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Canard limit cycles for piecewise linear Liénard systems with three zones*

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This paper deals with planar piecewise linear slow fast Liénard differential systems with three zones separated by two vertical lines. We show the existence and uniqueness of canard limit cycles for systems with a unique singular point located in the middle zone.

Keywords: Limit cycle; piecewise linear differential system; slow–fast system; Liénard system

1. Introduction and statement of main results

Piecewise linear Liénard differential systems are very important within the realm of nonlinear dynamical systems. On one hand there are many models coming from the real world which can be analyzed by piecewise linear differential systems. It seems that piecewise linear differential systems can have almost all the dynamical behaviours which occur in the smooth ones, see for instance [Bernardo *et al.*, 2007; Chen & Tang , 2017]. On the other hand it is showed that most of the piecewise linear differential systems can be written in the Liénard form after some linear changes of variables, see [Freire, Ponce & Torres , 2012] for example.

In this paper, we investigate the piecewise linear slow fast Liénard differential systems

$$\begin{aligned} \frac{dx}{dt} &= y - f(x), \\ \frac{dy}{dt} &= \varepsilon(a - x), \end{aligned} \tag{1}$$

*For the title, try not to use more than three lines. Typeset the title in 15 pt Times Roman, uppercase and boldface.
†Typeset names in 11 pt Times Roman. Use the footnote to indicate the present or permanent address of the author.
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where $\varepsilon > 0$ is sufficiently small and

$$f(x) = \begin{cases} k_1(x-1) + k_2 & \text{if } x \geqslant 1, \\ k_2 x & \text{if } -1 \leqslant x \leqslant 1, \\ k_3(x+1) - k_2 & \text{if } x \leqslant -1, \end{cases} \tag{2}$$

with $k_i \neq 0, i = 1, 2, 3$. Without loss of generality we assume that $k_2 < 0$. Otherwise we can do the change $X = x; Y = -y; T = -t$, and then obtain $k_2 < 0$.

Doing the time rescaling $\tau = \varepsilon t$, systems (1) become

$$\begin{aligned} \varepsilon \frac{dx}{d\tau} &= y - f(x), \\ \frac{dy}{d\tau} &= a - x. \end{aligned} \tag{3}$$

Systems (1) are the *fast systems*, and (3) are the *slow systems*. Let $\varepsilon = 0$ in systems (1) and (3), then we obtain the *layer problem*

$$\begin{aligned} \frac{dx}{dt} &= y - f(x), \\ \frac{dy}{dt} &= 0, \end{aligned} \tag{4}$$

and the *reduced problem*

$$\begin{aligned} 0 &= y - f(x), \\ \frac{dy}{d\tau} &= a - x, \end{aligned} \tag{5}$$

respectively.

We define the *critical manifold* of the slow-fast differential systems (1)

$$C = \{y = f(x)\}. \tag{6}$$

The different branches of C will be denoted

$$\begin{aligned} C_R &= \{y = k_1(x-1) + k_2, x > 1\}, \\ C_M &= \{y = k_2 x, -1 \leqslant x \leqslant 1\}, \\ C_L &= \{y = k_3(x+1) - k_2, x < -1\}. \end{aligned} \tag{7}$$

Thus $C = C_R \cup C_M \cup C_L$. It is obvious that C_R (resp. C_M, C_L) is an attracting critical manifold when k_1 (resp. k_2, k_3) positive, and C_R (resp. C_M, C_L) is a repelling critical manifold when k_1 (resp. k_2, k_3) negative.

We denote

$$\Delta_i = k_i^2 - 4\varepsilon, \quad i = 1, 2, 3. \tag{8}$$

If $k_1 > 0, k_2 < 0$ and $k_3 > 0$ satisfying $\Delta_1 > 0$ and $\Delta_3 > 0$, we say that the critical manifold C has *S shape*. If either $k_1 > 0, k_2 < 0$ and $k_3 < 0$ or $k_1 < 0, k_2 < 0$ and $k_3 > 0$ satisfying $\Delta_1 > 0$ and $\Delta_3 > 0$, we say that the critical manifold C has *U shape*, see Fig.1.

In this spirit we call a limit cycle of systems (1) satisfying (2) a *canard limit cycle* if it contains both attracting and repelling critical manifold. We obtain the following results on existence and uniqueness of canard limit cycles of systems (1) satisfying (2) when the critical manifold C has *S shape* and *U shape*.

