# A Chattering-Free Robust Finite-Time Output Feedback Control Scheme for a Class of Uncertain Nonlinear Systems

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**Abstract: In this paper, an innovative technique to design an observer-based Finite-Time Output Feedback Controller (FT-OFC) is proposed for a class of nonlinear systems. This controller aims to make the state variables converge to a small bound around the origin in a finite time. The main innovation of this paper is to transform the nonlinear system into a new time-varying form to achieve the finite-time boundedness criteria using the asymptotic stability methods. Moreover, without any prior knowledge of the upper bounds of the system uncertainties and/or disturbances, and only based on the output measurements, a novel time-varying extended state observer is designed to estimate the states of the nonlinear system as well as the uncertainties and disturbances in a finite time. In this way, the time-varying gains of the extended state observer are designed to converge the observation error to a neighborhood of zero while remaining uniformly bounded in finite-time. Subsequently, an observer-based time-varying control law is designed to make the system globally uniformly bounded in finite time. Finally, the efficiency of the proposed FT-OFC for a disturbed double integrator system with unknown measurement noise is illustrated by numerical simulations.**

## 1. Introduction

The stabilization problem is one of the main focuses in the control theory. Many results have been reported in this regard over the last decades [1, 2]. On the other hand, uncertainties exist in practical systems; thus, robust stabilization of uncertain systems is of both theoretical and practical significance. Robust controllers are designed to suppress uncertainties and the effect of external disturbances. Moreover, there are many papers considering asymptotic stability, *i.e.* stability over an infinite time. However, in many applications, Finite-Time Stability (FTS) is of particular importance; since finite-time controllers as well as guaranteeing the finite-time convergence of the state variables to the equilibrium point, provide a faster transient response, a higher precision, and a better disturbance rejection [3-7]. Moreover, in the presence of uncertainties and external disturbances, achieving asymptotic stability to the origin is impossible, and instead, the concept of ultimate boundedness is considered [7, 8]. In this paper, the definition of ultimate boundedness is used to describe the behavior of dynamical systems whose state variables approach a small neighborhood of zero in a finite time.

The concept of robust finite-time boundedness (FTB) has received considerable attention in the control theory. Most of the existing results on the robust FTB are based on the state feedback approach. To stabilize uncertain nonlinear systems via state feedback in a finite time, several schemes have been proposed in the literature [4-6], [9], and [10]. However, for instance, the approach of [10, 19] is applicable only for systems whose response is valid in a finite time interval. However, in many applications, it is important to have stability over a finite time as well as onwards. Another drawback of [10] is that its technique is restricted to matched uncertainties. These shortcomings will be overcome in this paper.

On the other hand, the main drawback of the state feedback controllers is that the value of state variables should be available online, which is not realistic in many practical situations. To overcome this restriction, an observer-based control scheme, which is a kind of output feedback controllers, has been proposed for a finite-time stabilization problem [11]. Indeed, in the context of finite-time output feedback design, a rather direct approach is to combine finite-time state feedback laws with finite-time observers. On the other hand, to estimate the unmeasured states and the uncertainties including the disturbances and/or unmodeled dynamics simultaneously, the extended state observer (ESO) has been proposed in [7, 8] and [11].

Although ESO-based finite-time controllers have been extensively studied by many authors, ESO-based finite-time controllers with a simple structure are relatively scarce [12-18]. The terminal sliding mode (TSM) method, as one of the most powerful techniques in finite-time control problems, has been studied in [12, 13]. However, the finite-time convergence has been achieved at the expense of singularity in some points and a slower convergence rate for the initial conditions far from the equilibrium point. To resolve the singularity and the convergence rate, nonsingular fast TSM controllers have been designed in [14, 15]. Besides, some finite-time controllers have been suggested using the adaptive control [16], and LMI-based methods [17]. In all of these methods, in addition to structural limitations in each approach, the design procedure is somewhat complicated. Therefore, the design of ESO-based finite-time controllers with a straightforward design procedure is still an open challenging problem. Also, the observer introduced in [20] is applicable only in systems with no internal dynamics.

Motivated by the above discussions, in this paper an alternative finite-time output-feedback control (FT-OFC) approach in a simple and straightforward manner is proposed. In this regard, the model of the system in normal form is transformed into a novel time-varying structure to make the FTB criteria possible using the asymptotic stability methods. In this paper, first, considering a known upper bound for uncertainties and/or disturbances, a nonsingular finite-time state feedback controller is designed to guarantee robust finite-time ultimate boundedness of the system. To make the controller independent from the value of the state variables, an observer-based control scheme, which is a kind of output feedback controller, is proposed. Then, to relax the design procedure from any knowledge of the upper bound of uncertainties and disturbances, a finite-time ESO is designed to estimate the state of the nonlinear system as well as uncertainties. The time-varying gains of the proposed ESO are designed to make the observation error converge to a neighborhood of zero in finite-time. Subsequently, a chattering-free ESO-based finite-time controller is proposed to guarantee the finite-time ultimate boundedness of the uncertain nonlinear system. Finally, the efficiency of the proposed method is compared with the existing approaches through two numerical simulations.

Compared with the previous research in this line, the contributions of this work are as follows:

(1) A novel time-varying conversion was introduced so that the state estimation with finite-time boundedness properties was guaranteed in a simple and straightforward manner.
(2) In order to estimate the full state of the nonlinear system as well as the uncertainties and/or disturbances, a novel continuous and free of chattering finite-time ESO was designed. In this regard, without neither any knowledge about the upper bound of uncertainties and/or disturbances nor its derivative, and only based on the output measurement, time-varying gains of the ESO are designed to make the observation error converge to a neighborhood of zero while remaining uniformly bounded in finite-time.
(3) A novel singularity-free FT-OFC is proposed to guarantee finite-time ultimate boundedness with a weak dependency on the initial conditions.

The remainder of this paper is organized as follows. Section 2 introduces the class of uncertain nonlinear systems and relevant definitions. In Section 3, a finite-time state feedback controller is proposed. Section 4 presents a finite-time ESO without any knowledge about the upper bounds of the generalized disturbance. In Section 5, a FT-OFC is designed. Simulation results are presented in Section 6. Finally, conclusions are presented in Section 7.

## 2. Problem Formulation

Consider the following uncertain nonlinear system:

$$\dot{x} = f_n(t,x) + \Delta f(t,x) + \big(g_n(t,x) + \Delta g(t,x)\big)[u + d(t,x)]$$
$$y = h(t,x) \qquad (1)$$

where $x \in D \subset R^n$, $u \in R$ and $y \in R$ are the state vector, input, and output signals, respectively. Moreover, based on [2], $d(t,x)$ is a bounded disturbance, $\Delta f(t,x) = f_n(t,x) + \Delta_1(t,x)$ and $\Delta g(t,x) = g_n(t,x) + \Delta_2(t,x)$ are the uncertain terms. Also, $f_n(t,x), g_n(t,x)$ and $h(t,x)$ are known nonlinear smooth functions.

Now, the nonlinear system (1) is transformed into the normal form using a diffeomorphism map. For this purpose, consider the relative degree $\rho$ for the system (1). Moreover, for every $x_0 \epsilon D \in R$, a neighborhood $\Omega$ of $x_0$ and the functions $\phi_i(t,x)$ for $i = 1, \dots, n-\rho$ exist, such that the map $T(x) = [\phi(t,x) | \Psi(t,x)]^T : \Omega \to R^n$ based on the Lie derivatives of $h(t,x)$ with respect to $f_n(t,x)$, is diffeomorphism on $\Omega$ [2], where $\phi(t,x) = [\phi_1(t,x) \ \cdots \ \phi_{n-\rho}(t,x)]$ and $\Psi(t,x) = \left[h(t,x) \quad L_{f_n(t,x)} h(t,x) \quad \cdots \quad L^{\rho-1}_{f_n(t,x)} h(t,x)\right]$. Then, according to [2], the function $g(x) = g_n(t,x) + \Delta g(t,x)$ will belong to the null space of $\frac{\partial \phi}{\partial x}$ and the transformation $[\eta \ \vdots \ \xi]^T = T(x)$ satisfies the following conditions [2]:

$$\frac{\partial T}{\partial x} f_n(t,x) = AT(x) - B\gamma(t,x)\alpha(t,x)$$
$$\frac{\partial T}{\partial x} g_n(t,x) = B\gamma(t,x) \qquad (2)$$

where $A = \begin{bmatrix} 0 & \boldsymbol{I}_{\rho-1\times\rho-1} \\ 0 & \boldsymbol{0}_{n-\rho+1} \end{bmatrix} \in R^{n\times n}$, $B = [0 \ \dots \ 0 \ \ 1]^T \in R^n$, $\gamma(t,x) = L_{g_n} L^{\rho-1}_{f_n} h(t,x) \neq 0$ and $\alpha(t,x) = \frac{-L^{\rho}_{f_n} h(t,x)}{L_{g_n} L^{\rho-1}_{f_n} h(t,x)}$.

Therefore, based on $T(x)$, the transformed version of (1) can be rewritten as

$$\begin{aligned} \dot{\eta} &= f_0(\eta,\xi) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\ \vdots \\ \dot{\xi}_{\rho-1} &= \xi_\rho \\ \dot{\xi}_\rho &= -\gamma(x)\alpha(x) + \gamma(x)[u + \delta(t,x,u)] \end{aligned} \qquad (3)$$

where the internal dynamic $\dot{\eta} = f_0(\eta,\xi)$ and the new uncertain term $\delta(t,x,u)$ are defined as:

$$f_0(\eta,\xi) = \frac{\partial \phi}{\partial x}[f_n(x) + \Delta f(x)]$$
$$\delta(t,x,u) = \Delta_1(x) + \Delta_2(x)[u + d(t,x)] + d(t,x) \qquad (4)$$

Moreover, based on [2], the smooth uncertainty $\Delta f(x)$ satisfies $\frac{\partial T}{\partial x}\Delta f(x) = B\gamma(t,x)\Delta_1(x)$.

**Remark 1:** The state variables $\eta$ (internal states) and the input variable $u$ appear in the $\xi -$ subsystem of (3). This assumption that the variables $\eta$ are not measurable has not been considered by the existing observers [11]. In this paper, the variables of the internal dynamics are considered as unknown perturbations in the dynamics of the $\xi -$subsystem. Thus, we restrict our analysis to the case where the internal dynamics of the system are input-to-state stable (ISS) when $\xi$ is considered as input [2]. According to [2], it is supposed that there is a Lyapunov function $V(\eta)$ satisfying, $\frac{\partial V(\eta)}{\partial \eta}\frac{\partial \phi}{\partial x}[f_n(x) + \Delta f(x)] \leq -W(\eta)$ for some positive definite functions $\mathrm{W}(\eta)$.

