a k-isomorphism ψ_0 with a non-singular hypersurface model defined over k. Thus, by [6, Theorems 1, 2], the cohomology class $[\varphi_0 \circ \psi_0]$ is represented by an $M \in \mathrm{PGL}_{n+1}(\overline{k}_f)$, which it is not since $[\varphi_0] \neq 1$). As a concrete example, we specify the above construction in $\mathbb{P}^2$, $\mathbb{P}^3$ and $\mathbb{P}^4$.

(1) In $\mathbb{P}^2$: take $k = \mathbb{Q}$ and consider the irreducible polynomial $f(t) = t^3 + 12t^2 - 64$ over k (thus $\lambda_0 = 64$). As we can check with SAGE [10], the discriminant of the field k_f is a power of 3, and the prime 2 becomes inert in k_f, hence is not a norm in k_f. Consequently, we can assume, for example, that $d = 9m - 12$, $\beta = 2$ and $\alpha = a_0 8^m$ with $m \geq 2$ an integer.

(2) In $\mathbb{P}^3$: take $k = \mathbb{Q}$ and consider the irreducible polynomial $f(t) = t^4 + t^3 + 2t^2 - 4t + 3$ over k (thus $\lambda_0 = 3$). Then k_f is a cyclic extension of $\mathbb{Q}$ with Galois group isomorphic to $\mathbb{Z}/4\mathbb{Z}$ (we can think of it inside $\mathbb{Q}(\zeta_{13})$), moreover, $\lambda_0 = 3$ splits completely in k_f, so it may be a norm. In the ring of integers $\mathcal{O}_{k_f}$ of k_f, $\beta = 17 = \mathfrak{p}_1 \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are certain prime ideals of $\mathcal{O}_{k_f}$, also one of the generators of $\mathfrak{p}_1$ as a fractional ideal, say γ, belongs to $k_f \setminus \mathbb{Q}$. It is easy to check that $N_{k_f/\mathbb{Q}}(\gamma) = 17^2$, and we can choose $\alpha := 3 \cdot 17^m \gamma \in k_f \setminus \mathbb{Q}$ with $m \geq 1$, giving degrees $d = 4m + 2$ (using Sage:

```
K.<a>=NumberField(x^4+x^3+2*x^2-4*x+3);
K.ideal(17).factor();
>(Fractional ideal(1/3*a^3-1/3*a+3))
                *(Fractional ideal(1/3*a^3-1/3*a-6));
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suggests taking $\gamma = \frac{1}{3}a^3 - \frac{1}{3}a + 3$).

(3) In $\mathbb{P}^4$: the polynomial $f(t) = t^5 - t^4 - 4t^3 + 3t^2 + 3t - 1$ is irreducible over $\mathbb{Q}$. Its splitting field $\mathbb{Q}_f$ is cyclic over $\mathbb{Q}$ of degree 5 (we can think $\mathbb{Q}_f = \mathbb{Q}(\cos(2\pi/11)) \subseteq \mathbb{Q}(\zeta_{11})$). In the ring of integers $\mathcal{O}_{\mathbb{Q}_f}$, the torsion units are ± 1, and the roots a_i of f are units in $\mathcal{O}_{\mathbb{Q}_f}$. Suppose that $m > 1$ is an integer satisfying $\gcd(m, n + 1 = 5) = 1$ and $\gcd(\varphi(m), 5) = 1$, where φ is the Euler function. Now, take $d = 5m$ and $k = \mathbb{Q}(\zeta_d)$ with ζ_d a fixed dth primitive root of unity inside $\overline{\mathbb{Q}}$. Note that $f(t)$ is irreducible over $k = \mathbb{Q}(\zeta_d)$, since $k \cap \mathbb{Q}_f = \mathbb{Q}$. Also, k_f does not contain torsion roots of unity other than $\langle \zeta_d \rangle$, therefore ζ_d is not a norm from k_f to k. In particular, we can set $\beta = \zeta_d$ and $\alpha = a_0$.