Theorem 1. *Consider the continuous piecewise linear slow-fast Liénard differential systems* (1) *satisfying* (2). *Assume that* $-1 < a < 1$, $k_2 < 0$, $\Delta_1 > 0$, $\Delta_3 > 0$, *and* $\varepsilon > 0$ *sufficiently small. Then the following statements hold.*

(I) If $k_1 < 0$ *and* $k_3 < 0$, *then there is no canard limit cycle.*

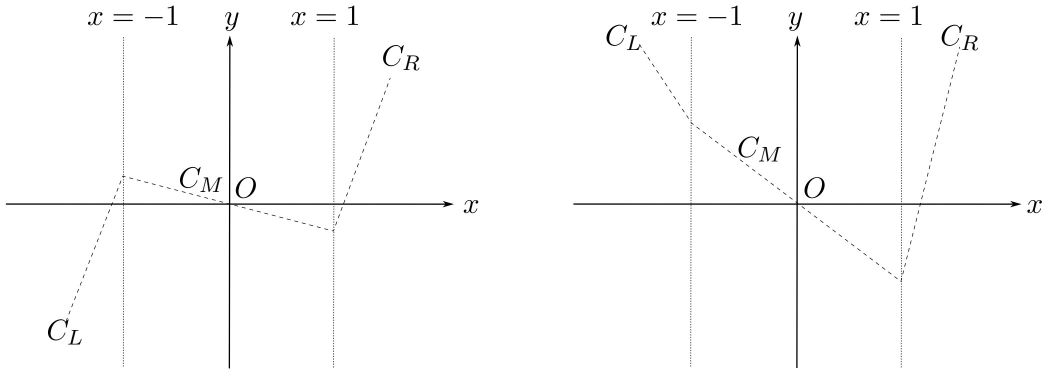

Fig. 1. The critical manifold of system (1) satisfying (2). Left: S shape. Right: U shape.

(II) For the case $k_1 > 0$ *and* $k_3 > 0$*, the singular point* (a, k_2a) *is surrounded by a canard limit cycle, which is unique and stable. Furthermore,*

(II.1) If $\Delta_2 \geqslant 0$*, then the canard limit cycle always has points in the three zones, which is a relaxation oscillation.*

(II.2) If $\Delta_2 < 0$ *and* $1 - a > 0$ *sufficiently small, then the canard limit cycle is contained only in the middle and right zones.*

(II.3) If $\Delta_2 < 0$ *and* $1 + a > 0$ *sufficiently small, then the canard limit cycle is contained only in the middle and left zones.*

(III) For the case $k_1 > 0$ *and* $k_3 < 0$*.*

(III.1) If $\Delta_2 \geqslant 0$*, then there is no canard limit cycle.*

(III.2) If $\Delta_2 < 0$*, then in the middle and right zones the singular point* (a, k_2a) *is surrounded by at most one canard limit cycle, which is stable if it exists.*

(IV) For the case $k_1 < 0$ *and* $k_3 > 0$*.*

(IV.1) If $\Delta_2 \geqslant 0$*, then there is no canard limit cycle.*

(IV.2) If $\Delta_2 < 0$*, then in the middle and left zones the singular point* (a, k_2a) *is surrounded by at most one canard limit cycle, which is stable if it exists.*

Statement (II) of Theorem 1 has been proved in the particular case $k_1 = k_3 > 0$ in the papers [Desroches *et al.*, 2012, 2016; Freire *et al.*, 2002; Garcia *et al.*, 2015; Llibre, Nuñez & Teruel , 2002]. In the paper [Garcia *et al.*, 2015] the authors investigated the existence of canard limit cycles for systems (1) with $k_1 = 1, k_3 = -1$ but the middle zone is very narrow. There are also the papers [Abdelouahab, Lozi & Chen , 2019; Nskano, Honda & Okazaki , 2005; Rotstein, Coombes & Gheorghe , 2012] generalizing the continuous piecewise linear slow–fast Liénard differential systems into FitzHugh–Nagumo type systems.

