The role of context for characterising students' strategies when estimating large numbers of elements on a surface

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Abstract *This article introduces a qualitative study to identify and characterise all possible resolution strategies that 16-year-old students developed to solve problems involving estimation of large numbers of elements on a bounded surface. Taking Realistic Mathematics Education as a reference, we presented the students with a sequence comprised of diverse problems with different contexts, a method that allows us to promote many different strategies. The main result of this study is a tree diagram containing the characterization of the students' strategies, which include different types of mathematical procedures induced by the characteristics of the different real-life contexts proposed.*

Key words *Large number estimation, problem solving, open tasks, tree diagram of a problem.*

Since the publication of the first issue of Educational Studies in Mathematics in the late 1970s, the relationship between problem solving and the real world has interested the research community in Mathematical Education. Work such as Pollack's (1969) triggered a movement that acknowledged the value of introducing classroom activities that demonstrate the powerful relationship between Mathematics and the world around us. This movement – originally called Realistic Mathematics Education (RME), initiated by Hans Freudenthal – is based on the use of realistic contexts as vehicles for the development of abstract concepts by working in concrete settings that promote students' free constructions and productions during teaching/learning processes (De Lange, 1996). Thus, in the framework of RME, the use of realistic contexts became one of the main characteristics of this educational approach (Van Den Heuvel-Panhuizen, 2003). Indeed, according to RME, during the process of learning mathematics, students should be given the opportunity to develop and apply mathematical concepts and procedures in situations that are relatively close to their own reality so that they can make sense of abstract concepts (Gravemeijer, 1994).

In previous work we have observed that problems that share the same mathematical background promote different approaches to resolution depending on the real-life context in which they are posed (Albarracín & Gorgorió, 2013). We have also seen that different problems related to the estimation of the number of elements on a surface can be raised by varying the context in which they are set (Albarracín & Gorgorió, 2014). These two facts lead us to explore different real-life contexts for the same type of problems as a methodological strategy to promote different ways of solving a particular type of problem and characterise them.

Thus, the study presented in this article focuses on the characterization of the solving strategies that 10th grade students generate to solve problems in which the number of objects on a surface has to be estimated. In order to promote variability of solving strategies, we use a sequence of problems with different real-life contexts. For this purpose, the problems given to the students are a type of Fermi problems that require the estimation of large numbers, initially set in contexts close to their own reality and that progressively become more distant. The most relevant contribution of this study is a tree diagram that characterises the different problem-solving strategies involving estimation of objects present on a surface. Each of the strategies is induced by a different context, which links different types of knowledge with mathematical procedures related to content, such as measurement and proportionality.

RME framework and contextualised tasks

Realistic Mathematics Education is characterised by promoting the presence of rich and realistic situations in the learning process, which give students the opportunity to construct and give meaning to mathematical concepts and procedures. To achieve this purpose, activities designed under the RME perspective follow six design principles (Treffers, 1978; Van den Heuvel-Panhuizen & Drijvers, 2014), included in the design of our study. The first principle is that students are the main actors in the learning process, thus, they will learn mathematics by actually doing mathematics. The second is the principle of reality, understanding that students should be able to use the mathematics they learn to solve real problems and, furthermore, that students can learn mathematics by studying reality from rich contexts. The third principle holds that promoted learning should go from informal to formalised knowledge, creating different levels of knowledge and introducing them progressively. The fourth principle proposes that mathematical content should not be presented in isolation, and that it should be possible to relate it to other types of knowledge. The fifth principle places mathematical learning in the social sphere and promotes group work and the sharing of mathematical strategies and inventions among students. The last principle is that learning should be guided so that students can reinvent their mathematical knowledge (Freudenthal, 1991) in a predetermined direction established by the educational programme based on the design of the activities or the teacher's intervention.

Thus, through studying complex real-life phenomena mathematically, students give meaning to mathematical concepts. Thus, the notion of real-life context is therefore very important in RME and plays a significant role as a learning starting point for students to explore mathematical notions in a situation that is *experientially real* for them (Gravemeijer & Doorman, 1999), also referred to as *relevant and essential context* in the literature (e.g., De Lange, 1995; Van den Heuvel-Panhuizen, 2005). According to Wijaya, Van den Heuvel-Panhuizen and Doorman (2015), mathematical tasks could have a realistic context, a camouflage context, or they could be purely mathematical tasks – that only use mathematical symbols. Tasks with a camouflaged context are problems which do not actually require considering the real context because the mathematical operations needed to solve the task are obvious (Wijaya et al., 2015). In order to fulfil this role of context as a promoter of knowledge development, it is essential that it provides all the necessary elements to place the student in the appropriate role to solve the problem. Using the words of Sriraman and Knott (2009) when referring to the context of a problem, "the word *realistic* refers not just to the connection with the real world, but also refers to problem solving situations which are real in the student's mind" (p. 206). Otherwise, students can try to solve the problem as if it had a camouflaged context from a phenomenon called suspension of sense making (Greer, 1997; Schoenfeld, 1991; Verschaffel & De Corte, 1997). In agreement with the latter, Palm (2008) describes the authenticity of a realistic problem as the degree to which the class task can be transferred to the real world, showing that students can take advantage of the characteristics of the task that connect with characteristics of reality. Palm (2006) describes the aspects of the task that are relevant to maintaining its authenticity, such as the type of event in which the statement is framed, the question and the data provided to students, the requirements that the solution must verify, and the purpose of the problem.

Fermi problems as contextualised problems

Fermi problems are named after the Nobel Prize winner in Physics, Enrico Fermi (1901-1954), who used them in his lessons to promote reasoning and to establish connections between theoretical aspects and laboratory practice. Following Ärlebäck (2009), Fermi problems are open and non-standard problems requiring the students to make assumptions about the problem situation and estimate relevant quantities before engaging in, often, simple calculations (p. 331). One of the defining singularities of Fermi problems lies in the way they are formulated: Fermi problems always seem diffuse in their statement and offer little concrete information to approach the solving process (Efthimiou & Llewellyn, 2007). According to Sriraman and Knott (2009), Fermi problems are estimation problems the aim of which is to get students to make educated guesses. Only the detailed analysis of the situation allows for decomposing the problem into simpler problems to arrive at the solution from reasoned conjectures (Carlson, 1997). However, the way we use Fermi problems in class goes beyond identifying relevant quantities and making educated estimates from previous knowledge. The implementation of the activities includes encouraging students to acquire this relevant information using their own methods, either from real-world measurements or using digital tools, such as spreadsheets, video cameras or Google Maps following the recommendations of other authors who work in the area of Mathematics Education (Keune & Henning, 2003; Tangney & Bray, 2013).