COROLLARY 4.1. *Let V be a smooth $\overline{k}$-hypersurface over a perfect field k of degree $d \geq 4$ in $\mathbb{P}^n_k$ such that $(n, d) \neq (3, 4)$. Then V is not necessarily a smooth hypersurface over k.*

4.2. Minimal fields of definition for non-singular hypersurface models. In this subsection, V is a smooth $\overline{k}$-hypersurface of degree $d \geq 4$ in $\mathbb{P}^n_k$ with $n \geq 2$ and $(n, d) \neq (3, 4)$. Accordingly, $\overline{V} := V \otimes_k \overline{k}$ has an n-dimensional linear series over $\overline{k}$ that allows us to embed $\Upsilon : \overline{V} \hookrightarrow \mathbb{P}^n_{\overline{k}}$ as a non-singular hypersurface, and such a linear series is unique modulo conjugation in $\mathrm{PGL}_{n+1}(\overline{k})$.

We first show

PROPOSITION 4.2. *Let V be a smooth $\overline{k}$-hypersurface over k of degree $d \geq 4$ in $\mathbb{P}_{\overline{k}}^n$, where $n \geq 2$ and $(n,d) \neq (3,4)$. There exists a non-singular hypersurface model over a field extension L/k of degree $[L:k] \mid n+1$.*

Proof. We know from [7, Lemma 4] that the set of k-morphisms (modulo automorphisms) to some n-dimensional Brauer–Severi variety over k are in bijection with the base-point free n-dimensional linear series over $\overline{k}$ which are invariant under the G_k-action. Consequently, we may apply Theorem 3.3 to obtain a k-morphism $g : V \hookrightarrow \mathcal{B}$ to a Brauer–Severi variety $\mathcal{B}$ of dimension n over k such that $g \otimes_k \overline{k} : \overline{V} \hookrightarrow \mathbb{P}_{\overline{k}}^n$ equals to Υ. Moreover, by [7, Theorem 13(5)], there exists a field extension L/k of index $[L:k] \mid n+1$ that splits $\mathcal{B}$ (this means that $\mathcal{B} \otimes_k L$ is L-isomorphic to $\mathbb{P}_L^n$). Hence, we reduce to an embedding of $V \otimes_k L$ into $\mathbb{P}_L^n$ as the smooth variety $g(V) \otimes_k L$. By assumption, $g(V) \otimes_k \overline{k}$ is a hypersurface inside $\mathbb{P}_{\overline{k}}^n$; then so does $g(V) \otimes_k \overline{L} \subset \mathbb{P}_{\overline{L}}^n$. Consequently, $g(V) \otimes_k L$ has dimension $n-1$ and therefore it is a non-singular hypersurface model for $V \otimes_k L$ over L. ∎

Second, we show

PROPOSITION 4.3. *Let V be a smooth $\overline{k}$-hypersurface over k of degree $d \geq 4$ in $\mathbb{P}_{\overline{k}}^n$, where $n \geq 2$ and $(n,d) \neq (3,4)$. Then V is a smooth hypersurface over k if $V(k) \neq \emptyset$, $\gcd(d, n+1) = 1$, or $\mathrm{Br}(k)[n+1]$ is trivial.*

Proof. Using [7, Theorems 13(1),(2)], we find that the base field k splits $\mathcal{B}$ when $V(k) \neq \emptyset$ or $\gcd(d, n+1) = 1$. On the other hand, let $[\mathcal{A}]$ be the image of $\mathcal{B}$ in the Brauer group $\mathrm{Br}(k)$ (in particular, $\mathcal{B}$ splits over a field extension L/k if and only if $\mathcal{A}$ does over L/k). Due to Châtelet's thesis [2, Chapter IV.1], the division algebra associated to $\mathcal{A}$ has dimension dividing $n+1$. That is, $\mathcal{B}$ corresponds to an element of the $(n+1)$-torsion of $\mathrm{Br}(k)$, since the order of a central simple algebra as an element of $\mathrm{Br}(k)$ divides its index, which is the square root of the dimension of the associated division algebra (see Corollary 3.7). Therefore, $\mathcal{B}$ also splits over k if $\mathrm{Br}(k)[n+1]$ is trivial, being associated to a trivial central simple algebra over k.

