TOPOLOGICAL MODULI SPACE FOR GERMS OF HOLOMORPHIC FOLIATIONS

DAVID MARÍN, JEAN-FRANÇOIS MATTEI AND ÉLIANE SALEM

Abstract. This work deals with the topological classification of germs of singular foliations on $(\mathbb{C}^2, 0)$. Working in a suitable class of foliations we fix the topological invariants given by the separatrix set, the Camacho-Sad indices and the projective holonomy representations and we compute the moduli space of topological classes in terms of the cohomology of a new algebraic object that we call group-graph. This moduli space may be an infinite dimensional functional space but under generic conditions we prove that it has finite dimension and we describe its algebraic and topological structure.

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1. Introduction

This work deals with the *topological classification* of germs of *singular foliations* on $(\mathbb{C}^2, 0)$. To every (possibly dicritical) foliation $\mathcal{F}$ we can associate the separatrix set $Sep_{\mathcal{F}}$, that is the collection of all germs at $0 \in \mathbb{C}^2$ of invariant irreducible analytic curves, called *separatrices*, its *minimal reduction map* $E_{\mathcal{F}} : (M_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}}) \to (\mathbb{C}^2, 0)$, cf. [5], and the *marked exceptional divisor*

$$\mathcal{E}^{\diamond}_{\mathcal{F}} = (\mathcal{E}_{\mathcal{F}}, \Sigma_{\mathcal{F}}, \imath_{\mathcal{F}}),$$

where $\Sigma_{\mathcal{F}} := \mathrm{Sing}(\mathcal{F}^{\sharp})$ is the finite set consisting of the singular points of the foliation $\mathcal{F}^{\sharp} := E^{*}_{\mathcal{F}}\mathcal{F}$ and $\imath_{\mathcal{F}}$ is the intersection pairing of $\mathcal{E}_{\mathcal{F}} = E^{-1}_{\mathcal{F}}(0)$ in $M_{\mathcal{F}}$. The topological class of $Sep_{\mathcal{F}}$ is clearly a topological invariant of $\mathcal{F}$. In this paper we will assume that $\mathcal{F}$ is a *generalized curve*, i.e. $\mathcal{F}^{\sharp}$ has no saddle-node singularities. The topological class $[\mathcal{E}^{\diamond}_{\mathcal{F}}]$ of $\mathcal{E}^{\diamond}_{\mathcal{F}}$ (as a marked intrinsic curve) is then a topological invariant of $\mathcal{F}$ because in this situation $E_{\mathcal{F}}$ is also the minimal desingularization map of $Sep_{\mathcal{F}}$, cf. [3].

We know [18] that under some assumptions the Camacho-Sad indices of $\mathcal{F}^{\sharp}$ at the points of $\Sigma_{\mathcal{F}}$ and the holonomy representations (up to inner automorphisms) of every component of $\mathcal{E}_{\mathcal{F}}$ are also topological invariants of the germ $\mathcal{F}$ at $0 \in \mathbb{C}^2$. Our purpose in this work is to describe the set of all other topological invariants and highlight its geometric and algebraic structure.

MAIN RESULT. *Under generic conditions,*

(a) there exists an analytic family of foliations parametrized by a finite dimensional space which gives all the topological types once we fix the topological class of the marked exceptional divisor, the Camacho-Sad indices and the holonomy representations;

(b) the quotient of this complete family by the topological equivalence relation is naturally isomorphic to the abelian group

$$\left(F \oplus B \oplus_{j=1}^{\lambda} (\mathbb{C}^*/\alpha_j^{\mathbb{Z}}) \oplus (\mathbb{C}^*)^{\nu}\right) / Z\,,$$

where $\alpha_j \in \mathbb{C}^*$, F is a finite abelian group, Z is a finite subgroup, B is a direct sum of β totally disconnected subgroups of the unitary group $\mathbb{U}(1)$ and the natural numbers β, λ, ν only depend on the (combinatorics of the) local types of the singularities inside the exceptional divisor.

We will also give an explicit characterization of those foliations satisfying Assertion (a) in the main result above, that we will call finite type foliations.

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2. Statement of results

2.1. **Marking of a foliation.** To give a precise sense to our problem let us call *marked divisor* any collection $\mathcal{E}^\diamond = (\mathcal{E}, \Sigma, \imath)$ consisting of a compact curve with normal crossings $\mathcal{E}$ whose irreducible components are biholomorphic to $\mathbb{P}^1$, a finite subset Σ of $\mathcal{E}$ and a symmetric map $\imath : \mathrm{Comp}(\mathcal{E})^2 \to \mathbb{Z}$, $\mathrm{Comp}(\mathcal{E})$ denoting the set of irreducible components of $\mathcal{E}$. We will denote by $\mathcal{E}^d \subset \mathcal{E}$ the union of the irreducible components of $\mathcal{E}$ that do not contain any point of Σ; we call them *dicritical components*.

A *marking* of a foliation $\mathcal{F}$ by $\mathcal{E}^\diamond$ will be a homeomorphism $f : \mathcal{E} \to \mathcal{E}_\mathcal{F}$ sending Σ onto $\Sigma_\mathcal{F}$ compatible with the intersection pairing:

$$\imath_\mathcal{F}(f(D), f(D')) = \imath(D, D').$$

In this way, the holonomy representations and the Camacho-Sad indices of all pairs $\mathcal{F}^\diamond := (\mathcal{F}, f)$ can now be associated to two common sets of indices: the set $C_{\mathcal{E}^\diamond} := \mathrm{Comp}(\overline{\mathcal{E} \setminus \mathcal{E}^d})$ of irreducible components of $\overline{\mathcal{E} \setminus \mathcal{E}^d}$ and the set

$$I_{\mathcal{E}^\diamond} := \{(D, s) \in C_{\mathcal{E}^\diamond} \times \Sigma \mid s \in D\}.$$

Indeed, we define

$$\mathrm{CS}^{\mathcal{F}^\diamond} := (\mathrm{CS}(\mathcal{F}^\diamond, D, s))_{(D,s)\in I_{\mathcal{E}^\diamond}}, \quad \mathrm{CS}(\mathcal{F}^\diamond, D, s) := \mathrm{CS}(\mathcal{F}^\sharp, f(D), f(s)),$$

$$\dot{\mathcal{H}}^{\mathcal{F}^\diamond} := (\dot{\mathcal{H}}_D^{\mathcal{F}^\diamond})_{D\in C_{\mathcal{E}^\diamond}}, \quad \mathcal{H}_D^{\mathcal{F}^\diamond} := \mathcal{H}_{f(D)}^{\mathcal{F}^\sharp} \circ f_* \ : \ \pi_1(D \setminus \Sigma, \cdot) \longrightarrow \mathrm{Diff}(\mathbb{C}, 0),$$

where $\mathcal{H}_{f(D)}^{\mathcal{F}^\sharp}$ is the $\mathcal{F}^\sharp$-holonomy representation of $\pi_1(f(D) \setminus \Sigma_\mathcal{F}, \cdot)$ in the group $\mathrm{Diff}(\mathbb{C}, 0)$ of germs of holomorphic automorphisms of $(\mathbb{C}, 0)$, $\dot{\mathcal{H}}_D^{\mathcal{F}^\diamond}$ is its class up to inner automorphisms, f_* is the isomorphism induced by f at the fundamental groups level and $\mathrm{CS}(\mathcal{F}^\sharp, f(D), f(s))$ is the Camacho-Sad index of $\mathcal{F}^\sharp$ along $f(D)$ at $f(s)$.

Let us denote by $\mathsf{Fol}(\mathcal{E}^\diamond)$ the set of germs of generalized curves $\mathcal{F}$ at $0 \in \mathbb{C}^2$ for which there exists a marking $f : \mathcal{E} \to \mathcal{E}_\mathcal{F}$ of $\mathcal{F}$ by $\mathcal{E}^\diamond$. Our general goal is to describe a generic subset of the quotient set

$$[\mathsf{Fol}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}$$

of the set $\mathsf{Fol}(\mathcal{E}^\diamond)$ by the equivalence relation:

- $\mathcal{F} \sim_{\mathcal{C}^0} \mathcal{G}$ *if* $\mathcal{F}$ *and* $\mathcal{G}$ *are topologically equivalent as germs at* $0 \in \mathbb{C}^2$.

2.2. **Globalization of topological equivalences.** Consider now the equivalence relation:

- $\mathcal{F} \sim_\mathcal{E} \mathcal{G}$ if $\mathcal{F}^\sharp$ and $\mathcal{G}^\sharp$ are topologically conjugated, as germs along the exceptional divisors, by a germ of a homeomorphism $(M_\mathcal{F}, \mathcal{E}_\mathcal{F}) \to (M_\mathcal{G}, \mathcal{E}_\mathcal{G})$ which is holomorphic at each point of $\Sigma_\mathcal{F} \setminus \mathcal{N}C_\mathcal{F}$,

$\mathcal{N}C_\mathcal{F}$ denoting the subset of the singular points of $\mathcal{E}_\mathcal{F}$, called *nodal corners*, where the Camacho-Sad index of $\mathcal{F}^\sharp$ is a strictly positive real number. Clearly relation $\sim_\mathcal{E}$ is stronger than $\sim_{\mathcal{C}^0}$, but they will coincide on a generic class of foliations when $\mathcal{E}^\diamond$ fulfills the following condition

(TC) *The closure of each connected component of* $\mathcal{E}\setminus\mathcal{E}^d$ *contains an irreducible component* D *with* $\mathrm{card}(D\cap\Sigma)\neq 2$*, i.e. there is no connected component of* $\overline{\mathcal{E}\setminus\mathcal{E}^d}$ *as in Figure 1.*

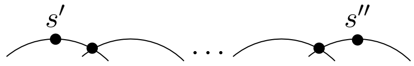

FIGURE 1. The only situation excluded by Condition (TC). Every divisor is non-dicritical and the elements of Σ are s', s'' and the intersection points of the divisors; dicritical components may intersect any component.

To specify the notion of genericity let us call *cut-component of* $\mathcal{E}_{\mathcal{F}}$ any closure $\mathcal{C}$ of a connected component of $\mathcal{E}_{\mathcal{F}}\setminus(\mathcal{E}^d_{\mathcal{F}}\cup\mathcal{N}C_{\mathcal{F}})$; if $\mathrm{card}(D\cap\Sigma)\leq 2$ for each $D\in\mathrm{Comp}(\mathcal{C})$ we will say that $\mathcal{C}$ is *exceptional*. Now consider the following transverse rigidity condition:

(TR) *Any non-exceptional cut-component of* $\mathcal{E}_{\mathcal{F}}$ *contains an irreducible component with topologically rigid*[1] *holonomy group.*

The Krull-open density in $\mathsf{Fol}(\mathcal{E}^\diamond)$ of the subset $\mathsf{Fol}_{\mathsf{tr}}(\mathcal{E}^\diamond)$ consisting of the foliations $\mathcal{F}$ fulfilling Condition (TR) is proven in [10].

Theorem A. *If* $\mathcal{E}^\diamond$ *satisfies condition (TC) then the relations* $\sim_{\mathcal{E}}$ *and* $\sim_{\mathcal{C}^0}$ *are equal on* $\mathsf{Fol}_{\mathsf{tr}}(\mathcal{E}^\diamond)$*.*

In other words

$$[\mathsf{Fol}_{\mathsf{tr}}(\mathcal{E}^\diamond)]_{\mathcal{E}}=[\mathsf{Fol}_{\mathsf{tr}}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}\ \subset\ [\mathsf{Fol}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}\ ,$$

$[X]_{\mathcal{E}}$ and $[X]_{\mathcal{C}^0}$ denoting the quotient of a subset $X\subset\mathsf{Fol}(\mathcal{E}^\diamond)$ by the relations $\sim_{\mathcal{E}}$ and $\sim_{\mathcal{C}^0}$ respectively. This result, proven in Appendix (Theorem 11.4), is an extended version of Main Theorem of [18].

Remark 2.1. Theorem A implies that under the hypothesis (TR) and (TC) the collection of Camacho-Sad indices at the singular points of $\mathcal{F}^\sharp$ is a topological invariant of the germ of $\mathcal{F}$ at 0. The topological classification of logarithmic foliations obtained by E. Paul shows [28, Théorème 3.5] that Condition (TR) is necessary for this. When Condition (TC) is not satisfied it is easy to construct topologically conjugated foliations with same separatrices but different Camacho-Sad indices. Indeed, on a neighborhood of a cut-component described in Figure 1, all the foliations with non-real Camacho-Sad indices are topologically conjugated. □

[1]We recall that a subgroup G of the group $\mathrm{Diff}(\mathbb{C},0)$ of germs of biholomorphisms of $\mathbb{C}$ at 0 is called *topologically rigid* if every topological conjugation between G and another subgroup $G'\subset\mathrm{Diff}(\mathbb{C},0)$ is necessarily conformal. This class contains the non-solvable groups [25] and the non-abelian groups with dense linear part [7, Théorème 2].

2.3. **The Teichmüller space of foliations.** In order to describe $[\mathsf{Fol}_{\mathsf{tr}}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}$ we consider the set $\mathsf{MFol}(\mathcal{E}^\diamond)$ of *marked by* $\mathcal{E}^\diamond$ *foliations* $\mathcal{F}^\diamond = (\mathcal{F}, f)$ and we adapt the equivalence relation $\sim_{\mathcal{E}}$ in $\mathsf{Fol}(\mathcal{E}^\diamond)$ to $\mathsf{MFol}(\mathcal{E}^\diamond)$ by means of

- $(\mathcal{F}, f) \sim_\diamond (\mathcal{G}, g)$ *if there is a germ of homeomorphism* $\Phi : (M_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}}) \to (M_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}})$ *that conjugates* $\mathcal{F}^\sharp$ *and* $\mathcal{G}^\sharp$*, is holomorphic at each point of* $\Sigma_{\mathcal{F}} \backslash \mathcal{N}C_{\mathcal{F}}$ *and its restriction to* $\mathcal{E}_{\mathcal{F}}$ *is isotopic to* $g \circ f^{-1}$ *by an isotopy fixing* $\Sigma_{\mathcal{F}}$*.*

We define the (topological) *Teichmüller space* as the quotient set

$$\mathrm{Mod}(\mathcal{E}^\diamond) := \mathsf{MFol}(\mathcal{E}^\diamond)/\sim_\diamond$$

so that the *Forgetful map*

$$\mathrm{Mod}(\mathcal{E}^\diamond) \longrightarrow [\mathsf{Fol}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}\ , \quad [\mathcal{F}, f] \mapsto [\mathcal{F}]_{\mathcal{C}^0}$$

is well defined. We consider the action

$$\dot{\varphi} \star [\mathcal{F}, f] := \left[\mathcal{F}, f \circ \varphi^{-1}\right]\ , \quad \dot{\varphi} \in \mathrm{Mcg}(\mathcal{E}^\diamond)\,, \quad (\mathcal{F}, f) \in \mathsf{MFol}(\mathcal{E}^\diamond)\,,$$

on $\mathrm{Mod}(\mathcal{E}^\diamond)$ of the *Mapping Class Group* $\mathrm{Mcg}(\mathcal{E}^\diamond)$ of $\mathcal{E}^\diamond$ defined as the group[2] of isotopy classes of $\mathcal{C}^0$-automorphisms of $\mathcal{E}$ leaving the set Σ and the intersection form $\imath$ invariant.

A direct consequence of Theorem A is:

Corollary 2.2. *If* $\mathcal{E}^\diamond$ *satisfies condition (TC) then the fibers of the Forgetful map over* $\mathsf{Fol}_{\mathrm{tr}}(\mathcal{E}^\diamond)$ *are exactly the orbits of the action of the mapping class group* $\mathrm{Mcg}(\mathcal{E}^\diamond)$*. Thus*

$$[\mathsf{Fol}_{\mathsf{tr}}(\mathcal{E}^\diamond)]_{\mathcal{C}^0} \simeq \mathrm{Mod}_{\mathrm{tr}}(\mathcal{E}^\diamond)/\mathrm{Mcg}(\mathcal{E}^\diamond)\,,$$

where $\mathrm{Mod}_{\mathrm{tr}}(\mathcal{E}^\diamond) := \{[\mathcal{F}, f] \in \mathrm{Mod}(\mathcal{E}^\diamond) \,|\, \mathcal{F} \in \mathsf{Fol}_{\mathrm{tr}}(\mathcal{E}^\diamond)\}$.

2.4. **Topological moduli space of a marked foliation.** In order to describe $[\mathsf{Fol}_{\mathrm{tr}}(\mathcal{E}^\diamond)]_{\mathcal{C}^0}$ we are led to study $\mathrm{Mod}_{\mathrm{tr}}(\mathcal{E}^\diamond)$. In fact, we give a description of the whole $\mathrm{Mod}(\mathcal{E}^\diamond)$ without the assumption (TC) on $\mathcal{E}^\diamond$. We obtain it by fixing the Camacho-Sad indices and the holonomy representations. In other words we give a description of each nonempty fiber of the well-defined map

$$\widetilde{\mathcal{H}} := \mathrm{CS} \times \dot{\mathcal{H}} : \mathrm{Mod}(\mathcal{E}^\diamond) \longrightarrow \mathbb{C}^{I_{\mathcal{E}^\diamond}} \times \dot{\mathcal{R}}_{\mathcal{E}^\diamond}\,, \quad [\mathcal{F}^\diamond] \mapsto (\mathrm{CS}^{\mathcal{F}^\diamond}, \dot{\mathcal{H}}^{\mathcal{F}^\diamond})\,,$$

where $\dot{\mathcal{R}}_{\mathcal{E}^\diamond}$ is the set of conjugacy classes of group morphisms from the free product of the groups $\pi_1(D \setminus \Sigma, \cdot)$ for all $D \in C_{\mathcal{E}^\diamond}$ with values in $\mathrm{Diff}(\mathbb{C}, 0)$.

Definition 2.3. *We call topological moduli space of* $[\mathcal{F}^\diamond] \in \mathrm{Mod}(\mathcal{E}^\diamond)$ *the fiber of* $\widetilde{\mathcal{H}}$ *above* $\widetilde{\mathcal{H}}([\mathcal{F}^\diamond])$*, that is the set*

$$\mathrm{Mod}([\mathcal{F}^\diamond]) := \left\{ [\mathcal{G}^\diamond] \in \mathrm{Mod}(\mathcal{E}^\diamond) \mid \mathrm{CS}^{\mathcal{F}^\diamond} = \mathrm{CS}^{\mathcal{G}^\diamond}, \dot{\mathcal{H}}^{\mathcal{F}^\diamond} = \dot{\mathcal{H}}^{\mathcal{G}^\diamond} \right\}.$$

Here the term topological does not refer to the $\mathcal{C}^0$-equivalence of foliations but to the equivalence relation $\sim_\diamond$ on the marked foliations. Indeed, as we have seen in Remark 2.1, if $\mathcal{E}^\diamond$ does not satisfy (TC) or $\mathcal{F} \in \mathsf{Fol}(\mathcal{E}^\diamond)$ does not satisfy Condition (TR) then there may exist $\mathcal{G} \in \mathsf{Fol}(\mathcal{E}^\diamond)$ such that $\mathcal{F} \sim_{\mathcal{C}^0} \mathcal{G}$ and $[\mathcal{F}, f] \neq [\mathcal{G}, g]$ for any markings f, g. However, the descriptions

[2] In fact, it is an extension of a finite group by a direct product of pure Artin groups.

of $\mathrm{Mod}([\mathcal{F}^\diamond])$ given in the following results do not assume conditions (TR) or (TC) anymore.

We will consider in Section 3 a new algebraic notion, which we call *group-graph*, that will be the key tool in the whole paper. It allows us, by combining Theorems 4.9 and 5.14, to obtain a bijection between this moduli space $\mathrm{Mod}([\mathcal{F}^\diamond])$ and the cohomology of a suitable group-graph defined in Section 5, namely the *symmetry group-graph* $\mathrm{Sym}^{\mathcal{F}^\diamond}$. If we call *$\mathcal{F}^\diamond$-cut-component* of $\mathcal{E}$ any inverse image $f^{-1}(\mathcal{C})$ of a cut-component $\mathcal{C}$ of $\mathcal{E}_{\mathcal{F}}$ and if we denote by $\mathsf{A}_{\mathcal{F}^\diamond}$ the dual graph of the disjoint union of all the *$\mathcal{F}^\diamond$-cut-components* of $\mathcal{E}$, then one can prove:

Theorem B. *If $\mathcal{F}^\diamond \in \mathsf{MFol}(\mathcal{E}^\diamond)$ then we have a natural bijection*

$$\mathrm{Mod}([\mathcal{F}^\diamond]) \xrightarrow{\sim} H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})\,.$$

Without any other assumption the computation of this cohomological space is difficult and the usefulness of this result is essentially theoretical. However it will allow us in Section 8 to construct examples for which $\mathrm{Mod}([\mathcal{F}^\diamond])$ is an infinite dimensional functional space. To get finiteness we shall need to restrict to some Krull open dense subsets of $\mathsf{Fol}(\mathcal{E}^\diamond)$ by requiring conditions on $\mathcal{H}^{\mathcal{F}^\diamond}$ depending only on a finite jet[3] of a differential 1-form defining $\mathcal{F}$.

2.5. **The generic case: non-degenerate foliations.** Let us call *singular chain*[4] of the dual graph of $\mathcal{E}_{\mathcal{F}}$ any sequence $D_0, \ldots, D_\ell$, $\ell \geq 1$, of invariant irreducible components of $\mathcal{E}_{\mathcal{F}}$ such that:

a) D_0 and D_ℓ contain at least 3 singular points of $\mathcal{F}^\sharp$,
b) $D_i \cap \Sigma_{\mathcal{F}} = \{s_i, s_{i+1}\}$ with $s_i = D_{i-1} \cap D_i$, if $1 \leq i \leq \ell - 1$.

At all the points s_i, $1 \leq i \leq \ell-1$, $\mathcal{F}^\sharp$ has the same property of normalization and we will say that the chain is linearizable, resonant normalizable or non-normalizable, non-resonant, if $\mathcal{F}^\sharp$ fulfills this property at these points s_i.

Definition 2.4. *A germ of a foliation $\mathcal{F}$ is called non-degenerate if it satisfies the following properties:*

(i) the holonomy group $\mathrm{Im}(\mathcal{H}_D^{\mathcal{F}^\sharp})$ of any invariant component D of $\mathcal{E}_{\mathcal{F}}$ with $\mathrm{card}(D \cap \Sigma_{\mathcal{F}}) \geq 3$, is non-abelian;
(ii) for any singular chain $D_0, \ldots, D_\ell$ in $\mathcal{E}_{\mathcal{F}}$, the local holonomies of $\mathcal{F}^\sharp$ at the singular points $s_i = D_{i-1} \cap D_i$, $i = 1, \ldots, \ell$, are non-periodic.

The subset of $\mathsf{Fol}(\mathcal{E}^\diamond)$ of all non-degenerate foliations will be denoted by $\mathsf{Fol}_{\mathsf{nd}}(\mathcal{E}^\diamond)$.

[3] It is well known that for p large enough all the foliations defined by a 1-form sharing the same p-jet have the same reduction map, the same singular points on the exceptional divisor, the same Camacho-Sad indices and the same dicritical components.

[4] Notice that a singular chain may not correspond to a chain of the dual graph of $\mathcal{E}$, in the usual sense. Indeed the interior vertices D_i, $0 < i < \ell$, may meet dicritical components and the number of their adjacent edges can be greater than two, and also D_0 or D_ℓ may have only two adjacent edges. Conversely a chain of the dual graph of $\mathcal{E}$ may not be a singular chain because there may exist points of Σ outside the singular locus of $\mathcal{E}$.

Remark 2.5. Theorem 6.2.1 of [24] claims the Krull open density, in the set of formal 1-forms defining foliations of second kind (in particular non-dicritical generalized curves), of the set of 1-forms defining foliations fulfilling conditions (i) and (ii) of Definition 2.4. The proof, which is given in the formal non-dicritical context, remains valid for all holomorphic foliations that are generalized curves. □

Theorem C. *Let $\mathcal{F}^\diamond = (\mathcal{F}, f) \in \mathsf{MFol}(\mathcal{E}^\diamond)$ be a marked foliation with $\mathcal{F}$ non-degenerate. Then we have an identification:*

$$\mathrm{Mod}([\mathcal{F}^\diamond]) \simeq \left(F \oplus B \oplus_{j=1}^{\lambda} (\mathbb{C}^*/\alpha_j^{\mathbb{Z}}) \oplus (\mathbb{C}^*)^\nu \right) \Big/ Z\,,$$

where $\alpha_j \in \mathbb{C}^*$, F is a finite abelian group, Z is a finite subgroup, B is a direct sum of β totally disconnected subgroups of the unitary group $\mathbb{U}(1)$ and λ, ν and β are respectively the number of linearizable, resonant normalizable and non-resonant non-linearizable singular chains contained in cut-components of $\mathcal{E}_\mathcal{F}$, the factor F corresponding to resonant non-normalizable chains. In particular, $\lambda + \nu$ is equal to the codimension $\tau_\mathcal{F}$ of $\mathcal{F}$ given in Definition 6.9.

The naturality of this identification will be explained by Assertion (b) in Theorem D below.

2.6. **Foliations of finite topological type.** We have seen that $\mathrm{Mod}([\mathcal{F}^\diamond])$ is endowed with a very specific structure of topological group of finite dimension if $\mathcal{F} \in \mathsf{Fol}_{\mathsf{nd}}(\mathcal{E}^\diamond)$. However this finiteness property continues to be valid for a larger class of foliations that we shall call *finite type foliations*. The set $\mathsf{Fol}_{\mathrm{ft}}(\mathcal{E}^\diamond) \supset \mathsf{Fol}_{\mathsf{nd}}(\mathcal{E}^\diamond)$ of these foliations is defined in Section 6 and it is optimal for finiteness as Example 3 in Section 8 shows. Furthermore we obtain *complete families* of marked foliations parametrized by finite dimensional spaces, completeness meaning that the family contains all the topological types of marked foliations by $\mathcal{E}^\diamond$ with prescribed Camacho-Sad indices and holonomies. Such a family of marked foliations is called SL-equisingular deformation, where SL stands for semi-local. We will give a precise formulation of this notion in Definition 10.2, Step (vi) of Section 10.

Theorem D. *Let $\mathcal{F}^\diamond = (\mathcal{F}, f) \in \mathsf{MFol}(\mathcal{E}^\diamond)$ be a marked foliation with $\mathcal{F}$ of finite type. Then $\mathrm{Mod}([\mathcal{F}^\diamond])$ admits an abelian group structure with identity element $[\mathcal{F}^\diamond]$ such that:*

(a) there is an exact sequence

$$\mathbb{Z}^p \to \mathbb{C}^{\tau_\mathcal{F}} \xrightarrow{\Lambda} \mathrm{Mod}([\mathcal{F}^\diamond]) \xrightarrow{\Gamma} \mathbf{D} \to 0 \qquad (1)$$

where $\mathbf{D}$ is a totally disconnected topological abelian group and $\tau_\mathcal{F}$ is the codimension of $\mathcal{F}$ given by Definition 6.9;

(b) given a section $i \mapsto [\mathcal{F}_i, f_i] \in \Gamma^{-1}(i)$ of Γ, there is a family parametrized by $i \in \mathbf{D}$ of SL-equisingular deformations $(\mathcal{F}^c_{i,t})_{t \in \mathbb{C}^{\tau_\mathcal{F}}}$ of $\mathcal{F}_i$ such that for all $t \in \mathbb{C}^{\tau_\mathcal{F}}$ we have $[\mathcal{F}^c_{i,t}, f^c_{i,t}] = \Lambda(t) \cdot [\mathcal{F}_i, f_i]$, $f^c_{i,t}$ being the marking induced by f_i and the dot $\cdot$ denoting the operation in the group $\mathrm{Mod}([\mathcal{F}^\diamond])$.