For simplicity and without loss of generality, let us define $\bar{\gamma}(x) = \gamma(x) - 1$. Thus,

$$\begin{aligned} \dot{\xi}_\rho &= -\big(1 + \bar{\gamma}(x)\big)\alpha(x) + \big(1 + \bar{\gamma}(x)\big)[u + \delta(t,x,u)] \\ &= -\big(1 + \bar{\gamma}(x)\big)\alpha(x) + u \\ &\quad + \big[\delta(t,x,u) + \bar{\gamma}(x)\big(u + \delta(t,x,u)\big)\big] \\ &= u + \Delta_{eq} \end{aligned} \qquad (5)$$

where $\Delta_{eq} = -(1+\bar{\gamma}(x))(\delta(t,x,u)-\alpha(x))+\bar{\gamma}(x)u$ is the summation of uncertainties and disturbances (the generalized disturbance). Assume that, with $u=\psi(t,\xi)+\vartheta$ the generalized disturbance $\Delta_{eq}$ is a function of time, state variables, and the input vector satisfying the inequality $\|\Delta_{eq}\| \le \Upsilon(t,\xi)+k\vartheta$ [2]. Here, $\psi(t,\xi)$ is the designed control law for the nominal system which is uniformly finite-time stable, $k\in[0,1)$ and the nonnegative continuous function $\Upsilon(t,\xi)$ is the only information that we need to know about the generalized disturbance $\Delta_{eq}$. In this way, the function $\Upsilon(t,\xi)$ is a measure of the size (*i.e.* an upper bound) of the generalized disturbance $\Delta_{eq}$. This function is not required to be small and it just should be initially known. Later in the proposed observer-based control scheme, this restriction will be overcome.

Throughout the paper, the following definitions will be used:

**Definition 1 [18]:** For a given positive-definite matrix function $X$, a positive-definite matrix $X_0$, and any positive constants $a,b$ $(0\le a\le b)$ and $\sigma$ if $x^TXx<b$, $t\in[0,T]$ whenever $x_0{}^TX_0x_0\le a$ and $\|d\|\le\sigma$, then, for a finite time $T>0$, the system $\dot{x}=f(t,x,d)$ is said to be finite-time bounded with respect to $(a,b,\sigma,T,X,X_0)$.

**Definition 2 [10]:** The system $\dot{x}=f(t,x,d)$ is said to be finite-time input-to-state stable (FT-ISS) with respect to $d$ in the finite time $T$, if the following inequality for a $KL$-class function $\beta$ and a $K$-class function $\gamma$ is guaranteed for any $t\ge t_0+T$:

$$\|x\| \le \beta(\|x_0\|,\bar{\vartheta})+\gamma(\|d\|) \quad (6)$$

where $\bar{\vartheta}$ is a time-varying function tending to infinity as $t\to t_0+T$. Note that an FT-ISS system, in the absence of the disturbance $d(t)$, will be finite-time stable (FTS) [18].

**Remark 2:** Consider the change of coordinates $C_1\to C_2$ as $C_2=\aleph C_1$, where $\aleph$ is a positive decreasing function, converging asymptotically to a neighborhood around zero and remaining uniformly small after the fixed time $t_0+T$. Then, if the variable $C_1$ remains stable (does not tend to infinity), the boundedness of $C_2$ as $t\to t_0+T$ is guaranteed.

## 3. Finite-Time State Feedback Control Design

In this section, based on the upper bound of the generalized disturbance $\Delta_{eq}$, a finite-time state feedback controller will be designed to stabilize the nonlinear system (3). The design procedure is composed of two steps: first, for finite-time stabilization, a time-varying transformation is proposed. Then, based on the sliding mode method, a time-varying state feedback controller is designed to stabilize this new system. The main results of the proposed approach and some more details of the procedure are stated as follows:

**Theorem 1:** The state variables of the closed-loop system composed of (3) and the time-varying state feedback controller (7), converge to a small neighborhood of the origin in finite-time and remain uniformly bounded if the sliding surface $s$ is designed as $s=\varpi_\rho+\sum_{i=1}^{\rho-1}a_i\varpi_i$, where $\varpi_i$'s are defined as $\varpi_i=\mu_c\xi_i$. Moreover, $\mu_c(t,\acute{T}_c)=\frac{1+e^{\frac{-t}{\acute{T}_c}}}{2e^{\frac{-t}{\acute{T}_c}}}$ is a positive time-varying function (for simplicity shown as $\mu_c$), $\acute{T}_c$ is a constant design parameter, and the control command is given by

$$u=-k_1 sgn(s)-\mu_c{}^{-1}\left(\left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c}+k_2\right)s+\sum_{i=1}^{\rho-1}a_i\varpi_{i+1}\right) \quad (7)$$

where for any $k\in[0,1)$, the controller gains are tuned as $k_1\ge\frac{\Upsilon(t,\xi)}{1-k}$, $k_2\ge 0$. Furthermore, the positive constants $a_i$ for $i=1,\dots,\rho-1$ are chosen to satisfy the differential Lyapunov inequality $\dot{P}+\Lambda_c^TP+P\Lambda_c\le-\alpha I$ for some $\alpha>0$ and the identity matrix $I$, where $P$ is a positive-definite bounded matrix and the time-varying matrix $\Lambda_c\in R^{\rho\times\rho}$ is defined as

$$\Lambda_c=\begin{bmatrix}\frac{\dot{\mu}_c}{\mu_c} & 1 & 0 & 0\\ 0 & \ddots & \ddots & 0\\ 0 & 0 & \frac{\dot{\mu}_c}{\mu_c} & 1\\ -a_1 & \cdots & -a_{\rho-2} & \frac{\dot{\mu}_c}{\mu_c}-a_{\rho-1}\end{bmatrix} \quad (8)$$

**Proof:** Consider the time-varying transformation $\xi_i\to\varpi_i$ defined as $\varpi_i=\mu_c\xi_i$, where $i=1,\dots,\rho$. The derivative of $\varpi_i$ along with the nonlinear system (3), (5), are

$$\begin{aligned}\dot{\varpi}_i&=\frac{\dot{\mu}_c}{\mu_c}\varpi_i+\varpi_{i+1}, \qquad i=1,\dots,\rho-1\\ \dot{\varpi}_\rho&=\frac{\dot{\mu}_c}{\mu_c}\varpi_\rho+\mu_c(u+\Delta_{eq})\end{aligned} \quad (9)$$

The candidate Lyapunov function is $V=\frac{1}{2}s^2$. Thus, the time-derivative of the Lyapunov function along (9) is

$$\dot{V}=s\dot{s}=s\left(\frac{\dot{\mu}_c}{\mu_c}\varpi_\rho+\mu_c(u+\Delta_{eq})\right)+s\left(\sum_{i=1}^{\rho-1}a_i\left(\frac{\dot{\mu}_c}{\mu_c}\varpi_i+\varpi_{i+1}\right)\right) \quad (10)$$

Substituting the controller (7) into (10) leads to

$$\dot{V}=\mu_c(-k_1|s|+s\Delta_{eq})-k_2s^2 \quad (11)$$

Since $\|\Delta_{eq}\|\le\Upsilon(t,\xi)+k\vartheta$ and $k_1\ge\frac{\Upsilon(t,\xi)}{1-k}, k_2\ge0$, the upper bound of $\dot{V}$ with $k_3=k_1-kk_1-\Upsilon(t,\xi)\ge0$ and $k\in[0,1)$ can be written as

$$\dot{V}\le-k_3\mu_c|s|-k_2s^2 \quad (12)$$

This inequality based on the Lyapunov function $V$, can be rewritten as follows:

$$\dot{V}\le-k_3\mu_c|s|-k_2s^2\le-k_3\mu_c(2V)^{0.5}-2k_2V \quad (13)$$

According to (13), the system trajectory reaches the sliding surface s = 0 in finite time and stays on it [2]. On this surface, the motion is governed by the following reduced-order model:

$$\begin{aligned}\dot{\varpi}_i&=\frac{\dot{\mu}_c}{\mu_c}\varpi_i+\varpi_{i+1}, \qquad i=1,\dots,\rho-2\\ \dot{\varpi}_{\rho-1}&=\frac{\dot{\mu}_c}{\mu_c}\varpi_{\rho-1}-\sum_{i=1}^{\rho-1}a_i\varpi_i\end{aligned} \quad (14)$$

The stability of (14) can be guaranteed via choosing the positive constants $a_i$, a positive-definite bounded matrix

$P$, and the matrix $\Lambda_c$ defined in (8) satisfying the differential Lyapunov inequality $\dot{P}+\Lambda_c^T P+P\Lambda_c \leq -\alpha I$. Consequently, the state variables tend to zero as $t$ tends to infinity along $s=0$. Subsequently, the sliding phase motion is ensured and the stability of the sliding surface $s$ is guaranteed [3]. Therefore, based on the stability of the reduced-order model (14) and the structure of $s$, it can be concluded that $\varpi_i$'s remain uniformly bounded. Finally, based on the inverse of the transformation $\varpi_i=\mu_c(t,\acute{T}_c)\xi_i$ and Remark 2, $\xi_i$ tends to the neighborhood of zero and remains uniformly bounded in finite-time. Furthermore, based on the boundedness stability results (*i.e.* $V(T)\neq 0$), an upper bound of the finite convergence time $T$ is obtained as $T\leq \frac{1}{k_2}\left(ln\frac{k_2V^{0.5}(0)+k_3\mu_c}{k_3\mu_c}-ln\frac{k_2V^{0.5}(T)+k_3\mu_c}{k_3\mu_c}\right)$ [21]. Finally, according to definitions 1 and 2, the FTB stability and/or FT-ISS of the system (3) (and subsequently the FT-ISS of the system (1)) is achieved. This completes the proof. ☐

As seen above, negative fractional powers do not occur in the design procedure and thus the singularity problem will not appear. Besides, since $\Lambda_c$ is time-varying, evaluation of the matrix $P$ may be difficult. In this regard, the next corollary is proposed to derive a sufficient condition to guarantee the stability of the system.