The way of working promoted by Fermi problems allows students to face the reality studied with their own possibilities, skills and knowledge. In this way, Fermi problems pose a different challenge to students of different ages, since students can use their knowledge differently at each educational level to describe reality. Previous investigations have studied Fermi problem-solving processes for Primary school students (Albarracín & Gorgorió, 2019; Haberzettl, Klett & Schukajlow, 2018; Peter-Koop, 2009), concluding that, at these ages, students can already generate their own solutions in a variety of ways and develop new mathematical knowledge during their solving processes. At the Secondary school level, Ärlebäck (2009) observed that students can use their extra-mathematical knowledge suggested by the context of a Fermi problem to generate new mathematical knowledge. In previous studies we analysed the problem-solving strategies present in the plans devised by secondary students to solve various Fermi problems that required estimating large numbers (Albarracín & Gorgorió, 2014; Ferrando, Albarracín, Gallart, García-Raffi & Gorgorió, 2017). Fermi problems have also been used to analyse complex resolution processes with university level students. In this sense, Czocher (2016, 2018)'s studies show that even apparently simple questions, based on complex realities, require advanced problem-solving and modelling skills. All of these research studies combined underline the versatility of Fermi problems by allowing students to rely on the richness of the real-life contexts that may arise.

Theoretical positioning

For this article we understand that a mathematical problem is a situation for which the student does not know a specific procedure. In this way we align ourselves with Schoenfeld (1985, 1992) when considering problems as opposed to routine mathematical activities. In our study, we take advantage of this lack of knowledge of concrete routines to solve Fermi problems by proposing contextualised realistic problems to the students, to promote different types of approximations. In this context, we understand that a strategy is a generic way of acting in order to reach a certain goal - for instance, to solve a problem - which takes different forms depending on the characteristics of the goal. When students try to solve a contextualised Fermi problem, they develop their own procedure based on their knowledge of the real world (Ärlebäck, 2009; Van den Heuvel-Panhuizen, 2005) and their previous mathematical knowledge. This procedure is explained as a chain of actions that allow the problem to be solved, moreover, the procedure is not unique, since Fermi problems are open tasks. In the case of contextualised problems that share the type of question - in our case, estimating the number of elements on a surface - students develop strategies that share the essential elements but that can be adapted to the characteristics of each situation studied (Albarracín & Gorgorió, 2014). In Gallart, Ferrando, García-Raffi, Albarracín and Gorgorió (2017) we presented an analytical tool to identify the coordinated set of actions and mathematical procedures that students develop to solve a Fermi problem in order to characterise the strategies used by students.

In order to promote the development of different strategies, we have designed a sequence of tasks with a context that evolves from the most familiar to the students (the schoolyard), to more distant yet known and accessible contexts – since they can use the google maps tool to take measurements. The aim of the sequence is for students to bring known mathematical concepts into play, as well as procedures that are linked to measurement and proportionality. Thus, following the work of Freudenthal (1991), we intend students to be active participants in their learning process, using mathematical tools and insights in a real-life context. To ensure that the problems have real meaning for the students, statements are composed of three elements. In the first place, we have the real space (accessible to the students or not) where the elements to be counted are arranged. Then, a short text is provided to them to introduce a situation related to making an estimation of the number of elements on the surface. Finally, we ask a specific question that gives the problem its final shape and provides a concrete role for the students as solvers.

In this way, we can study the students' resolution strategies by identifying the strategies used in the resolution of the sequence of tasks and representing them in the form of a diagram. By bringing together all the different ways of solving a problem, we constructed a problem tree. It consists of a tree-shaped structure, the branches of which show different strategies that students use to solve the problem and that can be used as an anticipation tool for the teacher (as in the work of Ferrer, Fortuny & Morera, 2014). The notion of the problem tree that gathers all the strategies identified in the resolution of an open task is related to work on multiple solution tasks. However, in this framework the problem tree refers to solution spaces of multiple solution tasks, see for instance Leikin and Levav-Waynberg (2008).

Aims of the study

In this study we propose a sequence of four problems with realistic contexts to 10th grade' students. All the problems consist of obtaining an estimation of the number of objects on a given surface. A qualitative analysis of the students' productions helps us to identify different strategies for each task, which we organise in a tree diagram. We progressively advance in the analysis in order to finally complete the tree to show a diagram that gathers all the strategies identified in the sequence. The research goals are the following:

1. To develop an analysis tool to identify and characterise the strategies developed by the students when solving a problem that requires estimating the number of elements on a surface.
2. To establish a tree diagram that gathers all the possible solving strategies of this type of task.

Methodology

The sequence of activities

For the purpose of this research, we designed a sequence of problems that could lead us to obtain rich data but should nevertheless remain activities that could be developed naturally in classrooms following the design principles of activities in the RME framework. The questions we raised in these activities made sense to the students, and the results could be contrasted among groups of students, and/or using external information to validate both the processes followed and the results obtained. In addition, we used activities that included different real-life contexts and that were complex tasks, far removed from conventional problems associated to already-defined problem-solving procedures. For the purpose of the research, we designed a sequence of activities consisting of four Fermi problems that addressed the same general problem – estimating the number of people or objects that can be placed on a given surface – but situated in different realistic contexts. In all four cases, the activities are composed of a text that describes a situation in line with the context expressed for each problem, the question posed here, and the real characteristics of the places where people or trees can be placed in each problem. These three elements are essential for shaping the problem statement. The activities are the following:

- **Activity A**: Context: An end-of-year party will be held at the school, including a concert, and we have to anticipate the maximum number of tickets to be sold. Question: How many people fit in the schoolyard[1]?

- **Activity B1**: Context: Popular music groups use sports halls as venues in which they can gather a large number of attendees to their concerts and it is necessary to determine their capacity in advance. Question: How many people fit in a concert – i.e. in a sports pavilion located in the students' city?