The superscript c in the deformation stands for *complete*. The group $\mathbf{D}$ will be specified in the proof (Step (i) of Section 10): it is a quotient of a product of a finite family of totally discontinuous subgroups of $\mathbb{U}(1)$, that can be uncountable. However, let us highlight that $\mathbf{D}$ is "generically finite" in the following sense:

- *there is a subset N of zero measure in the algebraic subset $\overline{\mathrm{CS}(\mathrm{Mod}(\mathcal{E}^\diamond))}$ of $\mathbb{C}^{I_{\mathcal{E}^\diamond}}$ such that if $\mathrm{CS}([\mathcal{F}^\diamond]) \notin N$, the formally linearizable singularities of $\mathcal{F}^\sharp$ are holomorphically linearizable, and in this case we can prove that $\mathbf{D}$ is finite.*

As a direct consequence of the proof we can see that if $\widetilde{\mathcal{H}}([\mathcal{F}^\diamond]) = \widetilde{\mathcal{H}}([\mathcal{G}^\diamond])$ then the sets $\mathrm{Mod}([\mathcal{F}^\diamond])$ and $\mathrm{Mod}([\mathcal{G}^\diamond])$ coincide. However their respective abelian group structures are related by the map $\mu \mapsto \gamma\mu$ where $\gamma = [\mathcal{G}^\diamond] \in \mathrm{Mod}([\mathcal{F}^\diamond])$.

The paper is organized as follows. Section 3 is devoted to general notions about group-graphs and their cohomology. In Section 4 we introduce our first group-graph $\mathrm{Aut}^{\mathcal{F}^\diamond}$ over $\mathsf{A}_{\mathcal{F}^\diamond}$ and we construct a natural bijection between $\mathrm{Mod}([\mathcal{F}^\diamond])$ and $H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$. In Section 5 we introduce a simpler group-graph $\mathrm{Sym}^{\mathcal{F}^\diamond}$ over $\mathsf{A}_{\mathcal{F}^\diamond}$ having the same cohomology as $\mathrm{Aut}^{\mathcal{F}^\diamond}$, obtaining in this way Theorem B. In Section 6 we prove that the group-graph $\mathrm{Sym}^{\mathcal{F}^\diamond}$ is abelian over a subgraph $\mathsf{R}_{\mathcal{F}^\diamond} \subset \mathsf{A}_{\mathcal{F}^\diamond}$. Then the cohomology $H^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$ is an abelian group. We also prove Theorem 6.8 asserting that under finite type hypothesis the cohomology of $\mathrm{Sym}^{\mathcal{F}^\diamond}$ over $\mathsf{A}_{\mathcal{F}^\diamond}$ and $\mathsf{R}_{\mathcal{F}^\diamond}$ coincide. Theorem C is proven in Section 7. Some applications of Theorems B and C are discussed in Section 8. In Section 9 we introduce the group-graphs $\mathrm{Exp}^{\mathcal{F}^\diamond}$ and $\mathrm{Dis}^{\mathcal{F}^\diamond}$ which allow us to compute the continuous and discrete parts of the cohomology group of $\mathrm{Sym}^{\mathcal{F}^\diamond}$ in Section 10. This computation jointly with Theorem B will conclude the proof of Theorem D. Finally, Theorem A is a direct consequence of Theorem 11.4 proven in Appendix.

3. Group-graphs

In this section we will introduce and study the algebraic notion of group-graph which differs in an essential way from the notion of graph of groups introduced by Serre in [34] and that will be a key tool of this work.

Let A be a finite graph with vertex set Ve_A and edge set Ed_A. In all the paper the graph we consider will be without loops, i.e. without edges $e \in \mathsf{Ed}_\mathsf{A}$ with $\partial e = \{v\}$ for some $v \in \mathsf{Ve}_\mathsf{A}$. Denote by

$$I_\mathsf{A} := \{(v,e) \in \mathsf{Ve}_\mathsf{A} \times \mathsf{Ed}_\mathsf{A} \,|\, v \in \partial e\}$$

the set of oriented edges of A.

Definition 3.1. *A group-graph G over A is the data of groups G_v and G_e for each vertex $v \in \mathsf{Ve}_\mathsf{A}$ and each edge $e \in \mathsf{Ed}_\mathsf{A}$, and of group morphisms $\rho_v^e : G_v \to G_e$ for each $(v,e) \in I_\mathsf{A}$ which are called* restriction maps. *A* morphim *$\alpha : F \to G$ between group-graphs over the same graph A is given by group morphisms $\alpha_v : F_v \to G_v$ and $\alpha_e : F_e \to G_e$ such that the diagram*

$$\begin{array}{ccc} F_v & \xrightarrow{\alpha_v} & G_v \\ \rho_v^e \downarrow & & \downarrow \rho_v^e \\ F_e & \xrightarrow{\alpha_e} & G_e \end{array}$$

commutes for each $(v, e) \in I_{\mathsf{A}}$*. A group-graph* G *is called abelian if all the groups* G_v *and* G_e *are abelian.*

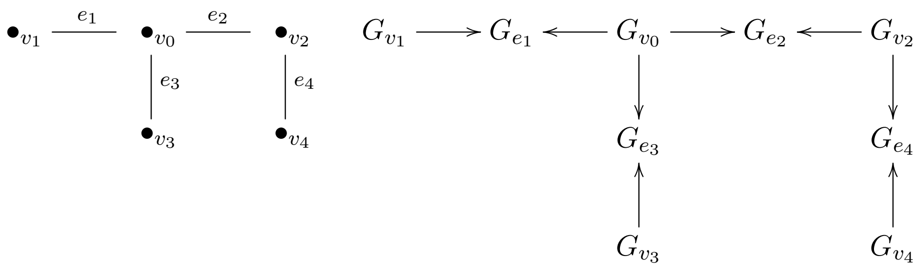

FIGURE 2. Dual tree of the desingularization of the curve $(y^2 - x^3)^2 - x^2y^3 = 0$ in the sense of Definition 4.1 and a group-graph G over it.

Remark 3.2. One can define in a natural way the notions of image and kernel of a group-graph morphism $\alpha : F \to G$, which are group-graphs over the same graph. In the abelian case the cokernel can also be defined as a group-graph. We also have an obvious notion of restriction of a group-graph over a graph to a subgraph. □

Definition 3.3. *Let* G *be a group-graph over a graph* A*. The cochain complex of* G *consists of*

$$C^0(\mathsf{A}, G) := \prod_{v \in \mathsf{Ve}_{\mathsf{A}}} G_v \qquad \text{and} \qquad C^1(\mathsf{A}, G) := \prod_{(v,e) \in I_{\mathsf{A}}} G_{v,e}\,, \quad G_{v,e} := G_e\,,$$

jointly with the right action $C^0(\mathsf{A}, G) \times C^1(\mathsf{A}, G) \to C^1(\mathsf{A}, G)$ *given by*

$$(g_v) \star_G (g_{v,e}) := (\rho^e_v(g_v)^{-1} g_{v,e} \rho^e_{v'}(g_{v'}))$$

where $\partial e = \{v, v'\}$*.*

The set of *0-cocycles* is the subset $H^0(\mathsf{A}, G)$ of $C^0(\mathsf{A}, G)$ of all elements (g_v) satisfying the relations $\rho^e_v(g_v) = \rho^e_{v'}(g_{v'})$ whenever $\partial e = \{v, v'\}$. Let us consider the set of 1-*cocycles*

$$Z^1(\mathsf{A}, G) := \{(g_{v,e}) \in C^1(\mathsf{A}, G) \,|\, g_{v,e} g_{v',e} = 1 \text{ when } \partial e = \{v, v'\}\} \subset C^1(\mathsf{A}, G)$$

which is invariant by the action of $C^0(\mathsf{A}, G)$ and its quotient, the 1-cohomology set:

$$H^1(\mathsf{A}, G) := Z^1(\mathsf{A}, G)/C^0(\mathsf{A}, G).$$

Remark 3.4. The sets of cochains $C^0(\mathsf{A}, G)$ and $C^1(\mathsf{A}, G)$ are groups but in general $H^0(\mathsf{A}, G)$ and $H^1(\mathsf{A}, G)$ are merely sets (although $Z^1(\mathsf{A}, G)$ is in bijection with $\prod_{e \in \mathsf{Ed}_{\mathsf{A}}} G_e$ which is a group). However, if the group-graph G is abelian then we can consider the group $C^2(\mathsf{A}, G) := \prod_{e \in \mathsf{Ed}_{\mathsf{A}}} G_e$ and the morphisms $\partial^0 : C^0(\mathsf{A}, G) \to C^1(\mathsf{A}, G)$ and $\partial^1 : C^1(\mathsf{A}, G) \to C^2(\mathsf{A}, G)$ given by

$$\partial^0(g_v) := (g_v) \star_G (1) = (\rho^e_v(g_v)^{-1} \rho^e_{v'}(g_{v'})) \quad \text{and} \quad \partial^1(g_{v,e}) = (g_{v,e} g_{v',e})$$

whenever $\partial e = \{v, v'\}$. It turns out that $\partial^1 \circ \partial^0 = 1$ and we obtain a complex

$$C^*(\mathsf{A}, G) : C^0(\mathsf{A}, G) \overset{\partial^0}{\to} C^1(\mathsf{A}, G) \overset{\partial^1}{\to} C^2(\mathsf{A}, G) \to 1$$

whose cohomology is $H^0(C^*(\mathsf{A}, G)) = H^0(\mathsf{A}, G) = \ker \partial^0$, $H^1(C^*(\mathsf{A}, G)) = H^1(\mathsf{A}, G) = Z^1(\mathsf{A}, G)/\partial^0 C^0(\mathsf{A}, G)$ and $H^2(C^*(\mathsf{A}, G)) = 1$ because $Z^1(\mathsf{A}, G) = \ker \partial^1$ and $\operatorname{coker} \partial^1 = 1$. □

The following result is straightforward.

Lemma 3.5 (Functoriality). *Every morphism $\alpha : F \to G$ of group-graphs over the same graph A induces well-defined maps $\alpha_i : H^i(\mathsf{A}, F) \to H^i(\mathsf{A}, G)$, for $i = 0, 1$, given by*

$$\alpha_0(g_v) = (\alpha_v(g_v)) \quad \textit{and} \quad \alpha_1([g_{v,e}]) = [\alpha_e(g_{v,e})].$$

Moreover, if F and G are abelian then α_i are morphisms.

A short sequence $1 \to F \overset{\alpha}{\to} G \overset{\beta}{\to} J \to 1$ of morphisms of group-graphs over the same graph A is exact if for all $a \in \mathsf{Ve}_\mathsf{A} \cup \mathsf{Ed}_\mathsf{A}$ the corresponding short sequence of groups $1 \to F_a \overset{\alpha_a}{\to} G_a \overset{\beta_a}{\to} J_a \to 1$ is exact. In the abelian case the complexes of abelian groups considered in Remark 3.4 fit into a short exact sequence $1 \to C^*(\mathsf{A}, F) \to C^*(\mathsf{A}, G) \to C^*(\mathsf{A}, J) \to 1$. The following results are classical.

Lemma 3.6 (Long exact sequence). *If $1 \to F \to G \to J \to 1$ is a short exact sequence of abelian group-graphs over the same graph A, then there is a long exact sequence $1 \to H^0(\mathsf{A}, F) \to H^0(\mathsf{A}, G) \to H^0(\mathsf{A}, J) \to H^1(\mathsf{A}, F) \to H^1(\mathsf{A}, G) \to H^1(\mathsf{A}, J) \to 1$.*

Lemma 3.7 (Mayer-Vietoris). *Let G be an abelian group-graph over a graph A. If A_0 and A_1 are subgraphs of A such that $\mathsf{A} = \mathsf{A}_0 \cup \mathsf{A}_1$ then there is an exact sequence $1 \to H^0(\mathsf{A}, G) \to H^0(\mathsf{A}_0, G) \oplus H^0(\mathsf{A}_1, G) \to H^0(\mathsf{A}_0 \cap \mathsf{A}_1, G) \to H^1(\mathsf{A}, G) \to H^1(\mathsf{A}_0, G) \oplus H^1(\mathsf{A}_1, G) \to H^1(\mathsf{A}_0 \cap \mathsf{A}_1, G) \to 1$.*

Proof. We have a short exact sequence of complexes of abelian groups

$$1 \to C^*(\mathsf{A}, G) \to C^*(\mathsf{A}_0, G) \oplus C^*(\mathsf{A}_1, G) \to C^*(\mathsf{A}_0 \cap \mathsf{A}_1, G) \to 1$$

and we consider the long exact sequence of cohomology. □

Definition 3.8. *The valency in A or the A-valency of a vertex v of A is the cardinality $\mathrm{val}_\mathsf{A}(v)$ of the set $\{e \in \mathsf{Ed}_\mathsf{A} \ ; \ v \in \partial e\}$. We say that v is an extremity of A if $\mathrm{val}_\mathsf{A}(v) = 1$; in that case we will write $v \in \partial\mathsf{A}$.*

A *partial dead branch* (M, v_0) *of* A is the data of a vertex v_0 of A called *attaching point* and a connected subgraph M of A such that:

- M contains an extremity v_0' of A,
- all its vertices are of valency 2 in A, except possibly its extremities that are v_0' and v_0.

Notice that M is always a chain. When $\mathsf{M} \neq \mathsf{A}$ and $\mathrm{val}_\mathsf{A}(v_0) \geq 3$ one says that M is a *dead branch of* A. We define a *total order* $<_\mathsf{M}$ on the sets of its vertices $\mathsf{Ve}_\mathsf{M} := \{v_0, \ldots, v_\ell := v_0'\}$ and of its edges $\mathsf{Ed}_\mathsf{M} := \{e_1, \ldots, e_\ell\}$ with $\partial e_j = \{v_{j-1}, v_j\}$, by setting $v_0 <_\mathsf{M} \cdots <_\mathsf{M} v_\ell$ and $e_1 <_\mathsf{M} \cdots <_\mathsf{M} e_\ell$ $j = 1, \ldots, \ell$.

Definition 3.9. *For a group-graph G, we say that a partial dead branch* M *is G-repulsive if the morphisms $\rho_v^e : G_v \to G_e$ are surjective for all $e \in \mathsf{Ed}_{\mathsf{M}}$ such that $\partial e = \{v, v'\}$ and $v' <_{\mathsf{M}} v$.*

Now we will give a process that will allow us to restrict a group-graph to a subgraph without changing its cohomology.

Definition 3.10. *If (M, v_0) is a partial dead branch of* A*, the pruning $\breve{\mathsf{A}}$ of* M *in* A*, at the attaching point v_0, is the subgraph $\breve{\mathsf{A}} = (\mathsf{A} \setminus \mathsf{M}) \cup \{v_0\}$.*

Theorem 3.11 (Pruning)**.** *Let G be a (not necessarily abelian) group-graph over* A *and* M *a G-repulsive partial dead branch of* A *then there is a natural bijection $H^1(\mathsf{A}, G) \xrightarrow{\sim} H^1(\breve{\mathsf{A}}, \breve{G})$ where $\breve{\mathsf{A}}$ is the pruning of* M *in* A *and $\breve{G}$ is the restriction of G to $\breve{\mathsf{A}}$. Moreover, if G is abelian, this bijection is an isomorphism of groups.*

Before giving the proof let us notice that the natural projections $\mathrm{pr}^i : C^i(\mathsf{A}, G) \to C^i(\breve{\mathsf{A}}, \breve{G})$, $i = 0, 1$, are group morphisms commuting with the actions $\star_G$ and $\star_{\breve{G}}$ and inducing a natural map $\mathrm{pr}^1_* : H^1(\mathsf{A}, G) \to H^1(\breve{\mathsf{A}}, \breve{G})$.

On the other hand, we have an "extension by 1" map $\mathrm{ext} : C^1(\breve{\mathsf{A}}, \breve{G}) \to C^1(\mathsf{A}, G)$ such that $\mathrm{pr}^1 \circ \mathrm{ext}$ is the identity map, $\mathrm{ext}(Z^1(\breve{\mathsf{A}}, \breve{G})) \subset Z^1(\mathsf{A}, G)$ and

$$\mathrm{ext} \circ \mathrm{pr}^1((g_{v,e})) = (g'_{v,e}), \quad \text{with} \quad g'_{v,e} = \begin{cases} g_{v,e} & \text{if} \quad (v,e) \in I_{\mathsf{A}} \setminus I_{\mathsf{M}} \\ 1 & \text{if} \quad (v,e) \in I_{\mathsf{M}} . \end{cases}$$

Proof of Theorem 3.11. First we will see that the morphism ext induces a map $\mathrm{ext}_* : H^1(\breve{\mathsf{A}}, \breve{G}) \to H^1(\mathsf{A}, G)$ that satisfies $\mathrm{pr}^1_* \circ \mathrm{ext}_*$ is the identity of $H^1(\breve{\mathsf{A}}, \breve{G})$. Indeed, by G-repulsivity of M we have the following diagram of groups and morphisms

$$G_{v_0} \overset{\rho^{e_1}_{v_0}}{\to} G_{e_1} \overset{\rho^{e_1}_{v_1}}{\twoheadleftarrow} G_{v_1} \overset{\rho^{e_2}_{v_1}}{\to} G_{e_2} \overset{\rho^{e_2}_{v_2}}{\twoheadleftarrow} G_{v_2} \to \cdots\cdots \twoheadleftarrow G_{v_{\ell-1}} \overset{\rho^{e_\ell}_{v_{\ell-1}}}{\to} G_{e_\ell} \overset{\rho^{e_\ell}_{v_\ell}}{\twoheadleftarrow} G_{v_\ell}$$

Let $\breve{h}^1 = (\breve{h}_{v,\ e})$ and $\breve{g}^1 = (\breve{g}_{v,e})$ be two cohomologous elements in $Z^1(\breve{\mathsf{A}}, \breve{G})$:

$$\breve{h}_{v,e} = \rho^e_v(\breve{h}_v)^{-1}\, \breve{g}_{v,e}\, \rho^e_{v'}(\breve{h}_{v'}) , \quad \breve{h}^0 := (\breve{h}_v) \in C^0(\breve{\mathsf{A}}, \breve{G}) .$$

We will determine $h^0 := (h_v) \in C^0(\mathsf{A}, G)$ such that $h^0 \star_G \mathrm{ext}(\breve{h}^1) = \mathrm{ext}(\breve{g}^1)$. We define $h_v = \breve{h}_v$ if $v \in \mathsf{Ve}_{\breve{A}}$. For $v \notin \mathsf{Ve}_{\breve{A}}$ it is sufficient to solve the system of ℓ equations:

$$\begin{cases} 1 &= \rho^{e_1}_{v_0}(h_{v_0})^{-1} \cdot 1 \cdot \rho^{e_1}_{v_1}(h_{v_1}) , \qquad h_{v_1} \in G_{v_1} \\ &\vdots \\ 1 &= \rho^{e_\ell}_{v_{\ell-1}}(h_{v_{\ell-1}})^{-1} \cdot 1 \cdot \rho^{e_\ell}_{v_\ell}(h_{v_\ell}) , \quad h_{v_\ell} \in G_{v_\ell} \end{cases}$$

This can be easily done using the surjectivity of the maps $\rho^{e_j}_{v_j}$, $j = 1, \ldots, \ell$. This shows that the map $\mathrm{ext}_* : H^1(\breve{\mathsf{A}}, \breve{G}) \to H^1(\mathsf{A}, G)$ is well defined. As $\mathrm{pr}^1 \circ \mathrm{ext}$ is the identity map on $C^1(\breve{\mathsf{A}}, \breve{G})$ we deduce that $\mathrm{pr}^1_* \circ \mathrm{ext}_*$ is the identity of $H^1(\breve{\mathsf{A}}, \breve{G})$.

Using again the G-repulsivity of M we now check that $\mathrm{ext}_* \circ \mathrm{pr}^1_*$ is the identity of $H^1(\mathsf{A}, G)$. To do that we prove that for each $(g_{v,e}) \in Z^1(\mathsf{A}, G)$

there is $(g_v) \in C^0(\mathsf{A}, G)$ such that:

$$\mathrm{ext} \circ \mathrm{pr}^1 \left((g_{v,e})\right) = \left(\rho_v^e(g_v)^{-1}\, g_{v,e}\, \rho_{v'}^e(g_{v'})\right) .$$

We define $g_v = 1$ when $v \in \mathsf{Ve}_{\breve{\mathsf{A}}}$, so that in particular $g_{v_0} = 1$ and therefore $\rho_{v0}^{e_1}(g_{v_0}) = 1$. As the maps $\rho_{v_j}^{e_j}$, $1 \leq j \leq \ell$, are surjective, the following system of ℓ equations has a solution with $g_{v_1} \in G_{v_1}, \ldots, g_{v_\ell} \in G_{v_\ell}$

$$\left\{\begin{array}{lcl} 1 & = & \rho_{v0}^{e_1}(g_{v_0})^{-1}\, g_{v_0,e_1}\, \rho_{v_1}^{e_1}(g_{v_1}) \\ 1 & = & \rho_{v_1}^{e_2}(g_{v_1})^{-1}\, g_{v_1,e_2}\, \rho_{v_2}^{e_2}(g_{v_2}) \\ & \vdots & \\ 1 & = & \rho_{v_{\ell-1}}^{e_\ell}(g_{v_{\ell-1}})^{-1}\, g_{v_{\ell-1},e_\ell}\, \rho_{v_\ell}^{e_\ell}(g_{v_\ell}) . \end{array}\right.$$

This proves that $\mathrm{ext}_* \circ \mathrm{pr}^1_*$ is the identity of $H^1(\mathsf{A}, G)$. The two maps ext_* and pr^1_* are inverses one of the other and therefore bijections.

When G is abelian the maps pr^1_* and ext_* are group morphisms, one the inverse of the other, and the map $H^1(\mathsf{A}, G) \xrightarrow{\sim} H^1(\breve{\mathsf{A}}, \breve{G})$ induced by pr^1_* is trivially an isomorphism. □

Remark 3.12. By repeating this process we obtain a subtree A_{pr} of A such that the restriction G_{pr} of G to A_{pr} has no G_{pr}-repulsive partial dead branches and the map $\mathrm{ext} : Z^1(\mathsf{A}_{\mathrm{pr}}, G_{\mathrm{pr}}) \to Z^1(\mathsf{A}, G)$ of extension by 1, induces an isomorphism $H^1(\mathsf{A}_{\mathrm{pr}}, G_{\mathrm{pr}}) \xrightarrow{\sim} H^1(\mathsf{A}, G)$. In particular if all the morphisms $\rho_v^e : G_v \to G_e$ are surjective, the subtree A_{pr} is reduced to a single vertex and $H^1(\mathsf{A}, G)$ is trivial. □

4. Automorphism group-graph

Let us fix once for all a marked divisor $\mathcal{E}^\diamond = (\mathcal{E}, \Sigma, \imath)$ and a marked foliation $\mathcal{F}^\diamond = (\mathcal{F}, f) \in \mathsf{MFol}(\mathcal{E}^\diamond)$. We recall that $\mathcal{E}^d$ is the union of *dicritical components* D of $\mathcal{E}^\diamond$, i.e. $D \cap \Sigma = \emptyset$, cf. Section 2.1.

As usual, the combinatorics of the reduction is encoded by the dual tree:

Definition 4.1. *The dual tree* $\mathsf{A}_{\mathcal{E}}$ *of* $\mathcal{E}$ *is the graph having* $\mathrm{Comp}(\mathcal{E})$ *and* $\mathrm{Sing}(\mathcal{E})$ *as sets* $\mathsf{Ve}_{\mathsf{A}_{\mathcal{E}}}$ *of vertices and* $\mathsf{Ed}_{\mathsf{A}_{\mathcal{E}}}$ *of edges respectively, with* $\partial s = \{D, D'\}$ *whenever* $D \cap D' = s$.

On the other hand, the dynamics of the marked foliation is organized along the following new subgraph of the dual tree:

Definition 4.2. *The subgraph of* $\mathsf{A}_{\mathcal{E}}$ *obtained by removing the vertices associated to* $\mathcal{E}^d$*, the edges attached with these vertices and the edges corresponding by* f *to nodal corners of* $\mathcal{F}^\sharp$*, is called the cut-graph of* $\mathcal{F}^\diamond$*. We denote it by* $\mathsf{A}_{\mathcal{F}^\diamond}$*, or more simply by* A *when there is not ambiguity. It is a finite graph without loops.*

The cut-graph of $\mathcal{F}^\diamond$ only depends on the class $[\mathcal{F}^\diamond]$ of $\mathcal{F}^\diamond$ by the equivalence relation $\sim_\diamond$. In fact, it is constant along the fiber $\mathrm{CS}^{-1}(\mathrm{CS}([\mathcal{F}^\diamond])) \subset \mathrm{Mod}(\mathcal{E}^\diamond)$ that contains $\mathrm{Mod}([\mathcal{F}^\diamond])$. This graph is a disjoint union of trees, denoted by $\mathsf{A}^i_{\mathcal{F}^\diamond}$, or more simply by A^i, that can be considered as the dual trees of the $\mathcal{F}^\diamond$-cut-components of $\mathcal{E}$ defined in Section 2.4. Notice that if G is a group-graph on A then

$$H^1(\mathsf{A}_{\mathcal{F}^\diamond}, G) = \prod_i H^1(\mathsf{A}^i_{\mathcal{F}^\diamond}, G_i)$$

where G_i is the restriction of G to $\mathsf{A}^i_{\mathcal{F}^\diamond}$.

Definition 4.3 (The group-graph $\mathrm{Aut}^{\mathcal{F}^\diamond}$)**.** *For $s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ and $D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$, let us denote by*

- $\mathrm{Aut}^{\mathcal{F}^\diamond}_s$ *the group of germs at $f(s)$ of holomorphic automorphisms of $\mathcal{F}^\sharp$,*
- $\mathrm{Aut}^{\mathcal{F}^\diamond}_D$ *the group of germs along $f(D)$ of continuous automorphisms of $\mathcal{F}^\sharp$ preserving $\mathcal{E}_\mathcal{F}$, that are holomorphic at each singular point of $\mathcal{F}^\sharp$ that is not a nodal corner (cf. Section 2.2) and whose restriction to $f(D) \setminus \mathrm{Sing}(\mathcal{F}^\sharp)$ is homotopic to the identity.*

We define by these data the automorphism group-graph $\mathrm{Aut}^{\mathcal{F}^\diamond}$ *over $\mathsf{A}_{\mathcal{F}^\diamond}$, the morphisms ρ^s_D, $s \in D$, being just the restriction maps.*

Remark 4.4. If D is not dicritical, the elements of $\mathrm{Aut}^{\mathcal{F}^\diamond}_D$ are transversely holomorphic at each point of $f(D) \setminus \Sigma_\mathcal{F}$, with $\Sigma_\mathcal{F} := \mathrm{Sing}(\mathcal{F}^\sharp)$, because they are holomorphic on an open set whose saturation by $\mathcal{F}^\sharp$ is a neighborhood of $f(D) \setminus \Sigma_\mathcal{F}$, cf. [19, Theorem A] or [4, Theorem 2]. □

Now we will assign to each topological class $\mathfrak{g} \in \mathrm{Mod}([\mathcal{F}^\diamond])$ a cohomology class $\mathbf{i}_{\mathcal{F}^\diamond}(\mathfrak{g}) \in H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$. To do that we fix a representative $(\mathcal{G}, g)$ of $\mathfrak{g}$.

