ABSTRACT BIVARIANT CUNTZ SEMIGROUPS II

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ABSTRACT. We previously showed that abstract Cuntz semigroups form a closed symmetric monoidal category. This automatically provides additional structure in the category, such as a composition and an external tensor product, for which we give concrete constructions in order to be used in applications.

We further analyse the structure of not necessarily commutative Cu-semirings and we obtain, under mild conditions, a new characterization of solid Cu-semirings R by the condition that $R \cong [\![R, R]\!]$.

1. INTRODUCTION

The Cuntz semigroup is a geometric refinement of K-theory that was introduced by Cuntz [Cun78] in his groundbreaking studies of simple C^*-algebras. It is a partially ordered semigroup that is constructed from positive elements in the stabilization of the algebra, in a similar way to how the Murray-von Neumann semigroup in K-theory is built from projections.

There has been an intensive use of Cuntz semigroups in C^*-algebra theory, particularly related to classification results. One of the most prominent instances of this appeared in Toms' example [Tom08] of two simple C^*-algebras that are indistinguishable by K-theoretic and tracial data, yet their Cuntz semigroups are not order-isomorphic. For the largest agreeable class of simple, separable, nuclear, finite and $\mathcal{Z}$-stable C^*-algebras that can be classified using K-theoretic and tracial information, the Cuntz semigroup features prominently since, suitably paired with K_1, it is functorially equivalent to the Elliott invariant; see [BPT08] and [ADPS14].

Further, for C^*-algebras of stable rank one, the Cuntz semigroup has additional fine structure that was key in solving a number of open problems, such as the Blackadar Handelman conjecture and the Global Glimm Halving Problem; see [APRT18].

The formal framework to study Cuntz semigroups of C^*-algebras is provided by the category Cu of abstract Cuntz semigroups, which was introduced by Coward, Elliott and Ivanescu in [CEI08], and which was studied in detail in [APT18] and later in [APT18b]. The objects of this category, called Cu-semigroups, are partially ordered semigroups that satisfy the order-theoretic analogues of being a complete topological space without isolated points. The Cu-morphisms are natural models for *-homomorphisms between C^*-algebras. The following result was established in [APT18] and [APT18b].

Theorem. *The category* Cu *of abstract Cuntz semigroups is a symmetric monoidal closed category.*

This means that there are bifunctors

$$_- \otimes {}_- : \mathrm{Cu} \times \mathrm{Cu} \to \mathrm{Cu}, \quad \text{and} \quad [\![{}_-, {}_-]\!] : \mathrm{Cu} \times \mathrm{Cu} \to \mathrm{Cu},$$

such that $\otimes$ is associative, symmetric, has as unit object the semigroup $\overline{\mathbb{N}} = \{0, 1, \ldots, \infty\}$ and, for any Cu-semigroup T, the functors $\_ \otimes T$ and $[\![T, \_]\!]$ are an adjoint pair. For Cu-semigroups S and T, the construction of the internal-hom $[\![S, T]\!]$ is based on paths of the so-called generalized Cu-morphisms, which are natural models for completely positive contractive order-zero maps between C^*-algebras; see Section 2 for more details. We refer to the semigroups $[\![S, T]\!]$ as bivariant Cu-semigroups.

Further, we show in [APT18c] that Cu is complete and cocomplete, and that the functor that assigns to each C^*-algebra its Cuntz semigroup is compatible with products and ultraproducts.

The fact that Cu is a closed category automatically adds additional features well known to category theory (see, for instance, [Kel05]). For example, one obtains a *composition product* given in the form of a Cu-morphism:

$$\circ \colon [\![T, P]\!] \otimes [\![S, T]\!] \to [\![S, P]\!],$$

which is the generalization of the composition of morphisms in a category to a notion of composition between internal-hom objects in a closed category; see the comments after [APT18b, Proposition 5.11].

Although the said features can be derived from general principles, in our setting they become concrete, and this is very useful in applications. In this direction, and bearing in mind that $[\![S, T]\!]$ is a semigroup built out of paths of morphisms from S to T, the composition product can be realized as the composition of paths. Another important example is the *evaluation map* which, for Cu-semigroups S and T is a Cu-morphism $\mathrm{e}_{S,T} \colon [\![S, T]\!] \otimes S \to T$ such that $\mathrm{e}_{S,T}(x \otimes a)$ can be interpreted as the evaluation of $x \in [\![S, T]\!]$ at $a \in S$. We therefore also write $x(a) := \mathrm{e}_{S,T}(x \otimes a)$. The evaluation map can be used to concretize the adjunction between the internal-hom bifunctor and the tensor product; see Proposition 3.22.

Likewise, the tensor product of generalized Cu-morphisms induces an *external tensor product*

$$\boxtimes \colon [\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!],$$

which is associative and, like in KK-Theory, compatible with the composition product; see Proposition 3.21. This means that, for elements $x_k \in [\![S_k, T_k]\!]$ and $y_k \in [\![T_k, P_k]\!]$ (for $k = 1, 2$), we have

$$(y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) = (y_2 \circ x_2) \boxtimes (y_1 \circ x_1).$$

In Section 4, we study the ideal structure of bivariant Cu-semigroups. Given an ideal J in S, and an ideal K in T, we show that there is a natural identification of $[\![S/J, K]\!]$ with an ideal in $[\![S, T]\!]$; see Propositions 4.2 and 4.5. However, in general, not every ideal of $[\![S, T]\!]$ arises this way. Indeed, in Example 4.7 we construct a simple Cu-semigroup S such that $[\![S, S]\!]$ is not simple.

In Section 5, we deepen our study of Cu-semirings and their semimodules, which was started in [APT18, Chapters 7 and 8]. Given a Cu-semigroup S, the composition product turns $[\![S, S]\!]$ into a Cu-semiring; see Proposition 5.1. Further, the evaluation map defines natural left $[\![S, S]\!]$-action on S; see Proposition 5.3. Finally, $[\![S, T]\!]$ has both a natural left $[\![T, T]\!]$-action and a compatible right $[\![S, S]\!]$-action; see Proposition 5.6.

For any Cu-semiring R, the internal-hom construction makes it possible to define a left regular representation-like map $\pi_R \colon R \to [\![R, R]\!]$, which is always a multiplicative order-embedding (and unital in case the unit of R is a compact element); see Definition 5.9, Theorem 5.12 and Proposition 5.16. We study when $\pi_R \colon R \to [\![R, R]\!]$ is an isomorphism; see Theorem 5.23. This is closely related to the property of being *solid*, which means that the multiplication defines an isomorphism between $R \otimes R$ and R; see [APT18, Definition 7.1.6].

2. PRELIMINARIES

Let S be a positively ordered semigroup. Recall (cf. [GHK$^+$03, Definition I-1.11, p.57]) that an *additive auxiliary relation* on S is a binary relation $\prec$ on S satisfying the following conditions for all $a, a', b, b' \in S$:

(1) If $a \prec b$ then $a \leq b$.
(2) If $a' \leq a \prec b \leq b'$ then $a' \prec b'$.
(3) We have $0 \prec a$.
(4) The relation $\prec$ is compatible with addition.

An important example of an auxiliary relation that we will use in the sequel is the *way-below relation*, originally coming from Domain Theory (see [GHK$^+$03]):

Let S be a positively ordered semigroup, and let $a, b \in S$. Recall that a is *way-below* b (we also say that a is *compactly contained in* b), denoted $a \ll b$, if whenever $(c_n)_n$ is an increasing sequence in S for which the supremum exists and that satisfies $b \leq \sup_n c_n$, then there exists $k \in \mathbb{N}$ with $a \leq c_k$.

We say that a is *compact* if $a \ll a$, and we let S_c denote the submonoid of compact elements in S.

Definition 2.1 ([CEI08]; see also [APT18, Definition 3.1.2])**.** A Cu-*semigroup*, also called *abstract Cuntz semigroup*, is a positively ordered semigroup S that satisfies the following axioms (O1)-(O4):

(O1) Every increasing sequence $(a_n)_n$ in S has a supremum $\sup_n a_n$ in S.
(O2) For every element $a \in S$ there exists a sequence $(a_n)_n$ in S with $a_n \ll a_{n+1}$ for all $n \in \mathbb{N}$, and such that $a = \sup_n a_n$.
(O3) If $a' \ll a$ and $b' \ll b$ for $a', b', a, b \in S$, then $a' + b' \ll a + b$.
(O4) If $(a_n)_n$ and $(b_n)_n$ are increasing sequences in S, then $\sup_n(a_n + b_n) = \sup_n a_n + \sup_n b_n$.

A Cu-*morphism* between Cu-semigroups S and T is an additive map $f \colon S \to T$ that preserves order, the zero element, the way-below relation and suprema of increasing sequences. In case f is not required to preserve the way-below relation, then we say it is a *generalized* Cu-*morphism*. The set of Cu-morphisms (respectively, generalized Cu-morphisms) is denoted by $\mathrm{Cu}(S, T)$ (respectively, by $\mathrm{Cu}[S, T]$).

We let Cu be the category whose objects are Cu-semigroups and whose morphisms are Cu-morphisms.

A notion central to the construction of tensor products is that of bimorphisms, which we now recall.

Definition 2.2 ([APT18, Definition 6.3.1])**.** Let S, T and P be Cu-semigroups. A map $\varphi\colon S \times T \to P$ is a Cu-*bimorphism* if it satisfies the following conditions:

(1) φ is a positively ordered monoid morphism in each variable.
(2) We have that $\sup_k \varphi(a_k, b_k) = \varphi(\sup_k a_k, \sup_k b_k)$, for every increasing sequences $(a_k)_k$ in S and $(b_k)_k$ in T.
(3) If $a', a \in S$ and $b', b \in T$ satisfy $a' \ll a$ and $b' \ll b$, then $\varphi(a', b') \ll \varphi(a, b)$.

The set of Cu-bimorphisms is denoted by $\mathrm{BiCu}(S \times T, P)$. Equipped with pointwise order and addition, this set is a positively ordered monoid. Similarly, the set of Cu-morphisms between two Cu-semigroups is also a positively ordered monoid.

Theorem 2.3 ([APT18, Theorem 6.3.3])**.** *Let S and T be* Cu-*semigroups. Then there exists a* Cu-*semigroup $S \otimes T$ and a* Cu-*bimorphism $\omega\colon S \times T \to S \otimes T$ such that for every* Cu-*semigroup P the following universal properties hold:*

(1) *For every* Cu-*bimorphism $\varphi\colon S \times T \to P$ there exists a (unique)* Cu-*morphism $\tilde{\varphi}\colon S \otimes T \to P$ such that $\varphi = \tilde{\varphi} \circ \omega$.*
(2) *If $\alpha_1, \alpha_2\colon S \otimes T \to P$ are* Cu-*morphisms, then $\alpha_1 \leq \alpha_2$ if and only if $\alpha_1 \circ \omega \leq \alpha_2 \circ \omega$.*

Thus, for every Cu-*semigroup P, the assignment*

$$(\alpha\colon S \otimes T \to P) \mapsto (\alpha \circ \omega\colon S \times T \to P)$$

defines a natural isomorphism of positively ordered monoids

$$\mathrm{Cu}\big(S \otimes T, P\big) \cong \mathrm{BiCu}\big(S \times T, P\big).$$

The existence of a natural tensor product turns Cu into a symmetric monoidal category; see [APT18, 6.3.7]. As mentioned above, the tensor product functor $- \otimes T$ has a right adjoint $[\![T, -]\!]$, and thus Cu is also a closed category. We recall some details; see [APT18b, Section 3] for a full account.

Let $(S, \prec)$ be an ordered semigroup equipped with an additive auxiliary relation $\prec$, and let $I_{\mathbb{Q}} = \mathbb{Q} \cap (0, 1)$. A *path* on S is a map $f\colon I_{\mathbb{Q}} \to S$ such that $f(\lambda') \prec f(\lambda)$ whenever $\lambda' < \lambda$. The set of paths on S is denoted by $P(S)$, which becomes a semigroup under pointwise addition. It is often the case that we write $f_\lambda = f(\lambda)$, and refer to a path as $\mathbf{f} = (f_\lambda)_{\lambda \in I_{\mathbb{Q}}}$.