- **Activity B2**: Context: The relevance of a demonstration can be quantified by the number of people who support the protest. Question: How many people fit in the Town Hall Square of your city during a demonstration?

- **Activitiy C**: Context: city parks allow their inhabitants to get in touch with nature, but the quality of these parks depends on their dimensions and especially on the number of trees that live in them. Question: How many trees are there in Central Park?

The first activity of the sequence – activity A – allowed conducting fieldwork in the school itself, so that students could come up with solving methods that they could then carry out on the ground. The fact that the school yard was rectangular was an essential part of the activity's contextualization, since it could facilitate the estimation of the value

[1] In both cases, the school yard is rectangular

of the area in comparison with the contexts of successive tasks (Ferrando et al., 2017). The goal in this activity was for students to develop their first strategies to estimate the number of elements in a bounded surface in which they could carry out actual measurements. The context of a concert set in a party in the school itself is authentic for students and connects with previous experiences in a place that is well known for them. The students worked during a 90-minute session on problem A, first on the schoolyard and then, in the classroom - in order to write down their solutions. After an intermediate 45-minutes session in which the groups of students shared their methods and solutions for activity A, they were presented with a set of activities – B1, B2, and C – that could not be addressed with a hands-on approach because the context described in the statement was inaccessible to them. The goal of this set of activities was to generate the need to apply, adapt, and/or reconstruct the strategies used when dealing with the first activity. Activities B1 and B2, like activity A, required estimating the number of people in a delimited area located in a context that was close to the students' everyday life. The number of people at concerts and demonstrations is a common way of expressing the success of musicians or the reasons for the demonstration. However, in this case, students could not actually go to the setting of these activities during the class. Activity C was different from the previous three; it was no longer about estimating the number of people, but the number of trees in a much larger setting, far removed from the reality of the students, but still in a realistic and authentic context, as environmental discussions are part of the society in which they live. The students worked in groups on activities B1, B2, and C during a 90-minutes class session.

This concretion of the sequence of problems allows us to comply with the design principles of the RME framework (Treffers, 1978; Van den Heuvel-Panhuizen & Drijvers, 2014). Students are responsible for defining the mathematical activity to be developed and can share the mathematical and extra-mathematical knowledge they produce when working in a group. The contexts and questions provided to the students have meaning for them, and they are aware that they are relevant to understand the situations given. Because of the open nature of Fermi problems, mathematical concepts appear intertwined, and dealing with different problems that share mathematical foundations allows students to consolidate the knowledge generated as well as to refine their resolution strategies.

Data collection

Our data comes from the development of the aforementioned sequence of activities in two schools, one in Sabadell (Barcelona, Spain) and another in Valencia (Spain). In each school, and during three class periods, the teachers requested their 10th grade' students (age 16) to solve activities A, B1, B2 and C. The 46 students were arranged into small teams of three or four students each, making a total of 13 working groups. In both cases, the regular mathematics teacher was in charge of the class and one of the researchers was supervising the development of the experience. Both teachers had experience in the use of contextualised open problems but none of the groups of students had faced this kind of tasks before. The first class proceeded as follows; the teacher presented the students with activity A and the students worked in small groups (in the playground, and later in the classroom). During the second session, the whole group shared and discussed their procedures and results. In the third session, students worked in groups for activities B1, B2 and C. We let the students work in small teams not only because that was the way they were used to solving problems, but also because it enables more students to participate and add their own ideas to the solving process (Albarracín & Gorgorió, 2014).

The teachers set the task, observed the work of the students, and moderated the group discussion and exchange. They did not in any case suggest ways of solving the activities, they merely clarified the meaning of an activity statement when necessary. They answered students' questions with other questions, but did not hint at problem-solving strategies, nor did they suggest that some may be more suitable than others. During the activity, the teachers took notes, as they usually did, writing down the discussions of the working groups that they later brought up in the final exchange with the whole class.

Our primary source of data consists of the written reports of 13 working groups when solving activities A, B1, B2, and C. We analysed them in the light of the notes taken during classroom observation by the first two authors – who were supervising the experience – and the teachers' class notes.

Analysis of student output and strategy trees for each problem

In this section, we detail the process of analysing student output and show the different strategies and procedures used by students. For each problem, we detailed the mathematical processes and strategies related, and identified and summarised them in a tree diagram containing all the strategies that appear in each of the teams' resolutions. In this way, for each problem we can obtain a specific strategy tree as a result of the analysis.

Strategy tree for Problem A

The written reports inform us about the concepts and procedures underlying the strategy used for solving this task. For example, let us consider the following explanation in which a group of students described how they reached an estimate for the number of people that could fit in the courtyard of their school:

> *"You take four people and place them in a straight line, from end to end of the yard. Then we do the same thing vertically. Then we multiply and get the result."*

The students explained their procedure, but there is no explicit information in their written report about the concepts involved in their reasoning. However, we inferred that the students had an image of people in the yard arranged in a grid pattern in which each intersection represented the location of a person. The classroom observation and the teachers' notes endorse our inference: the *distribution on a grid pattern* is the concept that articulates their solving process since it gives meaning to the procedure. In this strategy, students obtained an estimation of the number of people that can fit at each side of the playground. In order to obtain this first estimation, these students have used effective counting, because they used their own team members to estimate the number of people that fit at each side of the yard by counting people four at a time. In Albarracín & Gorgorió (2014), the *grid distribution* was one of the three strategies identified in the plans devised by the students to estimate the number of objects arranged on a surface. The other two were *population density* and *iteration of a unit*. We now observe that the students' models – while actually solving this first activity – also fall into one of the three categories above: grid distribution, density, or iteration of a unit. Moreover, students may even add complexity to these strategies by introducing variations that depend on the activity's setting.

Concerning the use of *population density*, we observe that not all students used the idea of density in the same way. Some of them considered density to be the same throughout the precinct. Others, considering the setting of the activity, realised that density is not the same in all regions of the enclosure. From this finding, they have used the idea of average density. When deciding on the playground's population density – globally or by area –

they determined it based on either an experimental justification (e.g. stating they used the tiles of the playground), or information provided without justification.