**Corollary 1:** Based on the definition of $\mu_c$, one has $\dot{\mu}_c/\mu_c={}^1/_{2\acute{T}_c}(1-\mu_c^{-1})$. Since $1\leq\mu_c\leq\infty$, therefore $0\leq \dot{\mu}_c/\mu_c\leq {}^1/_{2\acute{T}_c}$. On the other hand, since the time-varying matrix $\Lambda_c$ in (8) is continuous and bounded as $lim_{t\to\infty}\Lambda_c=\overline{\Lambda}_c$, the stability of the sliding surface $s$ is guaranteed if all the Eigenvalues of $\overline{\Lambda}_c$ are located on the open left-hand half of the complex plane [3] where $\overline{\Lambda}_c$ is defined as:

$$\overline{\Lambda}_c=\begin{bmatrix}\frac{1}{2T'_c} & 1 & 0 & 0\\ 0 & \ddots & \ddots & 0\\ 0 & 0 & \frac{1}{2T'_c} & 1\\ -a_1 & \cdots & -a_{\rho-2} & \frac{1}{2T'_c}-a_{\rho-1}\end{bmatrix} \qquad (15)$$

The controller (7) needs the value of state variables online, which is not realistic in many practical situations. Also, it is not possible to determine the coefficient $k_1$. To overcome these restrictions, in the next section, a finite-time extended state observer is designed to estimate the unmeasured states and the uncertainties simultaneously.

## 4. Finite-Time Extended State Observer Design

Now, a finite-time extended state observer (FT-ESO) is designed for the system (3). The estimations of state variables should track them where only the first variable $\xi_1$ as the output of the system is measurable. To design the ESO, based on (5), an auxiliary variable is introduced as $\xi_{\rho+1}=\Delta_{eq}$. Therefore, the system can be augmented as

$$\begin{aligned}\dot{\xi}_1&=\xi_2\\ \dot{\xi}_2&=\xi_3\\ &\vdots\\ \dot{\xi}_\rho&=\xi_{\rho+1}+u\\ \dot{\xi}_{\rho+1}&=\dot{\Delta}_{eq}=S\end{aligned} \qquad (16)$$

where $S$ is an unknown variable satisfying Remark 3.

**Remark 3:** Assume that the term $S$ satisfies $|S|\leq\sigma$. In this inequality $\sigma$ is a non-negative constant as an upper bound of the amplitude of $S$. Note that the parameter $\sigma$ may be unknown and it should just be bounded.

The proposed FT-ESO to estimate the state variable of the nonlinear system (16) is described as:

$$\begin{aligned}\dot{\hat{\xi}}_1&=\hat{\xi}_2+g_1(t,\acute{T}_o)\times(\xi_1-\hat{\xi}_1)\\ \dot{\hat{\xi}}_2&=\hat{\xi}_3+g_2(t,\acute{T}_o)\times(\xi_1-\hat{\xi}_1)\\ &\vdots\\ \dot{\hat{\xi}}_\rho&=\hat{\xi}_{\rho+1}+u+g_\rho(t,\acute{T}_o)\times(\xi_1-\hat{\xi}_1)\\ \dot{\hat{\xi}}_{\rho+1}&=g_{\rho+1}(t,\acute{T}_o)\times(\xi_1-\hat{\xi}_1)\end{aligned} \qquad (17)$$

where the time-varying gains $\{g_i(t,\acute{T}_o)\}_{i=1}^{\rho+1}$, depending on time and the constant parameter $\acute{T}_o$, are designed later to achieve finite-time convergence. Defining the observation error as $e=\xi-\hat{\xi}$, leads to the following error dynamics:

$$\begin{aligned}\dot{e}_1&=e_2-g_1(t,\acute{T}_o)\times e_1\\ \dot{e}_2&=e_3-g_2(t,\acute{T}_o)\times e_1\\ &\vdots\\ \dot{e}_\rho&=e_{\rho+1}-g_\rho(t,\acute{T}_o)\times e_1\\ \dot{e}_{\rho+1}&=S-g_{\rho+1}(t,\acute{T}_o)\times e_1\end{aligned} \qquad (18)$$

Or in a more compact form

$$\begin{aligned}\dot{e}_i&=e_{i+1}-g_i(t,\acute{T}_o)\times e_1 \qquad i=1,\dots,\rho\\ \dot{e}_{\rho+1}&=S-g_{\rho+1}(t,\acute{T}_o)\times e_1\end{aligned} \qquad (19)$$

To investigate the finite-time stability of (19), the time-varying gains are designed such that the observation errors $e_i$ approach to a small neighborhood around zero in a finite time. Theorem 2 is presented to guarantee uniformly boundedness of the observation errors without any knowledge about the upper bound of $\Delta_{eq}$.

**Theorem 2:** For the dynamical system (16), consider the observer (17), which yields the error dynamics (18). The observation error variables approach to a small neighborhood around zero as $t$ tends to the finite time $T$ if the observer gains $\{g_i(t,\acute{T}_o)\}$ are designed as

$$g_i=L_i+\bar{g}_{i,1}\frac{\rho+m+i}{2\acute{T}}(1-\mu_1^{-1})\mu_1^{i-1}-\bar{g}_{i+1,1}\mu_1^i-\sum_{j=1}^{i-1}\bar{g}_{i,j}\mu_1^{i-j}g_j \qquad (20)$$

for $i=1,\dots,\rho$ and,

$$g_{\rho+1}=L_{\rho+1}+\bar{g}_{\rho+1,1}\frac{2\rho+m+1}{2\acute{T}_o}(1-\mu_1^{-1})\mu_1^{\rho}-\sum_{j=1}^{\rho}\bar{g}_{\rho+1,j}\mu_1^{\rho-j+1}g_j \qquad (21)$$

where $\{\bar{g}_{i,j}\}$ for $i=2,\dots,\rho$ and $j=2,\dots,\rho+1$ are given as

$$\bar{g}_{i,j-1}=\bar{g}_{i,j}\frac{\rho+m+1}{2\acute{T}_o}(1+\mu_1^{-1})\mu_1^{-1}+\bar{g}_{i,j}\frac{i-j}{2\acute{T}_o}(1-\mu_1^{-1})\mu_1^{-1}+\bar{g}_{i+1,j} \qquad (22)$$

and for $i=\rho+1$,

$$\bar{g}_{\rho+1,j-1}=\bar{g}_{\rho+1,j}\frac{\rho+m+1}{2\acute{T}_o}(1+\mu_1^{-1})\mu_1^{-1}+\bar{g}_{\rho+1,j}\frac{\rho-j+1}{2\acute{T}_o}(1-\mu_1^{-1})\mu_1^{-1} \quad (23)$$

where $\{\bar{g}_{i,i}\}=1$ and for $i<j$ one has $\{\bar{g}_{i,j}\}=0$. Also, the scalar coefficients $\{L_i\}$ for $i=1,\dots,\rho$ should be chosen such that the $(\rho+1)\times(\rho+1)$ matrix $\Lambda_o=\begin{bmatrix}-L_1 & I_\rho\\ \vdots & \\ -L_{\rho+1} & 0\end{bmatrix}$ is Hurwitz. Then, there exists a positive constant $\varepsilon>1$, to make the error dynamic (18) FT-ISS as,

$$\|e\|\le v^{m+1}\underset{t}{Sup}\|\Gamma(v)\|\left[\varepsilon\, e^{\Lambda_o t}\,\|G(1)\|\,\|e(0)\|+\int_0^t e^{\Lambda_o(t-\tau)}\,\mu_o\,|S|\,d\tau\right] \quad (24)$$

where $\mu_1=\frac{1+e^{\frac{-t}{\acute{T}_o}}}{2e^{\frac{-t}{\acute{T}_o}}}$, $\mu_o={\mu_1}^{\rho+m+1}$ and $v=\mu_1^{-1}$ are positive time-varying functions. The integer $m$ is a design parameter and $G(\mu_1)$ is a lower triangular $(\rho+1)\times(\rho+1)$ matrix in the form of (25). Indeed, the element $i,j$ is given as $\bar{g}_{i,j}\mu_1^{\rho+i-j}$. Also, $\Gamma(v)$ is defined as $G(\mu_1)^{-1}$, where $G(\mu_1)\Gamma(v)=I$.

$$G(\mu_1)=\begin{bmatrix}\mu_1^{\rho} & 0 & 0 & \cdots & 0\\ \bar{g}_{2,1}\mu_1^{\rho+1} & \mu_1^{\rho} & 0 & \cdots & \vdots\\ \bar{g}_{3,1}\mu_1^{\rho+2} & \bar{g}_{3,2}\mu_1^{\rho+1} & \ddots & 0 & \vdots\\ \vdots & \vdots & \vdots & \ddots & 0\\ \bar{g}_{\rho+1,1}\mu_1^{2\rho} & \bar{g}_{\rho+1,2}\mu_1^{2\rho-1} & \dots & \bar{g}_{\rho+1,\rho}\mu_1^{\rho+1} & \mu_1^{\rho}\end{bmatrix} \quad (25)$$

**Proof:** Consider the transformation $e_i\to\omega_i$ defined as

$$\omega_i=\mu_o(t,\acute{T}_o)e_i \qquad i=1,\dots,\rho+1 \quad (26)$$

The derivatives of $\omega_i$'s along with the error dynamics (18) are as

$$\dot{\omega}_i=\frac{\dot{\mu}_o}{\mu_o}\omega_i+\omega_{i+1}-g_i\omega_1,\qquad i=1,\dots,\rho$$
$$\dot{\omega}_{\rho+1}=\frac{\dot{\mu}_o}{\mu_o}\omega_{\rho+1}-g_{\rho+1}\omega_1+\mu_o S \quad (27)$$

Besides, the transformation $\omega_i\to z_i$ is defined as

$$z_i=\sum_{j=1}^{i}\bar{g}_{i,j}\mu_1^{i-j}\,\omega_j \quad (28)$$

Therefore,

$$\dot{z}_i=\sum_{j=1}^{i}\bar{g}_{i,j}\mu_1^{i-j}\,\dot{\omega}_j+\sum_{j=1}^{i}\bar{g}_{i,j}(i-j)\mu_1^{i-j-1}\dot{\mu}_1\,\omega_j \quad (29)$$

