LIMIT CYCLES OF A PERTURBATION OF A POLYNOMIAL HAMILTONIAN SYSTEMS OF DEGREE 4 SYMMETRIC WITH RESPECT TO THE ORIGIN

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Abstract. We study the number of limit cycles bifurcating from the origin of a Hamiltonian system of degree 4. We prove using the averaging theory of order 7, that there are quartic polynomial systems close these Hamiltonian systems having 3 limit cycles.

1. Introduction and statement of the main result

One of the main open problems in the qualitative theory of planar differential systems is the determination of limit cycles. Closely related to the Hilbert's 16th problem is the study of the limit cycles from planar differential systems when we vary the parameters bifurcating from a center, or from its periodic solutions, and has been exhaustively studied in the last century. However there is no general method to solve completely this problem, the averaging theory as being largely studied in recent years in order to analyze the problem of the bifurcation of limit cycles, see for instance [4, 2, 7, 9, 8, 10, 13, 14, 17, 18, 15, 23]. For details about the averaging theory see the book of Sanders, Verhults and Murdock [21].

In this work we deal with polynomial differential systems in $\mathbb{R}^2$ of the form

$$\dot{x} = P(x,y), \quad \dot{y} = Q(x,y), \tag{1}$$

where the dot denotes derivative with respect to an independent real variable t, usually called the *time*. Assume that the origin O is an equilibrium point of system (1). When all the orbits of system (1) in a punctured neighborhood of the equilibrium point O are periodic, we say that the origin is a *center*. The study of the centers remain open in the present days and was started by Poincaré [20] and Dulac [6].

We focus on a polynomial differential system (1) having a center at the origin of linear type, i.e. after a linear change of variables and a scaling of the time variable, it can be written in the form:

$$\dot{x} = -y + P_2(x,y), \quad \dot{y} = x + Q_2(x,y),$$

where $P_2(x,y)$ and $Q_2(x,y)$ are polynomials without constant and linear terms.

This paper is a natural continuation of the work "Linear type centers of polynomial Hamiltonian systems with nonlinearities of degree 4 symmetric with respect to the y-axis" [16] where we consider the Hamiltonian systems

$$\dot{x} = -y - x^4 - 3bx^2y^2 - 5cy^4, \quad \dot{y} = x + 4x^3y + 2bxy^3, \tag{2}$$

of degree 4 with Hamiltonian function

$$H(x,y) = \frac{1}{2}(x^2+y^2) + x^4y + bx^2y^3 + cy^5, \tag{3}$$

and are classify all the phase portraits of these Hamiltonian systems in the Poincaré disk, see Figure 1.

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In this work we perturb the Hamiltonian systems (2). Thus we consider these class of all polynomial differential systems of degree 4, i.e.

$$
\begin{aligned}
\dot{x} = & \quad -y - x^4 - 3bx^2y^2 - 5cy^4 + \sum_{i=1}^{7} \varepsilon^i p_i(x,y), \\
\dot{y} = & \quad x + 4x^3y + 2bxy^3 + \sum_{i=1}^{7} \varepsilon^i q_i(x,y),
\end{aligned} \tag{4}
$$

where

$$
\begin{aligned}
p_i(x,y) = & \quad a_0^i + a_1^i x + a_2^i y + a_3^i x^2 + a_4^i xy + a_5^i y^2 + a_6^i x^3 + a_7^i x^2 y + \\
& \quad a_8^i xy^2 + a_9^i y^3 + a_{10}^i x^4 + a_{11}^i x^3 y + a_{12}^i x^2 y^2 + a_{13}^i xy^3 + a_{14}^i y^4, \\
q_i(x,y) = & \quad b_0^i + b_1^i x + b_2^i y + b_3^i x^2 + b_4^i xy + b_5^i y^2 + b_6^i x^3 + b_7^i x^2 y + \\
& \quad b_8^i xy^2 + b_9^i y^3 + b_{10}^i x^4 + b_{11}^i x^3 y + b_{12}^i x^2 y^2 + b_{13}^i xy^3 + b_{14}^i y^4.
\end{aligned} \tag{5}
$$

Our objective is to study the number of limit cycles bifurcating from the origin of system (4) using the averaging theory up to order 7. Our main result is the following one:

Theorem 1. *For $\varepsilon > 0$ sufficiently small the maximum number of small limit cycles of the differential system* (4) *bifurcating from the center* $(0,0)$ *obtained using the averaging theory of order*

- (a) *one and two is 0;*
- (b) *three and four is 1;*
- (c) *five and six is 2;*
- (d) *seven is 3.*

Theorem 1 is proved in Section 3. All the computations of this paper have been revised with the help of the algebraic manipulator Mathematica.

Thus the two main objectives of this paper are: First to illustrate how to use the averaging theory up to order 7 for compute periodic solutions, and second how to use the averaging theory for studying the periodic solutions which are born in a Hopf bifurcation. We note that if the objective of this paper was to estimate the bound of the maximum number of periodic solutions of the differential system (4), this can be done using the techniques of the papers [11, 12].

In section 2 we provide the notations, basic definitions and results which will allow to do this study.

2. Preliminary results

We consider the center at the origin of system (2), the global phase portraits of this system was detailed studied and the results are summarize in the next theorem proved in [16].

Theorem 2. *The phase portrait in the Poincaré disk of a linear type center of a polynomial Hamiltonian system with nonlinearities of degree* 4 *symmetric with respect to the y-axis is topologically equivalent to one of the 30 phase portraits of Figure 1.*

The averaging theory is fundamental to our study, so we introduce the main result for applying it, see [17]. Consider the system

$$
\dot{x} = \sum_{i=1} \varepsilon^i F_i(t,x) + \varepsilon^{k+1} R(t,x,\varepsilon), \tag{6}
$$

where $F_i : \mathbb{R} \mapsto \mathbb{R}^n$ for $i = 1, 2, \ldots, k$ and $R : \mathbb{R} \times I \times (-\varepsilon_0, \varepsilon_0) \mapsto \mathbb{R}^n$ are continuous functions and T-periodic in the first variable, I being an open subset of $\mathbb{R}^n$.