Definition 4.5. *A* good fibration *along an invariant component $g(D)$ of $\mathcal{E}_\mathcal{G}$ is a germ along $g(D)$ of a C^∞-submersion from a neighborhood of $g(D)$ to $g(D)$, that is holomorphic at each singular point of $\mathcal{G}^\sharp$, equal to the identity on $g(D)$, constant on each component adjacent to $g(D)$ and coincides with the projection given by linearizing holomorphic coordinates at nodal singularities.*

Clearly good fibrations along invariant components always exist and their fibers, except the adjacent components of $g(D)$, are transverse to the leaves of the foliation $\mathcal{G}^\sharp$ on a neighborhood of $g(D)$ thanks to the reducedness of the singularities of $\mathcal{G}^\sharp$.

Fix $D \in C_{\mathcal{E}^\diamond}$, a regular point $o_D \in D$ and good fibrations along $f(D)$ and $g(D)$. Since $[\mathcal{G}, g] \in \mathrm{Mod}([\mathcal{F}, f])$, cf. Definition 2.3, there is a biholomorphism ψ_Δ between the fibers Δ and Δ' of the good fibrations over the points $f(o_D)$ and $g(o_D)$ conjugating the holonomy representations $\mathcal{H}^{\mathcal{F}^\diamond}_D$ and $\mathcal{H}^{\mathcal{G}^\diamond}_D$.

Lemma 4.6 (Lifting path method)**.** *Up to performing an isotopy on f and g there is a unique germ of a transversely holomorphic homeomorphism*

$$\psi_D : (M_\mathcal{F}, f(D)) \to (M_\mathcal{G}, g(D)), \quad \psi_D(\mathcal{F}^\sharp) = \mathcal{G}^\sharp,$$

that conjugates the good fibrations and the foliations such that the restriction to $f(D)$ is $g \circ f^{-1}$ and the restriction to Δ is ψ_Δ. Moreover ψ_D is holomorphic at $\Sigma_\mathcal{F} \cap f(D)$.

Proof. We proceed in three steps. First, we extend ψ_Δ to a neighborhood of $f(D \setminus \Sigma)$ by the classical method of lifting to the leaves paths in $D \setminus \Sigma$ thus obtaining a transversely holomorphic conjugation ψ_D, see for instance [21] and [14].

Next, in order to extend ψ_D to the fibers over $\Sigma_{\mathcal{F}} = f(\Sigma)$ we must previously make an isotopy on f and g such that they become holomorphic at Σ. Hence $g \circ f^{-1}$ is holomorphic at $\Sigma_{\mathcal{F}}$. Since the good fibrations are holomorphic at $f(\Sigma)$ and $g(\Sigma)$ and ψ_D is transversely holomorphic we deduce that ψ_D is holomorphic on a neighborhood of $f(\Sigma)$ minus the adjacent components to $f(D)$. Using that the Camacho-Sad indices of $\mathcal{F}^\sharp$ and $\mathcal{G}^\sharp$ at the singular points corresponding by $g \circ f^{-1}$ are the same, the holomorphic extension of ψ_D to the adjacent components of $f(D)$ follows by [21] if the corresponding singular points are non-nodal.

Finally, at the nodal singularities $f(s)$ and $g(s)$ the good fibrations coincide with the projections given by linearizing holomorphic coordinates of the nodes and we can perform an isotopy on f and g such that the expression of $g \circ f^{-1}$ in the linearizing coordinates of the nodes is the identity. Thus, on the common linear model $yx^{-\lambda}$, $\lambda \in \mathbb{R}^+ \setminus \mathbb{Q}$, we have an automorphism which is the identity on $|x| = 1$. Since it commutes with the linear holonomy $y \mapsto ye^{2i\pi\lambda}$ we deduce that it is linear on the fibers and it extends to $|x| \leq 1$ by linearity. □

Thus, for each edge $s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$, the germs at $f(s)$

$$\varphi_{D,s} = \psi_D^{-1} \circ \psi_{D'}, \quad \varphi_{D',s} = \psi_{D'}^{-1} \circ \psi_D\,, \quad D \cap D' = s\,,$$

are holomorphic automorphisms of $\mathcal{F}^\sharp$ and the 1-cocycle $\mathbf{c}^1 := (\varphi_{D,s})$, with $(D,s) \in I_{\mathsf{A}_{\mathcal{F}^\diamond}}$, is an element of $Z^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$.

Definition 4.7. *The cohomology class of this cocycle* $\mathbf{c}^1 = (\varphi_{D,s})$ *is denoted by* $\mathbf{i}_{\mathcal{F}^\diamond}(\mathfrak{g})$.

Lemma 4.8. *The cohomology class* $\mathbf{i}_{\mathcal{F}^\diamond}(\mathfrak{g})$ *does not depend on the choice of a representative of the class* $\mathfrak{g}$ *nor on the good fibrations associated to it.*

Proof. If we choose another element $(\breve{\mathcal{G}}, \breve{g})$ in $\mathfrak{g}$, and good fibrations for $\breve{\mathcal{G}}$, taking in the same way homeomorphisms

$$\breve{\psi}_D : (M_{\mathcal{F}}, f(D)) \to (M_{\breve{\mathcal{G}}}, \breve{g}(D))\,, \quad \breve{\psi}_D(\mathcal{F}^\sharp) = \breve{\mathcal{G}}^\sharp\,,$$

we obtain another 1-cocycle

$$\breve{\mathbf{c}}^1 = (\breve{\varphi}_{D,s}) = (\breve{\psi}_D^{-1} \circ \breve{\psi}_{D'}) \in Z^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})\,.$$

Since $(\mathcal{G}, g) \sim_\diamond (\breve{\mathcal{G}}, \breve{g})$ there is a homeomorphism Φ between neighborhoods of the exceptional divisors $\mathcal{E}_{\mathcal{G}}$ and $\mathcal{E}_{\breve{\mathcal{G}}}$ of the reductions of these foliations that conjugates $\mathcal{G}^\sharp$ and $\breve{\mathcal{G}}^\sharp$ and is holomorphic at the singular points of $\mathcal{G}^\sharp$, except perhaps at the nodal corners; moreover, when restricted to $\mathcal{E}_{\mathcal{G}}$, Φ is isotopic to $\breve{g} \circ g^{-1}$.

Let us denote by Φ_D the germs of Φ along the invariant components D of $\mathcal{E}_{\mathcal{G}}$. One easily checks that the 0-cochain $\mathbf{c}^0 = (\psi_D^{-1} \circ \Phi_D^{-1} \circ \breve{\psi}_D) \in C^0(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$ fulfills $\mathbf{c}^0 \star \mathbf{c}^1 = \breve{\mathbf{c}}^1$. This proves that the cohomology class of $\mathbf{c}^1$ does not depend on the choice of the representative of the class $\mathfrak{g}$ nor on the good fibrations used to define it. □

Theorem 4.9. *The map*

$$\mathbf{i}_{\mathcal{F}^\diamond} : \mathrm{Mod}([\mathcal{F}^\diamond]) \xrightarrow{\sim} H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond}) \tag{2}$$

is bijective. Moreover, $\mathbf{i}_{\mathcal{F}^\diamond}([\mathcal{F}^\diamond]) = [(\mathrm{id})]$.

Proof. Let us recall that $\mathsf{A}_{\mathcal{F}^\diamond}$ is the common cut-graph to all marked foliations $(\mathcal{G}, g)$ with $[\mathcal{G}, g] \in \mathsf{Mod}([\mathcal{F}^\diamond])$; in this proof we will denote it by A and by A^i its connected components. Let us show first the injectivity of $\mathbf{i}_{\mathcal{F}^\diamond}$. If

$$\mathbf{i}_{\mathcal{F}^\diamond}([\mathcal{G}, g]) = [(\varphi_{D,s})] = [(\breve{\varphi}_{D,s})] = \mathbf{i}_{\mathcal{F}^\diamond}([\breve{\mathcal{G}}, \breve{g}])$$

then for each $D \in C_{\mathcal{E}^\diamond}$ there exists $\xi_D \in \mathrm{Aut}_D^{\mathcal{F}^\diamond}$ such that

$$\xi_D^{-1} \circ \varphi_{D,s} \circ \xi_{D'} = \breve{\varphi}_{D,s}.$$

Writing $\varphi_{D,s} = \psi_D^{-1} \circ \psi_{D'}$ and $\breve{\varphi}_{D,s} = \breve{\psi}_D^{-1} \circ \breve{\psi}_{D'}$, with $s = D \cap D'$, we deduce that

$$\breve{\psi}_D \circ \xi_D^{-1} \circ \psi_D^{-1} = \breve{\psi}_{D'} \circ \xi_{D'}^{-1} \circ \psi_{D'}^{-1}$$

defines conjugations $\Phi_i : W_i \to \breve{W}_i$ between the foliations $\mathcal{G}^\sharp$ and $\breve{\mathcal{G}}^\sharp$ restricted to some tubular neighborhoods W_i and $\breve{W}_i$ of $\mathcal{E}_{\mathcal{G}}^i := \bigcup_{D \in \mathsf{Ve}_{\mathsf{A}^i}} g(D) \subset \mathcal{E}_{\mathcal{G}}$ and $\bigcup_{D \in \mathsf{Ve}_{\mathsf{A}^i}} \breve{g}(D) \subset \mathcal{E}_{\breve{\mathcal{G}}}$ respectively. By composing Φ_i with suitable automorphisms of $\mathcal{G}^\sharp$ isotopic to the identity along the leaves and whose supports are disjoint from the singular locus of $\mathcal{G}^\sharp$, we can assume that Φ_i respects the attaching points of the adjacent dicritical components.

On the other hand, since the self-intersections of $g(D)$ and $\breve{g}(D)$ coincide for each dicritical component $D \subset \mathcal{E}^d$, there is a conjugation Φ_D between the foliations $\mathcal{G}^\sharp$ and $\breve{\mathcal{G}}^\sharp$ restricted to some tubular neighborhoods W_D and $\breve{W}_D$ of $g(D)$ and $\breve{g}(D)$ whose restriction to $g(D)$ is $\breve{g} \circ g^{-1}$.

In order to glue the conjugations Φ_i and Φ_D we use the following trick:

> *For each* $0 < \varepsilon < 1$*, any germ of biholomorphism of* $(\mathbb{C}^2, 0)$ *preserving the fibration* $(x, y) \mapsto x$ *and the curve* $\{y = 0\}$ *can be represented by a* $\mathcal{C}^1$*-diffeomorphism* F *from* $\mathbb{D}_1 \times \mathbb{D}_1$ *onto a neighborhood of* $(0, 0)$ *preserving the fibration and the curve with support in* $\{|x| < \varepsilon\}$*, where* $\mathbb{D}_1$ *is the open unit disc of* $\mathbb{C}$*.*

This implies that there is an automorphism F_D on a neighborhood of $g(D)$ preserving $g(D)$ and $\mathcal{G}^\sharp$, which is equal to $\Phi_D^{-1} \circ \Phi_i$ in a neighborhood of the attaching point $g(D) \cap \mathcal{E}_{\mathcal{G}}^i$ with support a polydisc centered at this point. Shrinking the domain of definition of $\Phi_D \circ F_D$ we obtain a conjugation of the pairs $(\mathcal{G}^\sharp, g(D))$ and $(\breve{\mathcal{G}}^\sharp, \breve{g}(D))$ which can be glued with Φ_i.

The gluing of Φ_i and Φ_j at the nodal corners is made by using linearizing coordinates for $(\mathcal{G}, g(s))$ and $(\breve{\mathcal{G}}, \breve{g}(s))$ as in [18, §8.5]. In this way we obtain a global conjugation $\Phi : M_{\mathcal{G}} \to M_{\breve{\mathcal{G}}}$ between the foliations $\mathcal{G}^\sharp$ and $\breve{\mathcal{G}}^\sharp$ which is holomorphic at the singular points.

By definition of $\mathrm{Aut}_D^{\mathcal{F}^\diamond}$, the restrictions of ξ_D to the divisor are isotopic to the identity. Hence the restriction of Φ to $\mathcal{E}_{\mathcal{G}}^\diamond$ is isotopic to $\breve{g} \circ g^{-1}$. Therefore $[\mathcal{G}, g] = [\breve{\mathcal{G}}, \breve{g}]$.

To prove the surjectivity of $\mathbf{i}_{\mathcal{F}^\diamond}$ we consider a cocycle $\mathbf{c} = (\varphi_{D,s})$ in a given class of $H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$. We define $\varphi_{D,s} = \mathrm{id}$ when $f(s)$ is a nodal corner or an attaching point of a dicritical component. By gluing open neighborhoods U_D of $f(D)$ using the local biholomorphisms $\varphi_{D,s}$ we obtain a complex manifold $M_{\mathbf{c}}$ endowed with a foliation $\mathcal{F}_{\mathbf{c}}$, a divisor $\mathcal{E}_{\mathbf{c}}$ and a biholomorphism between $\mathcal{E}_{\mathbf{c}}$ and $\mathcal{E}_{\mathcal{F}}$ sending the singular locus of $\mathcal{F}_{\mathbf{c}}$ onto $\Sigma_{\mathcal{F}}$. There is a composition of blow-ups $E' : M' \to (\mathbb{C}^2, 0)$ and a biholomorphism $g : M_{\mathbf{c}} \to M'$ sending $\mathcal{E}_{\mathbf{c}}$ onto the exceptional divisor $E'^{-1}(0)$, see for instance [20, p. 306]. We obtain a foliation $\mathcal{F}' = (E' \circ g)(\mathcal{F}_{\mathbf{c}})$ on $(\mathbb{C}^2, 0)$ and a biholomorphism $h : \mathcal{E}_{\mathcal{F}} \to \mathcal{E}_{\mathcal{F}'}$ satisfying $h(\Sigma_{\mathcal{F}}) = \Sigma_{\mathcal{F}'}$. We define $f' := h \circ f : \mathcal{E} \to \mathcal{E}_{\mathcal{F}'}$. By construction $\mathbf{i}_{\mathcal{F}^\diamond}([\mathcal{F}', f']) = [\mathbf{c}]$. □

5. Symmetry group-graph

We keep the notations and the fixed data of the previous section. In order to define the remaining group-graph associated to $\mathcal{F}^\diamond$, we moreover fix for each $D \in C_{\mathcal{E}^\diamond}$ a regular point $o_D \in D$ and a transverse section Δ_D to $f(D)$ passing through $f(o_D)$.

Definition 5.1. *For $s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ we say that $\phi \in \mathrm{Aut}_s^{\mathcal{F}^\diamond}$ fixes the leaves of $\mathcal{F}^\sharp$ if for every neighborhood V of $f(s)$ there is a neighborhood V' of $f(s)$ such that $\phi(V') \subset V$ and for all $p \in V'$ the points p and $\phi(p)$ belong to the same leaf of $\mathcal{F}^\sharp_{|V}$. We denote by $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$ the (normal) subgroup of $\mathrm{Aut}_s^{\mathcal{F}^\diamond}$ of these automorphisms.*

Remark 5.2. It is easy to see that an example of element of $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$ is provided by $\phi \in \mathrm{Aut}_s^{\mathcal{F}^\diamond}$ such that $\phi|_{f(D)} = \mathrm{id}_{f(D)}$ and $F_p \circ \phi = F_p$ for any local first integral F_p at every point $p \in f(D \setminus \Sigma)$ in a neighborhood of $f(s)$. The diffeomorphisms of the flow of a vector field tangent to the foliation fulfill this property for small times and, by composition, all the diffeomorphisms of the flow are in $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$. □

Remark 5.3. For a fundamental system (V_α) of open neighborhoods of $f(s)$ let us denote by $\mathcal{Q}^{\mathcal{F}}_{V_\alpha}$ the leaf space of the restriction of the foliation $\mathcal{F}^\sharp$ to $V_\alpha \setminus \mathcal{E}_{\mathcal{F}}$. The inclusion relation on the leaves induces an inverse system of continuous maps $\mathcal{Q}^{\mathcal{F}^\diamond}(s) := (\mathcal{Q}^{\mathcal{F}}_{V_\alpha} \leftarrow \mathcal{Q}^{\mathcal{F}}_{V_\beta})_{V_\beta \subset V_\alpha}$. Every $\psi \in \mathrm{Aut}_s^{\mathcal{F}^\diamond}$ defines an automorphism[5] of this inverse system $\widetilde{\psi} \in \mathrm{Aut}(\mathcal{Q}^{\mathcal{F}^\diamond}(s))$ and the map $\zeta : \psi \mapsto \widetilde{\psi}$ is a group morphism. It turns out that $\widetilde{\psi}$ is the identity if and only if $\psi \in \mathrm{Fix}_s^{\mathcal{F}^\diamond}$, i.e. we have an exact sequence:

$$1 \to \mathrm{Fix}_s^{\mathcal{F}^\diamond} \longrightarrow \mathrm{Aut}_s^{\mathcal{F}^\diamond} \xrightarrow{\ \zeta\ } \mathrm{Aut}(\mathcal{Q}^{\mathcal{F}^\diamond}(s))\,.$$

□

Definition 5.4. *For $s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ and $D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ we consider the groups*

$$\mathrm{Sym}_s^{\mathcal{F}^\diamond} := \mathrm{Aut}_s^{\mathcal{F}^\diamond} / \mathrm{Fix}_s^{\mathcal{F}^\diamond}$$

and

$$\mathrm{Sym}_D^{\mathcal{F}^\diamond} := \begin{cases} C(H_D) & \textit{if} \quad \mathrm{val}_\Sigma(D) \geq 3, \\ C(H_D)/H_D & \textit{if} \quad \mathrm{val}_\Sigma(D) \leq 2, \end{cases}$$

[5] The system $\mathcal{Q}^{\mathcal{F}^\diamond}(s)$ is an element of the category $\underleftarrow{\mathrm{Top}}$ of pro-objects associated to the category of topological spaces and continuous maps. The objects of this category are the inverse families of topological spaces and $\mathrm{Aut}(\mathcal{Q}^{\mathcal{F}^\diamond}(s))$ is the group of invertible elements of $\varprojlim_\beta \varinjlim_\alpha \mathcal{C}^0(\mathcal{Q}^{\mathcal{F}}_{V_\alpha}, \mathcal{Q}^{\mathcal{F}}_{V_\beta})$, cf. [8, §2.8] or [18, §3.1].

where $H_D \subset \mathrm{Diff}(\Delta_D, f(o_D))$ *is the holonomy group of* $\mathcal{F}^\sharp$ *along* $f(D)$*,* $C(H_D)$ *is its centralizer inside* $\mathrm{Diff}(\Delta_D, f(o_D))$ *and* $\mathrm{val}_\Sigma(D)$*, called here singular valency of* D*, is the number of elements of* $D \cap \Sigma$*.*

Notice that if $\mathrm{val}_\Sigma(D) \leq 2$ then $\pi_1(D \setminus \Sigma)$ is abelian and so is the holonomy group H_D. Thus, in this case we have $C(H_D) \supset H_D$.

In order to define maps $\rho_D^s : \mathrm{Sym}_D^{\mathcal{F}^\diamond} \to \mathrm{Sym}_s^{\mathcal{F}^\diamond}$, $s \in D$, we will need the following result:

Lemma 5.5. *If* $\psi \in \mathrm{Aut}_D^{\mathcal{F}^\diamond}$ *satisfies* $\psi_{|f(D)} = \mathrm{id}_{f(D)}$ *and* $\psi_{|\Delta_D} = \mathrm{id}_{\Delta_D}$*, then the germ of* ψ *at* $f(s)$ *belongs to* $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$*.*

Proof. For each $p \in f(D) \setminus \mathrm{Sing}(\mathcal{F}^\sharp)$ we choose a local holomorphic first integral F_p of $\mathcal{F}$ defined in a neighborhood of p. The set

$$\Omega := \{p \in f(D) \setminus \mathrm{Sing}(\mathcal{F}^\sharp) \,|\, F_p \circ \psi = F_p\}$$

is open and closed in $f(D) \setminus \mathrm{Sing}(\mathcal{F}^\sharp)$ and it contains $f(o_D) = \Delta_D \cap f(D)$. Hence $\Omega = f(D) \setminus \mathrm{Sing}(\mathcal{F}^\sharp)$ and we conclude thanks to Remark 5.2. □

By applying Lemma 4.6, with $\mathcal{F}^\diamond = \mathcal{G}^\diamond$, we obtain that each element ϕ of $C(H_D)$ can be extended to an element of $\mathrm{Aut}_D^{\mathcal{F}^\diamond}$. Thanks to Lemma 5.5, the class modulo $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$ of the germ at $f(s)$ of this extension does not depend on the way the extension is made and hence on the choice of the good fibrations. We define $\rho_D^s(\phi)$ as this class in case $\mathrm{val}_\Sigma(D) \geq 3$.

Before defining ρ_D^s for D with $\mathrm{val}_\Sigma(D) \leq 2$, we must make some preliminary considerations. Let us fix for each point $s \in \Sigma \cap D$ the image $\mathcal{K}_s$ of a holomorphic embedding of the closed unit disc $\overline{\mathbb{D}}_1$ into D sending 0 to s and 1 to o_D and satisfying $\Sigma \cap \mathcal{K}_s = \{s\}$. The simple loops γ_s that parametrize $\partial\mathcal{K}_s$ with the natural positive orientation, form a system of generators of the fundamental group $\pi_1(D \setminus \Sigma, o_D)$. Hence the holonomies $h_{D,s} \in \mathrm{Diff}(\Delta_D, f(o_D))$ of the foliation $\mathcal{F}^\sharp$ along $f \circ \gamma_s$, $s \in \Sigma \cap D$, generate H_D.

Definition 5.6. *The collection* $(\mathcal{K}_s)_{s \in D \cap \Sigma}$ *of such embedded discs is called appropriate compact discs system and the maps* $h_{D,s}$ *are called the local holonomies associated to it.*

Let us denote $K_s := f(\mathcal{K}_s)$ and fix a good fibration $\pi_D : W_D \to f(D)$ associated to $\mathcal{F}$ defined on an open neighborhood W_D of $f(D)$. The same arguments used in the proof of Lemma 4.6, with $\mathcal{F}^\diamond = \mathcal{G}^\diamond$, imply that any element ϕ of the *centralizer* $C(h_{D,s})$ of $h_{D,s}$ in $\mathrm{Diff}(\Delta_D, f(o_D))$ has a unique extension ϕ^{ext} as a germ along K_s of a homeomorphism that leaves invariant the foliation $\mathcal{F}^\sharp$ and each fiber of the fibration. Moreover ϕ^{ext} is necessarily holomorphic at $f(s)$, as it is shown in Lemma 4.6. Taking its germ at $f(s)$ we obtain a map

$$\mathrm{ext} : C(h_{D,s}) \longrightarrow \mathrm{Aut}_s^{\mathcal{F}^\diamond}, \quad \phi \mapsto \phi^{\mathrm{ext}} \tag{3}$$

and the property of uniqueness of the extensions implies that this map is a group morphism.

Let us consider the inverse system $\mathcal{Q}^{\mathcal{F}^\diamond}(\mathcal{K}_s) = (\mathcal{Q}_{W_\alpha}^{\mathcal{F}} \leftarrow \mathcal{Q}_{W_\beta}^{\mathcal{F}})_{W_\beta \subset W_\alpha}$, where $(W_\alpha)_\alpha$ is the fundamental system of neighborhoods of K_s and $\mathcal{Q}_{W_\alpha}^{\mathcal{F}}$

is the leaf space of the restriction of the foliation to $W_\alpha \setminus \mathcal{E}_\mathcal{F}$. Let $V_s \subset \overset{\circ}{K}_s$ be a small open disc centered at $f(s)$. Over $K_s \setminus V_s$ the foliation $\mathcal{F}^\sharp$ is a product foliation; thus we have an isomorphism of inverse systems (i.e. an isomorphism of the category $\overleftarrow{\text{Top}}$)

$$\mathcal{Q}^{\mathcal{F}^\diamond}(s) \xrightarrow{\sim} \mathcal{Q}^{\mathcal{F}^\diamond}(\mathcal{K}_s)\,. \tag{4}$$

We also consider the orbit spaces $\mathcal{Q}_\alpha^{h_{D,s}}$ of the pseudogroup defined by the restriction of $h_{D,s}$ to $\Delta_D \cap W_\alpha$; they form an inverse system $\mathcal{Q}^{h_{D,s}} = (\mathcal{Q}_\alpha^{h_{D,s}} \leftarrow \mathcal{Q}_\beta^{h_{D,s}})_{W_\beta \subset W_\alpha}$. We can choose each W_α such that there are retractions along the leaves from $W_\alpha \setminus \mathcal{E}_\mathcal{F}$ on $(W_\alpha \setminus \mathcal{E}_\mathcal{F}) \cap \pi_D^{-1}(\partial K_s)$, π_D being the good fibration fixed above; moreover we can require that $W_\alpha \cap \pi_D^{-1}(\partial K_s)$ is a set of *suspension type*, i.e. the union of all paths with origin in $\Delta_D \cap W_\alpha$ obtained by lifting via π_D the loop ∂K_s to the leaves of $\mathcal{F}^\sharp$, cf. [16, Definition 3.1.1]. This property implies that the leaf space of the restriction of the foliation to this set can be identified to the orbit space $\mathcal{Q}_\alpha^{h_{D,s}}$ of the restriction of $h_{D,s}$ to $\Delta_D \cap W_\alpha$. Hence using (4), the retractions induce isomorphisms

$$\tau : \mathcal{Q}^{\mathcal{F}^\diamond}(s) \xrightarrow{\sim} \mathcal{Q}^{h_{D,s}} \text{ and } \tau_* : \text{Aut}(\mathcal{Q}^{\mathcal{F}^\diamond}(s)) \xrightarrow{\sim} \text{Aut}(\mathcal{Q}^{h_{D,s}})\,,\ \tau_*(\varphi) := \tau \circ \varphi \circ \tau^{-1},$$

the inverse of τ being given by the inclusion relations of the orbits of $h_{D,s}$ in the leaves of the foliation on neighborhoods of K_s.

Each element ϕ of $C(h_{D,s})$ induces an automorphism $\xi(\phi) := \tau_*(\zeta(\phi^{\text{ext}}))$ of $\mathcal{Q}^{h_{D,s}}$ and the map $\xi : C(h_{D,s}) \to \text{Aut}(\mathcal{Q}^{h_{D,s}})$ is a group morphism.