By substituting (26)-(28) into (29) it can be shown that for $i=1,\dots,\rho$

$$\dot{z}_i=\left[\bar{g}_{i,1}\left(\frac{\rho+m+1}{2\acute{T}_o}\right)(1-\mu_1^{-1})\mu_1^{i-1}+\bar{g}_{i,1}\left(\frac{i-1}{2\acute{T}_o}\right)(1-\mu_1^{-1})\mu_1^{i-1}-g_i-\bar{g}_{i+1,1}\mu_1^{i}-\sum_{j=1}^{i-1}\bar{g}_{i,j}\mu_1^{i-j}g_j\right]z_1+z_{i+1}+\sum_{j=2}^{i}\left[-\bar{g}_{i,j}\left(\frac{\rho+m+1}{2\acute{T}_o}\right)(1+\mu_1^{-1})-\bar{g}_{i+1,j}\mu_1+\bar{g}_{i,j-1}\mu_1+\bar{g}_{i,j}\left(\frac{i-j}{2\acute{T}_o}\right)(1-\mu_1^{-1})\right]\mu_1^{i-j}\omega_j \quad (30)$$

and

$$\dot{z}_{\rho+1}=\left[\bar{g}_{\rho+1,1}\left(\frac{\rho+m+1}{2\acute{T}_o}\right)(1-\mu_1^{-1})\mu_1^{\rho}+\bar{g}_{\rho+1,1}\left(\frac{\rho}{2\acute{T}_o}\right)(1-\mu_1^{-1})\mu_1^{\rho}-g_{\rho+1}-\sum_{j=1}^{\rho}\bar{g}_{\rho+1,j}\mu_1^{\rho-j+1}g_j\right]z_1+\sum_{j=2}^{\rho+1}\left[-\bar{g}_{\rho+1,j}\left(\frac{\rho+m+1}{2\acute{T}_o}\right)(1+\mu_1^{-1})+\bar{g}_{\rho+1,j-1}\mu_1+\bar{g}_{\rho+1,j}\left(\frac{\rho-j+1}{2\acute{T}_o}\right)(1-\mu_1^{-1})\right]\mu_1^{\rho-j+1}\omega_j+\mu_o S \quad (31)$$

Then, by choosing the gains $\{g_i(t,\acute{T}_o)\}_{i=1}^{\rho+1}$ according to (20), (21), and the parameters $\{\bar{g}_{i,j}\}$ according to (22), (23), it is obtained that

$$\dot{z}_i=-L_i z_1+z_{i+1}$$
$$\dot{z}_{\rho+1}=-L_{\rho+1}z_1+\mu_o S \quad (32)$$

Or in the compact form as

$$\dot{z}=\Lambda_o z+B\mu_o S \quad (33)$$

where $B$ has been defined in Section 2. By choosing the constants $\{L_i\}$ to make the matrix $\Lambda_o$ Hurwitz, there exists a positive constant $\varepsilon>1$, such that,

$$\|z\|\le\varepsilon\|e^{\Lambda_o t}\|\|z(0)\|+\int_0^t e^{\Lambda_o(t-\tau)}\,\mu_o|S|d\tau \quad (34)$$

Now, combining the transformations (26) and (28) leads to

$$z=\mu_1^{m+1}G(\mu_1)e \quad (35)$$

Also, the inverse of the transformation (35) is as follows:

$$e=v^{m+1}\Gamma(v)z \quad (36)$$

Accordingly,

$$\|e\|\le v^{m+1}\underset{t}{Sup}\|\Gamma(v)\|\|z\| \quad (37)$$

Moreover, based on the transformation (35), one has

$$\|z(0)\|\le\|G(1)\|\|e(0)\| \quad (38)$$

where G(1) is the matrix $G(\mu_1)$ at the time $t=0$. Then, by substituting (33) into (29) and subsequently substituting this result into (32), one has

$$\|e\| \leq v^{m+1} \underset{t}{Sup} \|\Gamma(v)\| \left[ \varepsilon e^{\Lambda_o t} \|G(1)\| \|e(0)\| + \int_0^t e^{\Lambda_o (t-\tau)} \mu_o |S| d\tau \right] \quad (39)$$

As a result, since $G(\mu_1)$ is a positive definite matrix for any $t \geq 0$, the greatest lower bound of $G(\mu_1)$ (i.e. $\underset{t}{inf} |G(\mu_1)|$ ) exists, and consequently $\underset{t}{Sup} |\Gamma(v)|$ is bounded. Therefore, according to Remark 2, if the right-hand side of the inequality (39) (except $v^{m+1}$) remains bounded, based on the time behavior of $v^{m+1}$, clearly, $e$ will tend to the neighborhood of zero and will remain uniformly bounded in finite time. Therefore, according to definitions 1 and 2, the boundedness and/or FT-ISS of the error variables $e$ are achieved. This completes the proof. ☐

**Remark 4:** According to [3], if the eigenvalues of $\Lambda_o$ are placed on the left side of $-\frac{\rho+m+1}{\acute{T}_o}$, the right-hand side of the inequality (39) will remain bounded.

## 5. Finite-time output feedback synthesis

Now, we are in a position to design a global FT-OFC for the nonlinear system (16). Note that in Section 4, the FT-ESO was designed without a priori knowledge of the upper-bound of the generalized disturbance $\Delta_{eq}$. However, in this section, more information on this term can be considered. Indeed, based on Remark 3, it is assumed that the generalized disturbance $\Delta_{eq}$ is uniformly bounded with an unknown upper bound, and also Lipschitz continuous with a known Lipschitz constant $\dot{\Delta}_{eq} \leq \sigma$ [7].

Subsequently, to design the global FT-OFC, in (7) substituting $\xi_i$ $(i = 2, \dots, \rho)$ by its estimated values $\hat{\xi}_i$ from (17) leads to

$$u = -\left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c} + k_2\right)\left(a_1\xi_1 + \sum_{i=2}^{\rho-1} a_i\hat{\xi}_i + \hat{\xi}_\rho\right) - \sum_{i=1}^{\rho-1} a_i\hat{\xi}_{i+1} - \bar{\Theta} - \hat{\Delta}_{eq} \quad (40)$$

where only $\xi_1$ is measurable, $\hat{\Delta}_{eq} = \hat{\xi}_{\rho+1}$ is generated by the FT-ESO (17), and all the design parameters are the same as those in Theorems 1, 2. Also, the auxiliary term $\bar{\Theta}$ will be designed to suppress the effect of the estimation errors. In Theorem 3, based on the known upper bound of the derivative of $\Delta_{eq}$, the FTB property of the perturbed system (16) is ensured.

**Theorem 3:** The perturbed closed-loop system (16), with the FT-OFC law (40) and the variables $\hat{\xi}_i$ for $i = 2, \dots, \rho$ generated by the observer (17) will be robust FTB if the design parameters are selected as in Theorems 1 and 2.

**Proof:** Based on Theorem 2, we only know so far that the observation error converges to a finite ball around the origin, and there exists a finite time, (for instance $T_o$) such that all the error variables approach a small bound around the origin so that $\hat{\xi}_i = \xi_i - e_i$ for $t > T_o$. Thus, this ultimate boundedness should be explicitly considered in the proof; since it acts as a (small) uncertainty in the values of the state variables. As a result, the FT-OFC law (40) for all $t > T_o$ is as follows

$$u = -\left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c} + k_2\right)\left(a_1\xi_1 + \sum_{i=2}^{\rho-1} a_i(\xi_i - e_i) + \xi_\rho - e_\rho\right) - \sum_{i=1}^{\rho-1} a_i(\xi_{i+1} - e_{i+1}) - \bar{\Theta} - \hat{\Delta}_{eq} \quad (41)$$

and it coincides with the continuous observer-based form of the state feedback control law (7) with the additional terms of the ultimate boundedness effect (the estimation errors) as follows:

$$u = \underbrace{-\left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c} + k_2\right)\left(\xi_\rho + \sum_{i=1}^{\rho-1} a_i\xi_i\right) - \sum_{i=1}^{\rho-1} a_i\xi_{i+1}}_{Nominal\ form\ of\ (7)} - \left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c} + k_2\right)\left(-\sum_{i=2}^{\rho-1} a_i e_i - e_\rho\right) + \sum_{i=1}^{\rho-1} a_i e_{i+1} - \bar{\Theta} - \hat{\Delta}_{eq} \quad (42)$$

where the auxiliary term $\bar{\Theta}$ is designed to suppress the effect of the estimation errors. In the following, we must show that assuming these conditions, the closed-loop system remains ultimately bounded as well. Furthermore, if the system trajectory under the FT-OFC (40) does not escape during this time interval, according to Theorem 1 there exists a finite time, (for instance $T_c$), to make the system (16) FTB. Therefore, in the first step, it is enough to show that the closed-loop system under the FT-OFC (40) has not any finite escape time. In this regard, let us consider the following derivative of the time-varying sliding surface $s = \omega_\rho + \sum_{i=1}^{\rho-1} a_i\omega_i$ along with the trajectory (16) under the FT-OFC (40):

$$\begin{aligned}
\dot{s} &= \dot{\mu}_c\left(\xi_\rho + \sum_{i=1}^{\rho-1} a_i\xi_i\right) + \mu_c\left(\dot{\xi}_\rho + \sum_{i=1}^{\rho-1} a_i\dot{\xi}_i\right) \\
&= \dot{\mu}_c\left(\xi_\rho + \sum_{i=1}^{\rho-1} a_i\xi_i\right) \\
&+ \mu_c\Bigg(-\left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c} + k_2\right)\left(\xi_\rho + \sum_{i=1}^{\rho-1} a_i\xi_i\right) - \sum_{i=1}^{\rho-1} a_i\xi_{i+1} \\
&- \left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c} + k_2\right)\left(-\sum_{i=2}^{\rho-1} a_i e_i - e_\rho\right) + \sum_{i=1}^{\rho-1} a_i e_{i+1} - \bar{\Theta} \\
&- \hat{\Delta}_{eq} + \Delta_{eq} + \sum_{i=1}^{\rho-1} a_i\xi_{i+1}\Bigg)
\end{aligned} \quad (43)$$

It can be shown that

$$\dot{s} = -k_2 s + \mu_c\left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c} + k_2\right)\left(\sum_{i=2}^{\rho-1} a_i e_i + e_\rho\right) + \mu_c\sum_{i=1}^{\rho-1} a_i e_{i+1} - \mu_c\bar{\Theta} + \mu_c e_{\rho+1} \quad (44)$$