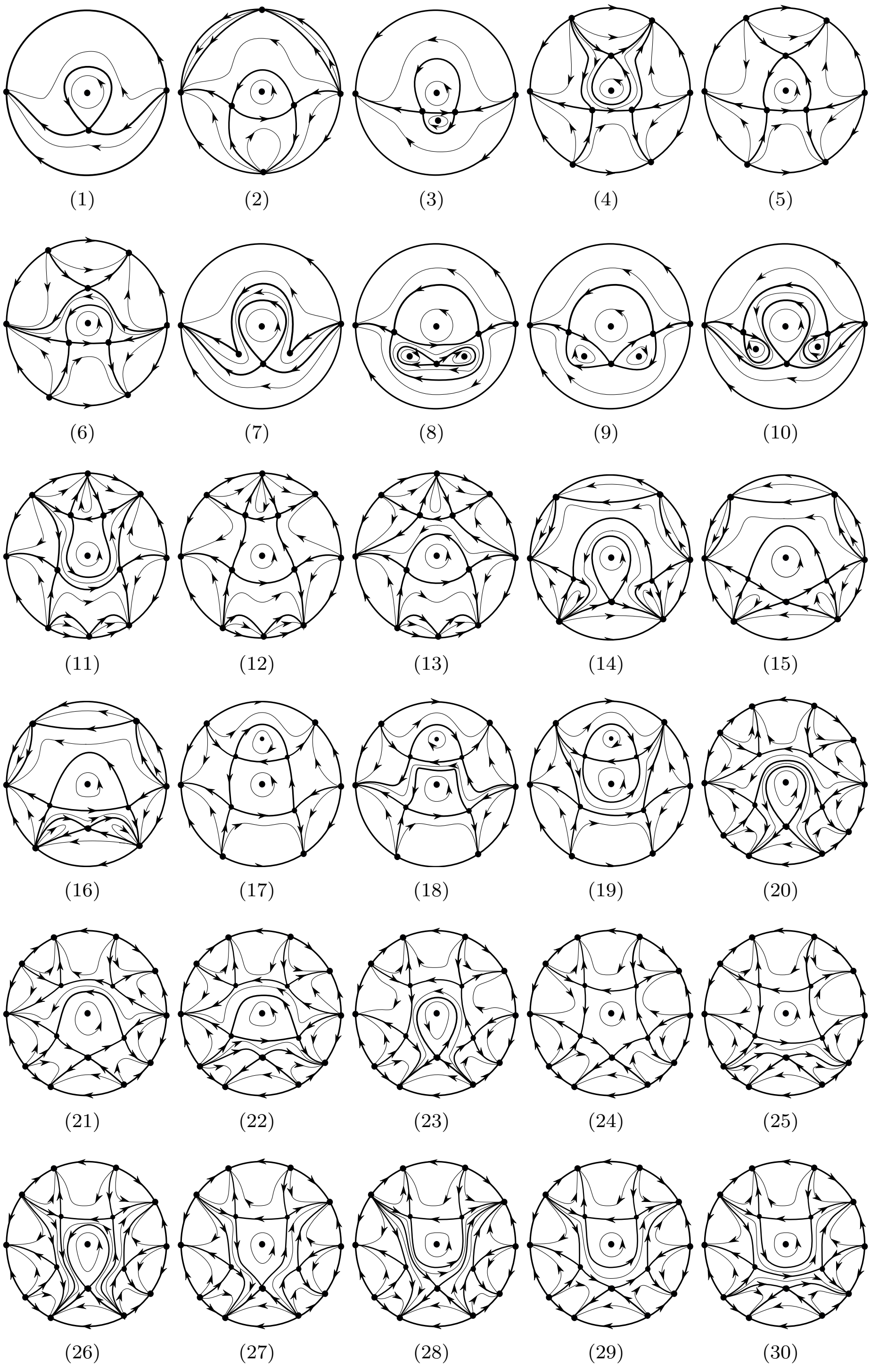

FIGURE 1. Phase portraits of the Hamiltonian systems (2). The separatrices are in bold.

For $i = 1, \dots, k$ we define the averaging function $f_i : I \mapsto \mathbb{R}^n$ of order i as

$$f_i(z) = \frac{y_i(T, z)}{i!}, \tag{7}$$

where $y_i : \mathbb{R} \times I \mapsto \mathbb{R}^n, i = 1, \dots, k-1$ are defined recurrently by the following integral equation

$$\begin{aligned} y_i(t,z) \quad &= i! \int_0^t \Big[F_i(s, \varphi(s,z)) + \\ &\sum_{l=1}^{i} \sum_{S_l} \frac{1}{b_1! b_2! 2!^{b_2} \cdots b_l! l!^{b_l}} \partial^L F_{i-l}(s, \varphi(s,z)) \prod_{j=1}^{l} y_j(s,z)^{b_j} \Big] ds \end{aligned} \tag{8}$$

where $\partial^L G(u, v)$ denote the derivative of order L of a function G with respect to the variable u, and S_l is the set of all l-tuples of non-negative integers $(b_1, b_2, \dots, b_l)$ satisfying $b_1 + 2b_2 + \cdots + lb_l = l$, and $L = b_1 + b_2 + \cdots + b_l$. The explicit expressions of the average functions for $i = 1, \dots, 7$ are given in the appendix A.

Now we can enunciate the following result, proved in section 3 of [17].

Theorem 3. *For the functions of* (6) *we assume the following conditions.*

1. *For each* $t \in \mathbb{R}$, $F_i(t, \cdot) \in \mathcal{C}^{k-i}$ *for* $i = 1, \cdots k, \partial^{k-i} F_i$ *is locally Lipschitz in the second variable for* $i = 1, \cdots, k$ *and* R *is a continuous function locally Lipschitz in the second variable.*
2. *Assume that* $f_i = 0$, $i = 1, \cdots, r-1$ *and* $f_r \neq 0, r \in \{1, \cdots, k\}$. *Moreover suppose that for some* $a \in I$ *with* $f_r(a) = 0$ *there exists a neighborhood* $V \subset I$ *of* a *such that* $f_r(z) \neq 0$, $\forall z \in \bar{V} \setminus a$ *and* $d_B(f_r(z), V, 0) \neq 0$ *(here* $d_B(f_r(z), V, 0)$ *denotes the Brouwer degree of* f_r *at* a*).*

Then for sufficiently small $|\varepsilon| > 0$ *there exists a* T*-periodic solution* $x(\cdot, \varepsilon)$ *of* (6) *such that* $x(0, \varepsilon) \to a$ *when* $\varepsilon \to 0$.

We note that when f_r is C^1 then the Brouwer degree of f_r at a is non-zero if the determinant of the Jacobian matrix $Df_r(a)$ is non-zero. For more details see [3, 19].

Another important tool is the Descartes Theorem about the number of zeros of a real polynomial (see [1]).

Theorem 4 (Descartes Theorem)**.** *Consider the real polynomial* $p(x) = a_{i_1} x i_1 + a_{i_2} x i_2 + \cdots + a_{i_r} x i_r$ *with* $0 < i_1 < i_2 < \cdots < i_r$ *and* $a_{i_j} \neq 0$ *real constants for* $j \in \{1, 2, \cdots, r\}$. *When* $a_{i_j} a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* m, *then* $p(x)$ *has at most* m *positive real roots. Moreover it is always possible to choose the coefficients of* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r - 1$ *positive real roots.*

Gauss showed later on that the number of allowable positive roots is $m, m-2, m-4, \cdots$ given that the sign changes m times.

3. Proof of Theorem 1

We shall use the averaging theory up to order seven for studying the number of small limit cycles of system (4) which can bifurcate from the origin of system (4) with $\varepsilon = 0$ when this system is perturbed with $\varepsilon \neq 0$ and small. First we do the rescaling of the variables $(x, y) \to (X, Y)$ with $x = \varepsilon X$, $y = \varepsilon Y$, then the initial differential system (4) becomes a differential system of the form

$$\dot{X} = -Y + \sum_{i=1}^{7} \varepsilon^i r_i(X.Y) + O(\varepsilon^8), \quad \dot{Y} = X + \sum_{i=1}^{7} \varepsilon^i s_i(X.Y) + O(\varepsilon^8). \tag{9}$$

After we pass to polar coordinates $(X, Y) \to (r, \theta)$ given by $X = r\cos\theta$, $Y = r\sin\theta$, and the previous differential system writes

$$\dot{r} = \sum_{i=1}^{7} \varepsilon^i R_i(r, \theta) + O(\varepsilon^8), \quad \dot{\theta} = 1 + \sum_{i=1}^{7} \varepsilon^i S_i(r, \theta) + O(\varepsilon^8). \tag{10}$$

Finally we take as independent variable the variable θ and the differential system becomes the differential equation

$$\frac{dr}{d\theta} = \sum_{i=1}^{7} \varepsilon^i T_i(r, \theta) + O(\varepsilon^8). \tag{11}$$

This differential equation is in the normal form for applying the averaging theory described in [14], which is summarized in section 2.

Thus we apply the averaging theory from order 1 to 7 in ε and we shall obtain the number of periodic solutions indicated in Theorem 1 for the different orders. More precisely, let

$$r(\theta, \varepsilon) = r^* + O(\varepsilon),$$

be a periodic solution of the differential equation (11) obtained from the averaging theory of order k, being r^* a simple zero of the averaged function of order k. Then this periodic solution provides the periodic solution

$$(r(t, \varepsilon), \theta(t, \varepsilon)) = (r^*, 1) + O(\varepsilon),$$

of the differential system (10). And this last periodic solution gives place to the periodic solution

$$(X(t, \varepsilon), Y(t, \varepsilon)) = (r^* \cos t, r^* \sin t) + O(\varepsilon),$$

of the differential system (9). Finally, we get the periodic solution

$$(x(t, \varepsilon), y(t, \varepsilon)) = (\varepsilon r^* \cos t, \varepsilon r^* \sin t) + O(\varepsilon^2),$$

of system (1).

In summary, all the periodic solutions $r(\theta, \varepsilon)$ obtained applying the averaging theory to the differential equation (4) provide periodic solutions $(x(t, \varepsilon), y(t, \varepsilon))$ of the differential system (1) which tends to the origin (i.e. to the center localized at the origin of coordinates) when $\varepsilon \to 0$. Therefore are periodic solutions bifurcating from the origin in a Hopf bifurcation.