By the above discussion, we can see that our variety V lives inside a trivial Brauer–Severi variety in any of the prescribed situations. This in turn gives a non-singular hypersurface model over k, and we conclude. ∎

COROLLARY 4.4. *Let V be a smooth $\overline{k}$-hypersurface over k of degree $d \geq 4$ in $\mathbb{P}_{\overline{k}}^n$, where $n \geq 2$ and $(n,d) \neq (3,4)$. Then V is a smooth hypersurface over k if*

(i) *k is an algebraically closed field;*
(ii) *k is a finite field;*
(iii) *k is the function field of an algebraic curve over an algebraically closed field;*

(iv) k *is an algebraic extension of* $\mathbb{Q}$, *containing all roots of unity; or*

(v) $k = \mathbb{R}$ *and* $n + 1$ *odd.*

Proof. In the following cases, every division algebra over a field k is k itself, so that the Brauer group $\mathrm{Br}(k)$ is trivial:

- k is an algebraically closed field (cf. [11, Example 4.2]).
- k is a finite field (Wedderburn's Little Theorem, cf. [8, p. 162]).
- k is the function field of an algebraic curve over an algebraically closed field (Tsen's Theorem, cf. [4, Theorem 6.2.8]). More generally, the Brauer group vanishes for any quasi-algebraically closed field.
- k is an algebraic extension of $\mathbb{Q}$, containing all roots of unity (cf. [8, p. 162]).

Finally, there are just two non-isomorphic real division algebras with center $\mathbb{R}$: $\mathbb{R}$ itself and the quaternion algebra $\mathbb{H}$. Since $\mathbb{H} \otimes \mathbb{H} \simeq M_4(\mathbb{R})$, the class of $\mathbb{H}$ has order two in the Brauer group. That is, the Brauer group $\mathrm{Br}(\mathbb{R})$ is the cyclic group of order two and then, for $n + 1$ odd, $\mathrm{Br}(\mathbb{R})[n + 1]$ does not contain non-trivial elements of order dividing $n + 1$.

Now, the result is an immediate consequence of Proposition 4.3. ∎

4.3. Twists which are smooth hypersurfaces over the base field k.
Now, assume that V is a smooth hypersurface over k, in particular V is k-isomorphic to a non-singular hypersurface model of degree $d \geq 4$ of the form

$$H_{F_V,d,n} : F_V(X_0, \ldots, X_n) := F_{M^{-1}\overline{V}}(X_0, \ldots, X_n)$$
$$= F_{\overline{V}}(M(X_0, \ldots, X_n)) \in k[X_0, \ldots, X_n]$$

for some $M \in \mathrm{PGL}_{n+1}(\overline{k})$ with $n \geq 2$ and $(n, d) \neq (3, 4)$. Since we have $\mathrm{Aut}(H_{F_V,d,n} \otimes_k \overline{k}) \subset \mathrm{PGL}_{n+1}(\overline{k})$ as G_k-groups, $\mathrm{Aut}(\overline{V})$ is naturally embedded into $\mathrm{PGL}_{n+1}(\overline{k})$ as G_k-groups, and we get a well-defined map

$$\Sigma : \mathrm{Twist}_k(V) = \mathrm{H}^1(k, \mathrm{Aut}(\overline{V})) \to \mathrm{H}^1(k, \mathrm{PGL}_{n+1}(\overline{k})).$$

Proof of Theorem 2.2 and Corollary 2.3. Let $[\mathbb{P}^n_k]$ denote the class of the trivial Brauer–Severi variety in $\mathrm{Twist}_k(\mathbb{P}^n_k) = \mathrm{H}^1(k, \mathrm{PGL}_{n+1}(\overline{k}))$. If a twist V'/k is k-isomorphic to a non-singular hypersurface model $F_{V'}(X_0, \ldots, X_n) = 0$ over k, then $F_{V'}(X_0, \ldots, X_n) = 0$ and $F_V(X_0, \ldots, X_n) = 0$ are isomorphic through some $M' \in \mathrm{PGL}_{n+1}(\overline{k})$ by [6, Theorem 1, 2] when $n \geq 3$ and it is well known when $n = 2$ in the case of plane curves of degree ≥ 4. Hence, the corresponding 1-cocycle $\sigma \mapsto M' \circ {}^\sigma(M')^{-1} \in \mathrm{Aut}(H_{F_V,d,n} \otimes_k \overline{k})$ is trivial in $\mathrm{H}^1(k, \mathrm{PGL}_{n+1}(\overline{k}))$, being cohomologous to the trivial 1-cocycle. Conversely, if $\Sigma([V'])$ is trivial (that is, V'/k lives inside a trivial Brauer–Severi variety of dimension n over k), then it must be given by a $\overline{k}$-isomorphism $\varphi : \{F_V(X_0, \ldots, X_n) = 0\} \to V'$ induced by some $\tilde{M} \in \mathrm{PGL}_{n+1}(\overline{k})$, i.e. V' is k-isomorphic to $F_{(\tilde{M})^{-1}V}(X_0, \ldots, X_n) = 0$. This would give a non-singular hypersurface model over k for V'.