Lemma 5.7. *The kernel of the morphism ξ is the cyclic group generated by $h_{D,s}$.*

Proof. Let us take $\phi \in \ker(\xi)$. This means that for each open neighborhood U of $f(o_D)$ in Δ_D there is an open set $V \supset U$ such that for each $z \in U$, z and $\phi(z)$ are in the same V-orbit of $h_{D,s}$. The V-orbit of z is the set of points z' of V such that either there exists $n \in \mathbb{N}$ fulfilling either $\phi(z), \ldots, \phi^n(z) \in V$ and $\phi^n(z) = z'$, or $\phi^{-1}(z), \ldots, \phi^{-n}(z) \in V$ and $\phi^{-n}(z) = z'$. Let us denote $\lambda = h'_{D,s}(f(o_D))$ and $\mu = \phi'(f(o_D))$. If $|\lambda| \neq 1$ there is a holomorphic coordinate z such that $h_{D,s}(z) = \lambda z$ and $\phi(z) = \mu z$. Hence $\phi(z) = \mu z = \lambda^{\nu(z)} z = h_{D,s}^{\nu(z)}(z)$ implies $\nu(z)$ is constant. To see that, let V_n be the set of points $z \in U$ such that $h_{D,s}^k(z) \in V$ for each $k = 0, \ldots, n$. There is an uncountable set K invariant by $h_{D,s}$ such that for all $n \in \mathbb{Z}$ it is contained in the connected component of V_n containing $f(o_D)$. If $h_{D,s}$ is linearizable (conjugated to a rotation) then we can take an invariant conformal disc as K. If $h_{D,s}$ is resonant non-linearizable then K is a union of petals contained in U. If $h_{D,s}$ is non-resonant non-linearizable we take as K the hedgehog associated to U, cf. [30].

For each $z \in K$ there is an integer $\nu(z)$ such that $\phi(z) = h_{D,s}^{\nu(z)}(z)$. Thus, there is $n \in \mathbb{Z}$ such that ϕ and $h_{D,s}^n$ coincide on an uncountable subset of K and by isolated-zero principle they coincide on the connected component of V_n containing $f(o_D)$. Then the germs of ϕ and $h_{D,s}^n$ at $f(o_D)$ are equal, that achieves the proof. □

Corollary 5.8. *The extension map (3) induces an isomorphism:*

$$[\mathrm{ext}] : C(h_{D,s})/\langle h_{D,s}\rangle \xrightarrow{\sim} \mathrm{Sym}_s^{\mathcal{F}^\diamond} .$$

Proof. By construction the following diagram is commutative

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathrm{Fix}_s^{\mathcal{F}^\diamond} & \longrightarrow & \mathrm{Aut}_s^{\mathcal{F}^\diamond} & \xrightarrow{\zeta} & \mathrm{Aut}(\mathcal{Q}^{\mathcal{F}^\diamond}(s)) \\ & & & & \uparrow{\scriptstyle \mathrm{ext}} & & \downarrow{\scriptstyle \tau_*} \\ 1 & \longrightarrow & \langle h_{D,s}\rangle & \longrightarrow & C(h_{D,s}) & \xrightarrow{\xi} & \mathrm{Aut}(\mathcal{Q}^{h_{D,s}}) \end{array}$$

Thanks to Remark (5.3) and Lemma (5.7) the lines are exact. Because τ is an isomorphism, ext induces an isomorphism between $C(h_{D,s})/\langle h_{D,s}\rangle$ and $\mathrm{Aut}_s^{\mathcal{F}^\diamond}/\mathrm{Fix}_s^{\mathcal{F}^\diamond} = \mathrm{Sym}_s^{\mathcal{F}^\diamond}$. □

Remark 5.9. Suppose that there are local coordinates u_1, u_2 at s for which $\mathcal{F}^\sharp$ is defined by a linear differential 1-form $\omega = \mu u_2 du_1 - u_1 du_2$ and $D_1 = \{u_2 = 0\}$, $D_2 = \{u_1 = 0\}$ are the components of $\mathcal{E}_{\mathcal{F}}$. On the transversals $\{u_i = 1\}$ the local holonomies are $h_{D_1,s}(u_2) = e^{2\pi i\mu}u_2$ and $h_{D_2,s}(u_1) = e^{2\pi i\frac{1}{\mu}}u_1$ and their centralizers are formed by the linear automorphisms in the coordinates u_i, $C(h_{D_i,s}) = \mathbb{C}^* u_i$. Therefore we have isomorphisms

$$\frac{C(h_{D_1,s})}{\langle h_{D_1,s}\rangle} \simeq \frac{\mathbb{C}}{2\pi i(\mathbb{Z}+\mu\mathbb{Z})} \xrightarrow{\tau_1} \mathrm{Sym}_s^{\mathcal{F}^\diamond} \xleftarrow{\tau_2} \frac{\mathbb{C}}{2\pi i(\mathbb{Z}+\frac{1}{\mu}\mathbb{Z})} \simeq \frac{C(h_{D_2,s})}{\langle h_{D_2,s}\rangle}$$

To describe $\tau_2^{-1}\circ\tau_1$ let us remark that the automorphisms $(u_1,u_2)\mapsto(e^t u_1, e^{\mu t}u_2)$, $t\in\mathbb{C}$ are elements of $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$. Thus the automorphisms $(u_1,u_2)\mapsto(e^t u_1, u_2)$ and $(u_1,u_2)\mapsto(u_1, e^{\mu t}u_2)$ in $\mathrm{Aut}_s^{\mathcal{F}^\diamond}$ that extend $h_{D_1,s}$ and $h_{D_2,s}$ respectively, define the same element of $\mathrm{Sym}_s^{\mathcal{F}^\diamond}$. It follows:

$$\tau_2^{-1}\circ\tau_1 : \mathbb{C}/2\pi i(\mathbb{Z}+\mu\mathbb{Z}) \longrightarrow \mathbb{C}/2\pi i(\mathbb{Z}+\frac{1}{\mu}\mathbb{Z}) , \quad \dot{t}\mapsto -\frac{1}{\mu}\dot{t} .$$

□

Remark 5.10. If $\mathrm{val}_\Sigma(D)\geq 3$, then $[\mathrm{ext}]^{-1}\circ\rho_D^s$ is the quotient map

$$\mathrm{Sym}_D^{\mathcal{F}^\diamond} \ = \ C(H_D) \hookrightarrow C(h_{D,s}) \to C(h_{D,s})/\langle h_{D,s}\rangle .$$

□

For D containing at most two singular points of $\mathcal{F}^\sharp$ we define

$$\rho_D^s := [\mathrm{ext}]\colon \mathrm{Sym}_D^{\mathcal{F}^\diamond} = C(h_{D,s})/\langle h_{D,s}\rangle \to \mathrm{Sym}_s^{\mathcal{F}^\diamond} .$$

Definition 5.11. *We call symmetry group-graph and we denote by* $\mathrm{Sym}^{\mathcal{F}^\diamond}$ *the group-graph consisting of the groups* $\mathrm{Sym}_D^{\mathcal{F}^\diamond}$, $\mathrm{Sym}_s^{\mathcal{F}^\diamond}$, *with* $D\in\mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$, $s\in\mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ *and the morphisms* ρ_D^s, $s\in D$.

Now, we are going to define a group-graph morphism $\alpha : \mathrm{Aut}^{\mathcal{F}^\diamond}\to\mathrm{Sym}^{\mathcal{F}^\diamond}$ which will induce an isomorphism on the 1-cohomology of $\mathsf{A}_{\mathcal{F}^\diamond}$. If $s\in\mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$, we define α_s as the quotient map $\mathrm{Aut}_s^{\mathcal{F}^\diamond}\to\mathrm{Sym}_s^{\mathcal{F}^\diamond}$. If $D\in\mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$, we define $\alpha_D : \mathrm{Aut}_D^{\mathcal{F}^\diamond}\to\mathrm{Sym}_D^{\mathcal{F}^\diamond}$ as follows. Fix $\Phi\in\mathrm{Aut}_D^{\mathcal{F}^\diamond}$ and take an homotopy $\phi_t : f(D\setminus\Sigma)\to f(D\setminus\Sigma)$, $t\in[0,1]$, between $\phi_0 := \Phi_{|f(D\setminus\Sigma)}$ and $\phi_1 := \mathrm{id}_{f(D\setminus\Sigma)}$, which exists by definition of $\mathrm{Aut}_D^{\mathcal{F}^\diamond}$. Consider the path $\beta(t) = \phi_t(o_D)$ and the holonomy map $h_\beta : (\Phi(\Delta_D), \Phi(o_D)) \to (\Delta_D, o_D)$ associated to it. Since ϕ_0 induces the identity on $\pi_1(D \setminus \Sigma)$ we have that $h_\beta \circ \Phi|_{\Delta_D}$ belongs to $C(H_D)$. If D has singular valency $\mathrm{val}_\Sigma(D) \geq 3$, the group consisting of the homeomorphisms of $f(D \setminus \Sigma)$ which are homotopic to the identity is simply connected [36]; consequently h_β does not depend on the chosen homotopy ϕ_t and we can put $\alpha_D(\Phi) := h_\beta \circ \Phi$. Finally, if $v(D) \leq 2$ then only the class $[h_\beta \circ \Phi_{|\Delta_D}]$ of $h_\beta \circ \Phi_{|\Delta_D}$ modulo H_D is well-defined and we put $\alpha_D(\Phi) := [h_\beta \circ \Phi_{|\Delta_D}]$.

Lemma 5.12. $\alpha : \mathrm{Aut}^{\mathcal{F}^\diamond} \to \mathrm{Sym}^{\mathcal{F}^\diamond}$ *is a group-graph morphism.*

Proof. We must see that the following diagram is commutative:

$$\begin{array}{ccc} \mathrm{Aut}_D^{\mathcal{F}^\diamond} & \xrightarrow{\alpha_D} & \mathrm{Sym}_D^{\mathcal{F}^\diamond} \\ {\scriptstyle \breve{\rho}_D^s}\downarrow & & \downarrow{\scriptstyle \rho_D^s} \\ \mathrm{Aut}_s^{\mathcal{F}^\diamond} & \xrightarrow{\alpha_s} & \mathrm{Sym}_s^{\mathcal{F}^\diamond} \end{array}$$

where $\breve{\rho}_D^s$, resp. ρ_D^s, denote the restriction maps of the group-graphs $\mathrm{Aut}^{\mathcal{F}^\diamond}$, resp. $\mathrm{Sym}^{\mathcal{F}^\diamond}$. Let us consider two cases, depending on the singular valency $\mathrm{val}_\Sigma(D)$. First let us assume $\mathrm{val}_\Sigma(D) \geq 3$ and let us fix $\phi \in \mathrm{Aut}_D^{\mathcal{F}^\diamond}$. Then $\alpha_D(\phi) = h_\beta \circ \phi_{|\Delta_D}$ with $h_\beta : \phi(\Delta_D) \to \Delta_D$ the holonomy along a path β. There exists $\phi' \in \mathrm{Aut}_D^{\mathcal{F}^\diamond}$ with compact support outside $f(D) \cap \mathrm{Sing}(\mathcal{F}^\sharp)$ whose restriction to Δ_D coincides with h_β. Indeed ϕ' can be constructed by composition of flows of tangent vector fields whose supports intersect the divisor $f(D)$ in holomorphically embedded discs disjoint from the singularities and which cover the image of β. Thus $\alpha_D(\phi) = \phi' \circ \phi_{|\Delta_D}$ and $\rho_D^s(\alpha_D(\phi))$ coincides with the class modulo $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$ of the germ of $\phi' \circ \phi$ at $f(s)$ thanks to Lemma 5.5. This germ is just the germ of ϕ at $f(s)$ because the support of ϕ' does not intersect the singularities. This achieves the proof in the case $\mathrm{val}_\Sigma(D) \geq 3$. If $\mathrm{val}_\Sigma(D) \leq 2$, the only difference is that only the class of $h_\beta \circ \phi_{|\Delta_D}$ modulo $H_D = \langle h_{D,s} \rangle$ is well-defined; but we can proceed analogously choosing arbitrarily h_β. □

Proposition 5.13 (Extension)**.** *Let W be a neighborhood of $f(s)$, $s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$, then each germ $\phi \in \mathrm{Fix}_s^{\mathcal{F}^\diamond}$ can be extended to a germ $\Phi \in \mathrm{Aut}_D^{\mathcal{F}^\diamond}$ along $f(D)$, whose support fulfills* $\mathrm{supp}(\Phi) \cap f(D) \subset W$.

Proof. At the point $f(s)$ let us fix local holomorphic coordinates (u, v), $u(f(s)) = v(f(s)) = 0$ such that the axes are invariant by the foliation, and $v = 0$ is a local equation of $f(D)$. We consider a holomorphic vector field tangent to the foliation of the form $Z = u\frac{\partial}{\partial u} + vB(u, v)\frac{\partial}{\partial v}$.

First, we will see that each germ of biholomorphism $\zeta : (f(D), f(s)) \to (f(D), f(s))$ can be extended as an element g of $\mathrm{Aut}_D^{\mathcal{F}^\diamond}$ whose germ at $f(s)$ belongs to $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$ and whose support intersect $f(D)$ inside W. This is easy to prove when ζ is embedded in the flow $(\psi_t)_t$ of a vector field $a(u)u\frac{\partial}{\partial u}$, i.e. $\zeta = \psi_1$. Indeed in this case, let us consider the real vector field Y whose flow is the flow of aZ, but with real times. Let us take a real smooth function ρ equal to 1 on an open neighborhood of $f(s)$, such that $\mathrm{supp}(\rho) \cap f(D)$ is contained in W and in the domain of definition of Y. Then ρY extends by zero along $f(D)$ and the elements Ψ_t of its flow induce homeomorphisms defined on neighborhoods of $f(D)$. Their supports are contained in the support of ρ and their germs at $f(s)$ are element of $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$, cf. Remark 5.2. Clearly the restriction of Ψ_1 to $f(D)$ is equal to ζ near $f(s)$. Now, when ζ is not embedded in a flow, we decompose $\zeta = \zeta_1 \circ \zeta_2$, with $|\zeta_1'(0)|, |\zeta_2'(0)| \neq 1$. Both ζ_1 and ζ_2 are linearizable. Thus they can both be embedded in a flow and have convenient extensions. Their composition extends ϕ along $f(D)$, and fulfills the required properties.

By replacing ϕ by $\phi \circ g^{-1}$ where g is the above extension of $\zeta := \phi_{|f(D)}$ we can suppose that the restriction of the germ ϕ to $f(D)$ is the identity. Let us choose $\varepsilon > 0$ such that the compact disc $\overline{\mathbb{D}_{2\varepsilon}} \subset f(D)$ defined by $|u| \leq 2\varepsilon$, is contained in W and in a definition domain of ϕ. Denote by C the compact annulus contained in $\overline{\mathbb{D}_{2\varepsilon}}$ given by $\varepsilon \leq |u| \leq 2\varepsilon$. By the implicit function theorem, there is a holomorphic function τ defined in an open neighborhood Ω of C, that verifies:

$$(u \circ \phi)(m) = u \circ \Phi^Z_{\tau(m)}(m) \quad \text{and} \quad \tau_{|f(D)} = 0\,,$$

Φ^Z_t being the flow of the previous vector field Z. Let us take a $\mathcal{C}^\infty$ function $\alpha : f(D) \to \mathbb{R}$ with compact support in $\Omega \cap f(D)$, that is equal to 1 on a neighborhood of C. The map

$$\xi : m \mapsto \xi(m) := \Phi^Z_{\alpha(u(m))\tau(m)}(m)$$

is a $\mathcal{C}^\infty$-diffeomorphism, because its restriction to $f(D)$ is the identity and moreover it is a local diffeomorphism. Indeed using coordinates (u, z) at each point of $f(D)$, with z a local first integral of the foliation, we easily see that the jacobian matrix of ξ is the identity. Clearly $\chi := \phi \circ \xi^{-1}$ coincides with ϕ on a neighborhood of $f(s)$, it preserves the foliation and it leaves invariant each fiber of u:

$$\Delta_c := \{u = c\}, \quad \varepsilon \leq |c| \leq 2\varepsilon\,.$$

Thus the restriction $\chi_{|\Delta_c}$ of χ to Δ_c leaves invariant the orbits of the holonomy map of $\mathcal{F}^\sharp$ around $f(s)$ represented on Δ_c -which is equal to the restriction of $\Phi^Z_{2\pi}$ to Δ_c. By Lemma 5.7, $\chi_{|\Delta_c}$ is an iteration of this holonomy map. Therefore, there exists an integer $p \in \mathbb{Z}$ such that χ coincides with $\Phi^Z_{2i\pi p}$ on a neighborhood of C.

Let us now take a continuous function $\sigma : [0, 2\varepsilon] \to \mathbb{R}^+$ vanishing on $[0, \varepsilon]$ and being equal to 1 on $[\frac{3}{2}\varepsilon, 2\varepsilon]$. The homeomorphism $\Theta : m \mapsto \Phi^Z_{2i\pi p\sigma(|u(m)|)}(m)$ is the identity on a neighborhood of $f(s)$, it coincides with χ for $\frac{3}{2}\varepsilon \leq |u| \leq 2\varepsilon$ and it leaves $\mathcal{F}^\sharp$ invariant. To end the proof, we define the required diffeomorpism Φ as the germ along $f(D)$ of the diffeomorphism equal to $\Theta^{-1} \circ \chi$ when $|u| \leq 2\varepsilon$ and equal to the identity otherwise. $\square$

Theorem 5.14. *The morphism of group-graphs* $\alpha : \mathrm{Aut}^{\mathcal{F}^\diamond} \to \mathrm{Sym}^{\mathcal{F}^\diamond}$ *induces a natural bijection*

$$\alpha_1 : H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond}) \xrightarrow{\sim} H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond}). \tag{5}$$

Proof. The surjectivity of α_1 follows easily from the surjectivity of $\alpha_s : \mathrm{Aut}_s^{\mathcal{F}^\diamond} \to \mathrm{Sym}_s^{\mathcal{F}^\diamond}$. For this we fix an orientation $\prec$ of the union of trees $\mathsf{A}_{\mathcal{F}^\diamond}$ and for $(c_{D,s}) \in Z^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$, we choose for each edge s with

$\partial s = \{D, D'\}$, $D \prec D'$, an element $\varphi_{D,s}$ such that $\alpha_s(\varphi_{D,s}) = c_{D,s}$ and we set $\varphi_{D',s} := \varphi_{D,s}^{-1}$. Clearly the family $(\varphi_{D,s})$ is an element of $Z^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$ defining a lift of $(c_{D,s})$.

To prove the injectivity of α_1 we consider $[\phi_{D,s}], [\widetilde{\phi}_{D,s}] \in H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$ such that $\alpha_1([\phi_{D,s}]) = [\alpha_s(\phi_{D,s})] = [\alpha_s(\widetilde{\phi}_{D,s})] = \alpha_1([\widetilde{\phi}_{D,s}])$. Then there is $(g_D) \in C^0(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$ such that

$$\alpha_s(\widetilde{\phi}_{D,s}) = \rho_D^s(g_D)^{-1} \circ \alpha_s(\phi_{D,s}) \circ \rho_{D'}^s(g_{D'}) \in \mathrm{Sym}_s^{\mathcal{F}^\diamond}$$

where $s = D \cap D'$. Let $\varphi_D \in \mathrm{Aut}_D^{\mathcal{F}^\diamond}$ be extensions of $g_D \in \mathrm{Sym}_D^{\mathcal{F}^\diamond}$ and let us denote by $(\varphi_D)_s \in \mathrm{Aut}_s^{\mathcal{F}^\diamond}$ their germs at s. Then

$$\alpha_s(\widetilde{\phi}_{D,s}) = \alpha_s((\varphi_D^{-1})_s) \circ \alpha_s(\phi_{D,s}) \circ \alpha_s((\varphi_{D'})_s)$$

and there is $F_s \in \mathrm{Fix}_s^{\mathcal{F}^\diamond}$ such that $\widetilde{\phi}_{D,s} = (\varphi_D^{-1})_s \circ \phi_{D,s} \circ (\varphi_{D'})_s \circ F_s$. Now we choose a map $\delta : \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}} \to \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ such that $s \in \delta(s)$ for each $s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ and we define $\bar{F}_D$ as the composition over the set $\{s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}} \mid \delta(s) = D\}$ of extensions of F_s to a neighborhood of $\delta(s)$ with disjoint supports given by Proposition 5.13. Finally putting $\bar{\varphi}_D = \varphi_D \circ \bar{F}_D \in \mathrm{Aut}_D^{\mathcal{F}^\diamond}$ we have that

$$\widetilde{\phi}_{D,s} = \bar{\varphi}_D^{-1} \circ \phi_{D,s} \circ \bar{\varphi}_{D'} \in \mathrm{Aut}_s^{\mathcal{F}^\diamond},$$

i.e. $[\phi_{D,s}] = [\widetilde{\phi}_{D,s}]$ in $H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$. □

Proof of Theorem B. It follows immediately from Theorem 4.9 and Theorem 5.14. □

6. Foliations of finite type

In this section we introduce the optimal condition on a germ of singular foliation $\mathcal{F}$ in order to have a finite dimensional moduli space $\mathrm{Mod}([\mathcal{F}^\diamond])$. We keep all the notations of previous sections.

Given a marked foliation $\mathcal{F}^\diamond = (\mathcal{F}, f)$ and a sheaf Q defined on a neighborhood of $\mathcal{E}_\mathcal{F}$ in the ambient space $M_\mathcal{F}$ of $\mathcal{F}^\sharp$, we can associate a group-graph, denoted by $Q^{\mathcal{F}^\diamond}$, over the cut-graph $\mathsf{A}_{\mathcal{F}^\diamond}$ as follows: if $s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ then $Q_s^{\mathcal{F}^\diamond}$ is the stalk of Q at $f(s)$ and if $D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$, then

$$Q_D^{\mathcal{F}^\diamond} := H^0(f(D), \iota_{f(D)}^{-1} Q),$$

$\iota_{f(D)}$ being the inclusion map of $f(D)$ in $M_\mathcal{F}$ and for $s \in D$ the morphism $\rho_D^s : Q_D^{\mathcal{F}^\diamond} \to Q_s^{\mathcal{F}^\diamond}$ being the canonical restriction.

Definition 6.1. *We call group-graph of transverse infinitesimal symmetries of $\mathcal{F}$ the group-graph $\mathcal{T}^{\mathcal{F}^\diamond}$ associated to the sheaf $\mathcal{T}^{\mathcal{F}^\sharp} := \mathcal{B}^{\mathcal{F}^\sharp}/\mathcal{X}^{\mathcal{F}^\sharp}$ on $M_\mathcal{F}$ equal to the quotient of the sheaf $\mathcal{B}^{\mathcal{F}^\sharp}$ of $\mathcal{F}^\sharp$-basic[6] holomorphic vector fields tangent to the $\mathcal{F}^\sharp$-invariant components of $\mathcal{E}_\mathcal{F}$, by the sheaf $\mathcal{X}^{\mathcal{F}^\sharp}$ of holomorphic vector fields tangent to $\mathcal{F}^\sharp$.*

[6] i.e. whose flow leaves $\mathcal{F}^\sharp$ invariant.

Remark 6.2. For each $(D,s)\in I_{\mathcal{E}^\diamond}$ let us consider the local holonomies $h_{D,s}$ as in Definition 5.6. There are linear isomorphisms, depending on the choice of an appropriate compact discs system,

$$\mathcal{T}_D^{\mathcal{F}^\diamond}\xrightarrow{\sim}\mathcal{T}_{H_D}\,,\qquad \mathcal{T}_s^{\mathcal{F}^\diamond}\xrightarrow{\sim}\mathcal{T}_{h_{D,s}}\,,$$

where $\mathcal{T}_{H_D}$ (resp. $\mathcal{T}_{h_{D,s}}$) is the vector space of germs at $f(o_D)$ of vector fields on the transversal disc Δ_D which are invariant by the holonomy group H_D of $\mathcal{F}^\sharp$ along $f(D)$ (resp. invariant by $h_{D,s}$), see [13, 24]. Moreover, if $\mathcal{X}^0_{\Delta_D}$ denotes the set of germs of vector fields on $(\Delta_D, f(o_D))$ vanishing at $f(o_D)$, then the exponential map $\exp:\mathcal{X}^0_{\Delta_D}\to \mathrm{Diff}(\Delta_D, f(o_D))$ sends $\mathcal{T}_{h_{D,s}}$ into $C(h_{D,s})$ and $\mathcal{T}_{H_D}$ into $C(H_D)$. □

Now we define *a coloring on* $\mathsf{A}_{\mathcal{F}^\diamond}$ by saying:

(1) $D\in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ *is green if the holonomy group* H_D *is finite,*
(2) $s\in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ *is green if for each* $D\in\partial s$ *the holonomy map* $h_{D,s}$ *is periodic,*
(3) $D\in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ *or* $s\in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ *are red otherwise.*

Let us denote by $\mathcal{J}^{\mathcal{F}^\diamond}$ the *group-graph of holomorphic first integrals* associated to the sheaf of germs of holomorphic first integrals of $\mathcal{F}^\sharp$. Because $\mathcal{F}^\sharp$ does not have saddle-node singularities, an element $a\in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}\cup \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ is green iff $\mathcal{J}_a^{\mathcal{F}^\diamond}\neq\mathbb{C}$, see [21]. Notice that if an edge $s=D\cap D'$ of $\mathsf{A}_{\mathcal{F}^\diamond}$ is red then the vertices D and D' are also red. We can therefore consider the following:

Definition 6.3. *The set of red elements of* $\mathsf{A}_{\mathcal{F}^\diamond}$ *is a subgraph called red graph of* $\mathcal{F}^\diamond$ *and denoted by* $\mathsf{R}_{\mathcal{F}^\diamond}$.