By doing a Taylor expansion truncated at the 7th order in ε we obtain an expression in the form (6) for $dr/d\theta$ with $k = 7$. The explicit expansion is

$$\frac{dr}{d\theta} = K_1\varepsilon + K_2\varepsilon^2 + K_3\varepsilon^3 + K_4\varepsilon^4 + K_5\varepsilon^5 + K_6\varepsilon^6 + K_7\varepsilon^7 + \cdots, \tag{12}$$

where considering $C = \cos\theta$ and $S = \sin\theta$ the first three coefficients are:

$K_1 = r(a_1^1C^2 + (a_2^1 + b_1^1)CS + b_2^1S^2)$,

$K_2 = r(a_1^2C^2 + a_3^1C^3r + a_2^2CS + b_1^2CS + a_4^1C^2rS + b_3^1C^2rS + b_2^2S^2 + a_5^1CrS^2 + b_4^1CrS^2 + b_5^1rS^3 + (-b_1^1C^2 + S(a_1^1C - b_2^1C + a_2^1S))(a_1^1C^2 + S((a_2^1 + b_1^1)C + b_2^1S)))$,

and

$K_3 = r(a_1^3C^2 + a_3^2C^3r + a_6^1C^4r^2 - C^5r^3 + a_2^3CS + b_1^3CS + a_4^2C^2rS + b_3^2C^2rS + a_7^1C^3r^2S + b_6^1C^3r^2S + b_2^3S^2 + a_5^2CrS^2 + b_4^2CrS^2 + a_8^1C^2r^2S^2 + b_7^1C^2r^2S^2 + 4C^3r^3S^2 + 3bC^3r^3S^2 + b_5^2rS^3 + a_9^1Cr^2S^3 + b_8^1Cr^2S^3 + b_9^1r^2S^4 + 2bCr^3S^4 + 5cCr^3S^4 + (-b_1^2C^2 + (b_1^1)^2C^4 - b_3^1C^3r + a_1^2CS - b_2^2CS + a_3^1C^2rS - b_4^1C^2rS + a_2^2S^2 + (a_1^1)^2C^2S^2 - 2a_1^1b_2^1C^2S^2 + (b_2^1)^2C^2S^2 + a_4^1CrS^2 - b_5^1CrS^2 + 2a_1^1a_2^1CS^3 - 2a_2^1b_2^1CS^3 + a_5^1rS^3 + (a_2^1)^2S^4 - 2b_1^1C^2S(a_1^1C - b_2^1C + a_2^1S))(a_1^1C^2 + S(a_2^1C + b_1^1C + b_2^1S)) + (-b_1^1C^2 + S(a_1^1C - b_2^1C + a_2^1S))(a_1^2C^2 + a_3^1C^3r + S(a_2^2C + b_1^2C + a_4^1C^2r + b_3^1C^2r + b_2^2S + a_5^1CrS + b_4^1CrS + b_5^1rS^2)))$.

The other coefficient are too longer and we do not provide them here.

In this work we consider the center at the origin. Our system (4) is a polynomial differential system so the functions F_i and R_i are analytic, and the variable θ is 2π-periodic because appear through sinus and cosinus. Therefore we can apply Theorem 3 setting the interval $I = \{r : 0 < \bar{r}\}$, for some $\bar{r} > 0$.

Applying Theorem 3 we obtain the average function of first order

$f_1(r) = \pi(a_1^1 + b_2^1)r.$

Clearly $f_1(r)$ has no solution in I. Therefore the averaging method of first order does not provide any small limit cycle bifurcating from the origin.

We set $b_2^1 = -a_1^1$ and obtain $f_1(r) = 0$. So we can apply the averaging theory of second order and we obtain the averaging function of order two:

$f_2(r) = \pi(a_1^2 + b_2^2)r.$

Newly the averaging function has not solution in I. Thus, the statement (a) of Theorem 1 is proved.

Doing $b_2^2 = -a_1^2$ we have $f_2(r) = 0$ and we can apply the averaging method of thirst order, the averaging function of order three is

$f_3(r) = r(A_1 + A_3r^2),$
where

$A_1 = \pi(a_1^3 + b_2^3), \quad A_3 = (3a_6^1 + a_8^1 + b_7^1 + 3b_9^1)\pi/4.$

Thus $f_3(r)$ has one positive real root $r^* = \sqrt{-A_1/A_3}$ in I if $0 < -A_1/A_3$ and in this case it holds that $(df_3/dr)(r^*) \neq 0$. Hence, applying the averaging theory of order three we can detect one limit cycle bifurcating from the center at the origin of system (4) with $\varepsilon = 0$.

In order to apply the averaging method of fourth order we set $b_2^3 = -a_1^3$ and $b_9^1 = -a_6^1 - a_8^1/3 - b_7^1/3$. So the averaging function of fourth order is

$f_4(r) = r(B_1 + B_3r^2),$

where $B_1 = \pi(a_1^4 + b_2^4)$ and $B_3 = (-a_3^1a_4^1 - a_4^1a_5^1 + 3a_2^1a_6^1 - 3a_6^2 - 2a_1^1a_7^1 - a_8^2 + 3a_6^1b_1^1 + 2a_3^1b_3^1 + b_3^1b_4^1 - 2a_5^1b_5^1 + b_4^1b_5^1 + a_2^1b_7^1 + b_1^1b_7^1 - b_7^2 - 2a_11b_8^1 - 3b_9^2)\pi/4$. Thus $f_4(r)$ has one positive real zero in I if $0 < -B_1/B_3$. Hence applying the averaging theory of order fourth we know that one small limit cycle bifurcates from the origin of system(4) with $\varepsilon = 0$. So statement (b) of Theorem 1 is proved.

To apply the averaging method of fifth order we first set $b_2^4 = -a_1^4$ and $b_9^2 = (-a_3^1a_4^1 - a_4^1a_5^1 + 3a_2^1a_6^1 - 3a_6^2 - 2a_1^1a_7^1 - a_8^2 + 3a_6^1b_1^1 + 2a_3^1b_3^1 + b_3^1b_4^1 - 2a_5^1b_5^1 + b_4^1b_5^1 + a_2^1b_7^1 + b_1^1b_7^1 - b_7^2 - 2a_1^1b_8^1)/3$ and then $f_4(r) = 0$. We continue applying the averaging method of fifth order where the averaging function is

$f_5(r) = r(C_1 + C_3r^2 + C_5r^4),$

where

$C_1 = (a_1^5 + b_2^5)\pi,$

$C_3 = (2a_1^1(a_3^1)^2 + a_3^2a_4^1 + a_1^1(a_4^1)^2 + a_3^1a_4^2 + 2a_1^1a_3^1a_5^1 + a_2^1a_4^1a_5^1 + a_4^2a_5^1 + a_4^1a_5^2 - 3a_2^2a_6^1 - 3a_2^1a_6^2 + 3a_6^3 + 2a_1^2a_7^1 + 2a_1^1a_7^2 + a_8^3 - a_3^1a_4^1b_1^1 + 3a_2^1a_6^1b_1^1 - 3a_6^2b_1^1 - 2a_1^1a_7^1b_1^1 + 3a_6^1(b_1^1)^2 - 3a_6^1b_1^2 + 2a_2^1a_3^1b_3^1 - 2a_3^2b_3^1 - a_1^1a_4^1b_3^1 + 4a_3^1b_1^1b_3^1 - 2a_3^1b_3^2 - a_1^1a_3^1b_4^1 + a_1^1a_5^1b_4^1 + a_2^1b_3^1b_4^1 + 2b_1^1b_3^1b_4^1 - b_3^2b_4^1 - a_1^1(b_4^1)^2 - b_3^1b_4^2 + a_1^1a_4^1b_5^1 + 2a_2^1a_5^1b_5^1 + 2a_5^2b_5^1 - 2a_1^1b_3^1b_5^1 + b_1^1b_4^1b_5^1 - b_4^2b_5^1 - 2a_1^1(b_5^1)^2 + 2a_5^1b_5^2 - b_4^1b_5^2 - a_2^2b_7^1 + a_2^1b_1^1b_7^1 + (b_1^1)^2b_7^1 - b_1^2b_7^1 - a_2^1b_7^2 - b_1^1b_7^2 + b_7^3 + 2a_1^2b_8^1 - 2a_1^1b_1^1b_8^1 + 2a_1^1b_8^2 + 3b_9^3)\pi/4,$

and

$C_5 = (-a_4^1 + a_4^1b - 4bb_3^1 - 2b_5^1 - 10bb_5^1 + 5a_4^1c + 10b_5^1c)\pi/8.$

The averaging function $f_5(r)$ can have at most 2 positive real zeros in I because C_1, C_3 and C_5 are linearly independent since C_1 depends only on a_1^5 and b_2^5, C_5 only depends on a_4^1, b_3^1 and b_5^1 and C_3 depends of these coefficients and on other more, for example only C_3 present the coefficient a_6^3, and by the Descartes Theorem 4 the averaging function f_5 can present two change of sing so almost there is two positive simple roots for f_5. We prove through the averaging method of fifth order that at most 2 limit cycles can bifurcate from the origin of system (4) with $\varepsilon = 0$ using this averaging theory.