Those using *the iteration of a unit* solved the activity by determining the area occupied by a single element and divided the total area by this value. We again observe that the use of the *iteration of a unit* strategy presents variations. In this first task, we find two ways to determine the area occupied by an element: using an estimate without justifying it or making an experimental measurement.

Thus, in the analysis of the resolutions of problem A, we observe that, first of all, the students can approach it directly (using the strategy based on the grid pattern model) or indirectly. In this case the students must find an estimated value of the area of the enclosure and then divide or multiply by the area occupied by an element or the estimated density, respectively. Moreover, we have observed that those who choose indirect resolution strategies took two different ways to calculate the area: some used standard measurement units (the square metre), but others chose other units that are more accessible for them (the tiles or their own steps). In addition, we have detected some elements of complexity in the proposed resolutions: some students removed the space occupied by obstacles from the total area (e.g. litter bins, a possible stage for the concert, or trees).

Next, we are going to show the transcriptions of the resolutions of some groups of students to clearly show how we have carried out the analysis that will allow us to establish the identified strategies and the relations between them.

> *We have used the square as the unit of measure. Strategies: using squares (tiles), removing the area corresponding to obstacles (where people cannot stand). This*

> *leaves approximately 4000 squares (125*200). Assuming that there is enough space for 6 people per square, 24000 people can fit there.*

In our analysis, we have considered that this group has tackled the problem through an indirect strategy, so they find the area of useful space using a non-standard unit of measure and then multiply this result by the estimated density which, in this case, is uniform. Another group of students made the following resolution:

> *We have estimated that the total area is 5100 square meters, but considering that there are benches, trees, litter bins and areas of low visibility, we approximated the area to 3900 square metres. Then, we have estimated that there should be between 5 and 8 people (without being too cramped), depending on the area of the enclosure (there will be more people in the centre than at the edge). Thus, we estimated an average density of 6.5 people per square meter. Finally, we multiplied and found that the yard fits about 25350 people.*

This group has used a similar strategy to the previous group. However, they consider an average density in their resolution, not a uniform density. On the other hand, other groups propose resolutions based on measuring the area of a single element. In the following, we transcribed one of these resolutions:

Step 1: calculate the area of the courtyard approximately, with steps.

Step 2: subtract the space occupied by the stage.

Step 3: Using steps, calculate the space occupied by one person.

Step 4: divide the total area by the area occupied by one person.

This group also chose to use the indirect way. The students calculated the (useful) area using non-standard units (steps) and used the iteration unit model by experimentally finding the area occupied by one person.

Once we have identified all the different procedures used by the different groups of learners, we represented them together in a diagram. This diagram is the strategy tree shown in Figure 1. We can see that the three different ways of dealing with the problem (grid distribution, density, and iteration of a unit) are shown and that the vertical choices made by the students can be followed. Specifically, we observed that the density and iteration of a unit strategies share the need to determine the area of the enclosure in which people are positioned and therefore we have placed the procedures related to the calculation of the area in the intersection of these two indirect strategies.

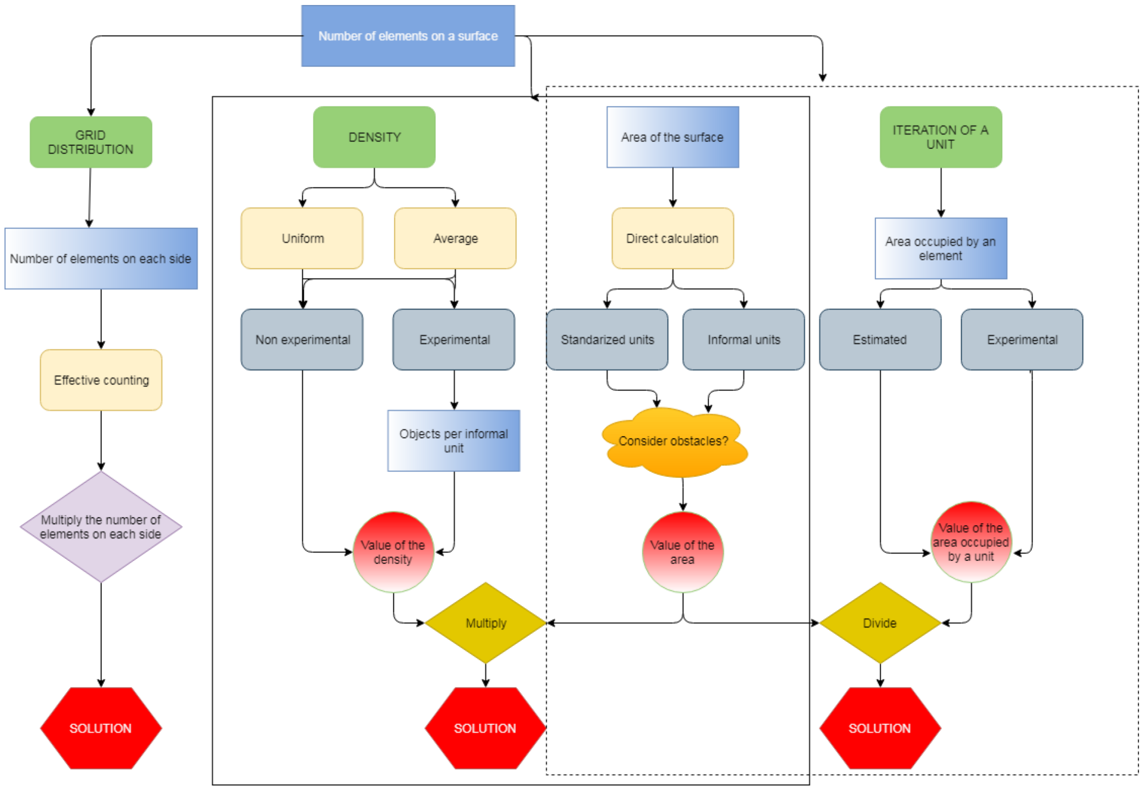

Figure 1. Tree diagram of the strategies identified in student resolutions for problem A

Strategy tree for Problem B1

Problem B1 consists of obtaining an estimate of the number of people that can fit in the town hall square of the city. In this case, the students can no longer solve the problem in situ, but work on scale maps. This implies that the strategies differ from those identified in the previous problem, as we will see below. In addition, the square is no longer rectangular, so we can no longer find a strategy based on the grid pattern model and, in the part related to surface area calculation, there are procedures that did not appear in problem A. Neither did we find the use of experimental procedures to estimate the density or area occupied by an element, nor the use of non-standard units of measurement.