Proposition 6.4. *Let* s *and* $D\in\partial s$ *be a green edge and a green vertex of* $\mathsf{A}_{\mathcal{F}^\diamond}$. *Then the following properties are equivalent:*

(1) the holonomy group H_D *is generated by* $h_{D,s}$*;*
(2) $\mathcal{J}_D^{\mathcal{F}^\diamond}\to\mathcal{J}_s^{\mathcal{F}^\diamond}$ *is surjective;*
(3) $\mathcal{T}_D^{\mathcal{F}^\diamond}\to\mathcal{T}_s^{\mathcal{F}^\diamond}$ *is surjective;*
(4) $\mathrm{Sym}_D^{\mathcal{F}^\diamond}\to\mathrm{Sym}_s^{\mathcal{F}^\diamond}$ *is surjective.*

Proof. Let D be a green vertex of $\mathsf{A}_{\mathcal{F}^\diamond}$ and let $z:(\Delta_D, f(o_D))\to\mathbb{C}$ be a linearizing coordinate of the holonomy group $H_D\subset\mathrm{Diff}(\mathbb{C},0)$ which is finite. For each singular point s of D (necessarily a green edge of $\mathsf{A}_{\mathcal{F}^\diamond}$) there is $n_{D,s}\in\mathbb{N}$ such that the local holonomies $h_{D,s}$ given in Definition 5.6 are $h_{D,s}(z)=\zeta_{D,s}\,z$ for some primitive $n_{D,s}$-root of unity $\zeta_{D,s}$. Let us denote by $n_D\in\mathbb{N}$ the least common multiple of $\{n_{D,s}\mid s\in D\cap\Sigma\}$ and by ζ_D a primitive n_D-root of unity. Because a first integral is completely determined by its restriction to the transversal Δ_D, we can consider $\mathcal{J}_D^{\mathcal{F}^\diamond}$ as subrings of $\mathbb{C}\{z\}$. In the same way, by extending the elements of $\mathcal{J}_s^{\mathcal{F}^\diamond}$ along the compact discs used in Definition 5.6 to define $h_{D,s}$, we can also consider $\mathcal{J}_s^{\mathcal{F}^\diamond}$ as a subring of $\mathbb{C}\{z\}$. With these identifications and using Remark 6.2, we have the following well known equalities and isomorphisms:

$$\mathcal{J}_D^{\mathcal{F}^\diamond}=\mathbb{C}\{z^{n_D}\}\,,\ \mathcal{T}_D^{\mathcal{F}^\diamond}\simeq\mathcal{T}_{H_D}=\mathcal{J}_D^{\mathcal{F}^\diamond}z\partial_z\,,\ C(H_D)=\{z(\alpha+\mathfrak{N}_D)\,|\,\alpha\in\mathbb{C}^*\}\,,$$

$$\mathcal{J}_s^{\mathcal{F}^\diamond}=\mathbb{C}\{z^{n_{D,s}}\}\,,\ \mathcal{T}_s^{\mathcal{F}^\diamond}\simeq\mathcal{T}_{h_{D,s}}=\mathcal{J}_s^{\mathcal{F}^\diamond}z\partial_z\,,\ C(h_{D,s})=\{z(\alpha+\mathfrak{N}_s)\,|\,\alpha\in\mathbb{C}^*\}\,,$$

with $\mathfrak{N}_D \subset \mathcal{J}_D^{\mathcal{F}^\diamond}$ and $\mathfrak{N}_s \subset \mathcal{J}_s^{\mathcal{F}^\diamond}$ being the maximal ideals. Furthermore H_D is cyclic, generated by $\zeta_D\, z$. The required equivalences follow immediately. □

If B is a nonempty connected subgraph of a connected component $\mathsf{A}^i_{\mathcal{F}^\diamond}$ of $\mathsf{A}_{\mathcal{F}^\diamond}$, then for every vertex $D \notin B$ of $\mathsf{A}^i_{\mathcal{F}^\diamond}$ or $D \in \partial B$ there is a unique geodesic $[D, B] \subset \mathsf{A}^i_{\mathcal{F}^\diamond}$ joining D to ∂B. We define the following *pre-order relation* on the set of vertices of the closure of $\mathsf{A}^i_{\mathcal{F}^\diamond} \setminus B$ by means of

$$D' \leq D \Longleftrightarrow D' \in [D, B].$$

Definition 6.5. *We say that B is repulsive in $\mathsf{A}^i_{\mathcal{F}^\diamond}$ if for each edge $s = D \cap D'$ of $\mathsf{A}^i_{\mathcal{F}^\diamond} \setminus B$ with $D' \leq D$, the restriction map $\mathrm{Sym}_D^{\mathcal{F}^\diamond} \to \mathrm{Sym}_s^{\mathcal{F}^\diamond}$ is surjective.*

Remark 6.6. Notice that when B is repulsive there is a filtration $B_0 = B \subset B_1 \subset \cdots \subset B_k = \mathsf{A}^i_{\mathcal{F}^\diamond}$ such that B_{j-1} is obtained from B_j by pruning a $\mathrm{Sym}^{\mathcal{F}^\diamond}$-repulsive partial dead branch M_j of B_j, cf. Definitions 3.9 and 3.10. □

A change of marking induces an isomorphism of colored graphs compatible with the repulsiveness property that gives sense to the following definition:

Definition 6.7. *The foliation $\mathcal{F}$ is of finite type if for each connected component $\mathsf{A}^i_{\mathcal{F}^\diamond}$ of $\mathsf{A}_{\mathcal{F}^\diamond}$ we have: either the subgraph $\mathsf{A}^i_{\mathcal{F}^\diamond} \cap \mathsf{R}_{\mathcal{F}^\diamond}$ is nonempty, connected and repulsive, or it is empty and there exists a green repulsive vertex in $\mathsf{A}^i_{\mathcal{F}^\diamond}$.*

As we have already pointed out, this finiteness property does not depend on the marking. In fact thanks to Theorem 11.4 of Appendix, under (TR) and (TC) conditions, it only depends on the topological class of the germ $\mathcal{F}$ at $0 \in \mathbb{C}^2$ and it is fulfilled by all the foliations $\mathcal{G}$ with $[\mathcal{G}^\diamond] \in \mathrm{Mod}([\mathcal{F}^\diamond])$ as soon as it holds for one of them.

Theorem 6.8. *If $\mathcal{F}$ is of finite type then the extension by 1 map*

$$Z^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond}) \to Z^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$$

induces a bijection

$$H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond}) \simeq H^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})\,. \qquad (6)$$

Proof. We apply Pruning Theorem 3.11 to each partial dead branch M_j considered in Remark 6.6, see also Remark 3.12. □

When the foliation $\mathcal{F}$ is of finite type let us now give the precise value of the integer $\tau_{\mathcal{F}}$ in the statement of Theorem D in Section 2.6. Let us consider the following subgraph $\mathsf{R}^0_{\mathcal{F}^\diamond}$ whose

- vertices correspond by f to the invariant irreducible components of $\mathcal{E}_{\mathcal{F}}$ whose holonomy groups do not leave invariant a non-trivial vector field,
- edges correspond by f to the resonant non-normalizable or non-resonant non-linearizable singularities of $\mathcal{F}^\sharp$.

As before changes of marking induce isomorphisms between the graphs $\mathsf{R}^0_{\mathcal{F}^\diamond}$; that allows us to put:

Definition 6.9. *If $\mathcal{F}$ is of finite type we call codimension of $\mathcal{F}$ the integer*

$$\tau_{\mathcal{F}} := \operatorname{rank} H_1(\mathsf{R}_{\mathcal{F}^\diamond}/\mathsf{R}^0_{\mathcal{F}^\diamond}, \mathbb{Z}),$$

where $\mathsf{R}_{\mathcal{F}^\diamond}/\mathsf{R}^0_{\mathcal{F}^\diamond}$ is the quotient graph obtained from $\mathsf{R}_{\mathcal{F}^\diamond}$ by collapsing $\mathsf{R}^0_{\mathcal{F}^\diamond}$ to a single vertex.

We will highlight now a group structure on $H^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$ when $\mathcal{F}$ is of finite type. Let us choose an arbitrary map $s \mapsto D_s$ from $\mathsf{Ed}_{\mathsf{R}_{\mathcal{F}^\diamond}}$ to $\mathsf{Ve}_{\mathsf{R}_{\mathcal{F}^\diamond}}$ with $D_s \in \partial s$. Since $H^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$ is the quotient of

$$Z^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond}) \simeq \bigoplus_{s\in \mathsf{Ed}_{\mathsf{R}_{\mathcal{F}^\diamond}}} \mathrm{Sym}^{\mathcal{F}^\diamond}_s \simeq \bigoplus_{s\in \mathsf{Ed}_{\mathsf{R}_{\mathcal{F}^\diamond}}} \frac{C(h_{D_s,s})}{\langle h_{D_s,s}\rangle}$$

by $C^0(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond}) = \bigoplus_{D\in \mathsf{Ve}_{\mathsf{R}_{\mathcal{F}^\diamond}}} \mathrm{Sym}^{\mathcal{F}^\diamond}_D$, we must pay attention to the centralizers $C(h)$ of the local holonomy transformations $h = h_{D,s} \in \mathrm{Diff}(\Delta_D, f(o_D))$.

Trivially h is of one and only one following type:

- (P) periodic;
- (L^1) linearizable and non-periodic;
- (L^0) formally linearizable but non-linearizable;
- (R^1) resonant non-linearizable but normalizable;
- (R^0) resonant non-linearizable and non-normalizable.

Classically, in the first three cases there exists a (only formal, in the case (L^0)) local coordinate u on $(\Delta_D, f(o_D))$, such that $u\circ h = \alpha u$, with $\alpha \in \mathbb{C}^*$. In these situations $h = \exp X$, with $X := \log(\alpha) u\partial_u$. In the resonant cases (R^0) and (R^1) there exists a coordinate u on Δ_D, only formal in the case (R^0), such that $h = \ell^r \circ \exp t_0 X$, $X := \frac{u^{p+1}}{1+\lambda u^p}\partial_u$, where $p+1$ is the contact order of h^k with the identity when $h'(0)^k = 1$, ℓ is the formal diffeomorphism defined by $u\circ \ell := e^{2i\pi/p}u$, $h'(0) = e^{2i\pi r/p}$, $r \in \{0, 1, \ldots, p-1\}$, $\lambda \in \mathbb{C}$, $t_0 \in \mathbb{C}^*$ and we can choose $t_0 = 1$, (remark that ℓ and $\exp X$ commute). In all cases u is unique up to multiplication by an element of $\mathbb{C}^*$.

Let us denote by $\widehat{C}(h)$ the centralizer of h inside the group $\widehat{\mathrm{Diff}}(\Delta_D, f(o_D))$ of formal diffeomorphisms of $(\Delta, f(o_D))$. Clearly $C(h) = \widehat{C}(h) \cap \mathrm{Diff}(\Delta_D, f(o_D))$. As in Remark 6.2 let us denote by $\mathcal{T}_h$ the space of germs of holomorphic vector fields on $(\Delta, f(o_D))$ invariant by h. The following result contains several well-known facts.

Proposition 6.10. *According to the type of $h \in \mathrm{Diff}(\Delta_D, f(o_D))$ we have:*

- (P) $C(h) = \{g \in \mathrm{Diff}(\Delta_D, f(o_D)) \mid u\circ g = u(\alpha + F(u^q)),\ \alpha\in\mathbb{C}^*,\ F \in u\mathbb{C}\{u\}\}$; $C(h)/\langle h\rangle \simeq \mathrm{Diff}(\mathbb{C}, 0)$; $\mathcal{T}_h = \mathbb{C}\{u^q\}u\partial_u$;
- (L) $\widehat{C}(h) = \{g \in \widehat{\mathrm{Diff}}(\Delta_D, f(o_D)) \mid u\circ g = \lambda g,\ \lambda\in\mathbb{C}^*\} = \exp \mathbb{C}X \simeq \mathbb{C}^*$;
- (R) $\widehat{C}(h) = \{\ell^n \circ \exp tX,\ n \in \mathbb{Z}/p\mathbb{Z},\ t\in\mathbb{C}\} = \langle \ell\rangle \oplus \exp \mathbb{C}X \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{C}$.

Moreover:

- (L^1) $\mathcal{T}_h = \mathbb{C}X$ *and* $C(h)/\exp \mathcal{T}_h = \{1\}$;
- (R^1) $\mathcal{T}_h = \mathbb{C}X$, $C(h)/\exp\mathcal{T}_h \simeq \mathbb{Z}/p\mathbb{Z}$ *and* $C(h)/\langle h, \exp \mathcal{T}_h\rangle \simeq \mathbb{Z}/\langle p, r\rangle$;
- (L^0) $\mathcal{T}_h = \{0\}$; $C(h) = \{g \in \mathrm{Diff}(\Delta_D, f(o_D)) \mid u \circ g = \lambda u, \lambda \in \mathbf{D}\} \simeq \mathbf{D}$, *where* $\mathbf{D} := \{\lambda \in \mathbb{C}^* \mid u^{-1}\circ(\lambda u)$ *is convergent*$\}$ *is a totally discontinuous subgroup of* $\mathbb{U}(1)$*, that can be uncountable* [29];

(R^0) $\mathcal{T}_h = \{0\}$, *there exists* $m \in \mathbb{N}^*$ *such that the sequence*

$$0 \to \mathbb{Z} \xrightarrow{\alpha} C(h) \xrightarrow{\beta} \mathbb{Z}/p\mathbb{Z}\,, \quad \alpha(t) = \exp \frac{t}{m} X\,, \quad \beta(g) = \frac{1}{2i\pi} \log g'(0)$$

is exact and $C(h)/\langle h \rangle$ *is finite.*

Proof. The periodic case has already been described in the proof of Proposition 6.4 except the isomorphism $C(h)/\langle h\rangle \simeq \mathrm{Diff}(\mathbb{C},0)$ which follows easily from the fact that every $g \in \mathrm{Diff}(\mathbb{C},0)$ commuting with a rotation $z \mapsto e^{2i\pi/q} z$ is equal to $(g^\flat(z^q))^{\frac{1}{q}}$ for a unique $g^\flat \in \mathrm{Diff}(\mathbb{C},0)$. The description of the formal centralizers is given for instance in [13, Proposition 1.3.2] or [5, p. 150], where it is also shown that $C(h) = \widehat{C}(h)$ in the normalizable cases (L^1) and (R^1). In addition, in the case (L^1), $\mathcal{T}_h = \mathbb{C}u\frac{\partial}{\partial u}$ and $\exp : \mathcal{T}_h \to C(h)$ can be canonically identified to the surjective morphism $\mathbb{C} \to \mathbb{C}^*$ given by $\mu \mapsto e^\mu$. In the case (R^1), $\mathcal{T}_h$ is equal to $\mathbb{C}X$ and

$$C(h)/\langle h, \exp \mathbb{C}X\rangle \simeq (\mathbb{Z}/p\mathbb{Z} \oplus \mathbb{C})/\langle \dot{r} \oplus 1, 0 \oplus \mathbb{C}\rangle \simeq \mathbb{Z}/\langle p, r\rangle,$$

where $\dot{r}$ is the class of r modulo p. Thanks to the description of $\widehat{C}(h)$, in the case (R^0) the kernel of β consists of the convergent elements of the flow of X and by the Écalle-Liverpool Theorem, see for instance [13, Corollary 2.8.2] or [11, Theorem 4] and [1, Theorems 7 and 10], it is equal to $\alpha(\mathbb{Z})$ for a suitable $m \in \mathbb{N}^*$. Indeed, Theorem 7 of [1] claims that if $\exp tX$ is convergent for $|t| \leq p$ then $\exp tX$ is bounded in $|t| \leq p$ and $|z| \leq \rho$; consequently in this case $X = \frac{d}{dt}\big|_{t=0} \exp(tX)$ would be convergent. Thus, the set $\{t \in \mathbb{C} : \exp tX \text{ is convergent}\}$, which contains $\mathbb{Z}$, is different from $\mathbb{C}$. By Theorem 10 of [1] this set is a lattice whose dimension is necessarily 1 by Theorem 4 of [11].

Finally in the case (L^0), let g be an element of $C(h) = \widehat{C}(h) \cap \mathrm{Diff}(\mathbb{C},0)$. Then $u \circ g = \lambda u$ with $|\lambda| = 1$. Indeed if $|\lambda| \neq 1$, g would be linearizable in a convergent coordinate and h, that commutes with g, would be also linearizable in the same coordinate, contradicting the assumption (R^0). On the other hand, $\mathbf{D}$ is a totally discontinuous subgroup of $\mathbb{U}(1)$ because otherwise $\mathbf{D} = \mathbb{U}(1)$ and h would be linearizable. □

It follows from this proposition and from Remark 3.4,

Theorem 6.11. *Given* $a \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}} \cup \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ *we have the equivalences:*

$$a \text{ is red} \Longleftrightarrow \mathrm{Sym}_a^{\mathcal{F}^\diamond} \text{ is abelian } \Longleftrightarrow \dim_{\mathbb{C}} \mathcal{T}_a^{\mathcal{F}^\diamond} < \infty$$

$$a \in \mathsf{Ve}_{\mathsf{R}^0_{\mathcal{F}^\diamond}} \cup \mathsf{Ed}_{\mathsf{R}^0_{\mathcal{F}^\diamond}} \Longleftrightarrow \mathcal{T}_a^{\mathcal{F}^\diamond} = 0\,.$$

Furthermore the abelian structure of the group-graph $\mathrm{Sym}^{\mathcal{F}^\diamond}$ *over* $\mathsf{R}_{\mathcal{F}^\diamond}$ *induces an abelian group structure on* $H^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$.

Remark 6.12. By following the natural bijections (2), (5) and (6) provided by Theorems 4.9, 5.14 and 6.8 respectively, one can check that if $\widetilde{\mathcal{H}}([\mathcal{F}^\diamond]) = \widetilde{\mathcal{H}}([\mathcal{G}^\diamond])$ then $\mathrm{Mod}([\mathcal{F}^\diamond])$ and $\mathrm{Mod}([\mathcal{G}^\diamond])$ coincide as sets, but their respective abelian group structures are related by the map $\mu \mapsto \gamma\mu$ where $\gamma = [\mathcal{G}^\diamond] \in \mathrm{Mod}([\mathcal{F}^\diamond])$. Indeed, the bijection (2) is given by $\mathbf{i}_{\mathcal{F}^\diamond}$ which sends $[\mathcal{F}^\diamond]$ to $[(\mathrm{id})] \in H^1(\mathsf{A}_{\mathcal{F}^\diamond}, \mathrm{Aut}^{\mathcal{F}^\diamond})$. □

We end this section by proving some properties of centralizers which will be useful in the sequel.

Lemma 6.13. *If g and h are non-periodic and $g \in C(h)$, then $C(g) = C(h)$.*

Proof. The group $C(h)$ in case (L) only depends on the formal coordinate u that linearizes h. Similarly in case (R) all the non-periodic elements of $C(h)$ have the same invariants p and λ, and if we fix these invariants, the centralizers of resonant diffeomorphisms only depend on the normalizing coordinate u. Thus the lemma follows from the fact that in both cases all the non-periodic elements of a centralizer can be linearized or normalized by using the same coordinate. □

Lemma 6.14. *Let H be a finitely generated subgroup of* $\mathrm{Diff}(\mathbb{C}, 0)$. *Suppose that H and its centralizer are infinite. Then H is abelian, it contains a non-periodic element h and $H \subset C(H) = C(h)$.*

Proof. We will first prove that H is abelian by contradiction. Take a non-trivial element $f \in [H, H]$ which is tangent to the identity: $f(z) = z + c_k z^k + \cdots$ with $k \geq 2$ and $c_k \neq 0$. Then $f^n(z) = z + nc_k z^k + \cdots$ is the identity only for $n = 0$. Thus f is non-periodic and $C(H) \subset C(f)$. By the description of $C(f)$ given in Proposition 6.10 there is a non-periodic element g in the infinite subgroup $C(H) \subset C(f)$. The fact that $g \in C(H)$ is equivalent to the inclusion $H \subset C(g)$. As $C(g) \subset \widehat{C}(g)$ is abelian by Proposition 6.10 we get a contradiction. Since $H = \langle h_1, \ldots, h_k \rangle$ is abelian, there is a non-periodic generator h_i, otherwise H would be finite. Let $h \in H$ be any non-periodic element. By Proposition 6.10, the group $C(h) \subset \widehat{C}(h)$ is abelian. Since H is abelian we have $H \subset C(H) \subset C(h)$. Since $C(h)$ is abelian every element of $C(h)$ commutes with H and consequently $C(h) \subset C(H)$. Hence $H \subset C(H) = C(h)$. □

It follows immediately:

Proposition 6.15. *Under the hypothesis of the previous lemma, all the non-periodic elements of H are of the same type (L^1), (L^0), (R^1) or (R^0).*

7. Non-degenerate foliations

A *$\mathcal{F}^\diamond$-singular chain $\mathcal{C}$ of $\mathcal{E}$* is a sequence $D_0, \ldots, D_{\ell_\mathcal{C}}$ of irreducible components of $\mathcal{E}$ defining a connected subgraph

$$(\mathcal{C}) \qquad \bullet_{D_0} \xrightarrow{s_1} \bullet_{D_1} \cdots \qquad \cdots \xrightarrow{s_{\ell_\mathcal{C}}} \bullet_{D_{\ell_\mathcal{C}}} \tag{7}$$

of $\mathsf{A}_{\mathcal{F}^\diamond}$ such that the singular valency (cf. Definition 5.4) of its *extremities* D_0 and $D_{\ell_\mathcal{C}}$ is at least three, and that of the others, called *interior vertices*, is exactly two. If $\ell_\mathcal{C} = 1$, then $\mathcal{C}$ consists of only two adjacent divisors of singular valency at least three: $\bullet_{D_0} \xrightarrow{s_1} \bullet_{D_1}$. The image by the marking map f of a $\mathcal{F}^\diamond$-singular chain of $\mathcal{E}$ is a singular chain of $\mathcal{E}_\mathcal{F}$ as considered in Section 2.5.

Proposition 7.1. *Let $\mathcal{F}^\diamond$ be a non-degenerate marked foliation. Then $\mathcal{F}$ is of finite type. Moreover, the union $\breve{\mathsf{R}}_{\mathcal{F}^\diamond}$ of all $\mathcal{F}^\diamond$-singular chains of $\mathcal{E}$*

is a subgraph of $\mathsf{R}_{\mathcal{F}^\diamond}$ *such that for each connected component* $\mathsf{A}^i_{\mathcal{F}^\diamond}$ *of* $\mathsf{A}_{\mathcal{F}^\diamond}$*,* $\breve{\mathsf{R}}_{\mathcal{F}^\diamond}\cap\mathsf{A}^i_{\mathcal{F}^\diamond}$ *is connected and repulsive in* $\mathsf{R}_{\mathcal{F}^\diamond}$*. Hence*

$$H^1(\mathsf{R}_{\mathcal{F}^\diamond},\mathrm{Sym}^{\mathcal{F}^\diamond})\simeq H^1(\breve{\mathsf{R}}_{\mathcal{F}^\diamond},\mathrm{Sym}^{\mathcal{F}^\diamond})\,.$$

In order to simplify the notations in the two proofs below, we will write R, $\breve{\mathsf{R}}$, Sym, instead of $\mathsf{R}_{\mathcal{F}^\diamond}$, $\breve{\mathsf{R}}_{\mathcal{F}^\diamond}$, $\mathrm{Sym}^{\mathcal{F}^\diamond}$.

Proof of Proposition 7.1. Clearly $\breve{\mathsf{R}}$ is connected in each connected component A^i of the cut-graph A and the closure of $\mathsf{R}\setminus\breve{\mathsf{R}}$ in R is exactly the union of all connected subgraphs $\mathcal{C}$ denoted as in (7) but with singular valencies satisfying $\mathrm{val}_\Sigma(D_0)\geq 3$, $\mathrm{val}_\Sigma(D_j)=2$ for $0<j<\ell_{\mathcal{C}}$ and $\mathrm{val}_\Sigma(D_{\ell_{\mathcal{C}}})=1$ or 2. By definition of the group-graph Sym, for $j\geq 1$ the morphisms $\rho^{s_j}_{D_j}:\mathrm{Sym}_{D_j}\to\mathrm{Sym}_{s_j}$ are bijective and $\breve{\mathsf{R}}$ is repulsive in R. Since $\mathcal{F}$ is non-degenerate all the vertices and edges of $\breve{\mathsf{R}}$ are red; thus $\mathsf{R}\cap\mathsf{A}^i$ is also repulsive and connected. By using Pruning Theorem 3.11 we obtain the group isomorphism $H^1(\mathsf{R},\mathrm{Sym})\simeq H^1(\breve{\mathsf{R}},\mathrm{Sym})$. □

Proof of Theorem C. To have uniqueness of the numbering in the notation (7) of a singular chain $\mathcal{C}$, we fix in the sequel an extremity $\breve{D}$ of $\breve{\mathsf{R}}$ and we prescript that D_0 belongs to the geodesic joining $D_{\ell_{\mathcal{C}}}$ to $\breve{D}$. We will say that D_0, resp. $s_{\ell_{\mathcal{C}}}$, is the *initial vertex*, resp. *terminal edge* of $\mathcal{C}$.

For an interior vertex D_j of $\mathcal{C}$ the morphisms $\rho^{s_j}_{D_j}$ and $\rho^{s_{j+1}}_{D_j}$ are bijective and by composition they induce isomorphisms

$$\xi_{D_j}:\mathrm{Sym}_{s_1}\simeq\mathrm{Sym}_{D_j}\,,\qquad \xi_{s_j}:\mathrm{Sym}_{s_1}\simeq\mathrm{Sym}_{s_j}\,,\quad 0<j<\ell_{\mathcal{C}}\,.$$

Let us consider the subgroups:

- $\widetilde{Z}^1(\breve{\mathsf{R}},\mathrm{Sym})\subset Z^1(\breve{\mathsf{R}},\mathrm{Sym})$ of the 1-cocycles $(\phi_{D,a})_{(D,a)\in I_{\breve{\mathsf{R}}}}$ such that $\phi_{D,a}=1$ if a is not the terminal edge of some singular chain,
- $\widetilde{C}^0(\breve{\mathsf{R}},\mathrm{Sym})\subset C^0(\breve{\mathsf{R}},\mathrm{Sym})$ of the 0-cochains $(\phi_D)_{D\in\mathsf{Ve}_{\breve{\mathsf{R}}}}$ such that $\phi_D=\xi_D\circ\rho^{s_1}_{D_0}(\phi_{D_0})$ for all interior vertices D of any singular chain, D_0 denoting its initial vertex.

Notice that the coboundary morphism ∂^0 defined in Remark 3.4 maps $\widetilde{C}^0(\breve{\mathsf{R}},\mathrm{Sym})$ into $\widetilde{Z}^1(\breve{\mathsf{R}},\mathrm{Sym})$, allowing us to define the group

$$\widetilde{H}^1(\breve{\mathsf{R}},\mathrm{Sym}):=\mathrm{coker}(\partial^0:\widetilde{C}^0(\breve{\mathsf{R}},\mathrm{Sym})\longrightarrow\widetilde{Z}^1(\mathsf{R},\mathrm{Sym}))\,.$$

We easily see that each element of $H^1(\breve{\mathsf{R}},\mathrm{Sym})$ can be represented by a cocycle belonging to $\widetilde{Z}^1(\breve{\mathsf{R}},\mathrm{Sym})$. We deduce that the morphism

$$\tau:\widetilde{H}^1(\breve{\mathsf{R}},\mathrm{Sym})\longrightarrow H^1(\breve{\mathsf{R}},\mathrm{Sym})$$

induced by the inclusion $\widetilde{Z}^1(\breve{\mathsf{R}},\mathrm{Sym})\subset Z^1(\breve{\mathsf{R}},\mathrm{Sym})$ is surjective. On the other hand if a cochain $\mathbf{c}^0:=(\phi_D)_{D\in\mathsf{Ve}_{\breve{\mathsf{R}}}}\in C^0(\breve{\mathsf{R}},\mathrm{Sym})$ satisfies $\partial^0(\mathbf{c}^0)\in\widetilde{Z}^1(\breve{\mathsf{R}},\mathrm{Sym})$, then for each singular chain $\mathcal{C}$ of lenght $\ell_{\mathcal{C}}\geq 2$ denoted as in (7) we have the equalities

$$\rho^{s_j}_{D_j}(\phi_{D_j})=\rho^{s_j}_{D_{j-1}}(\phi_{D_{j-1}})\,,\quad j<\ell_{\mathcal{C}}\,.$$

It follows that $\mathbf{c}^0\in\widetilde{C}^0(\breve{\mathsf{R}},\mathrm{Sym})$. Therefore $\ker(\tau)$ is trivial and τ is an isomorphism. To achieve the proof of Theorem C, let us first notice that the group $\widetilde{C}^0(\check{\mathsf{R}}, \mathrm{Sym})$ is finite. Indeed it is isomorphic to the direct sum of all centralizers of holonomy groups associated to the vertices of R having singular valency at least three. These holonomy groups being non-abelian by the non-degeneracy assumption, thanks to Lemma 6.14 their centralizers are finite. On the other hand, Proposition 6.10 gives a decomposition of $\widetilde{Z}^1(\check{\mathsf{R}}, \mathrm{Sym})$ as $F \oplus B \oplus_{j=1}^{\lambda} (\mathbb{C}^*/\alpha_j^{\mathbb{Z}}) \oplus (\mathbb{C}^*)^{\nu}$; that completes the proof of Theorem C. □

8. Examples

Before proving Theorem D in full generality let us motivate its statement by computing the moduli space of some non-trivial examples using the identification $\mathrm{Mod}([\mathcal{F}^\diamond]) \simeq H^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$.