At this moment we separate the study in three cases: $b \neq 0$, or $b = 0$ and $c \neq 1/5$, or $b = 0$ and $c = 1/5$.

3.1. **Case $b \neq 0$.**

Setting $b_2^5 = -a_1^5$, $b_3^1 = (-2b_5^1(1+5b-5c) + a_4 1(-1+b+5c))/(4b)$ (here we use that $b \neq 0$) and solving C_3 for b_9^3 we can apply the averaging method of sixth order. The sixth averaging function is

$$f_6(r) = r(D_1 + D_3 r^2 + D_5 r^4),$$

where

$$D_1 = (a_1^6 + b_2^6)\pi,$$

$$D_3 = \pi(2(a_1^1)^2 a_3^1 a_4^1 - 2a_2^2 a_3^1 a_4^1 - 2a_2^1 a_3^2 a_4^1 + 2a_3^3 a_4^1 + a_1^2 (a_4^1)^2 + a_1^1 a_4^1 a_4^2 + 8a_1^2 (a_3^1)^2 b + 16a_1^1 a_3^1 a_3^2 b + 10(a_1^1)^2 a_3^1 a_4^1 b + 2a_2^2 a_3^1 a_4^1 b + 2a_2^1 a_3^2 a_4^1 b + 2a_3^3 a_4^1 b + 3a_1^2 (a_4^1)^2 b + 4a_1^1 a_2^1 (a_4^1)^2 b + 4a_3^2 a_4^2 b + 7a_1^1 a_4^1 a_4^2 b + 4a_3^1 a_4^3 b + 8a_1^2 a_3^1 a_5^1 b + 8a_1^1 a_2^1 a_3^1 a_5^1 b + 8a_1^1 a_3^2 a_5^1 b + 4(a_1^1)^2 a_4^1 a_5^1 b + 4(a_2^1)^2 a_4^1 a_5^1 b + 4a_2^2 a_4^1 a_5^1 b + 4a_2^1 a_4^2 a_5^1 b + 4a_4^3 a_5^1 b + 8a_1^1 a_3^1 a_5^2 b + 4a_2^1 a_4^1 a_5^2 b + 4a_4^2 a_5^2 b + 4a_4^1 a_5^3 b - 12a_2^3 a_6^1 b - 12a_2^2 a_6^2 b - 12a_2^1 a_6^3 b + 12a_6^4 b + 8a_1^3 a_7^1 b + 8a_1^2 a_7^2 b + 8a_1^1 a_7^3 b + 4a_8^4 b + 4a_2^1 a_3^1 a_4^1 b_1^1 - 4a_3^2 a_4^1 b_1^1 - 2a_1^1 (a_4^1)^2 b_1^1 - 16a_1^1 (a_3^1)^2 b b_1^1 - 4a_2^1 a_3^1 a_4^1 b b_1^1 - 2a_1^1 (a_4^1)^2 b b_1^1 - 4a_3^1 a_4^2 b b_1^1 - 8a_1^1 a_3^1 a_5^1 b b_1^1 + 12a_2^2 a_6^1 b b_1^1 + 12a_2^1 a_6^2 b b_1^1 - 12a_6^3 b b_1^1 - 8a_1^2 a_7^1 b b_1^1 - 8a_1^1 a_7^2 b b_1^1 + 6a_3^1 a_4^1 (b_1^1)^2 - 2a_3^1 a_4^1 b (b_1^1)^2 - 12a_2^1 a_6^1 b (b_1^1)^2 + 12a_6^2 b (b_1^1)^2 + 8a_1^1 a_7^1 b (b_1^1)^2 - 12a_6^1 b (b_1^1)^3 - 4a_3^1 a_4^1 b_1^2 + 12a_2^1 a_6^1 b b_1^2 - 12a_6^2 b b_1^2 - 8a_1^1 a_7^1 b b_1^2 + 24a_6^1 b b_1^1 b_1^2 - 12a_6^1 b b_1^3 + 8a_2^1 a_3^1 b b_3^2 - 8a_3^2 b b_3^2 - 4a_1^1 a_4^1 b b_3^2 + 16a_3^1 b b_1^1 b_3^2 - 8a_3^1 b b_3^3 + (a_1^1)^2 a_4^1 b_4^1 - a_2^2 a_4^1 b_4^1 - 4a_1^2 a_3^1 b b_4^1 - 4a_1^1 a_3^2 b b_4^1 - (a_1^1)^2 a_4^1 b b_4^1 + a_2^2 a_4^1 b b_4^1 + 4a_1^2 a_5^1 b b_4^1 + 4a_1^1 a_2^1 a_5^1 b b_4^1 + 4a_1^1 a_5^2 b b_4^1 + 2a_2^1 a_4^1 b_1^1 b_4^1 + 8a_1^1 a_3^1 b b_1^1 b_4^1 - 2a_2^1 a_4^1 b b_1^1 b_4^1 - 4a_1^1 a_5^1 b b_1^1 b_4^1 + 3a_4^1 (b_1^1)^2 b_4^1 - 3a_4^1 b (b_1^1)^2 b_4^1 - 2a_4^1 b_1^2 b_4^1 + 2a_4^1 b b_1^2 b_4^1 + 4a_2^1 b b_3^2 b_4^1 + 8b b_1^1 b_3^2 b_4^1 - 4b b_3^3 b_4^1 - 4a_1^2 b (b_4^1)^2 + 8a_1^1 b b_1^1 (b_4^1)^2 - a_2^1 a_4^1 b_4^2 - 4a_1^1 a_3^1 b b_4^2 + a_2^1 a_4^1 b b_4^2 + 4a_1^1 a_5^1 b b_4^2 - 2a_4^1 b_1^1 b_4^2 + 2a_4^1 b b_1^1 b_4^2 - 4b b_3^2 b_4^2 - 8a_1^1 b b_4^1 b_4^2 + a_4^1 b_4^3 - a_4^1 b b_4^3 + 4(a_1^1)^2 a_3^1 b_5^1 - 4a_2^2 a_3^1 b_5^1 - 4a_2^1 a_3^2 b_5^1 + 4a_3^3 b_5^1 + 4a_1^2 a_4^1 b_5^1 + 2a_1^1 a_4^2 b_5^1 + 20(a_1^1)^2 a_3^1 b b_5^1 - 20a_2^2 a_3^1 b b_5^1 - 20a_2^1 a_3^2 b b_5^1 + 20a_3^3 b b_5^1 + 12a_1^2 a_4^1 b b_5^1 + 4a_1^1 a_2^1 a_4^1 b b_5^1 + 14a_1^1 a_4^2 b b_5^1 + 8(a_1^1)^2 a_5^1 b b_5^1 + 8(a_2^1)^2 a_5^1 b b_5^1 + 8a_2^2 a_5^1 b b_5^1 + 8a_2^1 a_5^2 b b_5^1 + 8a_5^3 b b_5^1 + 8a_2^1 a_3^1 b_1^1 b_5^1 - 8a_3^2 b_1^1 b_5^1 - 8a_1^1 a_4^1 b_1^1 b_5^1 + 40a_2^1 a_3^1 b b_1^1 b_5^1 - 40a_3^2 b b_1^1 b_5^1 - 20a_1^1 a_4^1 b b_1^1 b_5^1 + 12a_3^1 (b_1^1)^2 b_5^1 + 60a_3^1 b (b_1^1)^2 b_5^1 - 8a_3^1 b_1^2 b_5^1 - 40a_3^1 b b_1^2 b_5^1 - 8a_1^1 b b_3^2 b_5^1 + 2(a_1^1)^2 b_4^1 b_5^1 - 2a_2^2 b_4^1 b_5^1 - 2(a_1^1)^2 b b_4^1 b_5^1 - 10a_2^2 b b_4^1 b_5^1 + 4a_2^1 b_1^1 b_4^1 b_5^1 + 20a_2^1 b b_1^1 b_4^1 b_5^1 + 6(b_1^1)^2 b_4^1 b_5^1 + 26b(b_1^1)^2 b_4^1 b_5^1 - 4b_1^2 b_4^1 b_5^1 - 16b b_1^2 b_4^1 b_5^1 - 2a_2^1 b_4^2 b_5^1 - 10a_2^1 b b_4^2 b_5^1 - 4b_1^1 b_4^2 b_5^1 - 16b b_1^1 b_4^2 b_5^1 + 2b_4^3 b_5^1 + 6b b_4^3 b_5^1 + 4a_1^2 (b_5^1)^2 + 12a_1^2 b (b_5^1)^2 - 8a_1^1 a_2^1 b (b_5^1)^2 - 8a_1^1 b_1^1 (b_5^1)^2 - 32a_1^1 b b_1^1 (b_5^1)^2 + 2a_1^1 a_4^1 b_5^2 + 2a_1^1 a_4^1 b b_5^2 + 8a_2^1 a_5^1 b b_5^2 + 8a_5^2 b b_5^2 + 4b b_1^1 b_4^1 b_5^2 - 4b b_4^2 b_5^2 + 4a_1^1 b_5^1 b_5^2 + 4a_1^1 b b_5^1 b_5^2 + 8a_5^1 b b_5^3 - 4b b_4^1 b_5^3 - 4a_2^3 b b_7^1 + 4a_2^2 b b_1^1 b_7^1 - 4a_2^1 b (b_1^1)^2 b_7^1 - 4b (b_1^1)^3 b_7^1 + 4a_2^1 b b_1^2 b_7^1 + 8b b_1^1 b_1^2 b_7^1 - 4b b_1^3 b_7^1 - 4a_2^2 b b_7^2 + 4a_2^1 b b_1^1 b_7^2 + 4b (b_1^1)^2 b_7^2 - 4b b_1^2 b_7^2 - 4a_2^1 b b_7^3 - 4b b_1^1 b_7^3 + 4b b_7^4 + 8a_1^3 b b_8^1 - 8a_1^2 b b_1^1 b_8^1 + 8a_1^1 b (b_1^1)^2 b_8^1 - 8a_1^1 b b_1^2 b_8^1 + 8a_1^2 b b_8^2 - 8a_1^1 b b_1^1 b_8^2 + 8a_1^1 b b_8^3 + 12b b_9^4 - 10(a_1^1)^2 a_3^1 a_4^1 c + 10a_2^2 a_3^1 a_4^1 c + 10a_2^1 a_3^2 a_4^1 c - 10a_3^3 a_4^1 c - 5a_1^2 (a_4^1)^2 c - 5a_1^1 a_4^1 a_4^2 c - 20a_2^1 a_3^1 a_4^1 b_1^1 c + 20a_3^2 a_4^1 b_1^1 c + 10a_1^1 (a_4^1)^2 b_1^1 c - 30a_3^1 a_4^1 (b_1^1)^2 c + 20a_3^1 a_4^1 b_1^2 c - 5(a_1^1)^2 a_4^1 b_4^1 c + 5a_2^2 a_4^1 b_4^1 c - 10a_2^1 a_4^1 b_1^1 b_4^1 c - 15a_4^1 (b_1^1)^2 b_4^1 c + 10a_4^1 b_1^2 b_4^1 c + 5a_2^1 a_4^1 b_4^2 c + 10a_4^1 b_1^1 b_4^2 c - 5a_4^1 b_4^3 c - 20(a_1^1)^2 a_3^1 b_5^1 c + 20a_2^2 a_3^1 b_5^1 c + 20a_2^1 a_3^2 b_5^1 c - 20a_3^3 b_5^1 c - 20a_1^2 a_4^1 b_5^1 c - 10a_1^1 a_4^2 b_5^1 c - 40a_2^1 a_3^1 b_1^1 b_5^1 c + 40a_3^2 b_1^1 b_5^1 c + 40a_1^1 a_4^1 b_1^1 b_5^1 c - 60a_3^1 (b_1^1)^2 b_5^1 c + 40a_3^1 b_1^2 b_5^1 c - 10(a_1^1)^2 b_4^1 b_5^1 c + 10a_2^2 b_4^1 b_5^1 c - 20a_2^1 b_1^1 b_4^1 b_5^1 c - 30(b_1^1)^2 b_4^1 b_5^1 c + 20b_1^2 b_4^1 b_5^1 c + 10a_2^1 b_4^2 b_5^1 c + 20b_1^1 b_4^2 b_5^1 c - 10b_4^3 b_5^1 c - 20a_1^2 (b_5^1)^2 c + 40a_1^1 b_1^1 (b_5^1)^2 c - 10a_1^1 a_4^1 b_5^2 c - 20a_1^1 b_5^1 b_5^2 c)/(16b),$$