2. Proof of Theorem 1

Let $X = \sqrt{\varepsilon}(a - x), Y = y - k_2a$ and $T = \sqrt{\varepsilon}t$, then systems (1) satisfying (2) become

$$\begin{aligned} \frac{dX}{dT} &= F(X) - Y, \\ \frac{dY}{dT} &= X, \end{aligned} \tag{9}$$

where

$$F(X) = \begin{cases} T_R(X - v) + T_C v & \text{if } X \geqslant v, \\ T_C X & \text{if } -u \leqslant X \leqslant v, \\ T_L(X + u) - T_C u & \text{if } X \leqslant -u. \end{cases} \tag{10}$$

with

$$T_L = -\frac{k_1}{\sqrt{\varepsilon}}, T_C = -\frac{k_2}{\sqrt{\varepsilon}}, T_R = -\frac{k_3}{\sqrt{\varepsilon}}, u = (1-a)\sqrt{\varepsilon}, v = (1+a)\sqrt{\varepsilon}. \tag{11}$$

Note that systems (9) as well as function $F(X)$ of (10) are well defined when $\varepsilon \to 0$ because $X = \sqrt{\varepsilon}(a-x)$.

(I) For the case $k_i < 0, i = 1,2,3$. According to statement (a) of Theorem 5 in [Llibre, Ponce & Valls , 2015], systems (1) satisfying (2) have no canard limit cycles because $T_i, i = L, C, R$, are positive.

(II) For the case $k_1 > 0$, $k_2 < 0$ and $k_3 > 0$, the existence and uniqueness of canard limit cycles can be deduced directly from statement (b) of Theorem 5 in [Llibre, Ponce & Valls , 2015]. In the following we clarify the range of canard limit cycle according to the sign of Δ_2.

The eigenvalues of the linear part of systems (1) satisfying (2) at their singular point are

$$\lambda_i^+ = \frac{-k_i + \sqrt{\Delta_i}}{2}, \qquad \lambda_i^- = \frac{-k_i - \sqrt{\Delta_i}}{2}, \quad i = 1,2,3. \tag{12}$$

Recall that $\Delta_1 > 0$, then the singular point $E_R = (a, k_1(a-1)+k_2)$ is a stable virtual node with the linear invariant manifold

$$\Gamma_R: \quad y = \frac{k_1 + \sqrt{\Delta_1}}{2}(x-a) + k_1(a-1) + k_2, \quad x \geqslant 1. \tag{13}$$

Similarly, since $\Delta_3 > 0$ the singular point $E_L = (a, k_3(a+1)-k_2)$ is a stable virtual node with the linear invariant manifold

$$\Gamma_L: \quad y = \frac{k_3 + \sqrt{\Delta_3}}{2}(x-a) + k_3(a+1) - k_2, \quad x \leqslant -1. \tag{14}$$

(II.1) If $\Delta_2 \geqslant 0$, then the singular point $E_C = (a, k_2 a)$ is a real unstable node. The canard limit cycle must contain the portion of the linear invariant manifold

$$\Gamma_C: \quad y = \frac{k_2 \pm \sqrt{\Delta_2}}{2}(x-a) + k_2 a, \quad -1 \leqslant x \leqslant 1. \tag{15}$$

Therefore the canard limit cycle cannot be contained in two zones.

(II.2) If $\Delta_2 < 0$, then $E_C = (a, k_2 a)$ is a real unstable focus. Consider an orbit with initial value $(1, y_1), y_1 > 0$, it will reach a neighbourhood of Γ_R in an $O(\varepsilon)$ time, see Fig.2. Since the manifold Γ_R is attracting, the orbit target the central zone at the point $(1, y_2)$ for an $O(1)$ time. According to the Fenichel's first theorem [Fenichel , 1979], the distance between the orbit and Γ_R is of the order $O(e^{-c_1/\varepsilon})$, where $c_1 > 0$ is a constant depending on y_1. Thus the orbit reaches the central zone at the point $(1, y_2)$ which is close to $(1, y_R)$, that is,

$$y_2 = y_R + O(e^{-c_1/\varepsilon}), \tag{16}$$

with

$$y_R = \frac{k_1 + \sqrt{\Delta_1}}{2}(1-a) + k_1(a-1) + k_2. \tag{17}$$

This orbit must spiral up to the focus E_C, and intersects the switching line $x = 1$ again. If $1 - a > 0$ is sufficiently small, then the focus is so near the vertical line $x = 1$ that such an orbit does not enter the left zone. Thus we can build a positive invariant set containing the stable limit cycle. From the above analysis, the limit cycle only uses the right and middle zones when $1 - a > 0$ sufficiently small.