In general, any twist V' for V over k is again a smooth $\overline{k}$-hypersurface over k in $\mathbb{P}^n_{\overline{k}}$ of degree d. It only remains to apply Theorem 2.1 for V' to conclude that V' is a smooth hypersurface over k if $\gcd(d, n+1) = 1$ or if $\mathrm{Br}(k)[n+1]$ is trivial. In particular, Σ is the trivial map in both cases, which was to be shown. ∎

4.4. A smooth hypersurface over a number field k, having a twist which is not a smooth hypersurface over k. Let p be an odd prime number and fix ζ_p, a primitive pth root of unity inside $\overline{\mathbb{Q}}$. Assume that $H_{F,d,n} : F(X_0, \ldots, X_n) = 0 \subset \mathbb{P}^n_{\overline{k}}$ is a smooth hypersurface of degree $d = 2p$, not relatively prime to $n+1$, defined over the number field $k = \mathbb{Q}(\zeta_p)$. Suppose further that the projective matrix

$$
\phi := \begin{pmatrix}
0 & 0 & \ldots & 0 & \zeta_p \\
1 & 0 & \ldots & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 \\
\vdots & & & & \\
0 & 0 & \ldots & 1 & 0
\end{pmatrix}
$$

induces an automorphism of $H_{F,d=2p,n} \otimes_k \overline{k}$. For example, consider the following family parametrized by $a \in k$:

$$
H_{F_a,d=2p,n} : F_a(X_0, \ldots, X_n) = \sum_{i=0}^{n} X_i^d + \sum_{i=0}^{n-1} a X_i^p \sum_{j=i+1}^{n} X_j^p = 0.
$$

If $m \in k^* \setminus (k^*)^p$, then $x^p - m$ is irreducible in $k[x]$ by a theorem of Abel, the Galois field extension $L_m = k(\sqrt[p]{m})$ has Galois group $\mathrm{Gal}(L/k) = \langle \sigma \rangle \simeq \mathbb{Z}/p\mathbb{Z}$, where $\sigma(\sqrt[p]{m}) = \zeta_p \sqrt[p]{m}$. Define the 1-cocycle by

$$
\xi : \sigma \mapsto \phi \in \mathrm{H}^1(\mathrm{Gal}(L_m/k), \mathrm{PGL}_{n+1}(L_m)).
$$

Since no new primitive root of unity appears in L_m other than ζ_p, ζ_p is not a norm in L_m, $[\xi]$ is non-trivial in $\mathrm{H}^1(\mathrm{Gal}(L_m/k), \mathrm{PGL}_{n+1}(L_m))$ by [5, §1.1], and hence the image of ξ in $\mathrm{H}^1(k, \mathrm{PGL}_{n+1}(\overline{k}))$, which coincides with the inflation of $\sigma \mapsto \phi$, is not trivial. Consequently, it corresponds to a twist V' for $H_{F_a,d,n}$ over k living inside a non-trivial Brauer–Severi variety of dimension n over k (that is, $\Sigma([V']) \neq [\mathbb{P}^n_k]$).

4.5. Diagonal twists. Let $V/k : F_V(X_0, \ldots, X_n) = 0$ be a smooth hypersurface over a perfect field k. Assume that $\mathrm{Aut}(V \otimes_k \overline{k}) \hookrightarrow \mathrm{PGL}_{n+1}(\overline{k})$ is a non-trivial cyclic group of order m (relatively prime to the characteristic of k), generated by $\psi = \mathrm{diag}(1, \zeta_m^{a_1}, \ldots, \zeta_m^{a_n})$ for some $a_i \in \mathbb{N}$, where ζ_m is a fixed primitive mth root of unity in $\overline{k}$.