and

$$D_5 = \pi(20a_{10}^1 a_4^1 + 2a_{12}^1 a_4^1 - 120a_1^1 a_3^1 b + 36a_{11}^1 a_3^1 b + 28a_{13}^1 a_3^1 b - 40a_{10}^1 a_4^1 b + 2a_{12}^1 a_4^1 b + 12a_{14}^1 a_4^1 b - 12a_4^2 b + 12a_{11}^1 a_5^1 b + 20a_{13}^1 a_5^1 b + 12a_6^1 a_7^1 b + 8a_7^1 a_8^1 b - 36a_6^1 a_9^1 b + 312a_1^1 a_3^1 b^2 + 12a_2^1 a_4^1 b^2 + 12a_4^2 b^2 + 240a_1^1 a_5^1 b^2 - 24a_3^1 b b_{10}^1 + 12a_4^1 b^2 b_1^1 + 5a_4^1 b_{11}^1 - 13a_4^1 b b_{11}^1 + 16a_3^1 b b_{12}^1 + 8a_5^1 b b_{12}^1 + 3a_4^1 b_{13}^1 - 3a_4^1 b b_{13}^1 + 88a_3^1 b b_{14}^1 + 80a_5^1 b b_{14}^1 - 48b^2 b_3^2 - 60a_1^1 b b_4^1 + 8a_{13}^1 b b_4^1 - 60a_1^1 b^2 b_4^1 - 12b b_{10}^1 b_4^1 - 4b b_{12}^1 b_4^1 + 20b b_{14}^1 b_4^1 + 40a_{10}^1 b_5^1 + 4a_{12}^1 b_5^1 + 112a_{10}^1 b b_5^1 + 4a_{12}^1 b b_5^1 + 24a_{14}^1 b b_5^1 - 120a_2^1 b^2 b_5^1 - 120b^2 b_1^1 b_5^1 + 10b_{11}^1 b_5^1 + 22b b_{11}^1 b_5^1 + 6b_{13}^1 b_5^1 - 6b b_{13}^1 b_5^1 - 24b b_5^2 - 120b^2 b_5^2 - 36a_6^1 b b_6^1 - 4a_7^1 b b_7^1 - 12a_9^1 b b_7^1 - 12b b_6^1 b_7^1 + 12a_6^1 b b_8^1 + 8a_8^1 b b_8^1 - 4b b_7^1 b_8^1 - 100a_{10}^1 a_4^1 c - 10a_{12}^1 a_4^1 c + 120a_1^1 a_3^1 bc + 120a_2^1 a_4^1 bc + 60a_4^2 bc + 120a_4^1 b b_1^1 c - 25a_4^1 b_{11}^1 c - 15a_4^1 b_{13}^1 c + 60a_1^1 b b_4^1 c - 200a_{10}^1 b_5^1 c - 20a_{12}^1 b_5^1 c + 240a_2^1 b b_5^1 c + 240b b_1^1 b_5^1 c - 50b_{11}^1 b_5^1 c - 30b_{13}^1 b_5^1 c + 120b b_5^2 c)/(96b).$$

Therefore $f_6(r)$ can have two positive real zeros in I following the arguments used for f_5. Note that D_1, D_3 and D_5 are linearly independent functions. In fact D_1 only present the coefficients a_1^6 and b_2^6, only D_3 has the coefficients a_2^2, a_3^2, a_8^2 and b_1^2, and D_5 is the only with the coefficients b_{12}^1, b_{13}^1 and b_{14}^1. So applying the averaging theory of order 6 we can detect that at most two small limit cycles bifurcating from the center at the origin of system (4) with $\varepsilon = 0$ and this number can be reached. Thus, the statement (c) of Theorem 1 is proved in the case $b \neq 0$.