Since all observation errors are guaranteed to be FTB, and based on the upper bound of the derivative of the generalized disturbance as $|\dot{\Delta}_{eq}| \leq \sigma$, the additional term $\bar{\Theta}$ is designed such that

$$\bar{\Theta} \geq sup\left\{\left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c}+k_2\right)\left(\sum_{i=2}^{\rho-1} a_i e_i + e_\rho\right) + \sum_{i=1}^{\rho-1} a_i e_{i+1} + e_{\rho+1}\right\} \quad (45)$$

where this upper bound can be calculated via the inequality (39) using the upper bound $|S| \leq \sigma$. Therefore, $\dot{s} \leq -k_2 s - \mu_c \overline{\Pi}$; where $\overline{\Pi} = \bar{\Theta} - \left|\left(\frac{1-\mu_c{}^{-1}}{2\acute{T}_c}+k_2\right)\left(\sum_{i=2}^{\rho-1} a_i e_i + e_\rho\right) + \sum_{i=1}^{\rho-1} a_i e_{i+1} + e_{\rho+1}\right|$ is positive. For the convenience of the proof, depending on the variable $s$ two different cases may occur:

***Case 1* ($s \geq 0$):**

In this case, $\dot{s}$ satisfies the following inequality:

$$\dot{s} \leq -\mu_c \overline{\Pi} \quad (46)$$

***Case 2* ($s < 0$):**

In this case, $\dot{s}$ satisfies the following inequality:

$$\dot{s} > -\mu_c \overline{\Pi} \quad (47)$$

It follows from (46) and (47) that, $s$ as well as the state variables $\xi_i$ cannot escape in any finite time interval. From the above analysis, it can be concluded that the system (16) with the FT-OFC (40) has not any finite escape time. In the following, we must show that with these conditions, the closed-loop system remains ultimately bounded as well. In this regard, the inequality $\dot{s} \leq -k_2 s - \mu_c \overline{\Pi} < -k_2 s$ based on the Lyapunov function $V$, can be rewritten as $\dot{V} \leq -k_2 s^2 = -2k_2 V$. Finally, based on the analysis at the beginning of the proof, and based on the proof of Theorem 1, $\xi_i$ tends to the neighborhood of zero and remains uniformly bounded in finite-time, and the FTB of the closed-loop system (16) under the FT-ESO (17) and FT-OFC (40) is guaranteed. This completes the proof. □

**Remark 5:** One of the main concerns of the proposed scheme is the definition of the time-varying transformations $\mu_c$ and $\mu_o$. From their definition, it can be concluded that they tend to infinity when $t \to \infty$. However, the designed observer provides accurate values for the states and the control objective is achieved in finite time, therefore without loss of generality, the transformation is only needed during a first interval. Thus, the finite-time stability is achieved before any diverging effect could appear. However, to theoretically present a sound approach to these functions, they may be re-defined for theoretical purposes as:

$$\mu_c = \begin{cases} \dfrac{1+e^{\frac{-t}{\acute{T}_c}}}{2e^{\frac{-t}{\acute{T}_c}}} & t \leq \acute{T}_c \\ \mu_{cMax} & t > \acute{T}_c \end{cases}$$

$$\mu_o = \begin{cases} \left(\dfrac{1+e^{\frac{-t}{\acute{T}_o}}}{2e^{\frac{-t}{\acute{T}_o}}}\right)^{\rho+m+1} & t \leq \acute{T}_o \\ \mu_{oMax} & t > \acute{T}_o \end{cases} \quad (48)$$

where for positive real constants $w_1$ and $w_2$, the constants $\mu_{cMax}$ and $\mu_{oMax}$ are defined as $w_1 \frac{1+e^{-1}}{2e^{-1}}$ and $w_2 \left(\frac{1+e^{-1}}{2e^{-1}}\right)^{\rho+m+1}$, respectively. The constants $w_1$ and $w_2$ will be designed based on a trade-off between the finite-time performance and the observation errors. Subsequently, to adapt the Lyapunov-based stability proofs to this new definition, we consider two intervals $t \leq \acute{T}_c$ and $t > \acute{T}_c$ (and/or $t \leq \acute{T}_o$ and $t > \acute{T}_o$) to perform the analysis. For the first one, the previous proofs are valid. Then, for the second time intervals, we have already proved that the system's states are in a ball around the control objectives for the designed control law and observed states and they remain within the ball onwards. It is also important to notice that the control laws (7) and (40) remain bounded for all time no matter which definition of $\mu_x$ is taken.

## 6. Simulation example

In this section, numerical simulations are carried out on three examples to show the efficiency of the proposed FT-OFC compared with the superior existing approaches [22], [23], [24].

**Example 1:**

In this example, similar to [23], the simulation is carried out using the Euler method with a fixed sampling time equal to $10^{-4}$ seconds.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u + d(t) \\ y &= x_1 \end{aligned} \quad (49)$$

In this system, only the first state variable $x_1$ is considered measurable as the output. Thus, the relative degree of this system is $\rho = 2$, and it is in standard normal form. Moreover, according to Section 3, under $d(t) = \frac{1}{2}(sin(t) + cos(t))$ defines $x_3 = d(t)$ as the generalized disturbance. Despite the reference [23] dates back more than 3 years ago its approach has been used many times in recent years due to its proper performance, [7]. Thus, the simulations ensure the fairness of the comparison.

First, the FT-ESO performance compared with the fixed-time observer of [23] for the open-loop double integrator system (49) is shown. Simulation results are achieved under the design parameters $\acute{T}_c = \acute{T}_o = 5$, $L_1 = 0.5, L_2 = 2.5, L_3 = 5, a_1 = 1,\ k_2 = 2$, and $\bar{\Theta} = 25$. These parameters have been chosen to meet the mentioned conditions in the proofs; so that, it has the following steps:

(1) To achieve the finite-time convergence, first, the design parameters $\acute{T}_c$ and $\acute{T}_o$ are chosen. Increasing the value of $\acute{T}_c$ reduces the control effort considerably; however, it will increase the convergence time significantly and vice versa. Also, increasing the value of $\acute{T}_o$, increases the observer convergence time, and vice versa.

(2) To ensure the stability of the designed observer, the constant gains $L_1, L_2$, and $L_3$ are designed to make the matrix $\Lambda_o$, Hurwitz.

(3) Since the simulations are carried out for the double integrator system, with the relative degree $\rho = 2$, the constants $a_1$ and $k_2$ should be positive only. Increasing the values of $a_1$ and $k_2$ increases the control effort considerably, and vice versa.

Also, simulation results are provided to illustrate the effectiveness of the proposed approach compared to prominent existing references, which shows better performance compared to previously published papers.

Fig. 1 shows the time evolution of the real states and their estimations in the perturbed open-loop system (49) using the continuous and chattering-free FT-ESO (17). It can be concluded that despite the existing disturbance, the proposed FT-ESO has achieved an appropriate estimation performance. Moreover, it has less convergence time (less than 0.6 $s$) compared to the privileged source [23] (almost equal with 2.2 $s$). On the other hand, Fig. 2 shows the performance of the FT-ESO in the closed-loop system under the FT-OFC (40). The appropriate performance of the proposed FT-OFC as a finite-time controller compared to [23] is obvious.

Furthermore, it is worth noting that, it is impossible to achieve the identity $x_i \equiv 0$, due to the impacts of imperfections like the generalized disturbance [7, 8]. Therefore, regarding Definitions 1, 2, the proposed FT-OFC can achieve satisfactory FTB performances in the presence of disturbances.

Also, since observation errors $e_2$ and $e_3$ were guaranteed to be FTB; therefore, the upper bound of $\bar{\Theta}$ exists. Its bound can be confirmed based on the simulation results. Indeed, simulation results show a weak dependence of the observation errors on the initial conditions; therefore, the upper bound (45) can be evaluated via offline and/or open-loop tests. In this regard, based on the simulation results, $sup\{e_2\} = sup\{e_3\} = 5$; also, according to Remark 5, $sup\{\mu_c\} = \mu_{cMax} = \frac{1+e^{-1}}{2e^{-1}}$. Therefore, $\bar{\Theta} \geq 5sup\left\{\frac{1+e^{-1}}{20e^{-1}} + 4\right\} = 20.9296$.