Given paths f, g in $P(S)$, write $f \precsim g$ if for every $\lambda \in I_{\mathbb{Q}}$, there is $\mu \in I_{\mathbb{Q}}$ such that $f(\lambda) \prec g(\mu)$. Set $f \sim g$ provided $f \precsim g$ and $g \precsim f$, and let $\tau(S, \prec) := P(S)/\sim$. Let $[f]$ denote the equivalence class of a path f. Then $\tau(S, \prec)$ becomes an ordered semigroup by setting $[f] + [g] = [f + g]$ and $[f] \leq [g]$ provided $f \precsim g$. It was proved in [APT18b, Theorem 3.15] that, for S as above, the semigroup $\tau(S)$ is a Cu-semigroup.

Remark 2.4. The above construction is also referred to as the τ-*construction* in [APT18b]. It defines a functor $\tau\colon \mathcal{Q} \to \mathrm{Cu}$, where $\mathcal{Q}$ is the category of positively ordered semigroups S with an additive auxiliary relation $\prec$ that additionally satisfy axioms (O1) and (O4). In this way, τ is a coreflector of the inclusion functor $\iota\colon \mathrm{Cu} \to \mathcal{Q}$; see [APT18b, Theorem 4.12]. Objects (respectively, morphisms) in the category $\mathcal{Q}$ are termed $\mathcal{Q}$-semigroups (respectively, $\mathcal{Q}$-morphisms).

If now S and T are Cu-semigroups, it is clear that $\mathrm{Cu}[S, T]$ is also an ordered semigroup (with pointwise order and addition), and satisfies axioms (O1) and (O4) by taking pointwise suprema of increasing sequences. Given $\varphi, \psi \in \mathrm{Cu}[S, T]$, we define $\varphi \prec \psi$ provided $\varphi(a') \ll \psi(a)$ whenever $a' \ll a$. This is easily seen to be an additive auxiliary relation on $\mathrm{Cu}[S, T]$. Thus, $\mathrm{Cu}[S, T]$ is a $\mathcal{Q}$-semigroup in the sense of Remark 2.4.

Definition 2.5 ([APT18b, Definition 5.3]). Let S and T be Cu-semigroups. The *internal hom* from S to T is the Cu-semigroup

$$[\![S,T]\!] := \tau\big(\mathrm{Cu}[S,T], \prec\big).$$

We also call $[\![S,T]\!]$ the *bivariant* Cu-*semigroup*, or the *abstract bivariant Cuntz semigroup* of S and T.

The proof that Cu is a closed category requires the use of the so-called endpoint map, which is made precise below.

Definition 2.6 ([APT18b, Definition 5.5]). Let S and T be Cu-semigroups. We let $\sigma_{S,T}\colon [\![S,T]\!] \to \mathrm{Cu}[S,T]$ be defined by

$$\sigma_{S,T}([\mathbf{f}])(a) = \sup_{\lambda \in I_{\mathbb{Q}}} f_{\lambda}(a),$$

for a path $\mathbf{f} = (f_{\lambda})_{\lambda}$ in $\mathrm{Cu}[S,T]$ and $a \in S$. We refer to $\sigma_{S,T}$ as the *endpoint map*.

Theorem 2.7 ([APT18b, Theorem 5.9]). *Let S, T and P be* Cu-*semigroups. Then there are natural positively ordered monoid isomorphisms*

$$\mathrm{Cu}\big(S, [\![T,P]\!]\big) \cong \mathrm{BiCu}\big(S \times T, P\big) \cong \mathrm{Cu}\big(S \otimes T, P\big).$$

The first isomorphism is given by

$$(\alpha\colon S \to [\![T,P]\!]) \mapsto (\tilde{\alpha}\colon S \times T \to P),$$

where $\tilde{\alpha}(a,b) = \sigma_{T,P}(\alpha(a))(b)$, for $(a,b) \in S \times T$. The second is given by

$$(\beta\colon S \otimes T \to P) \mapsto ((a,b) \mapsto \beta(a \otimes b)), \text{ for } (a,b) \in S \times T.$$

3. Concretization of categorical constructions for Cu

In this section, we give concrete pictures of general constructions in closed, symmetric, monoidal categories for the category Cu. This will be used in the next section, and we start below with the analysis of the unit and counit maps.

Definition 3.1. Given Cu-semigroups S and T, the *unit map* is the Cu-morphism $\mathrm{d}_{S,T}\colon S \to [\![T, S \otimes T]\!]$ that under the identification

$$\mathrm{Cu}\big(S, [\![T, S \otimes T]\!]\big) \cong \mathrm{Cu}\big(S \otimes T, S \otimes T\big)$$

corresponds to the identity map on $S \otimes T$.

In the result below we shall use that, if S is a Cu-semigroup, and $a \in S$, then there is $(a_{\lambda})_{\lambda \in I_{\mathbb{Q}}}$ such that $a_{\lambda'} \ll a_{\lambda}$ whenever $\lambda' < \lambda$, $a_{\lambda} = \sup_{\lambda' < \lambda} a_{\lambda'}$, and $\sup_{\lambda} a_{\lambda} = s$; see [APT18b, Proposition 2.8].

Proposition 3.2. *Let S and T be* Cu-*semigroups, and let $a \in S$. Let $(a_{\lambda})_{\lambda \in I_{\mathbb{Q}}}$ be a path in $(S, \ll)$ with endpoint as. Then for each $\lambda \in I_{\mathbb{Q}}$, the map $a_{\lambda} \otimes_{-}\colon T \to S \otimes T$, sending $b \in T$ to $a_{\lambda} \otimes b$, is a generalized* Cu-*morphism. Moreover, $(a_{\lambda} \otimes_{-})_{\lambda \in I_{\mathbb{Q}}}$ is a path in $(\mathrm{Cu}[T, S \otimes T], \prec)$, and we have $\mathrm{d}_{S,T}(a) = [(a_{\lambda} \otimes_{-})_{\lambda}]$.*

Proof. The map $\omega\colon S \times T \to S \otimes T$, given by $\omega(s,t) = s \otimes t$, is a Cu-bimorphism. This implies that $s \otimes_{-}\colon T \to S \otimes T$ is a generalized Cu-morphism for each $s \in S$. Moreover, using that ω preserves the joint way-below relation, we obtain that $s' \otimes_{-} \prec s \otimes_{-}$ for $s', s \in S$ satisfying $s' \ll s$. In particular, if $(s_{\lambda})_{\lambda \in I_{\mathbb{Q}}}$ is a path in S, then $(s_{\lambda} \otimes_{-})_{\lambda \in I_{\mathbb{Q}}}$ is a path in $(\mathrm{Cu}[T, S \otimes T], \prec)$. We define $\alpha\colon S \to [\![T, S \otimes T]\!]$ by sending $s \in S$ to $[(s_{\lambda} \otimes_{-})_{\lambda}]$ for some choice of path $(s_{\lambda})_{\lambda}$ in S with endpoint as. It is straightforward to check that α is a well-defined Cu-morphism.

Let us show that $\alpha = \mathrm{d}_{S,T}$. Consider the bijections

$$\mathrm{Cu}\big(S, [\![T, S \otimes T]\!]\big) \cong \mathrm{BiCu}\big(S \times T, S \otimes T\big) \cong \mathrm{Cu}\big(S \otimes T, S \otimes T\big)$$

from Theorem 2.7. Under the first bijection, α corresponds to the Cu-bimorphism $\bar{\alpha}$ given by

$$\bar{\alpha}(s, t) = \sigma_{T, S \otimes T}(\alpha(s))(t),$$

for $s \in S$ and $t \in T$, where $\sigma_{T, S \otimes T}$ is the endpoint map. We compute

$$\bar{\alpha}(s, t) = \sigma_{T, S \otimes T}(\alpha(s))(t) = \sup_{\lambda \in I_{\mathbb{Q}}} (s_\lambda \otimes {}_-)(t) = \sup_{\lambda \in I_{\mathbb{Q}}} (s_\lambda \otimes t) = s \otimes t,$$

for every path $(s_\lambda)_\lambda$ with endpoint $s \in S$, and every $t \in T$. It follows that $\bar{\alpha}$ corresponds to $\mathrm{id}_{S \otimes T}$ under the second bijection. By definition of $\mathrm{d}_{S,T}$, this shows that $\alpha = \mathrm{d}_{S,T}$, as desired. $\qquad\square$

Notation 3.3. Given Cu-semigroups S, T and P, and a Cu-bimorphism $\alpha \colon S \times T \to P$, we shall often use the notation $\bar{\alpha} \colon S \to [\![T, P]\!]$ to refer to the Cu-morphism that corresponds to α under the identification in Theorem 2.7.

Corollary 3.4. *Let S and T be* Cu-*semigroups. Then the composition*

$$\sigma_{T, S \otimes T} \circ \mathrm{d}_{S,T} \colon S \xrightarrow{\mathrm{d}_{S,T}} [\![T, S \otimes T]\!] \xrightarrow{\sigma_{T, S \otimes T}} \mathrm{Cu}[T, S \otimes T].$$

satisfies $(\sigma_{T, S \otimes T} \circ \mathrm{d}_{S,T})(a) = a \otimes {}_-$, for every $a \in S$. In particular

$$(\sigma_{T, S \otimes T} \circ \mathrm{d}_{S,T})(a)(b) = a \otimes b,$$

for $a \in S$ and $b \in T$.

Proof. Let $a \in S$ and $b \in T$. Choose a path $(a_\lambda)_\lambda$ in S with endpoint a. Then $\mathrm{d}_{S,T}(a) = [(a_\lambda \otimes {}_-)_\lambda]$ by Proposition 3.2. The supremum of the maps $a_\lambda \otimes {}_-$ in $\mathrm{Cu}[S, T \otimes S]$ is the map $a \otimes {}_-$. Thus, $(\sigma_{T, S \otimes T} \circ \mathrm{d}_{S,T})(a) = a \otimes {}_-$, as desired. $\qquad\square$

Definition 3.5. Given Cu-semigroups S and T, the *counit map* (also called *evaluation map*) is the Cu-morphism $\mathrm{e}_{S,T} \colon [\![S, T]\!] \otimes S \to T$ that under the identification

$$\mathrm{Cu}\big([\![S, T]\!] \otimes S, T\big) \cong \mathrm{Cu}\big([\![S, T]\!], [\![S, T]\!]\big)$$

corresponds to the identity map on $[\![S, T]\!]$. Given $x \in [\![S, T]\!]$ and $a \in S$, we also use $x(a)$ to denote $\mathrm{e}_{S,T}(x \otimes a)$, but note that $x(a) = x'(a)$ for all $a \in S$ does not imply $x = x'$.

Proposition 3.6. *Let S and T be* Cu-*semigroups, let $x \in [\![S, T]\!]$, and let $a \in S$. Then $\mathrm{e}_{S,T}(x \otimes a) = \sigma_{S,T}(x)(a)$. Thus, if $\mathbf{f} = (f_\lambda)_\lambda$ is a path in $\mathrm{Cu}[S, T]$, then*

$$[\mathbf{f}](s) = \mathrm{e}_{S,T}([\mathbf{f}] \otimes a) = \sup_{\lambda < 1} f_\lambda(a).$$

Proof. Consider the bijections

$$\mathrm{Cu}\big([\![S, T]\!], [\![S, T]\!]\big) \cong \mathrm{BiCu}\big([\![S, T]\!] \times S, T\big) \cong \mathrm{Cu}\big([\![S, T]\!] \otimes S, T\big)$$

from Theorem 2.7. To simplify notation, we denote the identity map on $[\![S, T]\!]$ by id. Under the first bijection, id corresponds to the Cu-bimorphism $\bar{\mathrm{id}}$ satisfying

$$\bar{\mathrm{id}}(y, s) = \sigma_{S,T}(\mathrm{id}(y))(s),$$

for all $y \in [\![S, T]\!]$ and $s \in S$. We obtain that

$$\mathrm{e}_{S,T}(x \otimes a) = \bar{\mathrm{id}}(x, a) = \sigma_{S,T}(\mathrm{id}(x))(a) = \sigma_{S,T}(x)(a). \qquad\square$$

Remark 3.7. Let $\varphi \colon S \to T$ be a Cu-morphism, and let $a \in S$. Considering φ as an element of $[\![S, T]\!]$, the notation $\varphi(a)$ for $\mathrm{e}_{S,T}(\varphi \otimes a)$ is consistent with the usual notation of $\varphi(a)$ for the evaluation of φ at a.