We do $b_2^6 = -a_1^6$, and solving D_3 for b_9^4 and D_5 for b_5^2 we can apply the averaging theory of seven order.

The averaging function of order 7 is

$f_7(r) = r(E_1 + E_3 r^2 + E_5 r^4 + E_7 r^6)$,

where

$E_1 = (a_1^7 + b_2^7)\pi$,

$E_7 = (-15a_{11}^1 - 3a_{13}^1 - 180a_1^1 b + 9a_{11}^1 b + 13a_{13}^1 b + 36a_1^1 b^2 - 12bb_{10}^1 - 10b_{12}^1 - 6bb_{12}^1 - 12b_{14}^1 - 8bb_{14}^1 + 15a_{11}^1 c + 35a_{13}^1 c + 420a_1^1 bc + 10b_{12}^1 c + 140b_{14}^1 c)\pi/64$,

and we do not provided the explicit expressions of E_3 and E_5 because they are huge.

The averaging function $f_7(r)$ can have at most 3 positive real zeros in I. This because E_1, E_3, E_5 and E_7 are linearly independent, since E_1 depends on a_1^7 and b_2^7, only E_3 has the coefficients (for example) a_3^3, b_4^3 and b_4^4, the coefficient a_{13}^2 only appear in E_5 and E_7 depends of a_1^1, a_{11}^1, a_{13}^1, b_{10}^1, b_{12}^1 and b_{14}^1, and using Descartes Theorem 4 we can affirm that f_7 can has three changes of sing so the averaging function of seven order can has three different positive real zeros. Therefore we can detect through the averaging method of order seven that at most 3 limit cycles can bifurcate from the origin of system (4) with $\varepsilon = 0$. Hence statement (d) of Theorem 1 is proved for averaging function of order 7 in the case $b \neq 0$.

3.2. **Case $b = 0$ and $c \neq 1/5$.** Under these conditions system (4) becomes

$$\dot{x} = -y - x^4 - 5cy^4 + \sum_{i=1}^{7} \varepsilon^i p_i(x,y), \quad \dot{y} = x + 4x^3 y + \sum_{i=1}^{7} \varepsilon^i q_i(x,y). \tag{13}$$

The averaging function until order four are the same that before. So following the previous elections of coefficient, we continue applying the averaging method of fifth order where the averaging function is

$\tilde{f}_5(r) = r(C_1 + C_3 r^2 + \tilde{C}_5 r^4)$,

where C_1 and C_3 are the same that before and

$\tilde{C}_5 = C_5|_{b=0} = (a_4^1 + 2b_5^1)(-1 + 5c)\pi/8$.

Since C_1, C_3 and $\tilde{C}_5$ are linearly independent because C_1 depends on a_1^5 and b_2^5, $\tilde{C}_5$ only depends on a_4^1 and b_3^1 and C_3 depends of these coefficients and other more, for example only C_3 present the coefficients b_7^3, b_9^3 and a_6^3 and using Descartes Theorem 4 we have that the averaging function $f_5(r)$ can have at most 2 positive real zeros in I. We can detect through the averaging method of fifth order that at most 2 limit cycles can bifurcate from the origin of (2).

Note that if $c = 1/5$ the coefficient $\tilde{C}_5$ vanish, in this case the averaging function $f_5(r)$ can have at most one positive real root in I. This situation will we studied in detail in the next case.

Setting $b_2^5 = -a_1^5$, $b_5^1 = a_4^1/2$ and solving C_3 for b_9^3 we can apply the averaging method of sixth order. The averaging function of order six when $b = 0$ is

$\tilde{f}_6(r) = r(D_1 + \tilde{D}_3 r^2 + \tilde{D}_5 r^4)$, where

$$\begin{aligned}\tilde{D}_3 = {} & (4a_1^2(a_3^1)^2+8a_1^1a_3^1a_3^2+6(a_1^1)^2a_3^1a_4^1+2a_3^3a_4^1+2a_3^2a_4^2+3a_1^1a_4^1a_4^2+2a_3^1a_4^3+4a_1^2a_3^1a_5^1+4a_1^1a_2^1a_3^1a_5^1+4a_1^1a_3^2a_5^1+\\ & 2a_2^1a_4^2a_5^1+2a_4^3a_5^1+4a_1^1a_3^1a_5^2+2a_4^2a_5^2-6a_2^3a_6^1-6a_2^2a_6^2-6a_2^1a_6^3+6a_6^4+4a_1^3a_7^1+4a_1^2a_7^2+4a_1^1a_7^3+2a_8^4-8a_1^1(a_3^1)^2b_1^1-\\ & 2a_3^2a_4^1b_1^1-2a_3^1a_4^2b_1^1-4a_1^1a_3^1a_5^1b_1^1+6a_2^2a_6^1b_1^1+6a_2^1a_6^2b_1^1-6a_6^3b_1^1-4a_1^2a_7^1b_1^1-4a_1^1a_7^2b_1^1+2a_3^1a_4^1(b_1^1)^2-6a_2^1a_6^1(b_1^1)^2+6a_6^2(b_1^1)^2+\\ & 4a_1^1a_7^1(b_1^1)^2-6a_6^1(b_1^1)^3-2a_3^1a_4^1b_1^2+6a_2^1a_6^1b_1^2-6a_6^2b_1^2-4a_1^1a_7^1b_1^2+12a_6^1b_1^1b_1^2-6a_6^1b_1^3-4(a_1^1)^2a_3^1b_3^1+4a_2^2a_3^1b_3^1+4a_2^1a_3^2b_3^1-\\ & 4a_3^3b_3^1-2a_1^1a_4^2b_3^1-8a_2^1a_3^1b_1^1b_3^1+8a_3^2b_1^1b_3^1-12a_3^1(b_1^1)^2b_3^1+8a_3^1b_1^2b_3^1+4a_2^1a_3^1b_3^2-4a_3^2b_3^2+8a_3^1b_1^1b_3^2-4a_3^1b_3^3-2a_1^2a_3^1b_4^1-\\ & 2a_1^1a_3^2b_4^1+3(a_1^1)^2a_4^1b_4^1+2a_1^2a_5^1b_4^1+2a_1^1a_2^1a_5^1b_4^1+2a_1^1a_5^2b_4^1+4a_1^1a_3^1b_1^1b_4^1-2a_1^1a_5^1b_1^1b_4^1+a_4^1(b_1^1)^2b_4^1-a_4^1b_1^2b_4^1-2(a_1^1)^2b_3^1b_4^1+\\ & 2a_2^2b_3^1b_4^1-4a_2^1b_1^1b_3^1b_4^1-6(b_1^1)^2b_3^1b_4^1+4b_1^2b_3^1b_4^1+2a_2^1b_3^2b_4^1+4b_1^1b_3^2b_4^1-2b_3^3b_4^1-2a_1^2(b_4^1)^2+4a_1^1b_1^1(b_4^1)^2-2a_1^1a_3^1b_4^2+2a_1^1a_5^1b_4^2-\\ & a_4^1b_1^1b_4^2+2a_2^1b_3^1b_4^2+4b_1^1b_3^1b_4^2-2b_3^2b_4^2-4a_1^1b_4^1b_4^2+a_4^1b_4^3-2b_3^1b_4^3+6a_1^1a_4^1b_5^2+4a_2^1a_5^1b_5^2+4a_5^2b_5^2-4a_1^1b_3^1b_5^2+2b_1^1b_4^1b_5^2-\\ & 2b_4^2b_5^2+4a_5^1b_5^3-2b_4^1b_5^3-2a_2^3b_7^1+2a_2^2b_1^1b_7^1-2a_2^1(b_1^1)^2b_7^1-2(b_1^1)^3b_7^1+2a_2^1b_1^2b_7^1+4b_1^1b_1^2b_7^1-2b_1^3b_7^1-2a_2^2b_7^2+2a_2^1b_1^1b_7^2+\\ & 2(b_1^1)^2b_7^2-2b_1^2b_7^2-2a_2^1b_7^3-2b_1^1b_7^3+2b_7^4+4a_1^3b_8^1-4a_1^2b_1^1b_8^1+4a_1^1(b_1^1)^2b_8^1-4a_1^1b_1^2b_8^1+4a_1^2b_8^2-4a_1^1b_1^1b_8^2+4a_1^1b_8^3+6b_9^4)\pi/8,\end{aligned}$$