The case for $a + 1 > 0$ sufficiently small can be proved similarly.

(III) For the case $k_1 > 0$, $k_2 < 0$ and $k_3 < 0$, systems (1) satisfying (2) have at most two limit cycle by Theorem 1.3 of [Chen & Tang , 2019].

(III.1) For the subcase $\Delta_2 > 0$. First we prove that there is no canard limit cycle located in the three zones. Consider a point $(0, y_0)$ with $y_0 > 0$, see Fig.3. In forward time the orbit starting at $(0, y_0)$ will cross from the central zone to the right zone at the point $(1, y_1)$, and then reach a neighbourhood of Γ_R in $O(\varepsilon)$ time. After that since the invariant manifold Γ_R is attracting, the orbit go along this critical manifold and targets the central zone in a time of order $O(1)$. Conditions (16) and (17) hold by Fenichel's First Theorem.

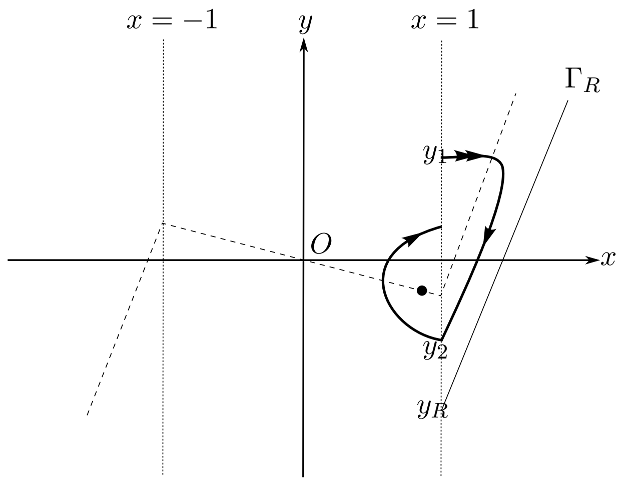

Fig. 2. The orbit of system (1) satisfying (2) with $k_1>0, k_2<0, k_3>0$ and $\Delta_1>0, \Delta_2<0, \Delta_3>0$.

Note that if $k_3<0$ and $\Delta_3>0$, then the singular point E_L is a virtual unstable node with the invariant manifold

$$\bar{\Gamma}_L: \quad y=\frac{k_3-\sqrt{\Delta_3}}{2}(x-a)+k_3(a+1)-k_2, \quad x \leqslant -1. \tag{18}$$

In backward time the orbit starting at the point $(0, y_0)$ will cross from the central zone to the left zone at the point $(-1, y_3)$. The orbit will reach a neighbourhood of $\bar{\Gamma}_L$ in $O(\varepsilon)$ time. After that, since $\bar{\Gamma}_L$ is repelling, the orbit goes near this critical manifold and targets the central zone in a time of order $O(1)$. According to the Fenichel First Theorem, the distance between the orbit and $\bar{\Gamma}_L$ is of the order $O(e^{-c_2/\varepsilon})$, where $c_2>0$ is a constant depending on y_3. Thus the orbit reaches the central zone at the point $(-1, y_4)$ which is exponentially close to $(-1, y_L)$, where

$$y_4=y_L+O(e^{-c_2/\varepsilon}), \tag{19}$$

with

$$y_L=\frac{k_3-\sqrt{\Delta_3}}{2}(-1-a)+k_3(a+1)-k_2. \tag{20}$$

Let $(x_C(t, y_2), y_C(t, y_2))$ denote the orbit of systems (1) satisfying (2) in the central zone with initial value $(1, y_2)$, then the zeros of the closing equations

$$\begin{aligned} F(t, y_2) &= x_C(t, y_2)+1=0, \\ G(t, y_2, y_4) &= y_C(t, y_2)-y_4=0, \end{aligned} \tag{21}$$

correspond to the canard limit cycles of systems (1) satisfying (2).