Lemma 3.8. *Let S be a* Cu-*semigroup. Let* $\mathrm{ev}_1 \colon \mathrm{Cu}[\overline{\mathbb{N}}, S] \to S$ *be given by* $\mathrm{ev}_1(f) = f(1)$ *for* $f \in \mathrm{Cu}[\overline{\mathbb{N}}, S]$. *Then* ev_1 *is an isomorphism of $\mathcal{Q}$-semigroups. That is,* ev_1 *is an additive order-isomorphism and we have* $f \prec g$ *if and only if* $\mathrm{ev}_1(f) \ll \mathrm{ev}_1(g)$, *for* $f, g \in \mathrm{Cu}[\overline{\mathbb{N}}, S]$.

It follows that $(\mathrm{Cu}[\overline{\mathbb{N}}, S], \prec)$ *is a* Cu-*semigroup (naturally isomorphic to S via* ev_1*). Moreover, the endpoint map* $\sigma_{\overline{\mathbb{N}}, S} \colon [\![\overline{\mathbb{N}}, S]\!] \to \mathrm{Cu}[\overline{\mathbb{N}}, S]$ *from Definition 2.6 is an isomorphism.*

Proof. It is straightforward to prove that ev_1 is an isomorphism of $\mathcal{Q}$-semigroups. By [APT18b, Proposition 4.10], the endpoint map of a Cu-semigroup is an isomorphism. Thus, the endpoint maps φ_S and $\varphi_{\mathrm{Cu}[\overline{\mathbb{N}}, S]}$ are isomorphisms. By definition, $\sigma_{\overline{\mathbb{N}}, S} = \varphi_{\mathrm{Cu}[\overline{\mathbb{N}}, S]}$. Since ev_1 is an isomorphism, so is $\tau(\mathrm{ev}_1)$. $\qquad\qquad\square$

Definition 3.9. Given a Cu-semigroup S, we let $\mathrm{i}_S \colon S \to [\![\overline{\mathbb{N}}, S]\!]$ be the Cu-morphism that under the identification

$$\mathrm{Cu}\big(S, [\![\overline{\mathbb{N}}, S]\!]\big) \cong \mathrm{Cu}\big(S \otimes \overline{\mathbb{N}}, S\big)$$

corresponds to the natural isomorphism $\mathrm{r}_S \colon S \otimes \overline{\mathbb{N}} \to S$.

We leave the proof of the following result to the reader.

Proposition 3.10. *Let S be a* Cu-*semigroup. Then* $\mathrm{i}_S \colon S \to [\![\overline{\mathbb{N}}, S]\!]$ *is an isomorphism. The inverse of* i_S *is* $\mathrm{ev}_1 \circ \sigma_{\overline{\mathbb{N}}, S}$, *where* ev_1 *is evaluation at 1 as in Lemma 3.8, and where* $\sigma_{\overline{\mathbb{N}}, S} \colon [\![\overline{\mathbb{N}}, S]\!] \to \mathrm{Cu}[\overline{\mathbb{N}}, S]$ *denotes the endpoint map from Definition 2.6.*

We now introduce and study the external tensor product map. To this end, let first S_k and T_k be Cu-semigroups, and let $\varphi_k \colon S_k \to T_k$ be (generalized) Cu-morphisms, for $k = 1, 2$. Recall from the comments after [APT18b, Theorem 2.10] that the map $\varphi_1 \times \varphi_2 \colon S_1 \times S_2 \to T_1 \otimes T_2$, defined by

$$(\varphi_1 \times \varphi_2)(a_1, a_2) := f_1(a_1) \otimes f_2(a_2),$$

for $a_1 \in S_1$ and $a_2 \in S_2$, is a (generalized) Cu-bimorphism. We denote the induced (generalized) Cu-morphism by $\varphi_1 \otimes \varphi_2 \colon S_1 \otimes S_2 \to T_1 \otimes T_2$, and we call it the *tensor product* of φ_1 and φ_2.

Next, we generalize this construction and define an external tensor product between elements of internal-homs.

Definition 3.11. Given Cu-semigroups S_1, S_2, T_1 and T_2, we define the *external tensor product map* $\boxtimes \colon [\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]$ as the Cu-morphism that under the identification

$$\mathrm{Cu}\big([\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!], [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]\big)$$
$$\cong \mathrm{Cu}\big([\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \otimes S_1 \otimes S_2, T_1 \otimes T_2\big),$$

corresponds to the composition

$$(\mathrm{e}_{S_1, T_1} \otimes \mathrm{e}_{S_2, T_2}) \circ (\mathrm{id}_{[\![S_1, T_1]\!]} \otimes \sigma \otimes \mathrm{id}_{S_2}),$$

where $\sigma \colon [\![S_2, T_2]\!] \otimes S_1 \to S_1 \otimes [\![S_2, T_2]\!]$ denotes the flip isomorphism.

Given $x_1 \in [\![S_1, T_1]\!]$ and $x_2 \in [\![S_2, T_2]\!]$, we denote the image of $x_1 \otimes x_2$ under this map by $x_1 \boxtimes x_2$, and we call it the *external tensor product* of x_1 and x_2.

Remark 3.12. Let $\varphi_1 \colon S_1 \to T_1$ and $\varphi_2 \colon S_2 \to T_2$ be Cu-morphisms. Using [APT18b, Proposition 5.11], we identify φ_1 with a compact element in $[\![S_1, T_1]\!]$, and similarly for φ_2. It is easy to see that the element $\varphi_1 \boxtimes \varphi_2$ from Definition 3.11 agrees with the compact element in $[\![S_1 \otimes S_2, T_1 \otimes T_2]\!]$ that is identified with the tensor product map $\varphi_1 \otimes \varphi_2 \colon S_1 \otimes S_2 \to T_1 \otimes T_2$ from the comments before the above definition.

Notice that there is a certain ambiguity with the notation $\varphi_1 \otimes \varphi_2$, in that it may refer to a Cu-morphism (identified with a compact element in $[\![S_1 \otimes S_2, T_1 \otimes T_2]\!]$), and also to an element in $[\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!]$. However, the precise meaning will be clear from the context.

Theorem 3.13. *Let S_1, S_2, T_1 and T_2 be Cu-semigroups, and let $\mathbf{f} = (f_\lambda)_\lambda$ and $\mathbf{g} = (g_\lambda)_\lambda$ be paths in $\mathrm{Cu}[S_1, T_1]$ and $\mathrm{Cu}[S_2, T_2]$, respectively. For each λ, consider the generalized Cu-morphism $f_\lambda \otimes g_\lambda \colon S_1 \otimes S_2 \to T_1 \otimes T_2$. Then $(f_\lambda \otimes g_\lambda)_\lambda$ is a path in $\mathrm{Cu}[S_1 \otimes S_2, T_1 \otimes T_2]$ and we have*

$$[\mathbf{f}] \boxtimes [\mathbf{g}] = [(f_\lambda \otimes g_\lambda)_\lambda].$$

Proof. To show that $(f_\lambda \otimes g_\lambda)_\lambda$ is a path, let $\lambda', \lambda \in I_\mathbb{Q}$ satisfy $\lambda' < \lambda$. To show that $f_{\lambda'} \otimes g_{\lambda'} \prec f_\lambda \otimes g_\lambda$, let $t', t \in S_1 \otimes S_2$ satisfy $t' \ll t$. By properties of the tensor product in Cu, we can choose $n \in \mathbb{N}$, elements $a'_k, a_k \in S_1$ and $b'_k, b_k \in S_2$ satisfying $a'_k \ll a_k$ and $b'_k \ll b_k$ for $k = 1, \ldots, n$, and such that

$$t' \leq \sum_{k=1}^n a'_k \otimes b'_k, \quad \text{and} \quad \sum_{k=1}^n a_k \otimes b_k \leq t.$$

We have $f_{\lambda'} \prec f_\lambda$ and $g_{\lambda'} \prec g_\lambda$, and therefore $f_{\lambda'}(a'_k) \ll f_\lambda(a_k)$ and $g_{\lambda'}(b'_k) \ll g_\lambda(b_k)$ for $k = 1, \ldots, n$. Using this at the third step we deduce that

$$\begin{aligned}
(f_{\lambda'} \otimes g_{\lambda'})(t') &\leq (f_{\lambda'} \otimes g_{\lambda'})\left(\sum_{k=1}^n a'_k \otimes b'_k\right) \\
&= \sum_{k=1}^n f_{\lambda'}(a'_k) \otimes g_{\lambda'}(b'_k) \\
&\ll \sum_{k=1}^n f_\lambda(a_k) \otimes g_\lambda(b_k) \\
&= (f_\lambda \otimes g_\lambda)\left(\sum_{k=1}^n a_k \otimes b_k\right) \leq (f_\lambda \otimes g_\lambda)(t).
\end{aligned}$$

Thus, given paths $\mathbf{p} = (p_\lambda)_\lambda$ and $\mathbf{q} = (q_\lambda)_\lambda$ in $\mathrm{Cu}[S_1, T_1]$ and $\mathrm{Cu}[S_2, T_2]$, respectively, then $(p_\lambda \otimes q_\lambda)_\lambda$ is a path in $\mathrm{Cu}[S_1 \otimes S_2, T_1 \otimes T_2]$. Moreover, it is tedious but straightforward to check that the map $[\![S_1, T_1]\!] \times [\![S_2, T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]$ that sends a pair $([\mathbf{p}] \otimes [\mathbf{q}])$ to $[(p_\lambda \boxtimes q_\lambda)_\lambda]$ is a well-defined Cu-bimorphism. We let $\alpha \colon [\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]$ be the induced Cu-morphism.

To show that $[\mathbf{f}] \boxtimes [\mathbf{g}] = [(f_\lambda \otimes g_\lambda)_\lambda]$, we will prove that the external tensor product $\boxtimes$ and the map α correspond to the same Cu-morphism under the bijection

$$\begin{aligned}
&\mathrm{Cu}\big([\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!], [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]\big) \\
&\cong \mathrm{Cu}\big([\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \otimes S_1 \otimes S_2, T_1 \otimes T_2\big)
\end{aligned}$$

from Theorem 2.7.

Let $\mathbf{p} = (p_\lambda)_\lambda$ and $\mathbf{q} = (q_\lambda)_\lambda$ be paths in $\mathrm{Cu}[S_1, T_1]$ and $\mathrm{Cu}[S_2, T_2]$, respectively, and let s_i be elements in S_i, for $i = 1, 2$. By definition of $\boxtimes$ (see Definition 3.11 and Notation 3.3), we have

$$\bar{\boxtimes}([\mathbf{p}] \otimes [\mathbf{q}] \otimes s_1 \otimes s_2) = [\mathbf{p}](s_1) \otimes [\mathbf{q}](s_2) = p_1(s_1) \otimes q_1(s_2).$$

Using Theorem 2.7 at the first step, we obtain that

$$\begin{aligned}
\bar{\alpha}([\mathbf{p}] \otimes [\mathbf{q}] \otimes s_1 \otimes s_2) &= \sigma_{S_1 \otimes S_2, T_1 \otimes T_2}(\alpha([\mathbf{p}] \otimes [\mathbf{q}]))(s_1 \otimes s_2) \\
&= \sigma_{S_1 \otimes S_2, T_1 \otimes T_2}([(p_\lambda \otimes q_\lambda)_\lambda])(s_1 \otimes s_2) \\
&= (p_1 \otimes q_1)(s_1 \otimes s_2) = p_1(s_1) \otimes q_1(s_2).
\end{aligned}$$

It follows that $\boxtimes = \alpha$ and therefore

$$[\mathbf{f}] \boxtimes [\mathbf{g}] = \boxtimes([\mathbf{f}] \otimes [\mathbf{g}]) = \alpha([\mathbf{f}] \otimes [\mathbf{g}]) = [(f_\lambda \otimes g_\lambda)_\lambda]. \qquad \square$$

The following result shows that the external tensor product is associative.