and

$$\begin{aligned}\tilde{D}_5 = {} & (-60a_1^1a_3^1+18a_{11}^1a_3^1+14a_{13}^1a_3^1+12a_{10}^1a_4^1+6a_{12}^1a_4^1-6a_4^2+6a_{11}^1a_5^1+10a_{13}^1a_5^1+6a_6^1a_7^1+4a_7^1a_8^1-18a_6^1a_9^1-\\ & 12a_3^1b_{10}^1+3a_4^1b_{11}^1+8a_3^1b_{12}^1+4a_5^1b_{12}^1+9a_4^1b_{13}^1+44a_3^1b_{14}^1+40a_5^1b_{14}^1-40a_{10}^1b_3^1-4a_{12}^1b_3^1-10b_{11}^1b_3^1-6b_{13}^1b_3^1-30a_1^1b_4^1+\\ & 4a_{13}^1b_4^1-6b_{10}^1b_4^1-2b_{12}^1b_4^1+10b_{14}^1b_4^1-12b_5^2-18a_6^1b_6^1-2a_7^1b_7^1-6a_9^1b_7^1-6b_6^1b_7^1+6a_6^1b_8^1+4a_8^1b_8^1-2b_7^1b_8^1+60a_1^1a_3^1c+\\ & 30a_4^2c+30a_1^1b_4^1c+60b_5^2c)\pi/48.\end{aligned}$$

As before (in the case $b \neq 0$) $f_6(r)$ can have two positive real zeros in I, because the coefficients of the averaging function of order 6, D_1, $\tilde{D}_3$ and $\tilde{D}_5$, are linearly independent functions, and then we can apply the Descartes Theorem 4. So applying the averaging theory of order 6 we can detect at most two small limit cycles bifurcating from the center at the origin and this number can be reached.

Solving D_1 for b_2^6, $\tilde{D}_3$ for b_9^4 and $\tilde{D}_5$ for b_5^2, we can apply the averaging theory of seventh order and the averaging function of order seven is

$$\tilde{f}_7(r) = r(E_1 + \tilde{E}_3 r^2 + \tilde{E}_5 r^4 + \tilde{E}_7 r^6),$$

where

$$\tilde{E}_7 = (-15a_{11}^1 - 3a_{13}^1 - 10b_{12}^1 - 12b_{14}^1 + 15a_{11}^1 c + 35a_{13}^1 c + 10b_{12}^1 c + 140b_{14}^1 c)\pi/64.$$

again we do not provide the explicit expressions of $\tilde{E}_3$ and $\tilde{E}_5$ because are very big. Under the hypothesis $b = 0$ and $c \neq 1/5$ the averaging function of order 7 associated to system (13) has at most three positive real zeros, becausethe coefficients of $\tilde{f}_7(r)$ are linearly independent and we can apply the Descartes Theorem.

Thus we can detect through the averaging method of order seven that at most 3 limit cycles can bifurcate from the origin of (4) with $\varepsilon = 0$. S o the statement (d) is proved when $b = 0$ and $c \neq 1/5$.

3.3. **Case** $b = 0,\ c = 1/5$.

In this case the averaging function of fifth order $f_5(r)$ is

$$\hat{f}_5(r) = r(C_1 + C_3 r^2).$$

The averaging function $\hat{f}_5(r)$ can have only at most 1 positive real zero in I. We detect through the averaging method of fifth order that at most 1 limit cycle can bifurcate from the origin of system (2) with $\varepsilon = 0$.

Solving C_1 for b_2^5 and C_3 for b_9^3. We can apply the averaging method of sixth order. The averaging function of order six is

$$\hat{f}_6(r) = r(D_1 + \hat{D}_3 r^2 + \hat{D}_5 r^4),$$

where $\hat{D}_3 = \tilde{D}_3 + (a_4^1 + 2b_5^1)(2(a_2^1)^2 a_5^1 + 2a_2^2 a_5^1 + 2a_2^1 a_5^2 + 2a_5^3 + (a_1^1)^2(2a_5^1 - 3b_4^1) - (b_1^1)^2 b_4^1 + b_1^2 b_4^1 + b_1^1 b_4^2 - b_4^3 + 2a_1^2(a_4^1 - b_3^1 - b_5^1) + a_1^1(2a_2^1 a_4^1 + a_4^2 - 2a_4^1 b_1^1 + 4b_1^1 b_3^1 - 2b_3^2 - 2a_2^1 b_5^1 + 2b_1^1 b_5^1 - 4b_5^2))\pi/8$,

and

$\hat{D}_5 = -(24a_1^1a_3^1 - 9a_{11}^1a_3^1 - 7a_{13}^1a_3^1 + 5a_{10}^1a_4^1 - a_{12}^1a_4^1 - 3a_{14}^1a_4^1 - 6a_2^1a_4^1 - 3a_{11}^1a_5^1 - 5a_{13}^1a_5^1 - 3a_6^1a_7^1 - 2a_7^1a_8^1 + 9a_6^1a_9^1 + 6a_3^1b_{10}^1 - 6a_4^1b_1^1 + 2a_4^1b_{11}^1 - 4a_3^1b_{12}^1 - 2a_5^1b_{12}^1 - 22a_3^1b_{14}^1 - 20a_5^1b_{14}^1 + 20a_{10}^1b_3^1 + 2a_{12}^1b_3^1 + 5b_{11}^1b_3^1 + 3b_{13}^1b_3^1 + 12a_1^1b_4^1 - 2a_{13}^1b_4^1 + 3b_{10}^1b_4^1 + b_{12}^1b_4^1 - 5b_{14}^1b_4^1 + 22a_{10}^1b_5^1 + 4a_{12}^1b_5^1 - 6a_{14}^1b_5^1 - 12a_2^1b_5^1 - 12b_1^1b_5^1 + 7b_{11}^1b_5^1 + 9b_{13}^1b_5^1 + 9a_6^1b_6^1 + a_7^1b_7^1 + 3a_9^1b_7^1 + 3b_6^1b_7^1 - 3a_6^1b_8^1 - 2a_8^1b_8^1 + b_7^1b_8^1)\pi/24.$

As for the previous cases ($b \neq 0$, and $b = 0$ with $c \neq 1/5$) the averaging function of sixth order $f_6(r)$ can have two real positive zeros in I, because D_1, $\hat{D}_3$ and $\hat{D}_5$ are linearly independent functions and we can apply Descartes Theorem 4. Therefore, through the averaging theory of order 6 we can detect that at most two small limit cycles bifurcating from the center at the origin and this number can be reached. So statement (c) is proved in the case $b = 0$ and $c = 1/5$.

Solving D_1 for b_2^6, $\hat{D}_3$ for b_9^4 and $\hat{D}_5$ for b_5^2, we can apply the averaging theory of seven order and the averaging function is

$$\hat{f}_7(r) = r(E_1 + \hat{E}_3r^2 + \hat{E}_5r^4 + \hat{E}_7r^6),$$

where

$$\hat{E}_1 = (a_1^7 + b_2^7)\pi,$$

$$\hat{E}_7 = -(1/16)(3a_{11}^1 - a_{13}^1 + 2b_{12}^1 - 4b_{14}^1)\pi.$$

we do not provide the explicit expressions of $\hat{E}_3$ and $\hat{E}_5$ because they are very long. Thus $\hat{f}_7(r)$ can have three positive real roots in I since the coefficients of $\hat{f}_7(r)$ are linearly independent and we can apply Descartes Theorem. So, applying the averaging theory of order 7 we can detect that at most three small limit cycles can bifurcate from the center at the origin and this number can be reached. This prove the statement (d) in Theorem 1.