We shall prove that the closing equations (21) have no zeros. First we claim that if the system

$$\begin{aligned} \tilde{F}(t, y_R) &= x_C(t, y_R)+1=0, \\ \tilde{G}(t, y_R, y_L) &= y_C(t, y_R)-y_L=0, \end{aligned} \tag{22}$$

has no zeros, then the closing equations (21) have no zeros. Now we shall prove the claim using the Implicit Function Theorem. Indeed, using (16) and (19) we write system (22) as

$$\begin{aligned} f_1(y_2, y_4, \varepsilon) &= \tilde{F}(t, y_R) = x_C(t, y_2)+1+O(e^{-c_1/\varepsilon})=0, \\ f_2(y_2, y_4, \varepsilon) &= \tilde{G}(t, y_R, y_L) = y_C(t, y_2)-y_4+O(e^{-c_3/\varepsilon})=0, \end{aligned}$$

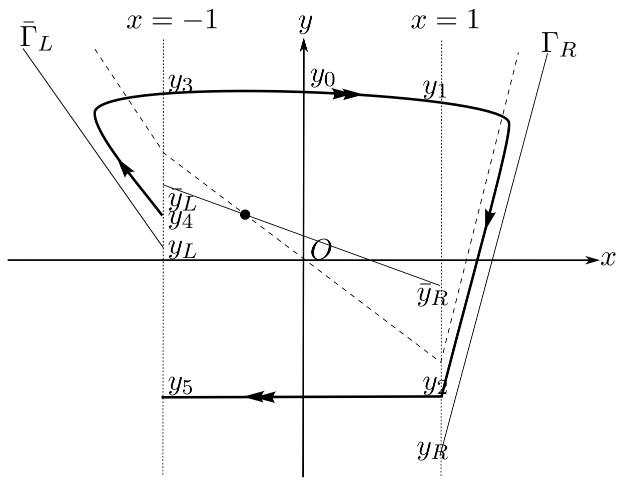

Fig. 3. The orbit of system (1) satisfying (2) with $k_1 > 0, k_2 < 0$, $k_3 < 0$ and $\Delta_i > 0, i = 1, 2, 3$.

The function (f_1, f_2) is C^1 in an open neighborhood of $\bar{y} = (\bar{y}_2, \bar{y}_4, 0)$ such that $(f_1, f_2)(\bar{y}_2, \bar{y}_4, 0) = 0$ satisfies that

$$\det \begin{pmatrix} \dfrac{\partial f_1}{\partial y_2}(\bar{y}) & \dfrac{\partial f_1}{\partial y_4}(\bar{y}) \\ \dfrac{\partial f_2}{\partial y_2}(\bar{y}) & \dfrac{\partial f_2}{\partial y_4}(\bar{y}) \end{pmatrix} = -\frac{\partial x_C}{\partial y_2}(t, \bar{y}_2) = \frac{2}{k_2} e^{-\frac{k_2 t}{2}} \sinh\left(\frac{k_2 t}{2}\right) \neq 0, \tag{23}$$

because

$$\begin{aligned} x_C(t, y_2) = {} & \frac{e^{-\frac{1}{2}t\left(\sqrt{k_2^2-4\varepsilon}+k_2\right)}}{2\sqrt{k_2^2-4\varepsilon}} \left(\sqrt{k_2^2-4\varepsilon}\left(2ae^{\frac{1}{2}t\left(\sqrt{k_2^2-4\varepsilon}+k_2\right)} - (a-1)\left(e^{t\sqrt{k_2^2-4\varepsilon}}+1\right)\right)\right. \\ & \left. -(a+1)k_2\left(e^{t\sqrt{k_2^2-4\varepsilon}}-1\right) + 2y_2\left(e^{t\sqrt{k_2^2-4\varepsilon}}-1\right)\right), \end{aligned}$$

and

$$\frac{\partial x_C(t, y_2)}{\partial y_2} = \frac{2}{\sqrt{k_2^2-4\varepsilon}} e^{-\frac{k_2 t}{2}} \sinh\left(\frac{1}{2}t\sqrt{k_2^2-4\varepsilon}\right).$$

Here t is the time that the solution $(x_C(t, y_2), y_C(t, y_2))$ starting at the point $(1, y_2)$ needs in order to arrive at the point $(1, y_4)$. Since the determinant (23) is nonzero the Implicit Function Theorem proves the claim.