In the following, we transcribed the resolution of one of the groups, which is accompanied by the illustration shown in Figure 2.

> *We have divided the area of the square in a triangle and a circle and, after scaling the measurements, in real metres, we have found the area of the two figures separately, then added them, obtaining the total area. If there is space for 4 people in a square metre, by multiplying the area by 4 we can find that there is room for 2515 people.*

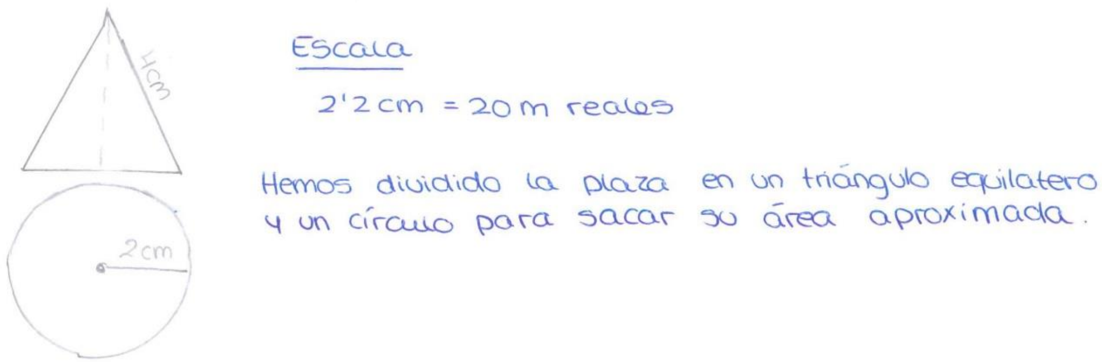

Fig. 2. A way to approximate a surface area by using simpler shapes[2].

In this case the students have based their strategy on finding an estimation of the density and multiplying this value by the area obtained by decomposing the original shape into two simple shapes.

Figure 3 shows a tree diagram showing the resolution strategies we have identified in the analysis of problem B1, we have used bold script in order to emphasize the new elements identified in the resolution of this task.

[2] Translation of the written text: "We have divided the town hall square into an equilateral triangle and a circle in order to find its approximate surface area"

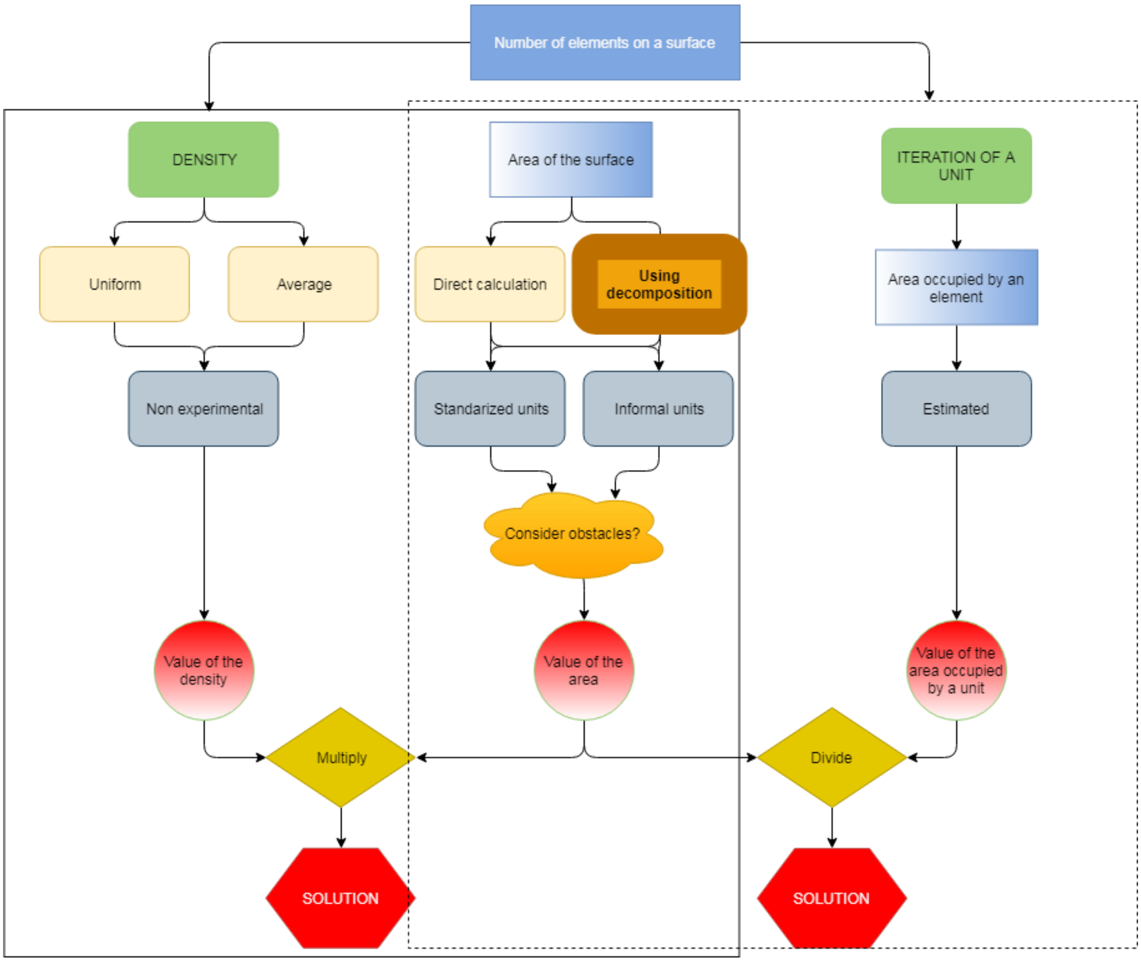

Figure 3. Tree diagram of the strategies identified in student resolutions for problem B1

Strategy tree for Problem B2

Problem B2 consists of obtaining an estimation of the number of people that can fit in a large enclosed space devoted to sport competitions or music concerts. Here, as in problem A, students have to deal with a rectangular shape, but this task has some particularities that help us to identify some new strategies and to discard others. In fact, the chosen space has differentiated zones, for instance, the stage stands in the case of music concerts and the court in the case of sport competitions. Thus, in this case, we did not find resolutions with strategies based on the grid pattern model and students did not consider obstacles or decompositions into simpler shapes to find the total area. However, as shown in the following transcription, students consider different densities depending on the area.