Proposition 3.14. *Let* S_1, S_2, T_1, T_2, P_1 *and* P_2 *be* Cu-*semigroups, let* $x \in [\![S_1, S_2]\!]$, $y \in [\![T_1, T_2]\!]$, *and let* $z \in [\![P_1, P_2]\!]$. *For* $k = 1, 2$, *we identify* $(S_k \otimes T_k) \otimes P_k$ *with* $S_k \otimes (T_k \otimes P_k)$ *using the natural isomorphism from the monoidal structure of* Cu *(see comments after* [APT18b, Theorem 2.10]*). Then*

$$(x \boxtimes y) \boxtimes z = x \boxtimes (y \boxtimes z).$$

Proof. Given $f \in \mathrm{Cu}[S_1, S_2]$, $g \in \mathrm{Cu}[T_1, T_2]$ and $h \in \mathrm{Cu}[P_1, P_2]$, it is straightforward to check that

$$(f \otimes g) \otimes h = f \otimes (g \otimes h),$$

as generalized Cu-morphisms $S_1 \otimes T_1 \otimes P_1 \to S_2 \otimes T_2 \otimes P_2$. The result follows by applying Theorem 3.13. $\qquad \square$

Problem 3.15. Study the order-theoretic properties of the external tensor product map $\boxtimes \colon [\![S_1, T_1]\!] \otimes [\![S_2, T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]$. In particular, when is this map an order-embedding, when is it surjective?

We recall below the definition of the composition product and analyse its relation with the external tensor product.

Definition 3.16. Given Cu-semigroups S, T and P, we define the *composition product*

$$\circ \colon [\![T, P]\!] \otimes [\![S, T]\!] \to [\![S, P]\!]$$

as the Cu-morphism that under the identification

$$\mathrm{Cu}\big([\![T, P]\!] \otimes [\![S, T]\!], [\![S, P]\!]\big) \cong \mathrm{Cu}\big([\![T, P]\!] \otimes [\![S, T]\!] \otimes S, P\big)$$

corresponds to the composition $\mathrm{e}_{T,P} \circ (\mathrm{id}_{[\![T,P]\!]} \otimes \mathrm{e}_{S,T})$. Given $x \in [\![S, T]\!]$ and $y \in [\![T, P]\!]$, we denote the image of $y \otimes x$ under the composition product by $y \circ x$.

Given $x \in [\![S, T]\!]$, we let $x^* \colon [\![T, P]\!] \to [\![S, P]\!]$ be given by $x^*(y) := y \circ x$ for $y \in [\![T, P]\!]$. Analogously, given $y \in [\![T, P]\!]$, we let $y_* \colon [\![S, T]\!] \to [\![S, P]\!]$ be given by $y_*(x) := y \circ x$ for $x \in [\![S, T]\!]$.

The composition product for the internal-hom satisfies the axioms for the hom sets in an enriched category. In particular, the product is associative and the identity element $\mathrm{id}_S \in \mathrm{Cu}(S, S) \subseteq [\![S, S]\!]$ acts as a unit for the composition product (see [Kel05, Section 1.6]). We recall these facts in the following proposition:

Proposition 3.17. *Let* S, T, P *and* Q *be* Cu-*semigroups, let* $x \in [\![S, T]\!]$, $y \in [\![T, P]\!]$, *and let* $z \in [\![P, Q]\!]$. *Then*

$$(z \circ y) \circ x = z \circ (y \circ x).$$

Further, for the identity Cu-*morphisms* $\mathrm{id}_S \in \mathrm{Cu}(S, S)$ *and* $\mathrm{id}_T \in \mathrm{Cu}(T, T)$, *we have*

$$\mathrm{id}_T \circ x = x = x \circ \mathrm{id}_S.$$

It follows that $[\![S, S]\!]$ and $[\![T, T]\!]$ are (not necessarily commutative) Cu-semirings and that $[\![S, T]\!]$ has a natural left $[\![S, S]\!]$- and right $[\![T, T]\!]$-semimodule structure; see Propositions 5.1 and 5.6 in the next section. In the following proposition we give an explicit description of the composition product.

Proposition 3.18. *Let* S, T, P *be* Cu-*semigroups, and let* $\mathbf{f} = (f_\lambda)_\lambda$ *and* $\mathbf{g} = (g_\lambda)_\lambda$ *be paths in* $\mathrm{Cu}[S, T]$ *and* $\mathrm{Cu}[T, P]$, *respectively. For each* λ, *consider the generalized* Cu-*morphism* $g_\lambda \circ f_\lambda \colon S \to P$. *Then* $(g_\lambda \circ f_\lambda)_\lambda$ *is a path in* $\mathrm{Cu}[S, P]$ *and*

$$[\mathbf{g}] \circ [\mathbf{f}] = [(g_\lambda \circ f_\lambda)_\lambda].$$

Proof. It is easy to check that $(g_\lambda \circ f_\lambda)_\lambda$ is a path. Moreover, it is tedious but straightforward to check that the map $[\![T,P]\!] \times [\![S,T]\!] \to [\![S,P]\!]$ that sends a pair $([\mathbf{p}],[\mathbf{q}])$ to $[(q_\lambda \circ p_\lambda)_\lambda]$ is a well-defined Cu-bimorphism. We let $\alpha\colon [\![T,P]\!] \otimes [\![S,T]\!] \to [\![S,P]\!]$ be the induced Cu-morphism.

To show that $[\mathbf{g}] \circ [\mathbf{f}] = [(g_\lambda \otimes f_\lambda)_\lambda]$, we will prove that the composition product $\circ$ and the map α correspond to the same Cu-morphism under the bijection

$$\mathrm{Cu}\big([\![T,P]\!] \otimes [\![S,T]\!], [\![S,P]\!]\big) \cong \mathrm{Cu}\big([\![T,P]\!] \otimes [\![S,T]\!] \otimes S, P\big)$$

from Theorem 2.7.

Let $\mathbf{p} = (p_\lambda)_\lambda$ and $\mathbf{q} = (q_\lambda)_\lambda$ be paths in $\mathrm{Cu}[S,T]$ and $\mathrm{Cu}[T,P]$, respectively, and let $s \in S$. Set $p_1 := \sup_{\lambda<1} p_\lambda$ and $q_1 := \sup_{\lambda<1} q_\lambda$. By definition, we have

$$\bar{\circ}([\mathbf{q}] \otimes [\mathbf{p}] \otimes s) = \mathrm{e}_{T,P} \circ (\mathrm{id}_{[\![T,P]\!]} \otimes \mathrm{e}_{S,T})([\mathbf{q}] \otimes [\mathbf{p}] \otimes s) = \mathrm{e}_{T,P}([\mathbf{q}] \otimes \mathrm{e}_{S,T}([\mathbf{p}] \otimes s))$$
$$= \mathrm{e}_{T,P}([\mathbf{q}] \otimes p_1(s)) = q_1(p_1(s)).$$

On the other hand, using Theorem 2.7 at the first step, we obtain that

$$\bar{\alpha}([\mathbf{q}] \otimes [\mathbf{p}] \otimes s) = \sigma_{S,P}(\alpha([\mathbf{q}] \otimes [\mathbf{p}]))(s) = \sigma_{S,P}([(q_\lambda \circ p_\lambda)_\lambda])(s)$$
$$= (q_1 \circ p_1)(s) = q_1(p_1(s)).$$

It follows that $\circ = \alpha$ and therefore

$$[\mathbf{g}] \circ [\mathbf{f}] = \circ([\mathbf{g}] \otimes [\mathbf{f}]) = \alpha([\mathbf{g}] \otimes [\mathbf{f}]) = [(g_\lambda \otimes f_\lambda)_\lambda]. \qquad \square$$

Note that, in Proposition 3.18, the composition product of two Cu-morphisms, viewed as compact elements in the internal-hom set, is the usual composition of morphisms as maps.

Next we show that the composition product is compatible with the evaluation map in the expected way. It will follow later that the evaluation map $\mathrm{e}_{S,S}\colon [\![S,S]\!] \otimes S \to S$ defines a natural left $[\![S,S]\!]$-semimodule structure on S; see Proposition 5.3.

Lemma 3.19. *Let S, T and P be* Cu-*semigroups, let $x \in [\![S,T]\!]$, and let $y \in [\![T,P]\!]$. Then*

$$\sigma_{S,P}(y \circ x) = \sigma_{T,P}(y) \circ \sigma_{S,T}(x).$$

Proof. Let $\mathbf{f} = (f_\lambda)_\lambda$ be a path in $\mathrm{Cu}[S,T]$ representing x, and let $\mathbf{g} = (g_\lambda)_\lambda$ be a path in $\mathrm{Cu}[T,P]$ representing y. Let $a \in S$. By Proposition 3.18, we have $y \circ x = [(g_\lambda \circ f_\lambda)_\lambda]$. Using this at the first step, we obtain that

$$\sigma_{S,P}(y \circ x)(a) = \sup_{\lambda \in I_\mathbb{Q}} (g_\lambda \circ f_\lambda)(a) = \sup_{\mu \in I_\mathbb{Q}} g_\mu \left(\sup_{\lambda \in I_\mathbb{Q}} f_\lambda(a) \right) = \sigma_{T,P}(y)\left(\sigma_{S,T}(x)(a)\right),$$

as desired. $\qquad \square$

By combining Lemma 3.19 with Proposition 3.6, we obtain:

Proposition 3.20. *Let S, T and P be* Cu-*semigroups, let $x \in [\![S,T]\!]$, let $y \in [\![T,P]\!]$, and let $a \in S$. Then*

$$(y \circ x)(a) = y(x(a)).$$

Moreover, for the identity Cu-*morphism* $\mathrm{id}_S \in \mathrm{Cu}(S,S)$, *we have* $\mathrm{id}_S(a) = a$.

The following result shows that the external tensor product and the composition product commute.

Proposition 3.21. *Let S_1, S_2, T_1 and T_2 be* Cu-*semigroups. Given $x_k \in [\![S_k,T_k]\!]$ and $y_k \in [\![T_k,P_k]\!]$ for $k = 1,2$, we have*

$$(y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) = (y_2 \circ x_2) \boxtimes (y_1 \circ x_1).$$

Proof. Let $\mathbf{f}^{(\mathbf{k})} = (f_\lambda^{(k)})_\lambda$ be a path in $\mathrm{Cu}[S_k, T_k]$ representing x_k, for $k = 1, 2$, and let $\mathbf{g}^{(\mathbf{k})} = (g_\lambda^{(k)})_\lambda$ be a path in $\mathrm{Cu}[T_k, P_k]$ representing y_k, for $k = 1, 2$. Given λ, it is straightforward to check that

$$(g_\lambda^{(2)} \otimes g_\lambda^{(1)}) \circ (f_\lambda^{(2)} \otimes f_\lambda^{(1)}) = (g_\lambda^{(2)} \circ f_\lambda^{(2)}) \otimes (g_\lambda^{(1)} \circ f_\lambda^{(1)}).$$

Using this at the second step, and using Theorem 3.13 and Proposition 3.18 at the first and last step, we obtain that

$$\begin{aligned}
(y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) &= \left[((g_\lambda^{(2)} \otimes g_\lambda^{(1)}) \circ (f_\lambda^{(2)} \otimes f_\lambda^{(1)}))_\lambda \right] \\
&= \left[((g_\lambda^{(2)} \circ f_\lambda^{(2)}) \otimes (g_\lambda^{(1)} \circ f_\lambda^{(1)}))_\lambda \right] \\
&= (y_2 \circ x_2) \boxtimes (y_1 \circ x_1). \qquad \square
\end{aligned}$$

In the last part of this section, we revisit the unit and counit maps, their functorial properties, and how they can be used to implement the adjunction between the tensor product and the internal-hom functors.