• *Example 0: a logarithmic generic multicusp*
Let $\mathcal{L}$ be the germ at $0 \in \mathbb{C}^2$ of the logarithmic foliation defined by the meromorphic form

$$\omega := \sum_{i=1}^{p} \alpha_i \frac{d(y^2 + a_i x^3)}{y^2 + a_i x^3} + \delta \frac{d(y-x)}{y-x} + \sum_{i=1}^{p} \beta_i \frac{d(x^2 + b_i y^3)}{x^2 + b_i y^3}\,,$$

with $a_i, b_i \in \mathbb{C}$ mutually distinct. We normalize the coefficients $\alpha_i, \beta_i, \delta \in \mathbb{C}^*$ by requiring $\delta + 2\sum_{i=1}^{p}(\alpha_i + \beta_i) = 1$. To simplify the exposition we suppose that $\mathcal{E}$ is equal to the exceptional divisor $\mathcal{E}_{\mathcal{L}}$, $\Sigma = \mathrm{Sing}(\mathcal{L}^\sharp)$, the marking being the identity map and $\mathcal{L}^\diamond = (\mathcal{L}, \mathrm{id}_{\mathcal{E}_{\mathcal{L}}})$. Clearly $\mathcal{E}$ is formed by five irreducible components, its dual tree is equal to the cut-graph $\mathsf{A}_{\mathcal{L}^\diamond}$

$$(\mathsf{A}_{\mathcal{L}^\diamond}) \qquad \bullet_{C'} \overset{s'_\infty}{\text{———}} \bullet_{D'} \overset{s'_0}{\text{———}} \bullet_{D} \overset{s''_0}{\text{———}} \bullet_{D''} \overset{s''_\infty}{\text{———}} \bullet_{C''}$$

$$\mathrm{val}_\Sigma(C') = \mathrm{val}_\Sigma(C'') = 1\,, \quad \mathrm{val}_\Sigma(D') = \mathrm{val}_\Sigma(D'') = p+2\,, \quad \mathrm{val}_\Sigma(D) = 3\,,$$

where val_Σ is the singular valency introduced in Definition 5.4. The cut graph $\mathsf{A}_{\mathcal{L}^\diamond}$ decomposes into one singular chain that is the red part $\mathsf{R}_{\mathcal{L}^\diamond}$ of the graph

$$(\mathsf{R}_{\mathcal{L}^\diamond}) \qquad \bullet_{D'} \overset{s'_0}{\text{———}} \bullet_{D} \overset{s''_0}{\text{———}} \bullet_{D''}$$

and two dead branches $\bullet_{C'} \overset{s'_\infty}{\text{———}} \bullet_{D'}$ and $\bullet_{D''} \overset{s''_\infty}{\text{———}} \bullet_{C''}$ necessarily green. Thus the restriction morphisms $\mathrm{Sym}^{\mathcal{L}^\diamond}_{C'} \to \mathrm{Sym}^{\mathcal{L}^\diamond}_{s'_\infty}$, $\mathrm{Sym}^{\mathcal{L}^\diamond}_{C''} \to \mathrm{Sym}^{\mathcal{L}^\diamond}_{s''_\infty}$ are surjective and by Pruning Theorem 3.11, the group $H^1(\mathsf{A}_{\mathcal{L}^\diamond}, \mathrm{Sym}^{\mathcal{L}^\diamond})$ is isomorphic to $H^1(\mathsf{R}_{\mathcal{L}^\diamond}, \mathrm{Sym}^{\mathcal{L}^\diamond})$. On $\mathsf{R} = \mathsf{R}_{\mathcal{L}^\diamond}$ the morphism ∂^0 defined in Remark 3.4 decomposes, with additive notations on abelian groups, as

$$\begin{array}{ccccccc}
C^0(\mathsf{R}, \mathrm{Sym}^{\mathcal{L}^\diamond}) = & C(H_{D'}) & \oplus & C(H_D) & \oplus & C(H_{D''}) \\
\downarrow \partial^0 & \searrow^{\xi_1} & & \swarrow^{\xi_2} \quad \searrow^{\xi_3} & & \swarrow^{\xi_4} \\
Z^1(\mathsf{R}, \mathrm{Sym}^{\mathcal{L}^\diamond}) = & & \mathrm{Sym}^{\mathcal{L}^\diamond}_{s'_0} & \oplus & \mathrm{Sym}^{\mathcal{L}^\diamond}_{s''_0} &
\end{array}$$

$$\partial^0(c_1 \oplus c_2 \oplus c_3) = (\xi_1(c_1) + \xi_2(c_2)) \oplus (\xi_3(c_2) + \xi_4(c_3))\,.$$

On the other hand, $\mathrm{Sing}(\mathcal{L}^\sharp) \cap D'$, resp. $\mathrm{Sing}(\mathcal{L}^\sharp) \cap D''$, is formed by s'_∞, s'_0, resp. s''_∞, s''_0, and the attaching points s'_i, resp. s''_i, of the strict transforms of the curve $\{y^2 + a_i x^3 = 0\}$, resp. $\{x^2 + b_i y^3 = 0\}$, $i = 1, \dots, p$; and $\mathrm{Sing}(\mathcal{L}^\sharp) \cap D$ is formed by s'_0, s''_0 and the attaching point s_1 of the strict transform of $\{y - x = 0\}$. The Camacho-Sad indices of $\mathcal{L}^\sharp$ at these points are

$$\mathrm{CS}(D', s'_\infty) = \mathrm{CS}(D'', s''_\infty) = -1/2\,, \quad \mathrm{CS}(D, s_1) = -\delta\,, \quad \mathrm{CS}(D', s'_0) = -\widetilde{\alpha}\,,$$
$$\mathrm{CS}(D'', s''_0) = -\widetilde{\beta}\,, \quad \mathrm{CS}(D', s'_j) = -\alpha_j \widetilde{\alpha}\,, \quad \mathrm{CS}(D'', s''_j) = -\beta_j \widetilde{\beta}\,,$$

with $\widetilde{\alpha} := (1 + \sum_{i=1}^p \alpha_i)^{-1}/2$, $\widetilde{\beta} := (1 + \sum_{i=1}^p \beta_i)^{-1}/2$. Assuming that $p \geq 3$, we choose α_i, β_i and δ sufficiently generic so that no Camacho-Sad index is a real number, except at the points s'_∞ and s''_∞. All the singularities of $\mathcal{L}^\sharp$ are linearizable and according to Proposition 6.10 the centralizers $C(H_{D'})$, $C(H_D)$, $C(H_{D''})$ are isomorphic to $\mathbb{C}^* = \mathbb{C}/2\pi i\mathbb{Z}$. Using Remark 5.9 we obtain:

$$\begin{array}{ccccccccc}
\frac{\mathbb{C}}{2i\pi\mathbb{Z}} & & \oplus & & \frac{\mathbb{C}}{2i\pi\mathbb{Z}} & & \oplus & & \frac{\mathbb{C}}{2i\pi\mathbb{Z}} \\
 & \searrow^{\xi_1} & & \swarrow^{\xi_2} & & \searrow^{\xi_3} & & \swarrow^{\xi_4} & \\
 & & \mathbb{C}/2i\pi(\mathbb{Z} + \widetilde{\alpha}\mathbb{Z}) & & \oplus & & \mathbb{C}/2i\pi(\mathbb{Z} + \widetilde{\beta}\mathbb{Z}) & &
\end{array}$$

moreover ξ_2 and ξ_3 are induced by the identity map, but ξ_1 is induced by $z \mapsto z/\widetilde{\alpha}$ and ξ_4 by $z \mapsto z/\widetilde{\beta}$. It immediately follows that ∂^0 is surjective and $H^1(\mathsf{R}, \mathrm{Sym}^{\mathcal{L}^\diamond}) = 0$. We conclude that $\mathrm{Mod}([\mathcal{L}^\diamond]) = \{[\mathcal{L}^\diamond]\}$. Thus, $\mathcal{L}$ is topologically SL-rigid which means that if a foliation $\mathcal{F}$ has the same separatrices, Camacho-Sad indices and holonomies than $\mathcal{L}$ then $\mathcal{F}$ is topologically conjugated to $\mathcal{L}$. It is worth to notice that the converse is not true, as [28, Théorème 3.5] shows, because condition (TR) is not satisfied.

Next Examples 1–4 will be suitable perturbations of this logarithmic foliation $\mathcal{L}$ that we have considered in Example 0.

• *Example 1: non-degenerate multicusps*
Let $\mathcal{F}_1$ be a foliation with the same reduction as the previous logarithmic foliation $\mathcal{L}$, same Camacho-Sad indices but such that the holonomy groups along D, D' and D'' are non-abelian. Such a foliation can be obtained by perturbing the holonomy groups of D, D' and D'' as in [22] and by using the realization theorem of [15]. In this case it is well-known that $\mathcal{F}_1$ satisfies condition (TR) and therefore we can compute $\mathrm{Mod}([\mathcal{F}_1^\diamond])$ by identifying it with $H^1(\mathsf{R}, \mathrm{Sym}^{\mathcal{F}_1^\diamond})$. According to Lemma 6.14 the centralizers of $H_{D'}$, H_D and $H_{D''}$ are finite groups F'_1, F_1 and F''_1 respectively, however $\mathrm{Sym}^{\mathcal{F}_1^\diamond}_{s'_0}$ and $\mathrm{Sym}^{\mathcal{F}_1^\diamond}_{s''_0}$ remain isomorphic to $\mathrm{Sym}^{\mathcal{L}^\diamond}_{s'_0}$ and $\mathrm{Sym}^{\mathcal{L}^\diamond}_{s''_0}$ because the singularities of $\mathcal{F}_1^\sharp$ at s'_0 and s''_0 are again linearizable.

$$\begin{array}{ccccccccc}
F'_1 & & \oplus & & F_1 & & \oplus & & F''_1 \\
 & \searrow^{\xi_1} & & \swarrow^{\xi_2} & & \searrow^{\xi_3} & & \swarrow^{\xi_4} & \\
 & & \mathbb{C}/2i\pi(\mathbb{Z} + \widetilde{\alpha}\mathbb{Z}) & & \oplus & & \mathbb{C}/2i\pi(\mathbb{Z} + \widetilde{\beta}\mathbb{Z}) & &
\end{array}$$

It follows:

- $\mathrm{Mod}([\mathcal{F}_1^\diamond])$ *is a finite quotient of a product of two elliptic curves.*

• *Example 2: partially degenerate multicusps*

Let $\mathcal{F}_2$ be a perturbation of the logarithmic foliation $\mathcal{L}$ with same Camacho-Sad indices, with non-abelian holonomy groups along D' and D'' (having finite centralizers F_2' and F_2''), but such that there is a biholomorphism between neighborhoods of D that conjugates $\mathcal{F}_2$ to $\mathcal{L}$. The existence of $\mathcal{F}_2$ is guaranteed by the same process as in Example 1, but without perturbing the holonomy group of D. We have

$$\begin{array}{ccccccccc} F_2' & & \oplus & & \mathbb{C}/2i\pi\mathbb{Z} & & \oplus & & F_2'' \\ & \searrow^{\xi_1} & & \swarrow^{\xi_2} & & \searrow^{\xi_3} & & \swarrow^{\xi_4} & \\ & & \mathbb{C}/2i\pi(\mathbb{Z}+\widetilde{\alpha}\mathbb{Z}) & & \oplus & & \mathbb{C}/2i\pi(\mathbb{Z}+\widetilde{\beta}\mathbb{Z}) & & \end{array}$$

where again ξ_2 and ξ_3 are induced by the identity map. We easily obtain the exact sequence

$$K \longrightarrow \mathbb{C}/2\pi i(\mathbb{Z}+\widetilde{\alpha}\mathbb{Z}+\widetilde{\beta}\mathbb{Z}) \longrightarrow \mathrm{Mod}([\mathcal{F}_2^\diamond]) \longrightarrow 0\,,$$

K being finite. If 1, $\widetilde{\alpha}$, $\widetilde{\beta}$ are $\mathbb{Z}$-independent then

- $\mathrm{Mod}([\mathcal{F}_2^\diamond])$ *is not a finite quotient of a product of elliptic curves; in particular, in the statement of Theorem D we cannot replace* $\mathbb{Z}^p$ *by a finite group in the exact sequence (1).*

• *Example 3: infinite type multicusps*

First we choose the coefficients α_i, β_i, δ in the expression of the 1-form ω, so that $\mathrm{CS}(D', s_0') \in \mathbb{Z}_{<0}$, the other Camacho-Sad indices being in $\mathbb{C}\setminus\mathbb{R}$, except for the points s_∞' and s_∞''. At s_0' the foliation $\mathcal{L}^\sharp$ possesses now a germ of holomorphic first integral, the local holonomy is a periodic rotation, thus $\mathrm{Sym}_{s_0'}^{\mathcal{L}^\diamond}$ is isomorphic to $\mathrm{Diff}(\mathbb{C},0)$. Then, as before we perform a perturbation $\mathcal{F}_3$ of $\mathcal{L}$ changing only the holonomy groups $H_{D'}$ and H_D that become non-abelian, but without changing $H_{D''}$ nor the local analytic types at any singular point. For such foliation $\mathcal{F}_3$ the group $\mathrm{Sym}_{s_0'}^{\mathcal{F}_3^\diamond}$ is always isomorphic to $\mathrm{Sym}_{s_0'}^{\mathcal{F}_3^\diamond} \simeq \mathrm{Diff}(\mathbb{C},0)$. The group-graph $\mathrm{Sym}^{\mathcal{F}_3^\diamond}$ is not abelian and its cohomology is no longer given by the cokernel of a morphism ∂^0. However

$$\xi_4 : \mathrm{Sym}_{D''}^{\mathcal{F}_3^\diamond} \simeq \mathbb{C}/2\pi i\mathbb{Z} \longrightarrow \mathrm{Sym}_{s_0''}^{\mathcal{F}_3^\diamond} \simeq \mathbb{C}/2\pi i(\mathbb{Z}+\widetilde{\beta}\mathbb{Z})$$

is always a submersion. Thus performing a new pruning we have an isomorphism $H^1(\mathsf{R}_{\mathcal{F}_3^\diamond}, \mathrm{Sym}^{\mathcal{F}_3^\diamond}) \simeq H^1(\mathsf{R}', \mathrm{Sym}^{\mathcal{F}_3^\diamond})$ with $\mathsf{R}' := \bullet_{D'} \overset{s_0'}{\longrightarrow} \bullet_D$. The centralizers of $H_{D'}$ and H_D being finite by Lemma 6.14, we obtain:

- $\mathrm{Mod}([\mathcal{F}_3^\diamond])$ *is a quotient of* $\mathrm{Diff}(\mathbb{C},0)$ *by the action of a finite group and* $\mathsf{R}_{\mathcal{F}_3^\diamond}$ *is not connected.*

• *Example 4: Cremer multicusps*

By gluing techniques and thanks to realization Theorem [31] and Pérez Marco's results [29] we can build a foliation $\mathcal{F}_4$ with same separatrices, thus same reduction, as in previous examples, with non-abelian holonomy groups $H_{D'}$, H_D, $H_{D''}$, but whose local holonomies at s_0' and s_0'' are Cremer with uncountable centralizers. In this case

- $\mathrm{Mod}([\mathcal{F}_4^\diamond])$ *is a finite quotient of a product of two uncountable totally discontinuous subgroups of* $\mathbb{U}(1)$.

• *Example 5: non-degenerate foliations with a single separatrix.*
For such a foliation $\mathcal{F}_5$, after pruning all dead branches of the dual graph of $\mathcal{E}_{\mathcal{F}_5}$, the obtained graph is the red graph $\mathsf{R}_{\mathcal{F}_5^\diamond}$ which is reduced to a geodesic segment $\bullet_{D_0} \overset{s_1}{\text{———}} \cdots \overset{s_\ell}{\text{———}} \bullet_{D_\ell}$. All Camacho-Sad indices are rational numbers. The singular chains in $\mathsf{R}_{\mathcal{F}_5^\diamond}$ are in two categories: the *normalizable chains* whose edges s correspond to normalizable resonant singularities of $\mathcal{F}_5^\sharp$ and the non-normalizable chains. For the first one the group $\mathrm{Sym}_s^{\mathcal{F}_5^\diamond}$ is isomorphic to $\mathbb{C}^*$ and for non-normalizable chains it is isomorphic to $\mathbb{Z}/m_s\mathbb{Z}$ for a suitable $m_s \in \mathbb{N}$. It follows:

- $\mathrm{Mod}([\mathcal{F}_5^\diamond]) \simeq (\bigoplus_{i=1}^{\mu} \mathbb{Z}/m_i\mathbb{Z} \oplus \mathbb{C}^{*\nu})/Z$, *with* Z *a finite subgroup,* μ, *resp* ν, *the number of non-normalizable, resp. normalizable singular chains; furthermore* $\mu+\nu$ *is equal to the number of Puiseux pairs of the unique separatrix.*

Another specificity of this foliation $\mathcal{F}_5$ is that the mapping class group of $\mathcal{E}_{\mathcal{F}_5}^\diamond$ is trivial because every singular point of $\mathcal{E}_{\mathcal{F}_5}$ is fixed[7] by $\mathrm{Mcg}(\mathcal{E}_{\mathcal{F}_5}^\diamond)$ and the pure mapping class group of $\mathbb{P}^1$ punctured at three points is trivial [9, Proposition 2.3]. From Corollary 2.2 we obtain that

$$\mathrm{Mod}([\mathcal{F}_5^\diamond]) \subset \mathrm{Mod}(\mathcal{E}_{\mathcal{F}_5}^\diamond) \simeq [\mathsf{Fol}_{\mathsf{tr}}(\mathcal{E}_{\mathcal{F}_5}^\diamond)]_{\mathcal{C}^0}.$$

• *Example 6: some topologically* SL-*rigid foliations.*
Whenever for a marked foliation $\mathcal{F}^\diamond$ the red part of any cut-component of $\mathsf{A}_{\mathcal{F}^\diamond}$ is reduced to one vertex, the moduli space $\mathrm{Mod}([\mathcal{F}^\diamond])$ is reduced to one element. In particular this is the case for:

- any non-dicritical foliation reduced after only one blow-up, its separatrices being smooth curves mutually transversal, or more generally any topologically quasi-homogeneous foliation, see [14],
- absolutely dicritical foliations of Cano-Corral [6],
- dicritical foliations that are non-singular after one blow-up, see [2] and [27].

9. Exponential and disconnected group-graphs

We keep all notations used in Section 6. For technical reasons the last group-graphs that we must consider will be defined uniquely over the red graph $\mathsf{R}_{\mathcal{F}^\diamond} \subset \mathsf{A}_{\mathcal{F}^\diamond}$. Recall that $\mathcal{X}^{\mathcal{F}^\diamond}$, $\mathcal{B}^{\mathcal{F}^\diamond}$ and $\mathcal{T}^{\mathcal{F}^\diamond}$ denote the group-graphs over $\mathsf{A}_{\mathcal{F}^\diamond}$ associated to the sheaves $\mathcal{X}^{\mathcal{F}^\sharp}$, $\mathcal{B}^{\mathcal{F}^\sharp}$ and $\mathcal{T}^{\mathcal{F}^\sharp} = \mathcal{B}^{\mathcal{F}^\sharp}/\mathcal{X}^{\mathcal{F}^\sharp}$ of tangent, basic and transverse holomorphic vector fields for $\mathcal{F}^\sharp$, respectively.

Lemma 9.1. *For* $s \in \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ *the exponential map* $\exp : \mathcal{B}_s^{\mathcal{F}^\diamond} \to \mathrm{Aut}_s^{\mathcal{F}^\diamond}$ *induces a well-defined map* $\exp_s^{\mathcal{F}^\diamond} : \mathcal{T}_s^{\mathcal{F}^\diamond} \to \mathrm{Sym}_s^{\mathcal{F}^\diamond}$.

Proof. We must prove that $\exp(Z+X) \equiv \exp(Z)$ modulo $\mathrm{Fix}_s^{\mathcal{F}^\diamond}$ if $Z \in \mathcal{B}_s^{\mathcal{F}^\diamond}$ and $X \in \mathcal{X}_s^{\mathcal{F}^\diamond}$. For that it suffices to show that for each neighborhood V of $f(s)$ there is another neighborhood U of $f(s)$ such that for each $p \in U$

[7] Each element of the mapping class group of $\mathcal{E}_{\mathcal{F}_5}^\diamond$ preserves dead branches so it must fix every singular point except maybe the attaching points of the two dead branches of the extremity divisor of singular valency 3. But these two points have different Camacho-Sad indices, as it can be easily deduced from [12, p. 164].

the curve $\alpha \in [0,1] \mapsto \exp(Z+\alpha X)(p)$ is contained in a leaf of $\mathcal{F}^{\sharp}_{|V}$. We choose $U \subset V$ such that the map $\phi : U \times \mathbb{D}_2 \times \mathbb{D}_2 \to V$ given by $\phi(p,t,\alpha) = \exp(t(Z+\alpha X))(p)$ is well-defined. Fix $p \in U$ and take a local holomorphic first integral F defined in a neighborhood W of p. If t is small enough then $\phi(p,t,\alpha) \in W$ and

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial \alpha}F(\phi(p,t,\alpha))\right) &= \frac{\partial}{\partial \alpha}\left(\frac{\partial}{\partial t}F(\phi(p,t,\alpha))\right)\\ &= \frac{\partial}{\partial \alpha}\left((Z+\alpha X)(F)\circ\phi(p,t,\alpha)\right)\\ &= [X(F)\circ\phi(p,t,\alpha)]\frac{\partial}{\partial \alpha}\phi(p,t,\alpha) = 0\end{aligned}$$

because X is tangent to $\mathcal{F}^{\sharp}$. Since $\phi(p,0,\alpha) = p$ does not depend on α we obtain that $\frac{\partial}{\partial \alpha}F(\phi(p,t,\alpha)) = 0$ for t small enough. As ϕ is holomorphic, we conclude that the curve $\alpha \mapsto \phi(p,1,\alpha)$ is contained in a leaf of $\mathcal{F}^{\sharp}_{|V}$. □

Remark 9.2. It can be checked that under the identifications $\mathcal{T}_s^{\mathcal{F}^\diamond} \simeq \mathcal{T}_{h_{D,s}}$ and $C(h_{D,s})/\langle h_{D,s}\rangle \simeq \mathrm{Sym}_s^{\mathcal{F}^\diamond}$ given by Remark 6.2 and Corollary 5.8, the morphism $\exp_s^{\mathcal{F}^\diamond}$ coincides with the composition of the restriction $\mathcal{T}_{h_{D,s}} \to C(h_{D,s})$ of the exponential map on the transverse section Δ_D and the quotient map $C(h_{D,s}) \to C(h_{D,s})/\langle h_{D,s}\rangle$. □

Motivated by the above remark, for $D \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ we define the map $\exp_D^{\mathcal{F}^\diamond} : \mathcal{T}_D^{\mathcal{F}^\diamond} \to \mathrm{Sym}_D^{\mathcal{F}^\diamond}$ as the composition $\mathcal{T}_D^{\mathcal{F}^\diamond} \simeq \mathcal{T}_{H_D} \overset{\exp}{\to} C(H_D) \to \mathrm{Sym}_D^{\mathcal{F}^\diamond}$. From Remark 9.2 it is clear that the following diagram is commutative:

$$\begin{array}{ccc}\mathcal{T}_D^{\mathcal{F}^\diamond} & \xrightarrow{\exp_D^{\mathcal{F}^\diamond}} & \mathrm{Sym}_D^{\mathcal{F}^\diamond}\\ \downarrow{\scriptstyle \rho_D^s} & & \downarrow{\scriptstyle \rho_D^s}\\ \mathcal{T}_s^{\mathcal{F}^\diamond} & \xrightarrow{\exp_s^{\mathcal{F}^\diamond}} & \mathrm{Sym}_s^{\mathcal{F}^\diamond}\end{array} \tag{8}$$

the vertical maps being the restriction maps of the group-graphs $\mathcal{T}^{\mathcal{F}^\diamond}$ and $\mathrm{Sym}^{\mathcal{F}^\diamond}$, written with the same notation.

Although the exponential map

$$\exp : \mathbb{C}\{z\}z\partial_z \to \mathrm{Diff}(\Delta_D, f(o_D)) \simeq \mathrm{Diff}(\mathbb{C},0),$$

$z : (\Delta_D, f(o_D)) \to (\mathbb{C},0)$ being a germ of coordinate, is not a morphism of groups, its restriction to a subspace of complex dimension ≤ 1 is. On the other hand it is well-known that $\dim_{\mathbb{C}} \mathcal{T}_s^{\mathcal{F}^\diamond} \leq 1$ if the stalk $\mathcal{J}_s^{\mathcal{F}^\diamond}$ of the sheaf of first integrals is equal to $\mathbb{C}$. Since $\mathcal{T}_D^{\mathcal{F}^\diamond} \subset \mathcal{T}_s^{\mathcal{F}^\diamond}$ for $s \in D$ we deduce that $\exp_D^{\mathcal{F}^\diamond}$ and $\exp_s^{\mathcal{F}^\diamond}$ define a morphism

$$\exp^{\mathcal{F}^\diamond} : \mathcal{T}^{\mathcal{F}^\diamond} \to \mathrm{Sym}^{\mathcal{F}^\diamond}$$

of abelian group-graphs over $\mathsf{R}_{\mathcal{F}^\diamond}$.

Definition 9.3. *The group-graph over* $\mathsf{R}_{\mathcal{F}^\diamond}$ *image of* $\exp^{\mathcal{F}^\diamond}$ *is called the exponential group-graph of* $\mathcal{F}^\diamond$. *We denote it by* $\mathrm{Exp}^{\mathcal{F}^\diamond}$.

At this point it is clear that the subset of $\mathsf{R}_{\mathcal{F}^\diamond}$ consisting of all $a \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}} \cup \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$ such that $\mathrm{Exp}_a^{\mathcal{F}^\diamond} = 0$ is just the subgraph $\mathsf{R}^0_{\mathcal{F}^\diamond}$ of $\mathsf{R}_{\mathcal{F}^\diamond}$ previously defined in Section 6 and characterized by the second equivalence in Theorem 6.11. Let us denote by $\mathsf{R}^1_{\mathcal{F}^\diamond}$ the *completion* of $\mathsf{R}_{\mathcal{F}^\diamond} \setminus \mathsf{R}^0_{\mathcal{F}^\diamond}$, i.e. the minimal subgraph of $\mathsf{R}_{\mathcal{F}^\diamond}$ containing $\mathsf{R}_{\mathcal{F}^\diamond} \setminus \mathsf{R}^0_{\mathcal{F}^\diamond}$.