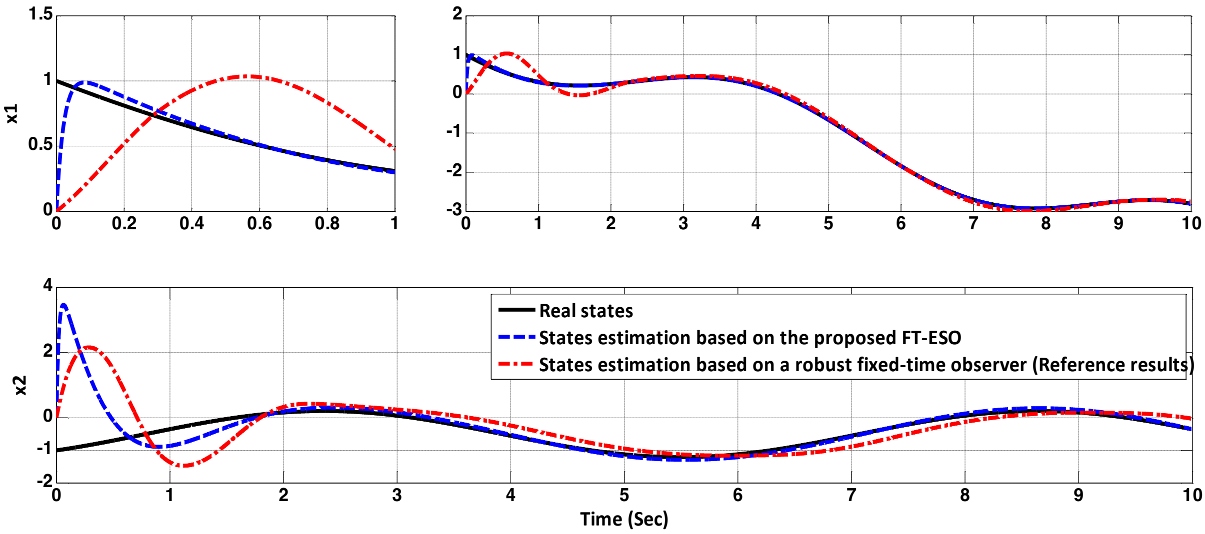


*Fig. 1. Time evolutions of the state estimations in the open-loop system.*

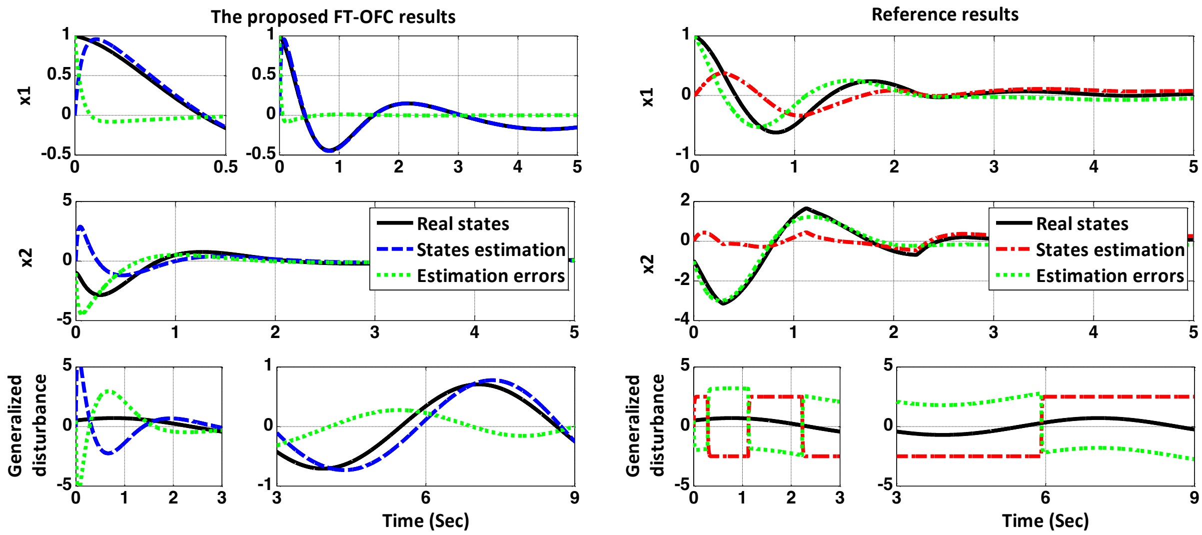


*Fig. 2. Time evolutions of state estimations in the closed-loop system.*

Figs. 3 and 4 show the results of the proposed chattering-free FT-OFC compared to [23] in terms of stability behavior and control efforts, respectively. In this regard, an acceptable convergence in both methods has been achieved. However, by defining $J_x = \left\| \int_0^{t_s} x^T x \, dt \right\|$ over the time interval $t \in [0, t_s = 10\ seconds]$, the comparative results are presented in Table 1.

**Table 1** Comparative results on the performance index $J_x$

| Methods | Performance index $J_x$ |
|---|---|
| Proposed FT-OFC | $1.323\text{E}+7$ |
| Fixed-time output feedback control [23] | $2.719\text{E}+7$ |

According to Table 1, the proposed FT-OFC demonstrates superior FTB performance compared to the fixed-time output feedback controller of [23]. Two other performances indices as $J_u = \left\| \int_0^{t_s} u^T u \, dt \right\|$ and $J_e = \left\| \int_0^{t_s} e^T e \, dt \right\|$ based on the input and the observation error ($u$ and $e$) are defined in Table 2 as energy indices of control effort and error variables.

**Table 2** Comparative results on the performance indices $J_u$ and $J_e$

| Methods | Performance index $J_u$ | Performance index $J_e$ |
|---|---|---|
| Proposed FT-OFC | 3.834E + 4 | 9.561E + 7 |
| Fixed-time output feedback control [23] | 5.993E + 4 | 2.369E + 9 |

According to Table 2, a significant difference between the performance indices is shown. Although an acceptable FTB performance of both controllers has been achieved, the controller of [23] has poor performance in the presence of disturbance; whereas the proposed FT-OFC demonstrates superior FTB performance.

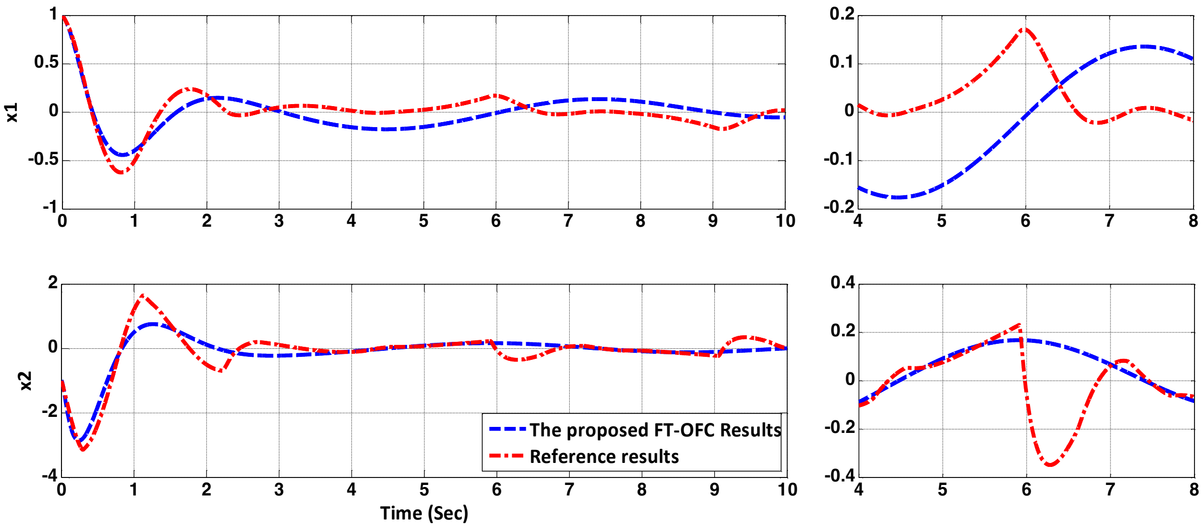


***Fig. 3.*** *Time evolutions of state variables of the closed-loop system.*

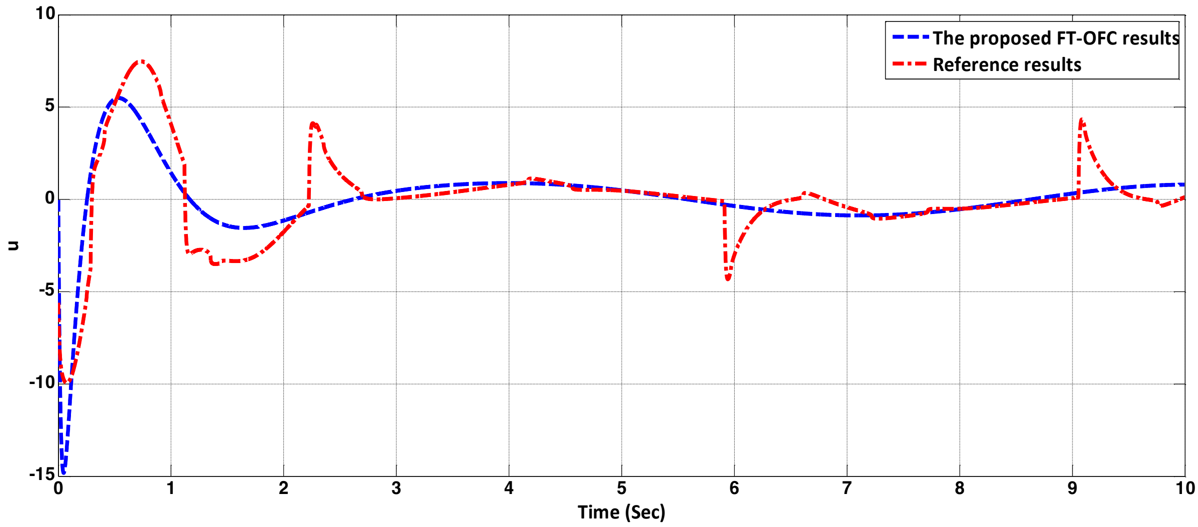


***Fig. 4.*** *Time evolutions of control inputs.*

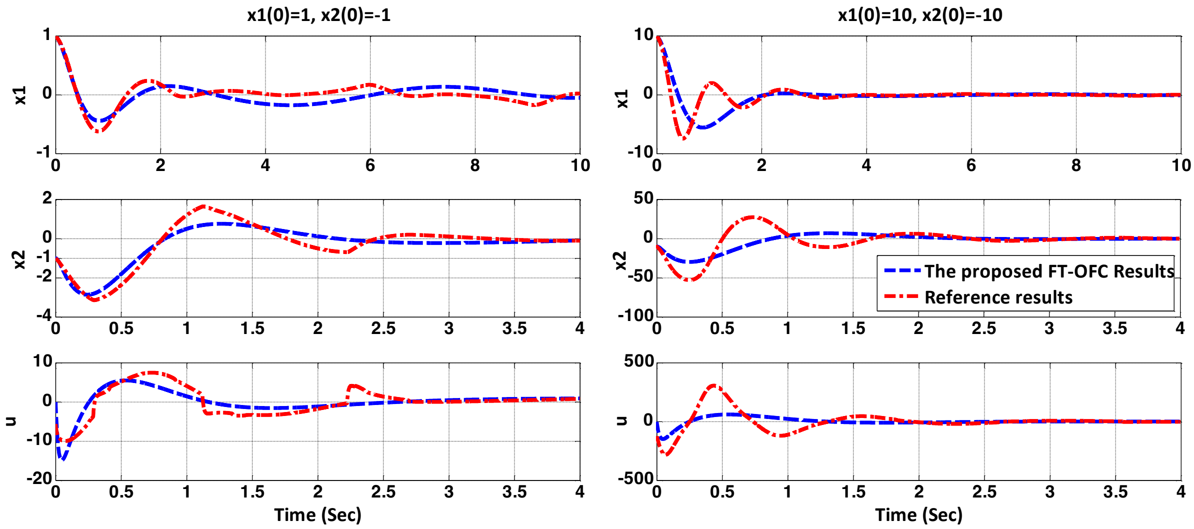


***Fig. 5.*** *Time evolutions of state variables of the closed-loop system for two values of the initial conditions.*

Fig. 5 shows that the proposed FT-OFC leads to the convergence of the state variables to a small neighborhood of zero regardless of the initial conditions. In this way, a weak dependence of the convergence time on the initial value is achieved. In this regard, Table 3 contains the performance indices of the proposed FT-OFC compared to [23] under different initial conditions.