Proposition 3.22. *Let S, T and P be* Cu*-semigroups. Then the bijection*

$$\mathrm{Cu}\big(S, [\![T, P]\!]\big) \cong \mathrm{Cu}\big(S \otimes T, P\big)$$

from Theorem 2.7 identifies a Cu*-morphism $f \colon S \to [\![T, P]\!]$ with*

$$\mathrm{e}_{T,P} \circ (f \otimes \mathrm{id}_T) \colon S \otimes T \xrightarrow{f \otimes \mathrm{id}_T} [\![T, P]\!] \otimes T \xrightarrow{\mathrm{e}_{T,P}} P.$$

Conversely, a Cu*-morphism $g \colon S \otimes T \to P$ is identified with*

$$g_* \circ \mathrm{d}_{S,T} \colon S \xrightarrow{\mathrm{d}_{S,T}} [\![T, S \otimes T]\!] \xrightarrow{g_*} [\![T, P]\!].$$

In particular, we have

$$f = (\mathrm{e}_{T,P} \circ (f \otimes \mathrm{id}_T))_* \circ \mathrm{d}_{S,T}, \quad \text{and} \quad g = \mathrm{e}_{T,P} \circ ((g_* \circ \mathrm{d}_{S,T}) \otimes \mathrm{id}_T).$$

Proof. Let $f \colon S \to [\![T, P]\!]$ be a Cu-morphism. Under the natural bijection from Theorem 2.7, f corresponds to the Cu-morphism $\bar{f} \colon S \otimes T \to P$ with

$$\bar{f}(s \otimes t) = \sigma_{T,P}(f(s))(t),$$

for a simple tensor $s \otimes t \in S \otimes T$. On the other hand, we have

$$(\mathrm{e}_{T,P} \circ (f \otimes \mathrm{id}_T))(s \otimes t) = \mathrm{e}_{T,P}(f(s) \otimes t) = \sigma_{T,P}(f(s))(t),$$

for a simple tensor $s \otimes t \in S \otimes T$. Thus $\bar{f}$ and $\mathrm{e}_{T,P} \circ (f \otimes \mathrm{id}_T)$ agree on simple tensors, and consequently $\bar{f} = \mathrm{e}_{T,P} \circ (f \otimes \mathrm{id}_T)$, as desired.

Let $g \colon S \otimes T \to P$ be a Cu-morphism. Set $\alpha := g_* \circ \mathrm{d}_{S,T}$. Under the natural bijection from Theorem 2.7, α corresponds to the Cu-morphism $\bar{\alpha} \colon S \otimes T \to P$ with

$$\bar{\alpha}(s \otimes t) = \sigma_{T,P}(\alpha(s))(t),$$

for a simple tensor $s \otimes t \in S \otimes T$. It is straightforward to verify that $\sigma_{T,P} \circ g_* = g_* \circ \sigma_{T,S \otimes T}$. Using this at the third step, and using Corollary 3.4 at the fourth step, we deduce that

$$\begin{aligned}
\bar{\alpha}(s \otimes t) = \sigma_{T,P}(\alpha(s))(t) &= (\sigma_{T,P} \circ g_* \circ \mathrm{d}_{S,T})(s)(t) \\
&= (g_* \circ \sigma_{T,S \otimes P} \circ \mathrm{d}_{S,T})(s)(t) = g(s \otimes t),
\end{aligned}$$

for every simple tensor $s \otimes t \in S \otimes T$. Thus, $\bar{\alpha} = g$, as desired. $\qquad \square$

Applying the previous result to the identity morphisms, we obtain:

Corollary 3.23. *Let S and T be* Cu*-semigroups. Then*

$$\mathrm{id}_{[\![S,T]\!]} = (\mathrm{e}_{S,T})_* \circ \mathrm{d}_{[\![S,T]\!],S}, \quad \text{and} \quad \mathrm{id}_{S \otimes T} = \mathrm{e}_{T,S \otimes T} \circ (\mathrm{d}_{S,T} \otimes \mathrm{id}_T).$$

Given Cu-semigroups S and T, we consider the unit map $\mathrm{d}_{S,T} \colon S \to [\![T, S \otimes T]\!]$ from Definition 3.1. Next, we introduce a more general form of the unit map.

Definition 3.24. Let S, T and T' be Cu-semigroups. We define *the general left unit map* $S \otimes [\![T', T]\!] \to [\![T', S \otimes T]\!]$ as the Cu-morphism that under the identification

$$\mathrm{Cu}\big(S \otimes [\![T', T]\!], [\![T', S \otimes T]\!]\big) \cong \mathrm{Cu}\big(S \otimes [\![T', T]\!] \otimes T', S \otimes T\big)$$

corresponds to the map $\mathrm{id}_S \otimes \mathrm{e}_{T',T}$. Given $a \in S$ and $x \in [\![T', T]\!]$, we denote the image of $a \otimes x$ under this map by $_a x$.

Analogously, we define *the general right unit map* $[\![T', T]\!] \otimes S \to [\![T', T \otimes S]\!]$ as the Cu-morphism that under the identification

$$\mathrm{Cu}\big([\![T', T]\!] \otimes S, [\![T', T \otimes S]\!]\big) \cong \mathrm{Cu}\big([\![T', T]\!] \otimes S \otimes T', T \otimes S\big)$$

corresponds to the map $(\mathrm{e}_{T',T} \otimes \mathrm{id}_S) \circ (\mathrm{id}_{[\![T',T]\!]} \otimes \sigma)$, where σ denotes the flip isomorphism. Given $a \in S$ and $x \in [\![T', T]\!]$, we denote the image of $x \otimes a$ under this map by x_a.

We leave the proof of the following result to the reader.

Proposition 3.25. *Let S, T and T' be* Cu-*semigroups, let a be an element in S, and let x be an element in $[\![T', T]\!]$. Let $\mathrm{i}_S \colon S \to [\![\overline{\mathbb{N}}, S]\!]$ be the isomorphism from Definition 3.9, and let $\mathrm{l}_{T'} \colon \overline{\mathbb{N}} \otimes T' \to T'$ and $\mathrm{r}_{T'} \colon T' \otimes \overline{\mathbb{N}} \to T'$ be the natural* Cu-*isomorphism. Then*

$$_a x = (\mathrm{i}_S(a) \otimes x) \circ \mathrm{l}_{T'}^{-1} = \mathrm{d}_{S,T}(a) \circ x = (\mathrm{id}_S \otimes x) \circ \mathrm{d}_{S,T'}(a).$$

and analogously $x_a = (x \otimes \mathrm{i}_S(a)) \circ \mathrm{r}_{T'}^{-1}$. Further, for the unit map $\mathrm{d}_{S,T} \colon S \to [\![T, S \otimes T]\!]$, we have $\mathrm{d}_{S,T}(a) = {}_a(\mathrm{id}_T)$ for every $a \in S$.

Finally, similar to KK-theory for C^*-algebras, we have a general form of the product that simultaneously generalizes the composition product and the external tensor product; see [Bla98, Section 18.9, p.180f].

Let P, S_1, S_2, T_1 and T_2 be Cu-semigroups. We let

$$\boxtimes_P \colon [\![S_1 \otimes P, T_1]\!] \otimes [\![S_2, P \otimes T_2]\!] \to [\![S_1 \otimes S_2, T_1 \otimes T_2]\!],$$

be the Cu-morphism that under the identification

$$\mathrm{Cu}\big([\![S_1 \otimes P, T_1]\!] \otimes [\![S_2, P \otimes T_2]\!], [\![S_1 \otimes S_2, T_1 \otimes T_2]\!]\big)$$
$$\cong \mathrm{Cu}\big([\![S_1 \otimes P, T_1]\!] \otimes [\![S_2, P \otimes T_2]\!] \otimes S_1 \otimes S_2, T_1 \otimes T_2\big)$$

corresponds to the composition

$$(\mathrm{e}_{S_1 \otimes P, T_1} \otimes \mathrm{id}_{T_2}) \circ (\mathrm{id}_{[\![S_1 \otimes P, T_1]\!]} \otimes \mathrm{e}_{S_2, P \otimes T_2}) \circ (\mathrm{id}_{[\![S_1 \otimes P, T_1]\!]} \otimes \sigma_{[\![S_2, P \otimes T_2]\!], S_1} \otimes \mathrm{id}_{S_2}),$$

where $\sigma_{[\![S_2, P \otimes T_2]\!], S_1}$ denotes the flip isomorphism.

Given $x \in [\![S_1 \otimes P, T_1]\!]$ and $y \in [\![S_2, P \otimes T_2]\!]$, we have

$$x \boxtimes_P y = (x \boxtimes \mathrm{id}_{T_2}) \circ (\mathrm{id}_{S_1} \boxtimes y).$$

Specializing to the case $P = \overline{\mathbb{N}}$, we obtain the external tensor product, after applying the usual isomorphisms $S_1 \otimes \overline{\mathbb{N}} \cong S_1$ and $\overline{\mathbb{N}} \otimes T_2 \cong T_2$.

Specializing to the case $T_2 = S_1 = \overline{\mathbb{N}}$, we obtain the composition product, after applying the natural isomorphisms $\overline{\mathbb{N}} \otimes P \cong P \cong P \otimes \overline{\mathbb{N}}$, $\overline{\mathbb{N}} \otimes S_2 \cong S_2$, and $T_1 \otimes \overline{\mathbb{N}} \cong T_1$.

4. Bivariant Cuntz semigroups of ideals and quotients

A *sub*-Cu-*semigroup* of a Cu-semigroup T is a submonoid $S \subseteq T$ that is a Cu-semigroup for the partial order inherited from T and such that the inclusion $S \to T$ is a Cu-morphism. It is easy to see that S is a sub-Cu-semigroup of T if and only if S is closed under passing to suprema of increasing sequences and if the way-below relation in S and T agree.

Lemma 4.1. *Let S and T be Cu-semigroups, and let $T' \subseteq T$ be a sub-Cu-semigroup. Then the inclusion map $\iota\colon T' \to T$ induces an order-embedding $\iota_*\colon [\![S, T']\!] \to [\![S, T]\!]$.*

Proof. Let $\mathbf{f} = (f_\lambda)_\lambda$ be a path in $\mathrm{Cu}[S, T']$. Then $\tilde{\mathbf{f}} := (\iota \circ f_\lambda)_\lambda$ is a path in $\mathrm{Cu}[S, T]$ and we have $\iota_*([\mathbf{f}]) = [\tilde{\mathbf{f}}]$; see the comments after Remark 5.4 in [APT18b].

To show that ι_* is an order-embedding, let $x, y \in [\![S, T']\!]$ with $\iota_*(x) \leq \iota_*(y)$. Choose paths $\mathbf{f}$ and $\mathbf{g}$ in $\mathrm{Cu}[S, T']$ representing x and y, respectively. We have $(\iota \circ f_\lambda)_\lambda \precsim (\iota \circ g_\lambda)_\lambda$. Thus, for every $\lambda \in I_\mathbb{Q}$, there is $\mu \in I_\mathbb{Q}$ such that $\iota \circ f_\lambda \prec \iota \circ g_\mu$. Using that $T' \subseteq T$ is a sub-Cu-semigroup, for such λ and μ we deduce that $f_\lambda \prec g_\mu$. (We use that for $a', a \in T'$ we have $a' \ll a$ in T' if and only if $\iota(a') \ll \iota(a)$ in T.) It follows that $\mathbf{f} \precsim \mathbf{g}$, and hence $x \leq y$, as desired. $\square$

Recall that an *ideal* of a Cu-semigroup S is a submonoid $J \subseteq S$ that is closed under passing to suprema of increasing sequences and that is downward-hereditary. Every ideal is in particular a sub-Cu-semigroup. (See [APT18, Section 5.1] for an account on ideals and quotients.)

Proposition 4.2. *Let S and T be Cu-semigroups, and let J be an ideal of T. Let $\iota\colon J \to T$ denote the inclusion map. Then the induced Cu-morphism $\iota_*\colon [\![S, J]\!] \to [\![S, T]\!]$ is an order-embedding that identifies $[\![S, J]\!]$ with an ideal of $[\![S, T]\!]$. Moreover, $x \in [\![S, T]\!]$ belongs to $[\![S, J]\!]$ if and only if for some (equivalently, for every) path $(f_\lambda)_\lambda$ representing x, each f_λ takes image in J.*

Proof. By Lemma 4.1, ι_* is an order-embedding. Hence, ι_* identifies $[\![S, J]\!]$ with a submonoid of $[\![S, T]\!]$ that is closed under passing to suprema of increasing sequences.

Let $x \in [\![S, T]\!]$ be represented by a path $\mathbf{f} = (f_\lambda)_\lambda$ in $\mathrm{Cu}[S, T]$. If each f_λ takes values in J, then we can consider $\mathbf{f}$ as a path in $\mathrm{Cu}[S, J]$ whose class is an element $x' \in [\![S, J]\!]$ satisfying $\iota_*(x') = x$. Conversely, assume that x belongs to $[\![S, J]\!]$. Then there is a path $\mathbf{g} = (g_\mu)_\mu$ in $\mathrm{Cu}[S, J]$ with $\iota_*([\mathbf{g}]) = x$. Let $\lambda \in I_\mathbb{Q}$. Since $\mathbf{f} \precsim \mathbf{g}$, we can choose $\mu \in I_\mathbb{Q}$ with $f_\lambda \prec g_\mu$. Since g_μ takes values in J, and since J is downward-hereditary, it follows that f_λ takes values in J, as desired.