Lemma 9.4. *If $(D, s) \in I_{\mathsf{R}_{\mathcal{F}^\diamond}}$ and $\mathcal{T}_D^{\mathcal{F}^\diamond} \neq 0$, then the restriction map ${\rho'}^s_D : \mathcal{T}_D^{\mathcal{F}^\diamond} \to \mathcal{T}_s^{\mathcal{F}^\diamond}$ is an isomorphism and all the red singular points in D share the same character resonant non-linearizable or linearizable; we will say that D is resonant or linearizable according to the case. Furthermore, the isomorphism class of the group $\mathrm{Exp}_D^{\mathcal{F}^\diamond}$ is given by the following table*

$\mathrm{Exp}_D^{\mathcal{F}^\diamond}$	$\mathrm{val}_\Sigma(D) \leq 2$	$\mathrm{val}_\Sigma(D) \geq 3$
D resonant non-linearizable	$\mathbb{C}/\mathbb{Z}$	$\mathbb{C}$
D linearizable	$\mathbb{C}^*/\alpha^{\mathbb{Z}}$	$\mathbb{C}^*$

(9)

The restriction morphism $\rho^s_D : \mathrm{Exp}_D^{\mathcal{F}^\diamond} \to \mathrm{Exp}_s^{\mathcal{F}^\diamond}$ is surjective and

$$\ker \rho^s_D \simeq \begin{cases} \mathbb{Z} & \textit{if} \quad \mathrm{val}_\Sigma(D) \geq 3, \\ 0 & \textit{if} \quad \mathrm{val}_\Sigma(D) \leq 2. \end{cases}$$

Proof. The homogeneity of singular types in D is given by Proposition 6.15. Notice that if a basic vector field for $\mathcal{F}^\sharp$ defined on a connected open set $U \subset M_{\mathcal{F}}$ is tangent to the foliation in a neighborhood of a point of U then it is tangent to the foliation on the whole U. Using this fact it is easy to see that if W is a connected subset of a $\mathcal{F}^\sharp$-invariant component of $\mathcal{E}_{\mathcal{F}}$, the stalk maps $\mathcal{T}^{\mathcal{F}^\sharp}(W) \to \mathcal{T}_m^{\mathcal{F}^\sharp}$, $m \in W$ are injective. Taking $W = D$ we deduce that the restriction map ${\rho'}^s_D : \mathcal{T}_D^{\mathcal{F}^\diamond} \to \mathcal{T}_s^{\mathcal{F}^\diamond}$ is injective. Since $\dim \mathcal{T}_D^{\mathcal{F}^\diamond} = \dim \mathcal{T}_s^{\mathcal{F}^\diamond} = 1$, ${\rho'}^s_D$ is an isomorphism. In fact, $\mathcal{T}_s^{\mathcal{F}^\diamond} \simeq \mathcal{T}_D^{\mathcal{F}^\diamond}$ can be identified to a line $\mathbb{C}X$ in the space of germs of vector fields on $(\Delta_D, f(o_D))$. Then Table (9) follows from Proposition 6.10.

On the other hand $\exp_s^{\mathcal{F}^\diamond} : \mathcal{T}_s^{\mathcal{F}^\diamond} \to \mathrm{Exp}_s^{\mathcal{F}^\diamond}$ being surjective by definition and the restriction map ${\rho'}^s_D$ being an isomorphism, it follows that $\rho^s_D \circ \exp_D^{\mathcal{F}^\diamond} = \exp_s^{\mathcal{F}^\diamond} \circ {\rho'}^s_D$ is surjective. Therefore ρ^s_D is also surjective. The last assertion follows from Remark 9.2 and the commutativity of the diagram (8). □

Thanks to Theorem 6.11, $H^1(\mathsf{R}_{\mathcal{F}^\diamond}, \mathrm{Sym}^{\mathcal{F}^\diamond})$ is an abelian group and the natural inclusion $\mathrm{Exp}^{\mathcal{F}^\diamond} \to \mathrm{Sym}^{\mathcal{F}^\diamond}$ is an injective morphism of abelian group-graphs over $\mathsf{R}_{\mathcal{F}^\diamond}$.

Definition 9.5. *The quotient group-graph over $\mathsf{R}_{\mathcal{F}^\diamond}$*

$$\mathrm{Dis}^{\mathcal{F}^\diamond} := \mathrm{Sym}^{\mathcal{F}^\diamond} / \mathrm{Exp}^{\mathcal{F}^\diamond}$$

given by $\mathrm{Dis}_a^{\mathcal{F}^\diamond} := \mathrm{Sym}_a^{\mathcal{F}} / \mathrm{Exp}_a^{\mathcal{F}}$ for every $a \in \mathsf{Ve}_{\mathsf{R}_{\mathcal{F}^\diamond}} \cup \mathsf{Ed}_{\mathsf{R}_{\mathcal{F}^\diamond}}$, is called the disconnected group-graph of $\mathcal{F}^\diamond$.

Obviously, we have over $\mathsf{R}_{\mathcal{F}^\diamond}$ the short exact sequence of abelian group-graphs:

$$0 \to \mathrm{Exp}^{\mathcal{F}^\diamond} \to \mathrm{Sym}^{\mathcal{F}^\diamond} \to \mathrm{Dis}^{\mathcal{F}^\diamond} \to 0. \tag{10}$$

The name "disconnected" is explained by the following proposition.

Proposition 9.6. *For $s \in \mathsf{Ed}_{\mathsf{R}_{\mathcal{F}^\diamond}}$ and $D \in \mathsf{Ve}_{\mathsf{R}_{\mathcal{F}^\diamond}}$ we have that*

(1) *the abelian group* $\mathrm{Dis}_s^{\mathcal{F}^\diamond} \simeq C(h_{D,s})/\langle h_{D,s},\, \exp(\mathcal{T}_{h_{D,s}})\rangle$ *is:*
   (a) *trivial if* $f(s)$ *is a linearizable (non-periodic) singularity*
   (b) *a finite abelian group if* $f(s)$ *is a resonant (non-periodic) singularity, cyclic in the normalizable case;*
   (c) *a cyclic quotient of a totally disconnected subgroup of* $\mathbb{U}(1)$ *if* $f(s)$ *is a non-resonant non-linearizable singularity;*

(2) *the abelian group* $\mathrm{Dis}_D^{\mathcal{F}^\diamond}$ *is*
   (a) *infinite and finitely generated if* H_D *is abelian and all the red singularities on* $f(D)$ *are resonant non-normalizable;*
   (b) *a cyclic quotient of a totally disconnected subgroup of* $\mathbb{U}(1)$ *if* H_D *is abelian and all the red singularities on* $f(D)$ *are non-resonant non-linearizable;*
   (c) *finite in all the remaining cases;*

(3) $\mathrm{Dis}_s^{\mathcal{F}^\diamond}$ *and* $\mathrm{Dis}_D^{\mathcal{F}^\diamond}$ *are finite if* $s \in \mathsf{Ed}_{\mathsf{R}^1_{\mathcal{F}^\diamond}}$ *and* $D \in \mathsf{Ve}_{\mathsf{R}^1_{\mathcal{F}^\diamond}}$.

Proof. Assertions (1) result directly from Proposition 6.10.

To obtain Assertions (2) we can suppose that the singular valency of D is at least three, otherwise $\mathrm{Dis}_D^{\mathcal{F}^\diamond} = \mathrm{Dis}_{\mathrm{s}}^{\mathcal{F}^\diamond}$ for $s \in D \cap \Sigma$ and Assertions (2) result again directly from Proposition 6.10. Now let us suppose also that $\mathrm{Dis}_D^{\mathcal{F}^\diamond}$ -thus also $C(H_D)$- is infinite. Because D is red, H_D is infinite and it follows from Lemmas 6.13 and 6.14 that the set H' of all non-periodic elements of H_D is nonempty and for every $h \in H'$ we have $C(H_D) = C(h)$, therefore $\mathrm{Dis}_D^{\mathcal{F}^\diamond} = C(h)/\exp(\mathcal{T}_h)$. By Proposition 6.10 the only case where this group is infinite is when h is resonant non-normalizable or non-resonant non-linearizable. To see that these two possibilities correspond to the cases (2a) and (2b) above, it is enough to notice that the local holonomies $h_{D,s}$, $s \in D \cap \Sigma$, that generate H_D, cannot be all periodic (otherwise by abelianity H_D would be finite), and to use Proposition 6.15.

Assertion (3) follows immediately from Assertions (1) and (2) except for $\mathrm{Dis}_{\mathrm{D}}^{\mathcal{F}^\diamond}$ when D is a common vertex of $\mathsf{R}^1_{\mathcal{F}^\diamond}$ and $\mathsf{R}^0_{\mathcal{F}^\diamond}$. In this case although $\mathrm{Exp}_D^{\mathcal{F}^\diamond} = 0$, at the meeting points s of D with components D' of $\mathsf{R}^1_{\mathcal{F}^\diamond}$ we have $\mathrm{Exp}_s^{\mathcal{F}^\diamond} \neq 0$ because $\mathrm{Exp}_{D'}^{\mathcal{F}^\diamond} \neq 0$. Therefore D does not correspond to case (2a) nor case (2b) and $\mathrm{Dis}_{\mathrm{D}}^{\mathcal{F}^\diamond}$ is finite according to (2c). □

10. Proof of Theorem D

In order to simplify the notations in the proofs below, we will write again A, R, Aut, Sym, Exp, Dis and τ, instead of $\mathsf{A}_{\mathcal{F}^\diamond}$, $\mathsf{R}_{\mathcal{F}^\diamond}$, $\mathrm{Aut}^{\mathcal{F}^\diamond}$, $\mathrm{Sym}^{\mathcal{F}^\diamond}$, $\mathrm{Exp}^{\mathcal{F}^\diamond}$, $\mathrm{Dis}^{\mathcal{F}^\diamond}$ and $\tau_{\mathcal{F}}$. We will also write R_0 and R_1 instead of $\mathsf{R}^0_{\mathcal{F}^\diamond}$, $\mathsf{R}^1_{\mathcal{F}^\diamond}$.

We have already shown that there are "natural" bijections:

$$\mathrm{Mod}([\mathcal{F}^\diamond]) \overset{(2)}{\simeq} H^1(\mathsf{A}, \mathrm{Aut}) \overset{(5)}{\simeq} H^1(\mathsf{A}, \mathrm{Sym}) \overset{(6)}{\simeq} H^1(\mathsf{R}, \mathrm{Sym})\,, \tag{11}$$

the bijection (6) being only valid when $\mathcal{F}$ is of finite type. Moreover Sym is an abelian group-graph over R and consequently $H^1(\mathsf{R}, \mathrm{Sym})$ is an abelian group, cf. Theorem 6.11. Recall that R_0 is the subgraph of R constituted by all the vertices and edges b satisfying $\mathrm{Exp}_b = 0$ and R_1 is the completion of $\mathsf{R} \setminus \mathsf{R}_0$. The rest of the proof is divided in several steps:

(i) The abelian group $H^1(\mathsf{R}, \mathrm{Sym})$ fits into an exact sequence
$$0 \to F \to H^1(\mathsf{R}, \mathrm{Exp}) \to H^1(\mathsf{R}, \mathrm{Sym}) \to \mathbf{D} \to 0,$$
where F is a finite abelian group and $\mathbf{D}$ is a totally disconnected topological abelian group.

(ii) We have group isomorphisms
$$H^1(\mathsf{R}, \mathrm{Exp}) \simeq H^1(\mathsf{R}_1, \mathrm{Exp}) \simeq \bigoplus_{\alpha} H^1(\mathsf{Z}^\alpha, \mathrm{Exp}),$$
with $\mathsf{R}_1 =: \bigcup_{\alpha \in \pi_0(\mathsf{R} \setminus \mathsf{R}_0)} \mathsf{Z}^\alpha$ where each *zone* Z^α is the completion of a connected component of $\mathsf{R} \setminus \mathsf{R}_0 = \mathsf{R}_1 \setminus \mathsf{R}_0$.

(iii) To simplify the computation of the cohomology groups $H^1(\mathsf{Z}^\alpha, \mathrm{Exp})$ we modify each zone (not reduced to a single vertex) without changing the number of its extremities nor its cohomology, by adding a vertex and an edge, for each of its extremities. The modified zones fulfill the following property:

$(*)$ *each extremity of* Z^α *is joined by its edge to a vertex of valency* 2 *in* Z^α.

(iv) We decompose each modified zone Z as $\mathsf{Z} = \mathsf{Z}_0 \cup \mathsf{Z}_1$ where Z_0 is either empty or a disjoint union of $n + 1 \geq 1$ segments $\bullet_{D'_i} — \bullet_{D_i}$ with $D'_i \in \mathsf{R}_0$, $D_i \in \mathsf{R}_1$ and $\mathsf{Z}_0 \cap \mathsf{Z}_1 = \{D_0, \ldots, D_n\}$. We prove that $H^1(\mathsf{Z}, \mathrm{Exp})$ is trivial if Z_0 is empty, and it is the quotient of $\bigoplus_{i=1}^{n} \mathrm{Exp}_{D_i}$ by a finitely generated subgroup if $\mathsf{Z}_0 \neq \emptyset$.

(v) Since Exp_{D_i} is isomorphic to $\mathbb{C}$ or $\mathbb{C}/\mathbb{Z} \simeq \mathbb{C}^*$ or $\mathbb{C}^*/\alpha^{\mathbb{Z}}$ by Lemma 9.4, we can construct a morphism $\Lambda : \mathbb{C}^\tau \to \mathrm{Mod}([\mathcal{F}^\diamond])$ with totally disconnected cokernel and finitely generated kernel.

(vi) We specify the notion of semi-local-equisingularity, denoted by SL-equisingularity. This notion was introduced in [24] for germs of deformations and in this paper we adapt it to the context of a global parameter space.

(vii) We construct SL-equisingular families of foliations $\mathcal{F}^U_{t,i}$ satisfying Theorem D.

Step (i). Consider the long exact sequences
$$\cdots \to H^0(\mathsf{R}, \mathrm{Dis}) \to H^1(\mathsf{R}, \mathrm{Exp}) \xrightarrow{\chi} H^1(\mathsf{R}, \mathrm{Sym}) \to H^1(\mathsf{R}, \mathrm{Dis}) \to 0 \quad (12)$$
and
$$\cdots \to H^0(\mathsf{R}_1, \mathrm{Dis}) \to H^1(\mathsf{R}_1, \mathrm{Exp}) \xrightarrow{\chi_1} H^1(\mathsf{R}_1, \mathrm{Sym}) \to H^1(\mathsf{R}_1, \mathrm{Dis}) \to 0$$
associated by Lemma 3.6 to the short exact sequence (10) of abelian group-graphs. By the first part of Proposition 9.6, $Z^1(\mathsf{R}, \mathrm{Dis})$ is a finite product of totally disconnected subgroups of $\mathbb{U}(1)$ and $H^1(\mathsf{R}, \mathrm{Dis})$ is thus a totally disconnected abelian topological group. Moreover when all the singularities of the foliation are resonant or linearizable, the case 1c) is excluded and $Z^1(\mathsf{R}, \mathrm{Dis})$ is finite. In order to conclude this step it only remains to prove that $\ker \chi$ is finite. Let us notice that $H^0(\mathsf{R}_0 \cap \mathsf{R}_1, \mathrm{Exp}) = 0$, $H^1(\mathsf{R}_0, \mathrm{Exp}) = 0$

and $H^1(\mathsf{R}_0 \cap \mathsf{R}_1, \mathrm{Exp}) = 0$. By applying Mayer-Vietoris Lemma 3.7 to the union $\mathsf{R} = \mathsf{R}_0 \cup \mathsf{R}_1$ we obtain the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^1(\mathsf{R}, \mathrm{Exp}) & \longrightarrow & 0 \oplus H^1(\mathsf{R}_1, \mathrm{Exp}) & \longrightarrow & 0 \\ & & \downarrow \chi & & \downarrow 0\oplus\chi_1 & & \\ \cdots & \longrightarrow & H^1(\mathsf{R}, \mathrm{Sym}) & \longrightarrow & H^1(\mathsf{R}_0, \mathrm{Sym}) \oplus H^1(\mathsf{R}_1, \mathrm{Sym}) & \longrightarrow & \cdots \end{array} \tag{13}$$

Thus $\ker \chi$ is isomorphic to a subgroup of $\ker \chi_1$ and it is sufficient to prove that $H^0(\mathsf{R}_1, \mathrm{Dis}^{\mathcal{F}})$ is finite. But using the second part of Proposition 9.6 we obtain that $C^0(\mathsf{R}_1, \mathrm{Dis}^{\mathcal{F}})$ is finite.

Step (ii). Diagram (13), coming from the Mayer-Vietoris sequence, gives us the isomorphism

$$H^1(\mathsf{R}, \mathrm{Exp}) \simeq H^1(\mathsf{R}_1, \mathrm{Exp})\,. \tag{14}$$

Clearly R_1 is a finite union of zones Z^α. Moreover, $\mathsf{Z}^\alpha \cap \mathsf{Z}^\beta$ is either empty or a single vertex of R_0. Therefore

$$H^0(\mathsf{Z}^\alpha \cap \mathsf{Z}^\beta, \mathrm{Exp}) = 0 \quad \text{and} \quad H^1(\mathsf{Z}^\alpha \cap \mathsf{Z}^\beta, \mathrm{Exp}) = 0.$$

By applying recursively Mayer-Vietoris Lemma 3.7 we deduce that

$$H^i(\mathsf{R}_1, \mathrm{Exp}) \simeq \bigoplus_{\alpha \in \pi_0(\mathsf{R} \setminus \mathsf{R}_0)} H^i(\mathsf{Z}^\alpha, \mathrm{Exp})\,, \quad i = 0, 1\,. \tag{15}$$

Step (iii). If a zone Z^α is reduced to a single vertex then $H^1(\mathsf{Z}^\alpha, \mathrm{Exp})$ is clearly trivial. If this is not the case, we modify Z^α in the following way: if v' is an extremity of Z^α and $v'' \in \mathsf{Ve}_{\mathsf{Z}^\alpha}$ is the unique vertex joined to v' by an edge e', we replace the segment $\bullet_{v''} \overset{e'}{\text{——}} \bullet_{v'}$ by $\bullet_{v''} \overset{e''}{\text{——}} \bullet_{v} \overset{e}{\text{——}} \bullet_{v'}$. We also extend the group-graph Exp to the new edges and vertices by defining

$$\mathrm{Exp}_{e''} := \mathrm{Exp}_e := \mathrm{Exp}_v := \mathrm{Exp}_{e'}\,, \qquad \rho_v^{e''} := \rho_v^e := \mathrm{id}_{\mathrm{Exp}_{e'}}\,, \tag{16}$$

$$\rho_{v''}^{e''} = \rho_{v''}^{e'}\,, \qquad \rho_{v'}^{e} = \rho_{v'}^{e'}\,.$$

We call this operation the *blow-up of the edge* e'. By performing these blow-ups for each extremity of Z^α we get a new graph $\widetilde{\mathsf{Z}}^\alpha$ called *a modified zone*. Clearly $\widetilde{\mathsf{Z}}^\alpha$ fulfills property $(*)$ of (iii).

By doing this process on each zone, we get a *modified graph* $\widetilde{\mathsf{R}}$ endowed with a group-graph still denoted by Exp. We define now a contraction map:

$$Z^1(\widetilde{\mathsf{R}}, \mathrm{Exp}) \overset{\flat}{\to} Z^1(\mathsf{R}, \mathrm{Exp})\,, \qquad c = (\phi_{ve}) \mapsto c^\flat = (\phi_{ve}^\flat)$$

where $\phi_{ve}^\flat = \phi_{ve}$ if e is not produced by a blow-up, and $\phi_{v''e'} := \phi_{v''e''}\phi_{ve} =: \phi_{v'e'}^{-1}$ if $\bullet_{v''} \overset{e''}{\text{——}} \bullet_{v} \overset{e}{\text{——}} \bullet_{v'}$ is given by the blow-up of $\bullet_{v''} \overset{e'}{\text{——}} \bullet_{v'}$. It is easy to see that this map induces group isomorphisms

$$H^1(\widetilde{\mathsf{Z}}^\alpha, \mathrm{Exp}) \simeq H^1(\mathsf{Z}^\alpha, \mathrm{Exp})\,, \quad H^1(\widetilde{\mathsf{R}}, \mathrm{Exp}) \simeq H^1(\mathsf{R}, \mathrm{Exp})\,. \tag{17}$$

Step (iv). Fix $\mathsf{Z} = \widetilde{\mathsf{Z}}^\alpha$ a modified zone of R_1. Let Z_1 be the maximal subgraph of Z with all vertices and edges b satisfying $\mathrm{Exp}_b \neq 0$. Denote by Z_0 the completion of $\mathsf{Z} \setminus \mathsf{Z}_1$. Clearly $\mathsf{Z} = \mathsf{Z}_0 \cup \mathsf{Z}_1$ and Z_0 is either empty or a disjoint union of $n+1 \geq 1$ segments $\bullet_{D'} — \bullet_D$ with $\mathrm{Exp}_{D'} = 0$, $\mathrm{Exp}_D \neq 0$, D' being an extremity of Z and $\mathrm{val}_{\mathsf{Z}}(D) = 2$. Notice that $H^1(\mathsf{Z}_1, \mathrm{Exp}) = 0$. Indeed the restriction morphisms of the group-graph Exp over Z_1 are surjective by Lemma 9.4. We apply recursively Pruning Theorem 3.11 and we conclude by Remark 3.12 that $H^1(\mathsf{Z}, \mathrm{Exp}) = 0$ if Z_0 is empty.

Now suppose that $\mathsf{Z}_0 \neq \emptyset$. We will apply Mayer-Vietoris Lemma 3.7 to $\mathsf{Z} = \mathsf{Z}_0 \cup \mathsf{Z}_1$. Using again Lemma 9.4 we see that $H^1(\mathsf{Z}_0, \mathrm{Exp}) = 0$ and $H^0(\mathsf{Z}_0, \mathrm{Exp}) = 0$ by construction of the modified zones. We obtain the exact sequence

$$H^0(\mathsf{Z}_1, \mathrm{Exp}) \xrightarrow{\sigma_\alpha} H^0(\mathsf{Z}_0 \cap \mathsf{Z}_1, \mathrm{Exp}) \xrightarrow{\delta_\alpha} H^1(\mathsf{Z}, \mathrm{Exp}) \to 0, \tag{18}$$

σ_α being the restriction map and δ_α the connecting map.

In the sequel we will choose one vertex D_0 in $\mathsf{Z}_0 \cap \mathsf{Z}_1 = \{D_0, \ldots, D_n\}$ and we will call the remaining vertices $D_1, \ldots, D_n$ the *active vertices of the zone* Z.

Lemma 10.1. *The projection* $\pi_\alpha : H^0(\mathsf{Z}_1, \mathrm{Exp}) \to H^0(D_0, \mathrm{Exp}) \simeq \mathrm{Exp}_{D_0}$, $(y_D)_D \mapsto y_{D_0}$ *is surjective and its kernel is a finitely generated abelian group.*

Proof. From Lemma 9.4 for each $(D, s) \in I_{\mathsf{Z}_1}$ the restriction map ρ_D^s is surjective with kernel 0 or $\mathbb{Z}$. Since $D_0 \in \mathsf{Z}_0 \cap \mathsf{Z}_1$ then $\mathrm{val}_{\mathsf{Z}}(D_0) = 2$ by Step (iv). Let s_0 be the edge of Z passing through D_0 and let D_0' be the other vertex of s_0. Let s_i, $i = 1, \ldots, \ell$, be the edges such that $D_0' \in \partial s_i$. For each $y_0 \in \mathrm{Exp}_{D_0}$ there are $y_i \in \mathrm{Exp}_{s_i}$, $i = 1, \ldots, \ell$, such that $\rho_{D_i}^{s_i}(y_i) = \rho_{D_0}^{s_i}(y_0)$, see Figure 3. Moreover, the different choices of y_i are parametrized by $\mathbb{Z}^k$ with $k \leq \ell$. By induction we easily deduce that π_α is surjective and its kernel is finitely generated. □

Denote by $\rho : H^0(\mathsf{Z}_0 \cap \mathsf{Z}_1, \mathrm{Exp}) \to H^0(D_0, \mathrm{Exp}) \simeq \mathrm{Exp}_{D_0}$ the projection map. Then we have the following morphism of exact sequences

$$\begin{array}{ccccccccc} & & H^0(\mathsf{Z}_1, \mathrm{Exp}) & \xrightarrow{\sigma_\alpha} & H^0(\mathsf{Z}_0 \cap \mathsf{Z}_1, \mathrm{Exp}) & \xrightarrow{\delta_\alpha} & H^1(\mathsf{Z}, \mathrm{Exp}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \pi_\alpha} & & \downarrow{\scriptstyle \rho} & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Exp}_{D_0} & \xrightarrow{\mathrm{id}} & \mathrm{Exp}_{D_0} & \longrightarrow & 0 & & \end{array}$$

Since $\mathsf{Z}_0 \cap \mathsf{Z}_1$ only contains the vertices D_i we have that $H^0(\mathsf{Z}_0 \cap \mathsf{Z}_1, \mathrm{Exp}) = \bigoplus_{i=0}^n \mathrm{Exp}_{D_i}$ and $\ker \rho$ is just $\bigoplus_{i=1}^n \mathrm{Exp}_{D_i}$. Because π_α is surjective, by applying the snake lemma we obtain the following exact sequence:

$$\ker(\pi_\alpha) \to \bigoplus_{i=1}^n \mathrm{Exp}_{D_i} \xrightarrow{\delta_\alpha} H^1(\mathsf{Z}, \mathrm{Exp}) \to \mathrm{coker}(\pi_\alpha)\,. \tag{19}$$

Since π_α is surjective and $\ker(\pi_\alpha)$ is finitely generated, thanks to Lemma 10.1 we get that $H^1(\mathsf{Z}, \mathrm{Exp})$ is a quotient of $\mathbb{C}^n$ by a finitely generated subgroup.

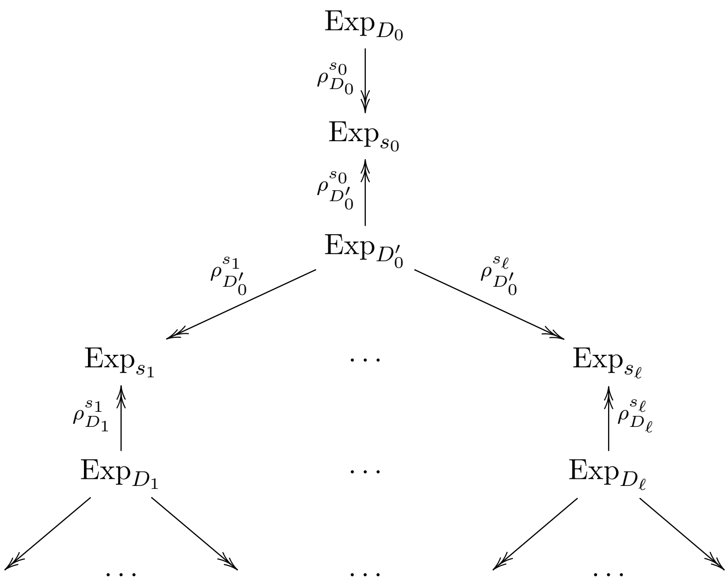

FIGURE 3. Schematic diagram for the group-graph Exp restricted to Z_1.