**Table 3** Comparative results of the performance indices $J_u$ and $J_e$ under different initial conditions

| Methods | Performance index $J_u$ | | Performance index $J_e$ | |
|---|---|---|---|---|
| initial cond. $x_0$ | $\begin{bmatrix}1\\-1\end{bmatrix}$ | $\begin{bmatrix}10\\-10\end{bmatrix}$ | $\begin{bmatrix}1\\-1\end{bmatrix}$ | $\begin{bmatrix}10\\-10\end{bmatrix}$ |
| Proposed FT-OFC | $3.83E+4$ | $3.83E+6$ | $9.56E+7$ | $9.2E+11$ |
| Controller of [23] | $5.99E+4$ | $3.15E+7$ | $2.37E+9$ | $1.4E+12$ |

According to Table 3, although there are significant differences between the performance indices under different initial conditions, the convergence time and the control effort have a weak dependency on the initial conditions, compared to the controller of [23] (Fig. 5).

To evaluate the effectiveness of the proposed FT-OFC scheme, in the next subsection, simulation studies will be addressed for the disturbed system (49) with unknown measurement noise.

### *6.1. Robustness to measurement noise*

In this section, the strong robustness of the proposed FT-OFC scheme to measurement noise is investigated. Assume that the output is available as $y = x_1 + N(t)$ where $N(t)$ is a Band-Limited White Noise. As seen in Figs. 6 and 7, the state variables $x_1$ and $x_2$ converge to a smaller neighborhood of zero compared to [23]. As Fig. 6 shows, despite the system is disturbed by a rather strong measurement noise, the system is ultimately bounded.

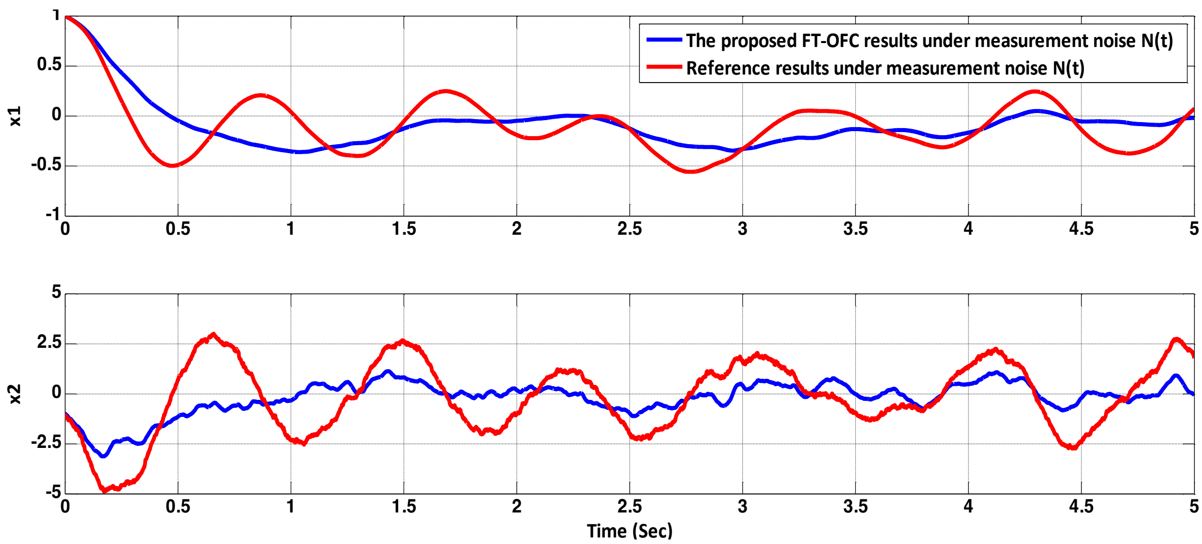


***Fig. 6.*** *Time evolutions of state variables of the closed-loop system under measurement noise.*

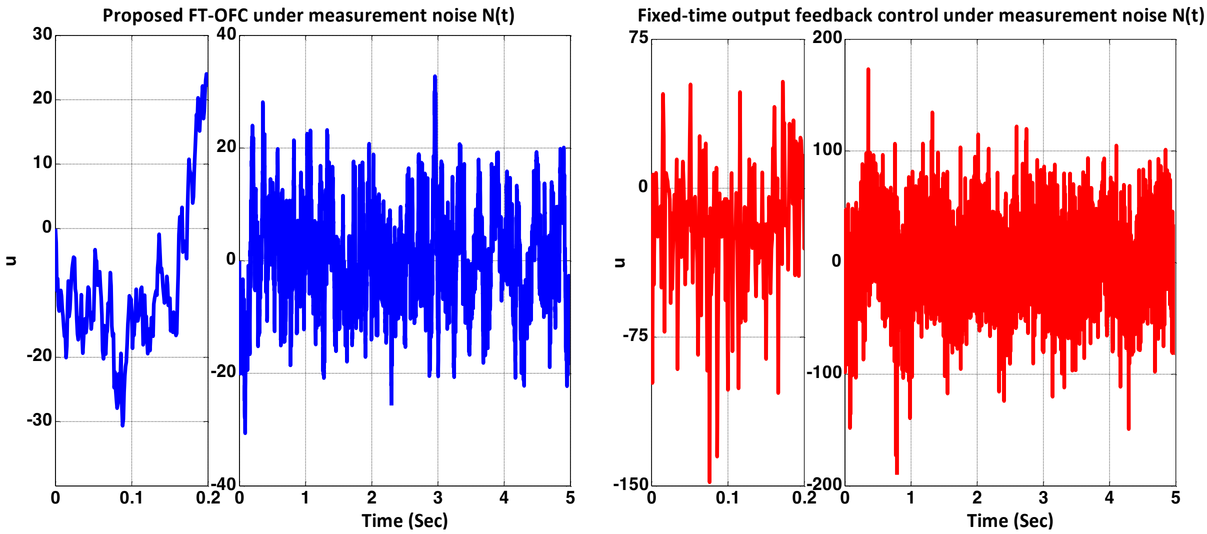


*Fig. 7. Time evolutions of control inputs under measurement noise.*

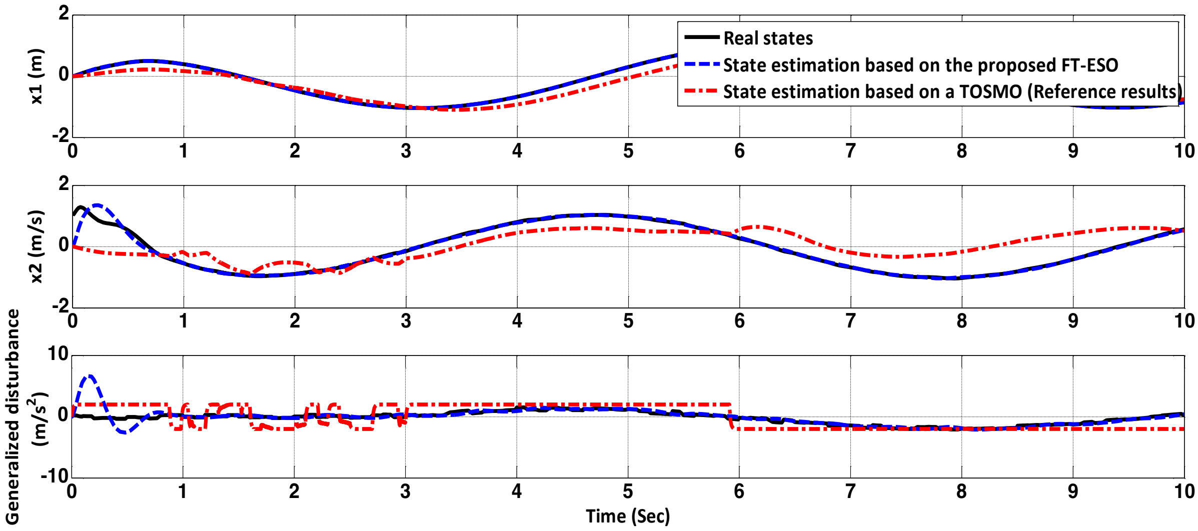


*Fig. 8. Time evolutions of the state estimations of the closed-loop system.*

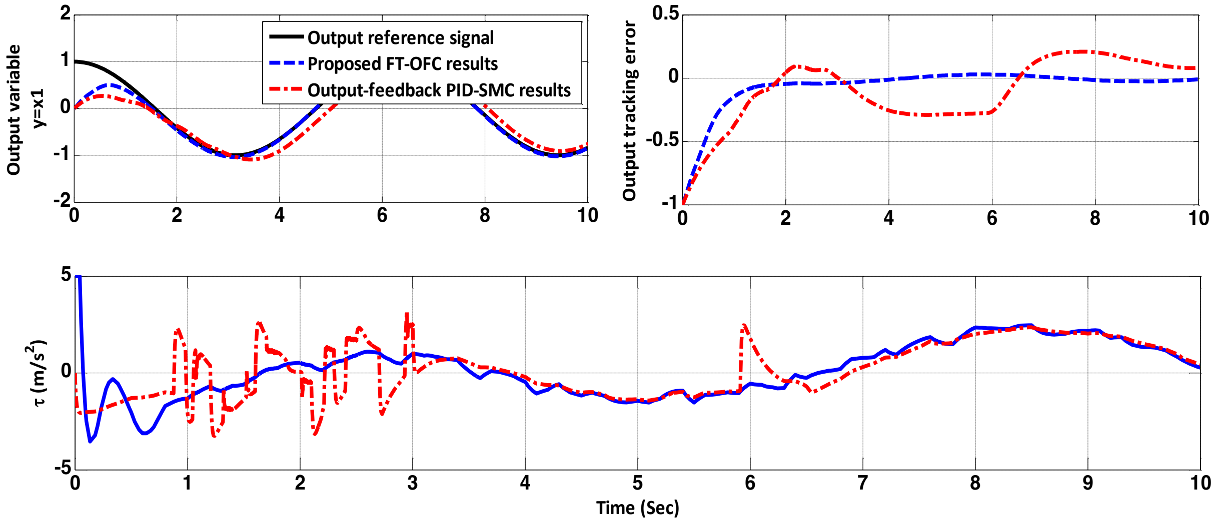


*Fig. 9. Time evolutions of the Reference signal, the output variables of the closed-loop system, the tracking errors, and the tracking control inputs.*

**Example 2:**

In this example, consider the following uncertain *1*-link robotic manipulator system [22]:

$$M(\alpha)\ddot{\alpha} + F(\alpha,\dot{\alpha}) + G(\alpha) = \tau - \tau_d + \Xi(t - T_f)\lambda(t) \quad (50)$$

where $\alpha, \dot{\alpha}, \ddot{\alpha} \in R$ represent the position, velocity, and acceleration of robotic joints, respectively. In this paper,