A similar argument shows that $[\![S, J]\!]$ is downward-hereditary in $[\![S, T]\!]$. $\square$

4.3. Given S, let us study whether the functor $[\![S, \_]\!]\colon \mathrm{Cu} \to \mathrm{Cu}$ is exact. More precisely, let $J \lhd T$ be an ideal, with inclusion map $\iota\colon J \to T$ and with quotient map $\pi\colon T \to T/J$. This induces the following Cu-morphisms:

$$[\![S, J]\!] \xrightarrow{\iota_*} [\![S, T]\!] \xrightarrow{\pi_*} [\![S, T/J]\!].$$

By Proposition 4.2, ι_* identifies $[\![S, J]\!]$ with an ideal in $[\![S, T]\!]$. Since $\pi \circ \iota$ is the zero map, so is $\pi_* \circ \iota_*$. Thus, π_* vanishes on the ideal $[\![S, J]\!] \lhd [\![S, T]\!]$. It follows that π_* induces a Cu-morphism

$$\widehat{\pi_*}\colon [\![S, T]\!]/[\![S, J]\!] \to [\![S, T/J]\!].$$

Problem 4.4. Study the order-theoretic properties of the Cu-morphism $\widehat{\pi_*}$ from Paragraph 4.3. In particular, when is $\widehat{\pi_*}$ an order-embedding, when is it surjective?

We are currently not aware of any example for S and $J \lhd T$ such that the map $\widehat{\pi_*}\colon [\![S, T]\!]/[\![S, J]\!] \to [\![S, T/J]\!]$ is not an isomorphism.

The following result and its proof are analogous to Proposition 4.2.

Proposition 4.5. *Let S and T be Cu-semigroups, let $J \lhd S$, and let $\pi \colon S \to S/J$ denote the quotient map. Then the induced Cu-morphism $\pi^* \colon [\![S/J, T]\!] \to [\![S, T]\!]$ is an order-embedding that identifies $[\![S/J, T]\!]$ with an ideal in $[\![S, T]\!]$. Moreover, $x \in [\![S, T]\!]$ belongs to $[\![S/J, T]\!]$ if and only if for some (equivalently, for every) path $(f_\lambda)_\lambda$ representing x, each f_λ vanishes on J.*

4.6. By Propositions 4.2 and 4.5, ideals in S and T naturally induce ideals in $[\![S, T]\!]$. More precisely, if $J \lhd S$ and $K \lhd T$, then we can identify $[\![S/J, K]\!]$ with an ideal in $[\![S, T]\!]$. Let $\mathrm{Lat}(P)$ denote the ideal lattice of a Cu-semigroup P. We obtain a natural map

$$\mathrm{Lat}(S)^{\mathrm{op}} \times \mathrm{Lat}(T) \to \mathrm{Lat}([\![S, T]\!]).$$

However, this map need not be injective. For example, consider $S = Z$ and $T = \overline{\mathbb{N}} \oplus Z$ with the ideal $J = 0 \oplus Z$. Note that every generalized Cu-morphism $Z \to \overline{\mathbb{N}} \oplus Z$ necessarily takes values in the ideal $0 \oplus Z$. It follows that in this case $[\![S, J]\!] = [\![S, T]\!]$.

The following example shows that the above map is also not surjective in general. In fact, the example shows that there exists a simple Cu-semigroup S such that $[\![S, S]\!]$ is not simple.

Example 4.7. Let $S := [0, 1] \cup \{\infty\}$, considered with order and addition as a subset of $\overline{\mathbb{P}}$, with the convention that $a + b = \infty$ whenever $a + b > 1$ in $\overline{\mathbb{P}}$. It is easy to check that S is a simple Cu-semigroup.

Given $t \in \{0\} \cup [1, \infty]$, let $\varphi_t \colon S \to S$ be the map given by $\varphi_t(a) := ta$, where ta is given by the usual multiplication in $\overline{\mathbb{P}}$ applying the above convention that an element is ∞ as soon as it is larger than 1. Then φ_t is a generalized Cu-morphism. One can show that every generalized Cu-morphism $S \to S$ is of this form. Hence, $\mathrm{Cu}[S, S]$ is isomorphic to $\{0\} \cup [1, \infty]$, identifying $\prec$ with $\leq$. It follows that

$$[\![S, S]\!] = \tau\big(\mathrm{Cu}[S, S], \prec \big) \cong \tau\big(\{0\} \cup [1, \infty], \leq \big) \cong \{0\} \sqcup [1, \infty] \sqcup (1, \infty],$$

which is a disjoint union of compact elements corresponding to $\{0\} \cup [1, \infty]$ and nonzero soft elements corresponding to $(1, \infty]$. (Similar to the decomposition of Z and R_q.) In particular, $[\![S, S]\!]$ contains a compact infinite element ∞, and a noncompact infinite element ∞'. The set $J := \{x : x \leq \infty'\}$ is an ideal in $[\![S, S]\!]$. We have $\infty \notin J$, which shows that $[\![S, S]\!]$ is not simple.

Problem 4.8. Characterize when $[\![S, T]\!]$ is simple. In particular, given simple Cu-semigroups S and T, give necessary and sufficient criteria for $[\![S, T]\!]$ to be simple.

5. Cu-SEMIRINGS AND Cu-SEMIMODULES

A (unital) Cu-*semiring* is a Cu-semigroup R together with a Cu-bimorphism $R \times R \to R$, denoted by $(r_1, r_2) \mapsto r_1 r_2$, and a distinguished element $1 \in R$, called the unit of R, such that $r_1(r_2 r_3) = (r_1 r_2) r_3$ and $r1 = r = 1r$ for all $r, r_1, r_2, r_3 \in R$. This concept was introduced and studied in [APT18, Chapter 7], where it is further assumed that the product is commutative. We will not make this assumption here. We let $\mu_R \colon R \otimes R \to R$ denote the Cu-morphism induced by multiplication in R.

The following result follows from the general properties of the composition product for the internal-hom (see [Kel05, Section 1.6]).

Proposition 5.1. *Let S be a Cu-semigroup. Then $[\![S, S]\!]$ is a Cu-semiring with product given by the composition product $\circ \colon [\![S, S]\!] \otimes [\![S, S]\!] \to [\![S, S]\!]$, and with unit element given by the identity map $\mathrm{id}_S \in [\![S, S]\!]$.*

Remark 5.2. Let S be a Cu-semigroup. The identity map $\mathrm{id}_S \colon S \to S$ is a Cu-morphism. Therefore, the unit of the Cu-semiring $[\![S, S]\!]$ is compact.

In Example 5.8, we will see that $[\![S, S]\!]$ is noncommutative in general.

Given a Cu-semiring R, a *left* Cu-*semimodule* over R is a Cu-semigroup S together with a Cu-bimorphim $R \times S \to S$, denoted by $(r, a) \mapsto ra$, such that for all $r_1, r_2 \in R$ and $a \in S$, we have $(r_1 r_2)a = r_1(r_2 a)$ and $1a = a$. We also say that S has a *left action* of R if S is a left Cu-semimodule over R. Right Cu-semimodules are defined analogously. If R_1 and R_2 are Cu-semirings, we say that a Cu-semigroup S is a (R_1, R_2)-Cu-*semibimodule* if it has a left R_1-action and a right R_2-action that satisfy $r_1(ar_2) = (r_1 a)r_2$ for all $r_1 \in R_1$, $r_2 \in R_2$ and $a \in S$.

We refer the reader to [APT18, Chapter 7] for a discussion on commutative Cu-semirings and their Cu-semimodules.

Proposition 5.3. *Let S be a* Cu-*semigroup. Then* $\mathrm{e}_{S,S} \colon [\![S, S]\!] \otimes S \to S$ *defines a left action of* $[\![S, S]\!]$ *on S.*

Proof. It follows directly from Proposition 3.20 that the action of $[\![S, S]\!]$ on S is associative and that id_S acts as a unit. $\qquad\square$

5.4. Let R be a Cu-semiring, let S be a Cu-semigroup, and let T be a Cu-semigroup with a left R-action $\alpha \colon R \otimes T \to T$. Consider the general left unit map $R \otimes [\![S, T]\!] \to [\![S, R \otimes T]\!]$ from Definition 3.24. Postcomposing with $\alpha_* \colon [\![S, R \otimes T]\!] \to [\![S, T]\!]$ we obtain a Cu-morphism that we denote by α_S:

$$\alpha_S \colon R \otimes [\![S, T]\!] \to [\![S, R \otimes T]\!] \xrightarrow{\alpha_*} [\![S, T]\!].$$

Let $r \in R$ and $x \in [\![S, T]\!]$. We denote $\alpha_S(r \otimes x)$ by rx. Choose a path $\mathbf{f} = (f_\lambda)_\lambda$ in $\mathrm{Cu}[S, T]$ representing x, and pick a path $(r_\lambda)_\lambda$ in R with endpoint r. For each λ, let $r_\lambda f_\lambda \colon S \to T$ be given by $s \mapsto r_\lambda f_\lambda(s)$. Then $(r_\lambda f_\lambda)_\lambda$ is a path in $\mathrm{Cu}[S, T]$ and

$$rx = [(r_\lambda f_\lambda)_\lambda].$$

Proposition 5.5. *Let R be a* Cu-*semiring with compact unit, let S be a* Cu-*semigroup, and let T be a* Cu-*semigroup with a left R-action $\alpha \colon R \otimes T \to T$. Then the map $\alpha_S \colon R \otimes [\![S, T]\!] \to [\![S, T]\!]$ from Paragraph 5.4 defines a left R-action on $[\![S, T]\!]$.*

Proof. Let $r, r' \in R$ and $x \in [\![S, T]\!]$. Choose a path $\mathbf{f} = (f_\lambda)_\lambda$ in $\mathrm{Cu}[S, T]$ representing x. Choose paths $(r_\lambda)_\lambda$ and $(r'_\lambda)_\lambda$ in R with endpoints r and r', respectively. Then $(r_\lambda r'_\lambda)_\lambda$ is a path in R with endpoint rr'. Using the description of the R-action on $[\![S, T]\!]$ from the end of Paragraph 5.4, we deduce that

$$(rr')x = [((r_\lambda r'_\lambda)f_\lambda)_\lambda] = [(r_\lambda(r'_\lambda f_\lambda))_\lambda] = r[(r'_\lambda f_\lambda)_\lambda] = r(r'x).$$

Let 1 denote the unit element of R. For every $f \in \mathrm{Cu}[S, T]$, we have $1f = f$. Since 1 is compact, the constant function with value 1 is a path in R with endpoint 1. It follows that

$$1x = [(1f_\lambda)_\lambda] = [(f_\lambda)_\lambda] = x. \qquad\square$$

Proposition 5.6. *Let S and T be* Cu-*semigroups. Then the composition product $\circ \colon [\![T, T]\!] \otimes [\![S, T]\!] \to [\![S, T]\!]$ defines a left action of the* Cu-*semiring $[\![T, T]\!]$ on $[\![S, T]\!]$. Analogously, we obtain a right action of $[\![S, S]\!]$ on $[\![S, T]\!]$. These actions are compatible and thus $[\![S, T]\!]$ is a $([\![T, T]\!], [\![S, S]\!])$-Cu-semibimodule.*

Proof. This follows directly from the associativity of the composition product; see Proposition 3.17. $\qquad\square$

Remark 5.7. Let S and T be Cu-semigroups. By Proposition 5.3, the evaluation map $\mathrm{e}_{T,T} \colon [\![T, T]\!] \otimes T \to T$ from Definition 3.5 defines a left action of $[\![T, T]\!]$ on T. By Proposition 5.5, this induces a left action of $[\![T, T]\!]$ on $[\![S, T]\!]$. This action agrees with that from Proposition 5.6.