We know that one square metre fits 3 people standing, but as some people are seated, let's assume that part of the space (half) is occupied by seats and that there are two per square metre. Thus, we will divide the total area in two parts, we will multiply one of them by three and another by two and we will add them up to obtain the total number.

Figure 4 shows a tree diagram showing the resolution strategies we have identified in the analysis of problem B2. We observed that, in this context, there are no resolutions considering informal units. Again, we have used bold script to emphasize those elements that have not previously appeared in the analysis of activities A and B1.

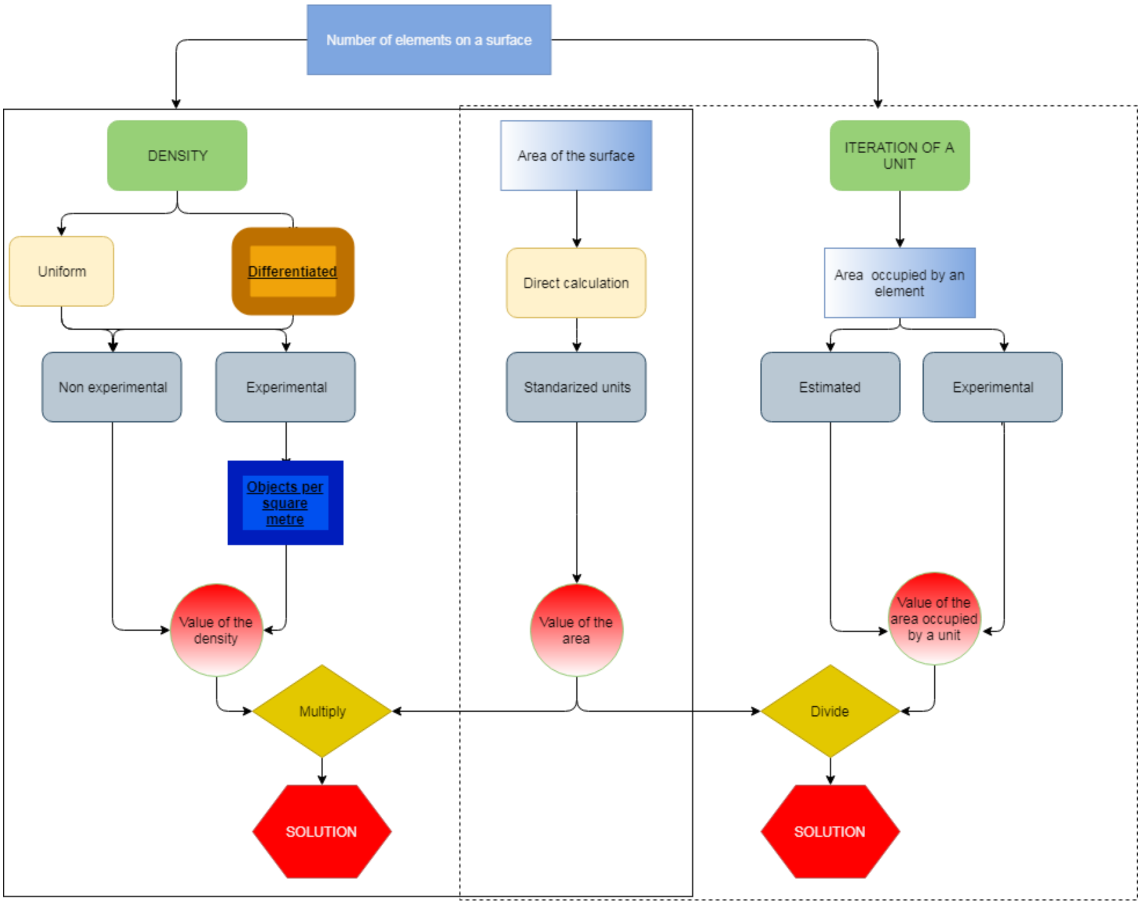

Figure 4. Tree diagram of the strategies identified in student resolutions for problem B2

Strategy tree for Problem C

The fourth problem of the context sequence differs greatly from the previous ones both due to the type of elements to be estimated and to the physical distance of the context from the reality of the students. Thus, in this problem we have identified fewer strategies than in the three previous problems. However, there is a new procedure linked to the strategy based on the iteration of a unit model: the aim is to approximate the area of the unit element through a simple figure. As we show in Figure 5, students estimated the area occupied by a tree, considering the radius of the tree and the distance between trees, ignoring the gaps between trees.

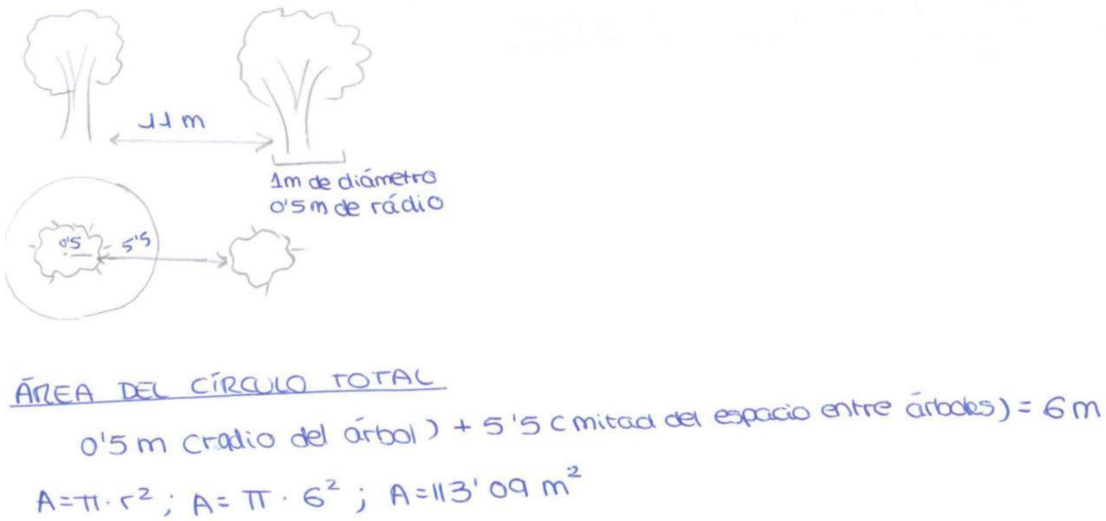

Figure 5. Calculation of the area occupied by an element in Activity C[3]

Thus, in the diagram in Figure 6 we gathered the strategies identified in the resolutions of the last task of the sequence. We used bold script to emphasize the new procedure identified in the resolution of this activity.