Recall that $\Delta_2 > 0$, the finite singular point $E_C = (a, k_2 a)$ is an unstable node. We obtain the explicit expression of the solution of the central zone as follows:

$$\begin{aligned} x_C(t, y_R) &= a - \frac{1}{2\sqrt{\Delta_2}} e^{\frac{t}{2}(-k_2+\sqrt{\Delta_2})} \left(k_2(a+1) - 2y_R - (1-a)\sqrt{\Delta_2}\right) \\ &\quad + \frac{1}{2\sqrt{\Delta_2}} e^{\frac{t}{2}(-k_2-\sqrt{\Delta_2})} \left(k_2(a+1) - 2y_R + (1-a)\sqrt{\Delta_2}\right), \\ y_C(t, y_R) &= k_2 a + \frac{1}{2\sqrt{\Delta_2}} e^{\frac{t}{2}(-k_2-\sqrt{\Delta_2})} \left((ak_2 - y_R)(k_2 - \sqrt{\Delta_2}) + 2(1-a)\varepsilon\right) \\ &\quad - \frac{1}{2\sqrt{\Delta_2}} e^{\frac{t}{2}(-k_2+\sqrt{\Delta_2})} \left((ak_2 - y_R)(k_2 + \sqrt{\Delta_2}) + 2(1-a)\varepsilon\right). \end{aligned} \tag{24}$$

Substitute (24) into (22), from the first equation of (22), we obtain that

$$a = 1 + \frac{4\sqrt{\Delta_2} e^{\frac{t}{2}(k_2+\sqrt{\Delta_2})}}{(e^{t\sqrt{\Delta_2}} - 1)(\sqrt{\Delta_1} - k_1 + k_2) + \sqrt{\Delta_2}(1 - 2e^{\frac{t}{2}(k_2+\sqrt{\Delta_2})} - e^{t\sqrt{\Delta_2}})}. \tag{25}$$

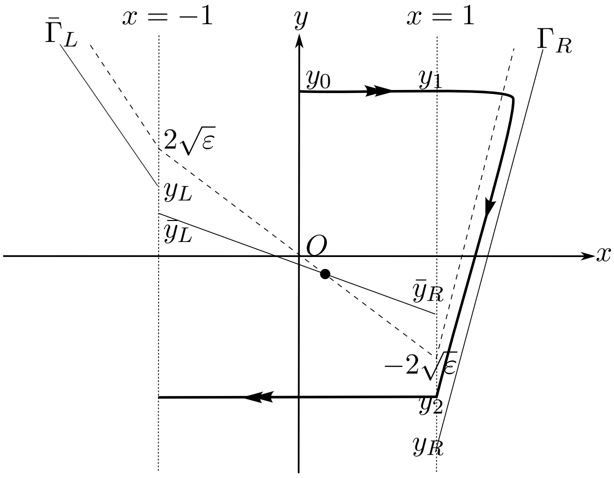

Fig. 4. The orbit of system (1) satisfying (2) with $k_1>0, k_2<0, k_3<0$ and $\Delta_1>0, \Delta_2=0, \Delta_3>0$.

Replace (25) and (20) into the second equation of (22), we have

$$\tilde{G}=\frac{-2e^{\frac{t}{2}\sqrt{\Delta_2}}G_1}{(e^{t\sqrt{\Delta_2}}-1)(\sqrt{\Delta_1}-k_1+k_2)+\sqrt{\Delta_2}(1-2e^{\frac{t}{2}(k_2+\sqrt{\Delta_2})}-e^{t\sqrt{\Delta_2}})} \tag{26}$$

with

$$\begin{aligned} G_1=&\left(\sqrt{\Delta_1}-k_1+\sqrt{\Delta_3}+k_3\right)\sqrt{\Delta_2}\mathrm{Cosh}\left(\frac{t}{2}\sqrt{\Delta_2}\right)\\ &+\left(k_1\left(k_2-k_3-\sqrt{\Delta_3}\right)+k_2\left(k_3-\sqrt{\Delta_1}+\sqrt{\Delta_3}\right)\right)\mathrm{Sinh}\left(\frac{t}{2}\sqrt{\Delta_2}\right)\\ &+\left(\left(k_3+\sqrt{\Delta_3}\right)\sqrt{\Delta_1}-4\varepsilon\right)\mathrm{Sinh}\left(\frac{t}{2}\sqrt{\Delta_2}\right). \end{aligned} \tag{27}$$

Note that $\Delta_2=k_2^2-4\varepsilon>0$ and $k_2<0$, we introduce a new parameter $K_2=k_2/\sqrt{\varepsilon}$, then $K_2<-2$. For $\varepsilon>0$ sufficiently small we can expand G_1 and obtain

$$G_1=\sqrt{K_2^2-4}\left(\frac{2}{k_3}-\frac{2}{k_1}-2t\right)\varepsilon^{3/2}+o(\varepsilon^{3/2}).$$

It is obvious that G_1 has no zeros because $k_1>0, k_3<0$ and $t>0$. From this analysis we can conclude that the closing equations (21) have no zeros, and thus systems (1) satisfying (2) have no canard limit cycles contained in the three zones.