Example 5.8. Given $k \in \mathbb{N}$, we let $\overline{\mathbb{N}}^k$ denote the Cuntz semigroup of the C^*-algebra $\mathbb{C}^k$. We think of $\mathbf{v} \in \overline{\mathbb{N}}^k$ as a tuple $(v_1, \ldots, v_k)^T$ with k entries in $\overline{\mathbb{N}}$. We let $\mathbf{e^{(1)}}, \ldots, \mathbf{e^{(k)}}$ denote the 'standard basis vectors' of $\overline{\mathbb{N}}^k$, such that $\mathbf{v} = \sum_{i=1}^k v_i \mathbf{e^{(i)}}$.

Let $k, l \in \mathbb{N}$. We claim that $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!]$ can be identified with $M_{l,k}(\overline{\mathbb{N}})$, the $l \times k$-matrices with entries in $\overline{\mathbb{N}}$, with order and addition defined entrywise. Thus, as a Cu-semigroup, $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!]$ is isomorphic to $\overline{\mathbb{N}}^{kl}$. However, the presentation as matrices allows to expatiate the composition product.

First, let $\varphi \colon \overline{\mathbb{N}}^k \to \overline{\mathbb{N}}^l$ be a generalized Cu-morphism. For each $j \in \{1, \ldots, k\}$, we consider the vector $\varphi(\mathbf{e^{(j)}})$ in $\overline{\mathbb{N}}^l$ and we let $x_{1,j}, \ldots, x_{l,j}$ denote its coefficients. This defines a matrix $\mathbf{x} = (x_{i,j})_{i,j}$ with $l \times k$ entries in $\overline{\mathbb{N}}$. It is then readily verified that the coefficients of $\varphi(\mathbf{v})$ are obtained by multiplication of the matrix $\mathbf{x}$ with the vector of coefficients of $\mathbf{v}$. We identify φ with the associated matrix $\mathbf{x}$ in $M_{l,k}(\overline{\mathbb{N}})$.

Let $\varphi, \psi \colon \overline{\mathbb{N}}^k \to \overline{\mathbb{N}}^l$ be generalized Cu-morphisms with associated matrices $\mathbf{x}$ and $\mathbf{y}$ in $M_{l,k}(\overline{\mathbb{N}})$. It is straightforward to check that $\varphi \prec \psi$ if and only if $x_{i,j}$ is finite and $x_{i,j} \leq y_{i,j}$ for each i, j, and thus the claim follows.

Given $k, l, m \in \mathbb{N}$, consider the composition product

$$[\![\overline{\mathbb{N}}^l, \overline{\mathbb{N}}^m]\!] \otimes [\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!] \to [\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^m]\!].$$

After identifying $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l]\!]$ with $M_{l,k}(\overline{\mathbb{N}})$, identifying $[\![\overline{\mathbb{N}}^l, \overline{\mathbb{N}}^m]\!]$ with $M_{m,l}(\overline{\mathbb{N}})$, and identifying $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^m]\!]$ with $M_{m,k}(\overline{\mathbb{N}})$, the composition product is given as a map

$$M_{m,l}(\overline{\mathbb{N}}) \otimes M_{l,k}(\overline{\mathbb{N}}) \to M_{m,k}(\overline{\mathbb{N}}).$$

It is straightforward to check that this map is induced by matrix multiplication. In particular, the Cu-semiring $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^k]\!]$ can be identified with the matrix ring $M_{k,k}(\overline{\mathbb{N}})$. Thus, for $k \geq 2$, the Cu-semiring $[\![\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^k]\!]$ is not commutative.

Given a Cu-semiring R, recall that $\mu_R \colon R \otimes R \to R$ denotes the Cu-morphism induced by multiplication.

Definition 5.9. Given a Cu-semiring R, we let $\pi_R \colon R \to [\![R, R]\!]$ be the Cu-morphism that corresponds to μ_R under the identification

$$\mathrm{Cu}\big(R, [\![R, R]\!]\big) \cong \mathrm{Cu}\big(R \otimes R, R\big).$$

Remark 5.10. The Cu-morphism π_R can be regarded as a kind of left regular representation of R.

Lemma 5.11. *We have $\pi_R = (\mu_R)_* \circ \mathrm{d}_{R,R}$ and $\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R) = \mu_R$.*

Proof. The first equality follows from Proposition 3.22. It is straightforward to show that $\mathrm{e}_{R,R} \circ ((\mu_R)_* \otimes \mathrm{id}_R) = \mu_R \circ \mathrm{e}_{R,R \otimes R}$. Further, we have $\mathrm{e}_{R,R \otimes R} \circ (\mathrm{d}_{R,R} \otimes \mathrm{id}_R) = \mathrm{id}_{R \otimes R}$ by Corollary 3.23. Using these equations, we deduce that

$$\begin{aligned}
\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R) &= \mathrm{e}_{R,R} \circ (((\mu_R)_* \circ \mathrm{d}_{R,R}) \otimes \mathrm{id}_R) \\
&= \mathrm{e}_{R,R} \circ ((\mu_R)_* \otimes \mathrm{id}_R) \circ (\mathrm{d}_{R,R} \otimes \mathrm{id}_R) \\
&= \mu_R \circ \mathrm{e}_{R,R \otimes R} \circ (\mathrm{d}_{R,R} \otimes \mathrm{id}_R) = \mu_R. \qquad \square
\end{aligned}$$

Theorem 5.12. *Let R be a Cu-semiring. Then $\pi_R \colon R \to [\![R, R]\!]$ is multiplicative. If the unit element of R is compact, then π_R is unital.*

Proof. Let $M \colon [\![R, R]\!] \otimes [\![R, R]\!] \to [\![R, R]\!]$ denote the composition map. We need to show that $M \circ (\pi_R \otimes \pi_R) = \pi_R \circ \mu_R$.

Given $r, s \in R$, choose paths $\mathbf{r} = (r_\lambda)_\lambda$ and $\mathbf{s} = (s_\lambda)_\lambda$ in $(R, \ll)$ with endpoints r and s, respectively. For each λ, let $f_\lambda \colon R \to R$ and $g_\lambda \colon R \to R$ be the generalized

Cu-morphism given by left multiplication with r_λ and s_λ, respectively. By Proposition 3.2, we have $\mathrm{d}_{R,R}(r) = [(r_\lambda \otimes \_)_\lambda]$, where $r_\lambda \otimes \_ \colon R \to R \otimes R$ is the map sending $t \in R$ to $r_\lambda \otimes t$. We also have $\mu_R \circ (r_\lambda \otimes \_) = f_\lambda$. Since $\pi_R = (\mu_R)_* \circ \mathrm{d}_{R,R}$ by Lemma 5.11, it follows that $\pi_R(r) = [(f_\lambda)_\lambda]$. Likewise, we deduce $\pi_R(s) = [(g_\lambda)_\lambda]$. By Proposition 3.18, we obtain $M(\pi_R(r) \otimes \pi_R(s)) = [(f_\lambda \circ g_\lambda)_\lambda]$.

As the product in R is associative, the composition $f_\lambda \circ g_\lambda$ is the generalized Cu-morphism h_λ defined by left multiplication with $r_\lambda s_\lambda$. Notice that $(r_\lambda s_\lambda)_\lambda$ is a path in $(R, \ll)$ with endpoint rs. Therefore, $\pi_R(rs) = [(h_\lambda)_\lambda]$. Altogether, we get the desired equality

$$M(\pi_R(r) \otimes \pi_R(s)) = [(f_\lambda \circ g_\lambda)_\lambda] = [(h_\lambda)_\lambda] = \pi_R(\mu_R(r \otimes s)).$$

To show the second statement, let us assume that the unit 1_R of R is compact. Then the constant function with value 1_R is a path in $(R, \ll)$ with endpoint 1_R. It follows easily as in the first part of the proof that $\pi_R(1_R) = [(\mathrm{id}_R)_\lambda] = \mathrm{id}_R$. $\qquad\square$

Definition 5.13. Let R be a Cu-semiring with unit 1_R. Then $\varepsilon_R \colon [\![R, R]\!] \to R$ is the generalized Cu-morphism given by

$$\varepsilon_R([\mathbf{f}]) = \sup_\lambda f_\lambda(1_R),$$

for a path $\mathbf{f} = (f_\lambda)_\lambda$ in $\mathrm{Cu}[R, R]$.

Remark 5.14. Let $\sigma_{R,R} \colon [\![R, R]\!] \to \mathrm{Cu}[R, R]$ denote the endpoint map as in Definition 2.6, Then $\varepsilon_R(x) = \sigma_{R,R}(x)(1_R)$ for every $x \in [\![R, R]\!]$.

Lemma 5.15. *We have* $\varepsilon_R \circ \pi_R = \mathrm{id}_R$.

Proof. Given $r \in R$, choose a path $(r_\lambda)_\lambda$ in $(R, \ll)$ with endpoint r, and for each λ let $f_\lambda \colon R \to R$ be given by left multiplication with r_λ. As in the proof of Theorem 5.12, we obtain $\pi_R(r) = [(f_\lambda)_\lambda]$, whence

$$\varepsilon_R(\pi_R(r)) = \varepsilon_R([(f_\lambda)_\lambda]) = \sup_\lambda f_\lambda(1_R) = \sup_\lambda(r_\lambda 1_R) = r. \qquad\square$$

Proposition 5.16. *Let R be a Cu-semiring. Then $\pi_R \colon R \to [\![R, R]\!]$ is a multiplicative order-embedding. Thus, in a natural way, R is a sub-semiring of $[\![R, R]\!]$. If the unit of R is compact, then R is even a unital sub-semiring of $[\![R, R]\!]$.*

Proof. By Lemma 5.15, we have $\varepsilon_R \circ \pi_R = \mathrm{id}_R$, which implies that π_R is an order-embedding. By Theorem 5.12, π_R is a (unital) multiplicative Cu-morphism. $\qquad\square$

An element a in a Cu-semigroup S is *soft* if for every $a' \in S$ with $a' \ll a$ there exists $k \in \mathbb{N}$ with $(k+1)a' \le ka$; see [APT18, Definition 5.3.1]. The following result will be used below.

Lemma 5.17. *Let S and T be Cu-semigroups, let $\varphi \colon S \to T$ be a generalized Cu-morphism, and let $a \in S$ be soft. Then $\varphi(a)$ is soft.*

Proof. To verify that $\varphi(a)$ is soft, let $x \in T$ satisfy $x \ll \varphi(a)$. Using that φ preserves suprema of increasing sequences, we can choose $a' \in S$ with $a' \ll a$ and $x \le \varphi(a')$. (Indeed, applying (O2) in S, choose a $\ll$-increasing sequence $(a_n)_n$ in S with supremum a. Then $\varphi(a) = \sup_n \varphi(a_n)$, whence there is n with $x \le \varphi(a_n)$.) Since a is soft, we can choose $k \in \mathbb{N}$ such that $(k+1)a' \le ka$. Then

$$(k+1)x \le (k+1)\varphi(a') = \varphi((k+1)a') \le \varphi(ka) = k\varphi(a). \qquad\square$$

Let $\overline{\mathbb{P}} = [0, \infty]$, with natural order and addition. Recall that $\overline{\mathbb{P}}$ is isomorphic to the Cuntz semigroup of the Jacelon-Razak algebra (see [Jac13] and [Rob13]). The usual multiplication of real numbers extends to $\overline{\mathbb{P}}$. This gives $\overline{\mathbb{P}}$ the structure of a commutative Cu-semiring.

Example 5.18. Let $M_1 = [0, \infty) \sqcup (0, \infty]$, a disjoint union of compact elements $[0, \infty)$ and nonzero soft elements $(0, \infty]$. Recall that M_1 denotes the Cuntz semigroup of a II_1-factor; see [APT18b, Example 4.14] and [APT18b, Proposition 4.15]. We identify $\overline{\mathbb{P}} = [0, \infty]$ with the sub-Cu-semigroup of soft elements in M_1, and we define the Cu-morphism $\varrho \colon M_1 \to \overline{\mathbb{P}} \subseteq M_1$ by fixing all soft elements and by sending a compact to the soft element of the same value.

We define a product on M_1 as follows: We equip the compact part $[0, \infty)$ with the usual multiplication of real numbers, and similarly for the product in $(0, \infty]$. The product of any element with 0 is 0. Given a nonzero compact element a and a nonzero soft element b, their product is defined as the soft element $ab := \varrho(a)b$.