[3] Translation of the written text: 1m diameter; 0.5m radius; TOTAL AREA OF THE CIRCLE: 0.5m (radius of the tree) + 5.5 (half of the distance between trees) = 6 m

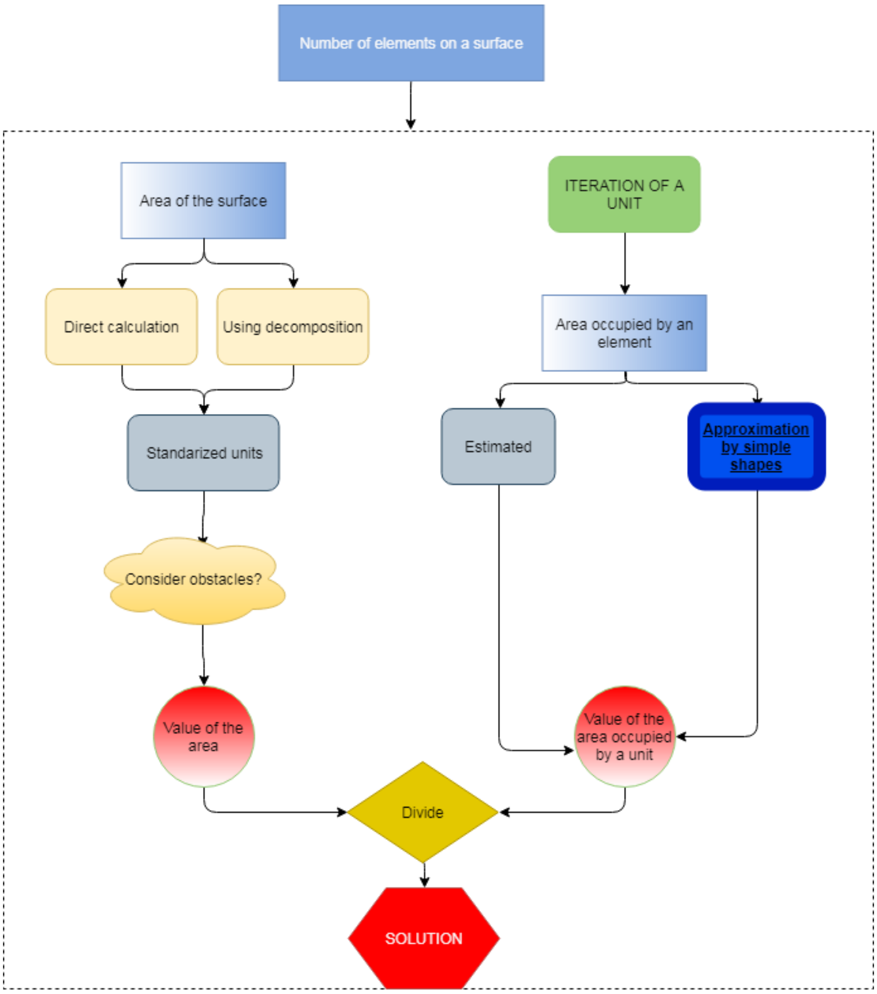

Figure 6. Tree diagram of the strategies identified in student resolutions for problem C

Results: The strategy tree for the "elements on a surface" problem

The series of strategies collected in the analyses of the four problems results in a diagram describing all the possible strategies identified in the sequence. Figure 7 shows this final diagram.

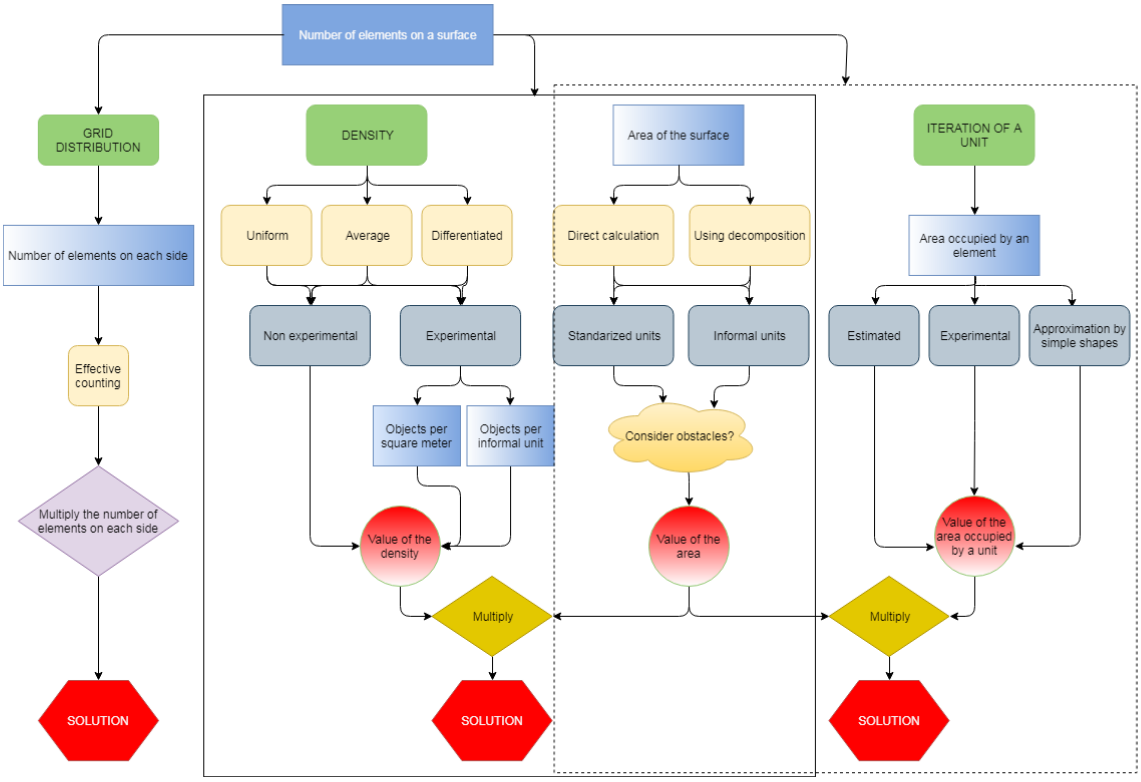

Figure 7. Tree diagram summarising the models in the solving processes.