Proof of Theorems 2.5 and 2.6. It suffices to notice that the embedding $\mathrm{Aut}(V \otimes_k \overline{k}) \hookrightarrow \mathrm{PGL}_{n+1}(\overline{k})$ factors through $\mathrm{GL}_{n+1}(\overline{k})$. Thus the map Σ in Theorem 2.2 factors as follows:

$$\Sigma : \mathrm{H}^1(k, \mathrm{Aut}(V \otimes_k \overline{k})) \to \mathrm{H}^1(k, \mathrm{GL}_{n+1}(\overline{k})) \to \mathrm{H}^1(k, \mathrm{PGL}_{n+1}(\overline{k})).$$

Moreover, $\mathrm{H}^1(k, \mathrm{GL}_{n+1}(\overline{k})) = 1$ by Hilbert's 90th Theorem, so the map Σ is trivial. By Theorem 2.2, any twist has a non-singular plane model of the form $F_{P^{-1}V}(X_0, \ldots, X_n) = 0$ over k, for some $P \in \mathrm{PGL}_{n+1}(\overline{k})$. Since $P \circ {}^{\sigma}(P^{-1}) \in \mathrm{Aut}(V \otimes_k \overline{k}) = \langle \mathrm{diag}(1, \zeta_m^{a_1}, \ldots, \zeta_m^{a_n}) \rangle$ for any $\sigma \in \mathrm{G}_k$, we have ${}^{\sigma}P = P \circ \mathrm{diag}(1, v_1, \ldots, v_n)$ for some mth roots of unity v_i. Writing $P = (a_{i,j})$, one easily deduces that $\sigma(a_{i,j}) = v_j a_{i,j}$ for all i, j. Hence, for any fixed integer j, we have $\sigma(a_{i,j}) a_{i,j}^{-1} = \sigma(a_{i',j}) a_{i',j}^{-1}$. That is, $a_{i,j} a_{i',j}^{-1}$ is G_k-invariant, which in turn gives $a_{i,j} = m_i a_{i',j}$ for some $m_i \in k$. In particular, P reduces to MD for some diagonal projective $(n + 1) \times (n + 1)$ matrix D and $M \in \mathrm{PGL}_{n+1}(k)$. This proves that all the twists are diagonal. However, the non-singular hypersurface model $F_{(MD)^{-1}V}(X_0, \ldots, X_n) = 0$ over k is k-isomorphic through M to $F_{D^{-1}V}(X_0, \ldots, X_n) = 0$. Consequently, $F_{D^{-1}V}(X_0, \ldots, X_n) = 0$ defines a non-singular hypersurface model over k for the twist. ∎

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Abstract (will appear on the journal's web site only)

Given a smooth projective variety of dimension $n - 1 \geq 1$ defined over a perfect field k that admits a non-singular hypersurface model in $\mathbb{P}^n_{\overline{k}}$ over $\overline{k}$, a fixed algebraic closure of k, it does not necessarily have a non-singular hypersurface model defined over the base field k. We first show an example of such phenomenon: a variety defined over k admitting non-singular hypersurface models but none defined over k. We also determine under which conditions a non-singular hypersurface model over k may exist. Now, even assuming that such a smooth hypersurface model exists, we wonder about the existence of non-singular hypersurface models over k for its twists. We introduce a criterion to characterize twists possessing such models and we also show an example of a twist not admitting any non-singular hypersurface model over k, i.e. for any $n \geq 2$, there is a smooth projective variety of dimension $n - 1$ over k which is a twist of a smooth hypersurface variety over k, but itself does not admit any non-singular hypersurface model over k. Finally, we obtain a theoretical result to describe all the twists of smooth hypersurfaces with cyclic automorphism group having a model defined over k whose automorphism group is generated by a diagonal matrix.

The particular case $n = 2$ for smooth plane curves was studied by the authors jointly with E. Lorenzo García in [Math. Comp. 88 (2019)], and we deal here with the problem in higher dimensions.