This gives M_1 the structure of a commutative Cu-semiring. Moreover, we may identify $\overline{\mathbb{P}}$ with the (nonunital) sub-Cu-semiring of soft elements in M_1. The map $\varrho \colon M_1 \to \overline{\mathbb{P}}$ is multiplicative. One can show that the map $\pi_{M_1} \colon M_1 \to [\![M_1, M_1]\!]$ is an isomorphism.

Example 5.19. We have $[\![\overline{\mathbb{P}}, \overline{\mathbb{P}}]\!] \cong M_1$. The map $\pi_{\overline{\mathbb{P}}} \colon \overline{\mathbb{P}} \to [\![\overline{\mathbb{P}}, \overline{\mathbb{P}}]\!]$ embeds $\overline{\mathbb{P}}$ as the sub-Cu-semiring of soft elements in M_1. In particular, $\pi_{\overline{\mathbb{P}}}$ is not unital.

Proof. We have $[\![\overline{\mathbb{P}}, \overline{\mathbb{P}}]\!] \cong M_1$ by [APT18b, Proposition 5.13]. By Proposition 5.16, $\pi_{\overline{\mathbb{P}}}$ is a multiplicative order-embedding. Note that every element of $\overline{\mathbb{P}}$ is soft. By Lemma 5.17, a generalized Cu-morphism maps soft elements to soft elements. Thus, the image of $\pi_{\overline{\mathbb{P}}}$ is contained in the soft elements of M_1. It easily follows that $\pi_{\overline{\mathbb{P}}}$ identifies $\overline{\mathbb{P}}$ with the soft elements in M_1. Since the unit of M_1 is compact, it also follows that $\pi_{\overline{\mathbb{P}}}$ is not unital. $\qquad\square$

We finally turn our attention to solid semirings. Recall from [APT18, Definition 7.1.5] that a Cu-semiring R is said to be *solid* if $\mu_R \colon R \otimes R \to R$ is an isomorphism. In [APT18], all Cu-semirings were required to be commutative, and thus a solid Cu-semiring was assumed to be commutative. Next, we show that this assumption is not necessary since a Cu-semiring is automatically commutative as soon as μ_R is injective.

Lemma 5.20. *Let R be a Cu-semiring such that $\mu_R \colon R \otimes R \to R$ is injective. Then R is commutative and μ_R is an isomorphism (and consequently R is solid.)*

Proof. To show that R is commutative, let $a, b \in R$. We have

$$\mu_R(1 \otimes a) = a = \mu_R(a \otimes 1),$$

and thus $1 \otimes a = a \otimes 1$ in $R \otimes R$. Consider the shuffle Cu-morphism $\alpha \colon R \otimes R \otimes R \to R \otimes R \otimes R$ that satisfies $\alpha(x \otimes y \otimes z) = y \otimes x \otimes z$ for every $x, y, z \in R$. Then

$$1 \otimes b \otimes a = \alpha(b \otimes 1 \otimes a) = \alpha(b \otimes a \otimes 1) = a \otimes b \otimes 1$$

in $R \otimes R \otimes R$. By the associativity of the product in R, we get $ba = ab$, as desired.

Thus, R is commutative and $1 \otimes a = a \otimes 1$ in $R \otimes R$, for every $a \in R$. Using [APT18, Proposition 7.1.6], this implies that R is solid. $\qquad\square$

Let R be a solid Cu-semiring, and let S be a Cu-semigroup. It was shown in [APT18, Corollary 7.1.8] that any two R-actions on S agree. (Since R is commutative, we need not distinguish between left and right R-actions.) Thus, S either has a (unique) R-action, or it does not admit any R-action, which means that having an R-action is a property rather than an additional structure for S, which justifies the following definition.

Definition 5.21. Let R be a solid Cu-semiring, and let S be a Cu-semigroup. We say that S is *R-stable* if S has an R-action.

Remark 5.22. In [APT18], we said that S 'has R-multiplication' if it has an R-action. Given a solid ring R, it was shown [APT18, Theorem 7.1.12] that S is R-stable if and only if $S \cong R \otimes S$.

Recall that a C^*-algebra A is said to be $\mathcal{Z}$-*stable* if $A \cong \mathcal{Z} \otimes A$, and similarly one defines being UHF-stable and $\mathcal{O}_\infty$-stable. Thus, the terminology of being 'R-stable' for Cu-semigroups is analogous to the terminology used for C^*-algebras.

Theorem 5.23. *Given a* Cu-*semiring R, consider the following statements:*

   (1) R is solid, that is, $\mu\colon R \otimes R \to R$ is an isomorphism.
   (2) The map $\mathrm{e}_{R,R}\colon [\![R,R]\!] \otimes R \to R$ is an isomorphism.
   (3) The map $\pi_R \otimes \mathrm{id}_R\colon R \otimes R \to [\![R,R]\!] \otimes R$ is an isomorphism.
   (4) The map $\pi_R\colon R \to [\![R,R]\!]$ is an isomorphism.
   (5) The map $\varepsilon_R\colon [\![R,R]\!] \to R$ is an isomorphism.

Then the following implications hold:

$$(1) \Leftarrow (2) \Rightarrow (3) \Leftarrow (4) \Leftrightarrow (5).$$

Further, if R satisfies (1) and (3), then it satisfies (2). The Cu-*semiring $\overline{\mathbb{P}}$ satisfies (1),(2) and (3), but not (4); see Example 5.19. The* Cu-*semiring M_1 satisfies (3) and (4) but neither (1) nor (2); see Example 5.18.*

Proof. By Lemma 5.15, we have $\varepsilon_R \circ \pi_R = \mathrm{id}_R$. It follows that ε_R is an isomorphism if and only if π_R is, which shows the equivalence of (4) and (5). It is obvious that (4) implies (3). To show that (2) implies (1), assume that $\mathrm{e}_{R,R}$ is an isomorphism. We have

$$(\varepsilon_R \otimes \mathrm{id}_R) \circ (\pi_R \otimes \mathrm{id}_R) = \mathrm{id}_R \otimes \mathrm{id}_R,$$

which shows that $\pi_R \otimes \mathrm{id}_R$ is an order-embedding. Hence, $\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R)$ is an order-embedding. By Lemma 5.11, we have $\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R) = \mu_R$, whence μ_R is an order-embedding. By Lemma 5.20, this implies that R is solid.

Using again that $\mathrm{e}_{R,R} \circ (\pi_R \otimes \mathrm{id}_R) = \mu_R$, if any two of the three maps $\mathrm{e}_{R,R}$, $\pi_R \otimes \mathrm{id}_R$ and μ_R are isomorphisms, then so is the third. This shows that (2) implies (3), and that the combination of (1) and (3) implies (2). $\qquad\square$

Question 5.24. Given a solid Cu-semiring R, is the evaluation map $\mathrm{e}_{R,R}\colon [\![R,R]\!] \otimes R \to R$ an isomorphism?

Remark 5.25. Let R be a solid Cu-semiring. The answer to Question 5.24 is 'yes' in the following cases:

   (1) If the unit of R is compact; see Remark 5.30 below.
   (2) If R satisfies (O5) and (O6). This follows from the classification of solid Cu-semirings with (O5) obtained in [APT18, Theorem 8.3.13] which shows that each such Cu-semiring is either isomorphic to $\overline{\mathbb{P}}$ or has a compact unit. In either case, Question 5.24 has a positive answer.

In particular, a Cu-semiring R with compact unit is solid if and only if the evaluation map $\mathrm{e}_{R,R}\colon [\![R,R]\!] \otimes R \to R$ is an isomorphism.

Theorem 5.26. *Let R be a solid* Cu-*semiring with compact unit, and let S and T be* Cu-*semigroups. Assume that T is R-stable. Then $[\![S,T]\!]$ is R-stable, and hence $[\![S,T]\!] \cong R \otimes [\![S,T]\!]$.*

Proof. Since the unit of R is compact, it follows from Proposition 5.5 that $[\![S,T]\!]$ has a left R-action. Since R is solid, this implies that $[\![S,T]\!]$ is R-stable. $\qquad\square$

Lemma 5.27. *Let R be a solid* Cu-*semiring, let S and T be* Cu-*semigroups, and let $f,g\colon R \otimes S \to T$ be a generalized* Cu-*morphisms. Assume that T is R-stable. Then $f \leq g$ if and only if $f(1 \otimes a) \leq g(1 \otimes a)$ for all $a \in S$.*

If the unit of R is compact, then $f \prec g$ if and only if $f(1 \otimes a') \ll g(1 \otimes a)$ for all $a', a \in S$ with $a' \ll a$.

Proof. The forward implications are obvious. To show the converse of the first statement, assume that $f(1 \otimes a) \leq g(1 \otimes a)$ for all $a \in S$. To verify $f \leq g$, it is enough to show that $f(r \otimes a) \leq g(r \otimes a)$ for all $r \in R$ and $a \in S$. Note that $R \otimes S$ and T are R-stable. Since R is solid, every generalized Cu-morphism between R-stable Cu-semigroups is automatically R-linear; see [APT18, Proposition 7.1.6]. Thus, given $r \in R$ and $a \in S$, we obtain

$$f(r \otimes a) = f(r(1 \otimes a)) = rf(1 \otimes a) \leq rg(1 \otimes a) = g(r \otimes a).$$

To show the converse of the second statement, assume that $f(1 \otimes a') \ll g(1 \otimes a)$ for all $a', a \in S$ with $a' \ll a$. To verify $f \prec g$, it is enough to show that $f(r' \otimes a') \ll g(r \otimes a)$ for all $r', r \in R$ and $a', a \in S$ with $r' \ll r$ and $a' \ll a$. Given such r', r, a' and a, we use at the second step that multiplication in R preserves the joint way-below relation, to deduce

$$f(r' \otimes a') = r'f(1 \otimes a') \ll rg(1 \otimes a) = g(r \otimes a). \qquad \square$$

Proposition 5.28. *Let R be a solid* Cu-*semiring with compact unit, and let S and T be* Cu-*semigroups. Assume that T is R-stable. Let $\alpha \colon S \to R \otimes S$ be the* Cu-*morphism given by $\alpha(a) = 1 \otimes a$, for $a \in S$. Then the induced map $\alpha^* \colon [\![R \otimes S, T]\!] \to [\![S, T]\!]$ is an isomorphism.*

Proof. Consider $\alpha_{\mathcal{Q}}^* \colon \mathrm{Cu}[R \otimes S, T] \to \mathrm{Cu}[S, T]$ given by sending a generalized Cu-morphism $f \colon R \otimes S \to T$ to the generalized Cu-morphism $\alpha_{\mathcal{Q}}^*(f)$ given by

$$\alpha_{\mathcal{Q}}^*(f)(a) = f(1 \otimes a),$$

for $a \in S$. It follows from Lemma 5.27 that $\alpha_{\mathcal{Q}}^*$ is an isomorphism of $\mathcal{Q}$-semigroups. Since α^* is obtained by applying the functor τ to α_Q^* (see the comments after [APT18b, Remark 5.4]), it follows that α^* is an isomorphism, as desired. $\square$

Corollary 5.29. *Let R be a solid* Cu-*semiring with compact unit, and let T be an R-stable* Cu-*semigroup. Then there is a natural isomorphism $[\![R, T]\!] \cong T$.*

Proof. Applying Proposition 5.28 for $S := \overline{\mathbb{N}}$, we obtain $[\![R, T]\!] \cong [\![\overline{\mathbb{N}}, T]\!]$. By Proposition 3.10, we have a natural isomorphism $[\![\overline{\mathbb{N}}, T]\!] \cong T$. $\square$

Remark 5.30. Let R be a solid Cu-semiring with compact unit. Since R is R-stable itself, it follows from Corollary 5.29 that $[\![R, R]\!] \cong R$. It follows that the evaluation map $\mathrm{e}_{R,R} \colon [\![R, R]\!] \otimes R \to R$ is an isomorphism.

For the solid Cu-semiring $\overline{\mathbb{P}}$, we have seen in [APT18b, Proposition 5.13] that $[\![\overline{\mathbb{P}}, \overline{\mathbb{P}}]\!] \cong M_1 \not\cong \overline{\mathbb{P}}$. This shows that Proposition 5.28 and Corollary 5.29 cannot be generalized to solid Cu-semirings without compact unit.

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