$x = [\alpha, \dot{\alpha}]^T$ is considered as the state vector and thus the robot dynamical system (50) can be rewritten in state-space form as:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(M_0 + \Delta M)^{-1}[F + G_0 + \cdots \\ &\cdots + \Delta G + \tau_d + \Xi(t - T_f)\lambda - \tau \end{aligned} \quad (51)$$

where the constants $M = 32, F = 0.8\,\dot{\alpha} + 1.2\cos(3\alpha)$ and $G = \pm\varrho g$ denote the inertia, friction, and gravitational force terms, respectively. Also, $\lambda = 2\cos(0.8t)\,, \tau$ and $\tau_d = 1.2\sin(0.95\dot{\alpha})$ denote the unknown faults, the input torque, and a bounded external disturbance, respectively. Moreover, $\varrho \in [0,1]$ and $g = 9.806\ m/s^2$ are respectively a random variable presenting the model uncertainties and the acceleration of gravity. In this paper, it is assumed that the unknown fault term $\lambda$ is the same as the disturbance term $\tau_d$; but the fault occurrence time $T_f$, may change stochastically with the time profile $\Xi(t - T_f)$ that is defined as:

$$\Xi(t-3) = \begin{cases} 0 & t \le 3 \\ 1 - e^{-(t-3)} & t \ge 3 \end{cases} \quad (52)$$

where the simulation results are given with $T_f = 3$. The goal is to implement the proposed FT-OFC law for robot system (51); so that, the position output $y = x_1$ tracks the time-varying reference signal $y_r = \cos(t)$ in finite time. The comparison results are shown in figures 8 and 9.
In Figure 9, the proposed FT-OFC is compared to a third-order sliding mode controller (TOSMC) [22] in terms of the tracking error and control efforts.

**Example 3:**

Consider the following uncertain nonlinear system [2]:

$$\begin{aligned} \dot{x}_1 &= -2x_1 + x_2 - x_3 \\ \dot{x}_2 &= -x_1 x_3 - 2x_2 + u \\ \dot{x}_3 &= -x_1 + u + d(t) \\ y &= x_3 \end{aligned} \quad (53)$$

where $d(t) = sin\,(0.8\pi t)$ and the relative degree of the system is $\rho = 1$. Thus, using a diffeomorphism map $T(x) = [\eta_1 \quad \eta_2 \quad \vdots \quad \xi_1]^T = [x_1 \quad x_2 - x_3 \quad \vdots \quad x_3]^T$ and based on Section 2 [2], the augmented normal form of the system is as follows:

$$\begin{aligned} \dot{\eta}_1 &= -2\eta_1 + \eta_2 \\ \dot{\eta}_2 &= \eta_1 - 2\eta_2 - 2\xi_1 - \eta_1\xi_1 \\ \dot{\xi}_1 &= \xi_2 + u \\ \dot{\xi}_2 &= S \end{aligned} \quad (54)$$

The zero dynamics are $\dot{\eta} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\eta$. The characteristic equation is $s^2 + 4s + 3$ that is Hurwitz. Therefore, the internal dynamics of the system are ISS. Thus, system (54) is minimum phase and Remark 1 is guaranteed. This is presented in the zero dynamic phase-plane plots (Figure 10). On the other hand, $\xi_2 = -\eta_1 + d(t)$ is the generalized disturbance, thus based on the ISS result of the internal dynamic, Remark 3 is met.

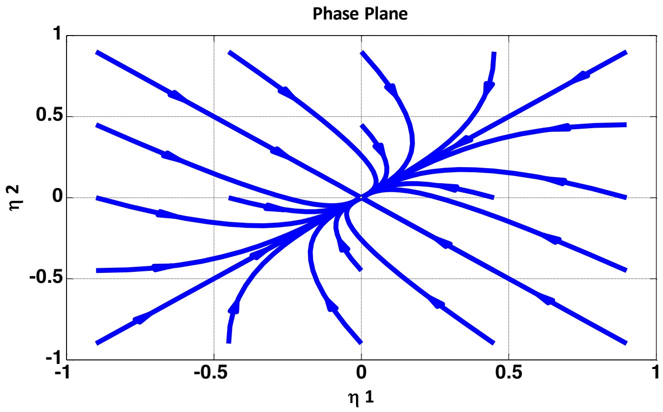


***Fig. 10.*** *Phase-plane plot of zero dynamic.*

The comparison study is not limited to the finite time control methods in order to ensure the wideness property of the simulations. Thus results are shown compared to the time-varying exponential approach [24]. Fig. 11 shows the time evolution of the states and states estimation. The proposed finite-time ESO achieves an acceptable estimation performance in the presence of disturbances and immeasurable internal dynamics in terms of convergence time and estimation errors. It has a smaller convergence time compared to [24].

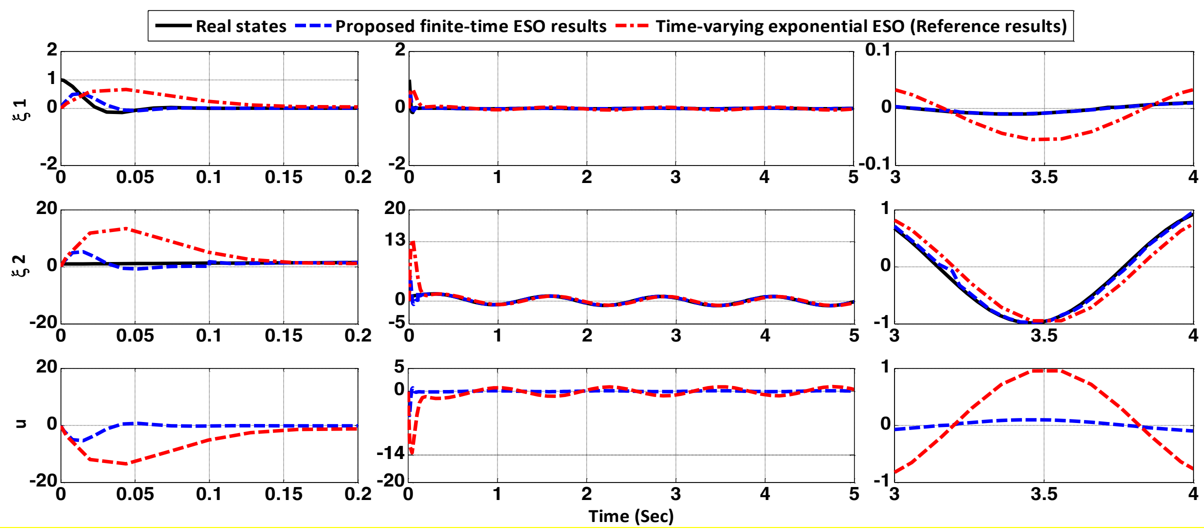


***Fig. 11.*** *Time evolutions of the state estimations and control input of the closed-loop system.*

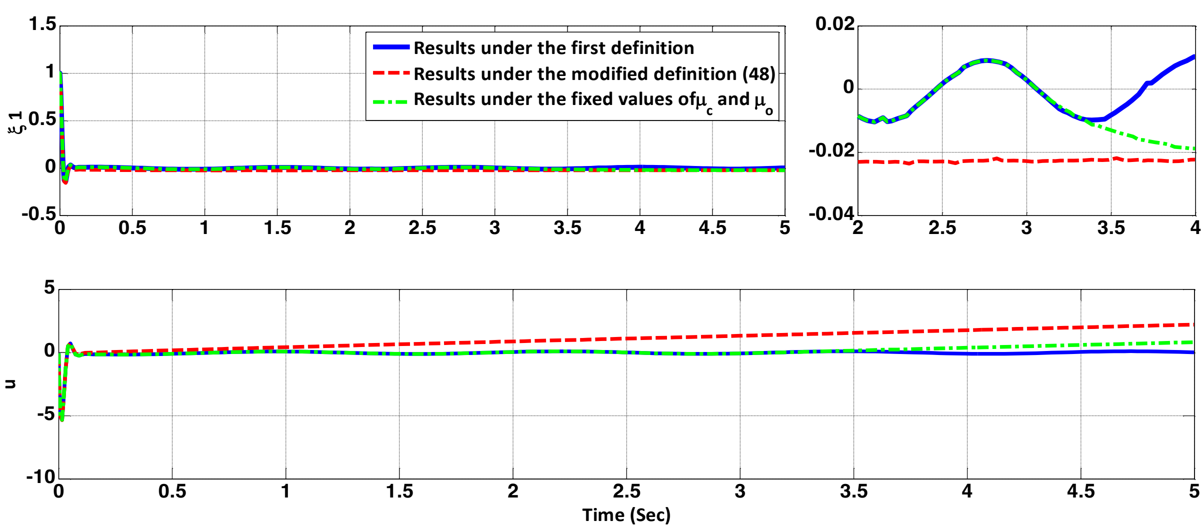


***Fig. 12.*** *Time evolutions of the state estimations and control input of the closed-loop system.*

Moreover, the new definition (48) is providing us with an opportunity to further clarify certain aspects of this work. The figure 12 shows that the finite-time performance has been achieved for all three cases (under the first definition, under the modified definition (48), and under the fixed-values based on the reviewer comment); but, according to the third part of this figure, the desired behavior of the control input has been achieved for the first two cases (under the first definition, under the modified definition (48)). In fact, if the maximum values are always set for two parameters of (48), the control input will increase significantly. Therefore, the two parameters of (48) should not always set as theirs maximum values.

## 7. Conclusion

The main purpose of this paper was to design a finite-time output feedback controller for a class of uncertain nonlinear systems. In the proposed approach, novel conversions were used to make the finite-time objectives possible. This enabled us to design the output feedback controller straightforwardly with the FTB properties. In this design procedure, the convergence of the state variables to a small neighborhood around zero was achieved in a finite time, without any knowledge about the upper bounds of the generalized disturbance. Moreover, it was shown that the state variables will not escape to infinity in finite time before the convergence of the observer error, and finally, the presented simulation results demonstrated the effectiveness of the proposed methods. In general, performance of the fixed-time stability is better than one of the finite-time stability. However, regardless of the highlighted results, the proposed scheme has valuable potential for competition with fixed-time approaches. In future work, it is suggested to use auxiliary terms to decrease the produced oscillations.

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