As shown in the previous sections, not all problems contribute to the final diagram with the same strategies. This fact highlights the need to vary the context in order to achieve resolutions that use strategies with different characteristics. This tree can be complemented with more detailed procedures for some of the strategies depending on the specific training or age of the students participating in the study, however, the diagram shows that there are three main ideas for solving the problem: grid distribution, iteration of a unit, and density. These three general strategies take many forms and some specific procedures are shared by different strategies. Following the sequence, sometimes these shared aspects are used by different student groups for the same problem and on other occasions the same group will apply them to different problems. The anticipation of the richness and variety of problem-solving approaches in the sequence gives the chance to delve into aspects related to choosing the best possible strategy in each case, and working on flexibility when dealing with multiple solution problems (Leikin & Levav-Waynberg, 2008).

Discussion and conclusion

We have analysed the resolution strategies used to solve a sequence of tasks from the written productions of 16-year-old students who have never dealt with open and contextualised activities, all of them consisting of the same mathematical activity – estimating the number of elements in a delimited enclosure.

First, the main contribution of this study is the tree diagram that gathers the elements identified in the problem-solving strategies of counting elements on a surface. We initially challenge the students with a simple but realistic activity, and only after sharing their initial solving strategies, difficulties, and limitations are they presented with other more complex activities by changing the problem context, thus scaffolding their solving processes. The schemes of the resolution strategies used by the students reflect the integration of different concepts and procedures linked to different curriculum areas. In fact, geometry, measurement, and arithmetic are comprehensively dealt with. We emphasize that the students not only use diverse contents in their resolutions, but the different groups use different strategies from each other that allow them to confront the use of those mathematical contents in each case. For example, we observe that several groups opted for using population density as a way of representing the distribution of people on the plane, but other working groups work with the same precision using the iteration of the unit. Thus, we can confirm that the sequence conforms to the intertwinement principle (Treffers, 1978) and that the students coordinate the mathematical contents that they use to generate their resolution of the problem.

The tree diagram offers a characterization of the mathematical strategies developed to solve a series of modelling activities that require estimating the number of elements that fit into a precinct with characteristics that vary from one activity to the other. Replicating our research by adapting the problem in other scenarios could allow for a more complete tree diagram since the context of the activity statement conditions the models generated. If this were to be done, the tree diagram that we have created would facilitate the design of these new mathematical procedures. The tree diagram could also be completed at a theoretical level, on the basis of sound mathematical knowledge, and then compared with the way students solve the different activities that specify the problem. Moreover, the idea of creating the tree diagram of a problem is a powerful one, since it may apply to any problem, generating a useful resource, both for research and for teaching. In further research, the tree diagram of the problem is not only a result in itself, but it could also be a powerful analytical tool for analysis.

We understand that from the methodological point of view, posing diverse problems with the same mathematical foundation but with different contexts allows us to explore a great variety of approaches of the students to solving the problem. This seems to be an interesting path to explore in order to obtain broad and precise descriptions of the methods and strategies generated by the students, which should allow them to be connected with the formal contents included in the mathematics curricula. In Albarracín and Gorgorió (2015) we presented proposals for sequences of Fermi problems organised by types of problems proposed (objects on the plane, in space, or problems related to travel and movement). Studying the informal knowledge that students use to solve these problems using sequences of problems with different contexts seems to be an interesting way to promote learning that can be adequately formalised, in line with the principles of RME. In this sense we highlight the role of Fermi problems as open activities that allow students to point out the essential elements of the real context studied in order to connect them with the mathematical content they are familiar with. Since the statements provided refer to authentic situations, students treat them as such by interpreting the situation in its real context.

Another aspect that we have detected in our analysis is that there seems to be an interplay between the context of the problem and the strategies used. In fact, in the playground task, in which they should estimate the number of people in a regular and medium sized space, we obtain a much richer resolution diagram than in the rest of the cases. We think that there should be a certain relationship between certain variables associated with the context (i.e. relative size of the elements and the enclosure, shape of the enclosure, shape of the elements or even arrangement of the elements on the surface) and the associated resolution strategies. However, the sample collected and the analysis carried out is not enough to draw conclusions on this aspect. But we consider that, in future research, it may be interesting to investigate this aspect since it would allow us to classify the problems of estimating the number of elements in a delimited enclosure in terms of the function of these variables and, thus, to design sequences that promote student inter-task flexibility, in the sense of Elia, Van den Heuvel-Panhuizen and Kolovou (2009). One of the conclusions derived from this study is that one way to design activity sequences that promote student-generated knowledge and the contrast of mathematical methods to study a reality is the use of activities that are similar from the mathematical point of view and that have different contexts. In this sense it seems relevant to study the possibilities of this approach in order to decide whether researchers and teachers will be able to anticipate and infer the procedures that students will use while designing their activities.

As a final remark, we would like to claim that our research results could also be useful for teachers. Chapman (2006) pointed out that a clear majority of them are too straightforward and rigid in their approach on introducing problems requiring the mathematization of a real context, not allowing for a narrative analysis of the situations posed. Doerr (2006) claimed that teachers would benefit from having access to a well-developed scheme of the different types of responses that students may produce. From this point of view, the tree diagram of the problem could be a tool to interpret the strategies used by the students, helping teachers to prepare or reorient lessons and anticipating students' difficulties preparing the resources that can support their mathematical work in advance.

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