In summary, for the averaging theory of order 1 and 2 we can not detect the existence of small limit cycles bifurcating from the center at the origin. For the averaging theory of order 3 and 4 we can detect that at most one small limit cycle bifurcating from the origin of system (4) with $\varepsilon = 0$. For the averaging theory of order 5 and 6 we can detect at most two small limit cycles bifurcating from the center at the origin and this number can be reached. Finally for order seven we detect trough the averaging theory at most three limit cycles. This complete the proof of Theorem 1.

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Appendix A. Averaging functions

We present explicitly the averaging function until order 7.

$$
\begin{aligned}
y_1(t,z) = \;& \textstyle\int_0^t F_1(s,\varphi(s,z))ds,\\
y_2(t,z) = \;& \textstyle\int_0^t \Bigl(2F_2(s,\varphi(s,z)) + 2\partial F_1(s,\varphi(s,z))y_1(s,z)\Bigr)ds,\\
y_3(t,z) = \;& \textstyle\int_0^t \Bigl(6F_3(s,\varphi(s,z)) + 6\partial F_2(s,\varphi(s,z))y_1(s,z)\\
& +3\partial^2 F_1(s,\varphi(s,z))y_1(s,z)^2 + 3\partial F_1(s,\varphi(s,z))\, y_2(s,z)\Bigr)ds,\\
y_4(t,z) = \;& \textstyle\int_0^t \Bigl(24F_4(s,\varphi(s,z)) + 24\partial F_3(s,\varphi(s,z))y_1(s,z)\\
& +12\partial^2 F_2(s,\varphi(s,z))y_1(s,z)^2 + 12\partial F_2(s,\varphi(s,z))y_2(s,z)\\
& +12\partial^2 F_1(s,\varphi(s,z))y_1(s,z)y_2(s,z)\\
& +4\partial^3 F_1(s,\varphi(s,z))y_1(s,z)^3 + 4\partial F_1(s,\varphi(s,z))y_3(s,z)\Bigr)ds,\\
y_5(t,z) = \;& \textstyle\int_0^t \Bigl(120F_5(s,\varphi(s,z)) + 120\partial F_4(s,\varphi(s,z))y_1(s,z)\\
& +60\partial^2 F_3(s,\varphi(s,z))y_1(s,z)^2 + 60\partial F_3(s,\varphi(s,z))y_2(s,z)\\
& +60\partial^2 F_2(s,\varphi(s,z))y_1(s,z)y_2(s,z) + 20\partial^3 F_2(s,\varphi(s,z))y_1(s,z)^3\\
& +20\partial F_2(s,\varphi(s,z))y_3(s,z) + 20\partial^2 F_1(s,\varphi(s,z))y_1(s,z)y_3(s,z)\\
& +15\partial^2 F_1(s,\varphi(s,z))y_2(s,z)^2 + 30\partial^3 F_1(s,\varphi(s,z))y_1(s,z)^2 y_2(s,z)\\
& +5\partial^4 F_1(s,\varphi(s,z))y_1(s,z)^4 + 5\partial F_1(s,\varphi(s,z))y_4(s,z)\Bigr)ds,\\
y_6(t,z) \;& = \textstyle\int_0^t \Bigl(720F_6(s,\varphi(s,z)) + 720\partial F_5(s,\varphi(s,z))y_1(s,z)\\
& +360\partial^2 F_4(s,\varphi(s,z))y_1(s,z)^2 + 360\partial F_4(s,\varphi(s,z))y_2(s,z)\\
& +120\partial^3 F_3(s,\varphi(s,z))y_1(s,z)^3 + 360\partial^2 F_3(s,\varphi(s,z))y_1(s,z)y_2(s,z)\\
& +120\partial F_3(s,\varphi(s,z))y_3(s,z) + 30\partial^4 F_2(s,\varphi(s,z))y_1(s,z)^4\\
& +180\partial^3 F_2(s,\varphi(s,z))y_1(s,z)^2 y_2(s,z) + 120\partial^2 F_2(s,\varphi(s,z))y_1(s,z)y_3(s,z)\\
& +90\partial^2 F_2(s,\varphi(s,z))y_2(s,z)^2 + 30\partial F_2(s,\varphi(s,z))y_4(s,z)\\
& +60\partial^4 F_1(s,\varphi(s,z))y_1(s,z)^3 y_2(s,z) + 60\partial^3 F_1(s,\varphi(s,z))y_1(s,z)^2 y_3(s,z)\\
& +90\partial^3 F_1(s,\varphi(s,z))y_1(s,z)y_2(s,z)^2 + 30\partial^2 F_1(s,\varphi(s,z))y_1(s,z)y_4(s,z)\\
& +60\partial^2 F_1(s,\varphi(s,z))y_2(s,z)y_3(s,z) + 6\partial^5 F_1(s,\varphi(s,z))y_1(s,z)^5\\
& +6\partial F_1(s,\varphi(s,z))y_5(s,z)\Bigr)ds,\\
y_7(t,z) = \;& \textstyle\int_0^t (5040F_7(s,\varphi(s,z)) + 5040\partial F_6(s,\varphi(s,z))y_1(s,z)\\
& +2520\partial^2 F_5(s,\varphi(s,z))y_1(s,z)^2 + 2520\partial F_5(s,\varphi(s,z))y_2(s,z)\\
& +2520\partial^2 F_4(s,\varphi(s,z))y_1(s,z)y_2(s,z) + 840\partial^3 F_4(s,\varphi(s,z))y_1(s,z)^3\\
& +840\partial F_4(s,\varphi(s,z))y_3(s,z) + 840\partial^2 F_3(s,\varphi(s,z))y_1(s,z)y_3(s,z)\\
& +630\partial^2 F_3(s,\varphi(s,z))y_2(s,z)^2 + 1260\partial^3 F_3(s,\varphi(s,z))y_1(s,z)^2 y_2(s,z)\\
& +210\partial^4 F_3(s,\varphi(s,z))y_1(s,z)^4 + 210\partial F_3(s,\varphi(s,z))y_4(s,z)\\
& +210\partial^2 F_2(s,\varphi(s,z))y_1(s,z)y_4(s,z) + 420\partial^3 F_2(s,\varphi(s,z))y_1(s,z)^2 y_3(s,z)\\
& +420\partial^4 F_2(s,\varphi(s,z))y_1(s,z)^3 y_2(s,z) + 630\partial^3 F_2(s,\varphi(s,z))y_2(s,z)^2 y_1(s,z)\\
& +42\partial^5 F_2(s,\varphi(s,z))y_1(s,z)^5 + 420\partial^2 F_2(s,\varphi(s,z))y_2(s,z)y_3(s,z)\\
& +42\partial F_2(s,\varphi(s,z))y_5(s,z) + 630\partial^3 F_2(s,\varphi(s,z))y_2(s,z)^2 y_1(s,z)\\
& +7\partial^6 F_1(s,\varphi(s,z))y_1(s,z)^6 + 105\partial^5 F_1(s,\varphi(s,z))y_1(s,z)^4 y_2(s,z)\\
& +140\partial^4 F_1(s,\varphi(s,z))y_1(s,z)^3 y_3(s,z) + 630\partial^4 F_1(s,\varphi(s,z))y_1(s,z)^2 y_2(s,z)^2\\
& +105\partial^3 F_1(s,\varphi(s,z))y_1(s,z)^2 y_4(s,z) + 42\partial^2 F_1(s,\varphi(s,z))y_1(s,z)y_5(s,z)\\
& +420\partial^3 F_1(s,\varphi(s,z))y_1(s,z)y_2(s,z)y_3(s,z)\\
& +105\partial^3 F_1(s,\varphi(s,z))y_2(s,z)^3 + 105\partial^2 F_1(s,\varphi(s,z))y_2(s,z)y_4(s,z)\\
& +70\partial^2 F_1(s,\varphi(s,z))y_3(s,z)^2 + 7\partial F_1(s,\varphi(s,z))y_6(s,z)\bigr)\, ds.
\end{aligned}
$$

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