Second we consider canard limit cycles contained in two zones. It is obvious that there is no canard limit cycles located in the middle and right zones because the invariant manifolds (15) exists. Note that the traces of systems (1) satisfying (2) have the same sign in the left and in the middle zones, so canard limit cycles also can not exist in there two zones.

(III.2) For the subcase $\Delta_2=0$, that is $k_2=-2\sqrt{\varepsilon}$. First we consider the canard limit cycles contained in three zones. Since $\Delta_1>0$ and $\Delta_3>0$, the invariant straight lines Γ_R and $\bar{\Gamma}_L$ given by (13) and (18) respectively exist in the external zones. In the middle zone the finite singular point $E_C=(a,k_2a)$ is an unstable node with the invariant manifold

$$\bar{\Gamma}_C: \quad y=-(x+a)\sqrt{\varepsilon}, \quad -1\leqslant x\leqslant 1, \tag{28}$$

see Fig.4.

Let $\bar{y}_R$ and $\bar{y}_L$ denote the intersection points of $\bar{\Gamma}_C$ with $x=1$ and $x=-1$ respectively. From (28) we have

$$\bar{y}_R=-(1+a)\sqrt{\varepsilon}, \qquad \bar{y}_L=(1-a)\sqrt{\varepsilon}. \tag{29}$$

For the fixed values $k_1 > 0$, $k_3 < 0$ and $-1 < a < 1$, since $\varepsilon > 0$ is sufficiently small, substituting $k_2 = -2\sqrt{\varepsilon}$ into (17) and (20), we can expand y_R and y_L as the following series

$$\begin{aligned} y_R &= -2\sqrt{\varepsilon} - \frac{1-a}{k_1}\varepsilon + o(\varepsilon), \\ y_L &= 2\sqrt{\varepsilon} + \frac{1+a}{k_3}\varepsilon + o(\varepsilon). \end{aligned} \tag{30}$$

Note that $y_R < \bar{y}_R$ and $y_L > \bar{y}_L$ by (29) and (30). Suppose that there is a canard limit cycle contain in the three zones, then it should contain the invariant line $\bar{\Gamma}_C$ given by (28). Since at all the points $(-1, y)$ with $y \in (\bar{y}_L, y_L)$ the vector filed associated to systems (1) satisfying (2) points to left, there is a contradiction.

In a similar way to the proof of statement (III.1), we can prove that there are no canard limit cycles contained in two zones when $\Delta_2 = 0$.

(III.3) For the subcase $\Delta_2 < 0$. The singular point $E_C = (a, k_2 a)$ is an unstable focus. We claim that there is a canard limit cycle contain in two zones. It is obvious that a canard limit cycle can not exist in the middle and left zones, so we just need to consider the middle and right zones. Let $X = x - 1, Y = k_2 - y$ and $T = -t$, then systems (1) satisfying (2) become

$$\begin{aligned} \frac{dX}{dT} &= Y - F(X), \\ \frac{dY}{dT} &= \varepsilon(A - X), \end{aligned} \tag{31}$$

where $A = a - 1$ and

$$F(X) = \begin{cases} -k_1 X & \text{if } X \geqslant 0, \\ -k_2 X & \text{if } -2 \leqslant X \leqslant 0. \end{cases} \tag{32}$$

For the case $\Delta_2 < 0$, note that $A < 0$ and $\Delta_1 \geqslant 0$. According to statements (IV) and (V) of Corollary 3 of [Li & Llibre , 2019], there is a canard limit cycle surrounding the singular point $(a, k_2 a)$, which is unique and stable.

(IV) The proof of statement (IV) follows from (III) doing the linear change of variables $X = -x, Y = -y$.

This completes the proof of Theorem 1.

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