Step (v). First we notice that the number of active vertices of a modified zone $\mathsf{Z} = \widetilde{\mathsf{Z}}^\alpha$ is equal to the rank of the homology groups $H_1(\mathsf{Z}/\mathsf{Z}_0, \mathbb{Z}) \simeq H_1(\mathsf{Z}^\alpha/(\mathsf{Z}^\alpha \cap \mathsf{R}_0), \mathbb{Z})$ of the corresponding quotient graphs. We easily deduce that the number of all active vertices a_r for all the modified zones is equal to the rank $\tau := \mathrm{rank}\, H_1(\mathsf{R}/\mathsf{R}_0, \mathbb{Z})$ introduced in Definition 6.9.

Each active vertex a_r, $r = 1, \ldots, \tau$, belonging to some modified zone, is produced by the blow-up of an edge $\bullet_{v''_r} \overset{s_r}{\text{---}} \bullet_{v'_r}$, v'_r being an extremity of the zone. By construction, $\mathrm{Exp}_{a_r} = \mathrm{Exp}_{s_r}$, cf. (16). With this identification and thanks to the isomorphisms (14), (15) and (17) it can be easily checked, using the proof of Lemma 3.7, that the map $\delta = \oplus_\alpha \delta_\alpha$ given by the connecting maps δ_α of the Mayer-Vietoris exact sequences (18) is the surjective morphism

$$\delta \ : \ \bigoplus_{r=1}^{\tau} \mathrm{Exp}_{a_r} = \bigoplus_{r=1}^{\tau} \mathrm{Exp}_{s_r} \ \ni \ (\varphi_r) \longmapsto [(\phi_{ve})] \in H^1(\mathsf{R}, \mathrm{Exp})$$

with $\phi_{v''_r s_r} := \varphi_r$, $\phi_{v'_r s_r} := \varphi_r^{-1}$ and ϕ_{ve} is trivial otherwise.

Now for each $r = 1, \ldots, \tau$ we choose a local holomorphic basic vector field X_r transverse to the foliation $\mathcal{F}^\sharp$ and defined on a neighborhood of $f(s_r)$ in the ambient space of $\mathcal{F}^\sharp$. We define the group morphism $\Lambda : \mathbb{C}^\tau \to \mathrm{Mod}([\mathcal{F}^\diamond])$ of Theorem D, as the composition

$$\mathbb{C}^\tau \overset{\xi}{\twoheadrightarrow} \bigoplus_{r=1}^{\tau} \mathrm{Exp}_{s_r} \overset{\delta}{\twoheadrightarrow} H^1(\mathsf{R}, \mathrm{Exp}) \overset{\chi}{\to} H^1(\mathsf{R}, \mathrm{Sym}) \overset{(11)}{\simeq} \mathrm{Mod}([\mathcal{F}^\diamond])\,,$$

where $\xi(t) = (\dot{\exp} t_1 X_1, \ldots, \dot{\exp} t_\tau X_\tau)$, $\dot{\exp} t_r X_r$ denotes the class of $\exp t_r X_r$ in $\mathrm{Aut}_{s_r}/\mathrm{Fix}_{s_r} = \mathrm{Sym}_{s_r}$ and χ is induced by the natural inclusion of group-graphs $\mathrm{Exp} \hookrightarrow \mathrm{Sym}$, see (10). The last bijection (11) induces an abelian group structure on $\mathrm{Mod}([\mathcal{F}^\diamond])$. Moreover, if we define $\mathbf{D} := H^1(\mathsf{R}, \mathrm{Dis})$ and $\Gamma : \mathrm{Mod}([\mathcal{F}^\diamond]) \to \mathbf{D}$ as the composition of the isomorphism (11) and the last arrow in the sequence (12), then the sequence

$$\mathbb{C}^\tau \xrightarrow{\Lambda} \mathrm{Mod}([\mathcal{F}^\diamond]) \xrightarrow{\Gamma} \mathbf{D} \to 0$$

is exact. It remains to check that $\ker \Lambda = \ker(\chi \circ \delta \circ \xi)$ is finitely generated. Since $\ker \chi$ is finite by step (i) and $\ker \delta$ is finitely generated thanks to Lemma 10.1 and (19), it suffices to see that $\ker \xi$ is also finitely generated. In fact, we will conclude by proving that the kernel of each group morphism $\xi_r : \mathbb{C} \to \mathrm{Sym}_{s_r}$, $t \mapsto \dot{\exp}\, t_r X_r$ is finitely generated. Since Remark 9.2 allows us to work on a transversal, we can use Proposition 6.10 to describe $\ker(\xi_r)$ as the kernel of the morphism $\mathbb{C} \to C(h_r)/\langle h_r \rangle$ given by $t \mapsto [\exp t X_r]$, which is finitely generated thanks to Lemma 9.4.

Step (vi). Let P be a complex connected manifold and t_0 a point of P. A *deformation of* $\mathcal{F}$ with parameter space the manifold P pointed at t_0, is a germ along all $\{0\} \times P$ of a 1-dimensional holomorphic singular foliation $\mathcal{F}_P$ defined on an open neighborhood of $\{0\} \times P$ in $\mathbb{C}^2 \times P$, which is tangent to the fibers of the projection $\pi_P : \mathbb{C}^2 \times P \to P$ and such that $\mathcal{F}$ is equal to the restriction of $\mathcal{F}_P$ to $\mathbb{C}^2 \times \{t_0\}$, with the identification $\mathbb{C}^2 \xrightarrow{\sim} \mathbb{C}^2 \times \{t_0\}$, $(x, y) \mapsto (x, y, t_0)$.

We say that $\mathcal{F}_P$ is *equireducible* if there exists a map $E_{\mathcal{F}_P} : \mathcal{M} \to \mathbb{C}^2 \times P$ obtained by composition of blow-up maps $E_j : \mathcal{M}_{j+1} \to \mathcal{M}_j$ fulfilling:

(1) each center of blow-up $C_j \subset \mathcal{M}_j$ of E_j is biholomorphic to P by the map $\pi_j := \pi_P \circ E_0 \circ E_1 \circ \cdots \circ E_{j-1} : \mathcal{M}_j \to P$,
(2) the singular locus of the foliation $E^*_{\mathcal{F}_P}\mathcal{F}_P$ is smooth, contained in the exceptional divisor $\mathcal{E}_{\mathcal{F}_P} := E^{-1}_{\mathcal{F}_P}(\{0\} \times P)$ and the restriction of $\pi_P \circ E_{\mathcal{F}_P}$ to each of its connected component is a biholomorphism onto P,
(3) the restriction of $E_{\mathcal{F}_P}$ to $\mathcal{M}_t := (\pi_P \circ E_{\mathcal{F}_P})^{-1}(t)$ is exactly the minimal reduction map of the foliation $\mathcal{F}_t$ on $\mathbb{C}^2 \times \{t\}$ induced by $\mathcal{F}_P$;

Notice that $\mathcal{E}_{\mathcal{F}_P}$ is a topological product over P, i.e. there is a homeomorphism $\Phi_P : \mathcal{M}_{t_0} \times P \xrightarrow{\sim} \mathcal{M}$ such that $\pi_P \circ \Phi_P$ is the second projection map. By identifying $\mathbb{C}^2 \times \{t_0\}$ with $\mathbb{C}^2$, each marking $f : \mathcal{E} \to \mathcal{E}_{\mathcal{F}_{t_0}}$ of $\mathcal{F}_{t_0}$ by $\mathcal{E}^\diamond$ extends via Φ_P to markings $f_t : \mathcal{E} \to \mathcal{E}_{\mathcal{F}_t} \subset \mathcal{M}_t$ of $\mathcal{F}_t$, $t \in P$, defining in this way a map

$$P \to \mathrm{Mod}([\mathcal{F}^\diamond]), \quad t \mapsto [\mathcal{F}_t, f_t].$$

On the other hand, given a point $t' \in P$, for each base point o_D in a component D of $\mathcal{E}$ introduced at the beginning of Section 5, let us choose a $(1 + \dim P)$-dimensional submanifold $\Delta_{P,D}$ of $\mathcal{M}$, transverse to $f_{t'}(D)$ at the point $f_{t'}(o_D)$. The representation of $\mathcal{F}_P$-holonomy of the leaf $f_{t'}(D \setminus \Sigma)$ defines a representation $\mathcal{H}^{t'}_{P,D}$ of the fundamental group $\pi_1(D \setminus \Sigma, o_D)$ in the group $\mathrm{Diff}(\Delta_{P,D}, f_{t'}(o_D))$ of germs of holomorphic automorphisms of $(\Delta_{P,D}, f_{t'}(o_D))$.

Definition 10.2. *We say that $\mathcal{F}_P$ is SL-equisingular at a point t' of P if*

(1) $\mathcal{F}_P$ is equireducible,
(2) for $t \in P$ sufficienty close to t' and for each $(D, s) \in \mathsf{Ve}_{\mathsf{A}_{\mathcal{F}^\diamond}} \times \mathsf{Ed}_{\mathsf{A}_{\mathcal{F}^\diamond}}$, $D \in \partial s$, the Camacho-Sad indices $CS(E_t^* \mathcal{F}_t, f_t(D), f_t(s))$ do not depend on t;
(3) there is a germ of biholomorphism $\psi : (\Delta_{P,D}, f_{t'}(o_D)) \xrightarrow{\sim} (\mathbb{C} \times P, (0, t'))$ such that
   (a) the composition of ψ with the second projection $\mathbb{C} \times P \to P$ is equal to $\pi_P \circ E_{\mathcal{F}_P}$ restricted to $\Delta_{P,D}$;
   (b) for all $\gamma \in \pi_1(D \setminus \Sigma, o_D)$ the biholomorphism $(z, t) \mapsto \psi \circ \mathcal{H}^{t'}_{P,D}(\gamma) \circ \psi^{-1}(z, t)$ does not depend on t.

We say that $\mathcal{F}_P$ is SL-equisingular if it is SL-equisingular at each point of P.

Step (vii). We now do a construction similar to the one given in the proof of Theorem 4.9 after a suitable choice of cocycles in $Z^1(\mathsf{A}, \mathrm{Aut})$.

Consider the elements $[\mathcal{F}_i, f_i]$ of $\mathrm{Mod}([\mathcal{F}^\diamond])$, $i \in \mathbf{D}$, given in the statement of Theorem D. By isomorphism (11) they are represented by cocycles $\mathbf{c}^i = (c^i_{D,s}) \in Z^1(\mathsf{R}, \mathrm{Sym})$. Now we fix an orientation $\prec$ of A and as in the proof of Theorem 5.14, we lift this cocycle to a cocycle $(\varphi^i_{D,s}) \in Z^1(\mathsf{R}, \mathrm{Aut})$ and we continue to denote by $s_1, \ldots, s_\tau$ the edges associated to the active vertices $a_1, \ldots, a_\tau$ chosen in step (v). We define then $(\varphi^{i,t}_{D,s}) \in Z^1(\mathsf{A}, \mathrm{Aut})$ by setting

$$\varphi^{i,t}_{D,s} = \begin{cases} \mathrm{id} & \text{if} \quad s \in \mathsf{Ed}_\mathsf{A} \setminus \mathsf{Ed}_\mathsf{R}\,, \\ \varphi^i_{D,s} & \text{if} \quad s \in \mathsf{Ed}_\mathsf{R} \setminus \{s_1, \ldots, s_\tau\}\,, \\ \varphi^i_{D,s} \circ \exp t_j X_j & \text{if} \quad s = s_j \in \{s_1, \ldots, s_\tau\}\,, \end{cases}$$
$$\varphi^{i,t}_{D',s} = (\varphi^{i,t}_{D,s})^{-1}\,,$$

for $s \in \mathsf{Ed}_\mathsf{A}$ with $\partial s = \{D, D'\}$ and $D \prec D'$, where $\exp t_j X_j$ is defined in Step (v).

Using these cocycles, for each $i \in \mathbf{D}$ we glue suitable neighborhoods W_D of $D \times \mathbb{C}^\tau$ inside $M_\mathcal{F} \times \mathbb{C}^\tau$ obtaining

(i) a manifold $\mathcal{M}_i$ endowed with a submersion map π_i onto $\mathbb{C}^\tau$,
(ii) a normal crossing divisor such that the restriction of π_i to each irreducible component is a locally trivial fibration with fiber $\mathbb{P}^1$ and the restriction of π_i to the singular locus of the divisor is a covering over $\mathbb{C}^\tau$,
(iii) and a foliation by curves tangent to the fibers of the submersion π_i and to the divisor.

By the same arguments used in Theorem 4.9 we obtain an open neighborhood of $\{0\} \times \mathbb{C}^\tau$ in $\mathbb{C}^2 \times \mathbb{C}^\tau$ and on this neighborhood an holomorphic vector field defining a one-dimensional equireducible foliation tangent to the fibers of the projection onto $\mathbb{C}^\tau$, whose singular locus is $\{0\} \times \mathbb{C}^\tau$. By construction, after equireduction the exceptional divisor, as an intrinsic analytic space, is holomorphicaly trivial over $\mathbb{C}^\tau$ and along each of its irreducible components the reduced foliation is holomorphically trivial. Hence we have obtained a SL-equisingular deformation $\mathcal{F}^c_{i,t}$ of $\mathcal{F}_i$, see [23], and biholomorphisms $h_{i,t} : \mathcal{E}_{\mathcal{F}_i} \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}^c_{i,t}}$, $i \in \mathbf{D}$. The superscript c stands for *complete*. We define the markings $f^c_{i,t} : \mathcal{E} \xrightarrow{\sim} \mathcal{E}_{\mathcal{F}^c_{i,t}}$ by $f^c_{i,t} := h_{i,t} \circ f_i$.

Notice that by the construction of Λ in step (v), $\Lambda(t)$ is represented in $H^1(\mathsf{A}, \mathrm{Aut})$ by the following 1-cocycle with support in R_1:

$$a^t_{D,s} := \begin{cases} \mathrm{id} \ \text{ if } \ s \in \mathsf{Ed}_\mathsf{A} \setminus \mathsf{Ed}_\mathsf{R}\,, \\ \mathrm{id} \ \text{ if } \ s \in \mathsf{Ed}_\mathsf{R} \setminus \{s_1, \ldots, s_\tau\}\,, \\ \exp t_j X_j \ \text{ if } \ s = s_j\,. \end{cases}$$

Thanks to Theorems 6.8 and 6.11 we have in $H^1(\mathsf{R}, \mathrm{Sym})$ the equality

$$\left[(\overset{\bullet}{\varphi}{}^{i,t}_{D,s})\right] = \left[\mathbf{c}^i\right] \cdot \left[(\overset{\bullet}{a}{}^t_{D,s})\right]\,,$$

where $\overset{\bullet}{\varphi}{}^{i,t}_{D,s}$ and $\overset{\bullet}{a}{}^t_{D,s}$ denote the classes of $\varphi^{i,t}_{D,s}$ and $a^t_{D,s}$ in Sym_s. The abelian group structure on $\mathrm{Mod}([\mathcal{F}^\diamond])$ being induced by the one on $H^1(\mathsf{R}, \mathrm{Sym}^\mathcal{F})$ by (11), the previous equality proves that in $\mathrm{Mod}([\mathcal{F}^\diamond])$ we have

$$[\mathcal{F}^c_{i,t}, f^c_{i,t}] = \Lambda(t) \cdot [\mathcal{F}_i, f_i]\,, \quad i \in \mathbf{D}\,.$$

11. Appendix

Let us denote by $B_r \subset \mathbb{C}^2$ the closed ball $\{|x|^2 + |y|^2 \leq r\}$. For a curve $S \ni 0$ in $\mathbb{C}^2$ let us call *Milnor ball* any ball $B = B_R$ such that $S \cap B \setminus \{0\}$ is regular and meets transversely each sphere ∂B_r, $0 < r \leq R$. We fix a germ $\mathcal{F}$ at $0 \in \mathbb{C}^2$ of a singular holomorphic foliation.

Definition 11.1. *A germ of an invariant curve S at $0 \in \mathbb{C}^2$ will be called $\mathcal{F}$-appropriate if S is invariant by $\mathcal{F}$, contains all the isolated separatrices*[8] *and its strict transform by the reduction of $\mathcal{F}$ meets any dicritical component D with* $\mathrm{card}(D \cap \mathrm{Sing}(\mathcal{E}_\mathcal{F})) = 1$.

The following incompressibility property is proved under some additional assumptions in [16], [19] and an optimal version was obtained by L. Teyssier in [35]:

Theorem 11.2. *Let $\mathcal{F}$ be a generalized curve and let S be an $\mathcal{F}$-appropriate curve in a Milnor ball B. Then there exists a fundamental system $(U_n)_{n\in\mathbb{N}}$ of open neighborhoods of S in B such that for each $n \in \mathbb{N}$*

(1) the inclusion map $U_n \hookrightarrow B$ induces an isomorphism between the fundamental groups of $U_n \setminus S$ and $B \setminus S$;

(2) for each leaf L of the foliation $\mathcal{F}_{|(U_n\setminus S)}$ the inclusion map $L \hookrightarrow U_n \setminus S$ induces an injective morphism $\pi_1(L, \cdot) \hookrightarrow \pi_1(U_n \setminus S, \cdot)$;

(3) there is a finite union of curves in $U_n \setminus S$ whose preimage Ω in the universal covering $\widetilde{U}_n$ of $U_n \setminus S$ is a disjoint union of holomorphically embedded open discs Ω_α such that each leaf L of the foliation $\widetilde{\mathcal{F}}_n$ induced by $\mathcal{F}$ on $\widetilde{U}_n$ meets Ω and $\mathrm{card}(L \cap \Omega_\alpha) \leq 1$ *for any α.*

Remark 11.3. Two direct consequences of this result are the simple connectedness of the leaves of the foliation $\widetilde{\mathcal{F}}_n$ and a structure of (non-Hausdorff) Riemann surface on the leaf space $\widetilde{\mathcal{Q}}^\mathcal{F}_{U_n}$, an atlas being given by the transversals Ω_α . □

Previous Theorem 11.2 will allow us to extend Theorem 1.6 of [18] with weaker assumptions:

[8] i.e. their strict transforms meet invariant components of the exceptional divisor.

Theorem 11.4. *Let $\mathcal{F}$ and $\mathcal{G}$ be two topologically equivalent germs of foliations at $0 \in \mathbb{C}^2$ and suppose the existence of a germ of homeomorphism that conjugates them:*

$$\psi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)\,, \qquad \psi^* \mathcal{G} = \mathcal{F}\,. \tag{20}$$

If $\mathcal{F}$ is a generalized curve fulfilling Conditions (TC) and (TR) stated in Section 2, then there exists a germ of homeomorphism $\phi : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that:

(1) the lifting $E_{\mathcal{G}}^{-1} \circ \phi \circ E_{\mathcal{F}}$ of ϕ through the reduction maps of $\mathcal{F}$ and $\mathcal{G}$ extends to the exceptional divisor as a germ of homeomorphism $\Phi : (M_{\mathcal{F}}, \mathcal{E}_{\mathcal{F}}) \to (M_{\mathcal{G}}, \mathcal{E}_{\mathcal{G}})$ along the exceptional divisors;
(2) Φ is holomorphic at each singular point of $\mathcal{F}^\sharp := E_{\mathcal{F}}^* \mathcal{F}$ which is not a nodal corner;
(3) Φ is transversely holomorphic at each point of the exceptional divisor which is regular for $\mathcal{F}^\sharp$ and not contained in a dicritical component.

The rest of this appendix is devoted to the proof of this theorem, which is similar to that of Main Theorem of [18]. The new difficulties lie in the fact that ψ may not be transversely holomorphic on a whole neighborhood of 0, and this for three reasons:

- on the union of leaves meeting a dicritical component of $\mathcal{E}_{\mathcal{F}}$ there are $\mathcal{C}^0$-automorphisms of the foliation $\mathcal{F}$ which are not transversally holomorphic;
- there are also such automorphisms near nodal singular points with support in nodal separators, cf. [18, page 406];
- exceptional cut-components[9] introduced in Section 2.2 of $\mathcal{E}_{\mathcal{F}}$ are not excluded by Condition (TC) and at such a component $\mathcal{C}$ the property (TR) is ineffective. Indeed, every irreducible component of $\mathcal{C}$ contains at most two singular points, therefore its holonomy group, being monogenous or trivial, cannot be topologically rigid.

To prove Theorem 11.4 we will proceed in five steps:

(i) we prove that the one-to-one correspondence induced by ψ between the irreducible components of $\mathcal{E}_{\mathcal{F}}$ and those of $\mathcal{E}_{\mathcal{G}}$ and between the singular points of both foliations $\mathcal{F}^\sharp$ and $\mathcal{G}^\sharp$ preserves the Camacho-Sad indices;
(ii) we recall the notion of monodromy and two key results given in [18] that remain valid in our more general setting;
(iii) we construct a conjugation Φ' between $\mathcal{F}^\sharp$ and $\mathcal{G}^\sharp$ along the non-exceptional cut-components of $\mathcal{E}_{\mathcal{F}}$, except at the nodal corners and at the intersection with the dicritical components;
(iv) we construct a conjugation Φ'' along the exceptional cut-components, except at the nodal corners and at the intersection with the dicritical components;
(v) we extend and glue Φ' and Φ'' at the nodal corners and along the dicritical components.

[9] which cannot exist if $\mathcal{F}$ is non-dicritical by Theorem 11.5 below.

11.1. **Step (i): Equality of Camacho-Sad indices.** Since $\mathcal{F}$ and therefore $\mathcal{G}$ are generalized curves [3], there is a unique one-to-one correspondence

$$D \mapsto D' \quad \text{and} \quad s \mapsto s' \tag{21}$$

between the irreducible components of the exceptional divisors $\mathcal{E}_{\mathcal{F}}$ and $\mathcal{E}_{\mathcal{G}}$, the strict transforms of the isolated separatrices of $\mathcal{F}$ and $\mathcal{G}$, and between the points of $\mathrm{Sing}(\mathcal{E}_{\mathcal{F}}) \cup \mathrm{Sing}(\mathcal{F}^\sharp)$ and $\mathrm{Sing}(\mathcal{E}_{\mathcal{G}}) \cup \mathrm{Sing}(\mathcal{G}^\sharp)$, such that:

- if s is the intersection point of an isolated separatrix S with $\mathcal{E}_{\mathcal{F}}$, then s' is the intersection point of $\psi(S)$ with $\mathcal{E}_{\mathcal{G}}$,
- we have equalities of intersection numbers

$$(D'_1, D'_2) = (D_1, D_2) \quad \text{for} \quad D_1 \mapsto D'_1, \quad D_2 \mapsto D'_2\,. \tag{22}$$

Indeed the reduction map of a generalized curve foliation coincides with the reduction map of the curve formed by all its isolated separatrices and two dicritical separatrices for each dicritical component of the exceptional divisor. Thus the above properties follow from classical topological properties for germs of curves.

Let us point out that property (2) in Theorem 11.4 implies equality of Camacho-Sad indices of these foliations. These equalities will be strongly used in the proof of the above theorem and in fact we need to prove them first. We will use the following result [4, Theorem 9 and Remark 11]:

Theorem 11.5. *For any cut-component there is a strict transform of an isolated separatrix which meets it at a non-nodal singular point.*

Lemma 11.6. *Under the assumptions of Theorem 11.4, $\mathcal{F}$ fulfilling again Conditions (TR) and (TC), if $s \in D \subset \mathcal{E}_{\mathcal{F}}$ and $s' \in D' \subset \mathcal{E}_{\mathcal{G}}$ correspond by (21), then*

$$\mathrm{CS}(\mathcal{F}^\sharp, D, s) = \mathrm{CS}(\mathcal{G}^\sharp, D', s')\,. \tag{23}$$

As discussed in Remark 2.1, Conditions (TR) and (TC) are necessary to get this lemma.

Proof. The induction process given in [18, §7.3], which is based on the Camacho-Sad Index Formula, proves that equalities (23) hold at all singularities in a cut-component $\mathcal{C}$ if they are satisfied at every intersection point s of the strict transform of a separatrix with $\mathcal{C}$. The existence of such a point s being assured by Theorem 11.5, we distinguish three possibilities.

a) $\lambda := \mathrm{CS}(\mathcal{F}^\sharp, D, s)$ *is an irrational real number.* If λ is positive, s is a nodal singular point, and equality (23) was obtained by R. Rosas in [33, Proposition 13]. Another proof is given in [19, Theorem 1.12] that remains valid for $\lambda < 0$.
b) $\mathcal{C}$ *is non-exceptional and s is not a nodal singular point.* Then by using (TR) and thanks to an extended version [18] of the rigidity theorem in [32], ψ is transversely holomorphic on the image by $E_{\mathcal{F}}$ of a neighborhood of $\mathcal{C}$ and specifically at the points of the separatrix S. In this case the proof of equality (23) given in [18, §2] remains valid.
c) $\mathcal{C}$ *is exceptional.* Then $\mathcal{C}$ is a union of components of $\mathcal{E}_{\mathcal{F}}$, $\mathcal{C} = D_1 \cup \cdots \cup D_\ell$, $\ell \geq 1$, D_i meeting D_{i+1} in one point s_i, and $D_i \cap D_j = \emptyset$ if $|i - j| \neq 1$.

Perhaps $\mathcal{C}$ meets several dicritical components of $\mathcal{E}_{\mathcal{F}}$, but we only have two possibilities fulfilling Assumption (TC):

(i) $s \in D_1$ and the other singular points of $\mathcal{F}^\sharp$ belonging to $\mathcal{C}$ are $s_1, \ldots, s_{\ell-1}$, see Figure 4;

(ii) $s \in D_1$, D_ℓ contains a nodal singular point $s_\ell \neq s_{\ell-1}$, the other singular points of $\mathcal{F}^\sharp$ belonging to $\mathcal{C}$ being $s_1, \ldots, s_{\ell-1}$, see Figure 5.

Indeed, Theorem 11.5 implies that singularity s is not nodal. In case (ci) using the classical index formula, we see that $\mathrm{CS}(\mathcal{F}^\sharp, D_1, s)$ is given by a continuous fraction whose coefficients are the self-intersections (D_i, D_i), $i = 1, \ldots, \ell$; thus (23) follows from (22). In the same way we obtain in case (cii) that $\mathrm{CS}(\mathcal{F}^\sharp, D_1, s)$ is an irrational (negative) real number, but this case was already examined in case (a).

□

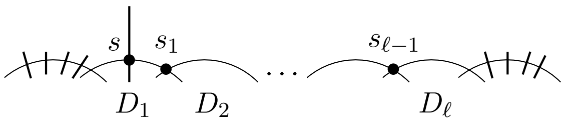

FIGURE 4. Situation (ci) with two dicritical components.

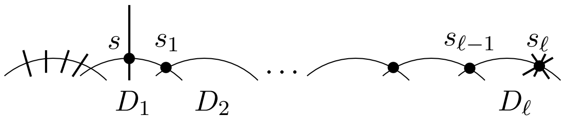

FIGURE 5. Situation (cii) with a dicritical component and a nodal singularity s_ℓ.

Since the dicriticity of an irreducible component D can be characterized by the vanishing of the Camacho-Sad indices along all the adjacent components at their intersection points with D, we have:

Corollary 11.7. *The image by correspondence (21) of a dicritical component, an exceptional cut-component, a non-exceptional cut-component of $\mathcal{F}$ is respectively a dicritical component, an exceptional cut-component, a non-exceptional cut-component of $\mathcal{G}$.*

11.2. **Step (ii): Monodromy and holonomy.** Let us now fix a $\mathcal{F}$-appropriate curve S, Milnor balls B and B' for S and the $\mathcal{G}$-appropriate curve $S' := \psi(S)$, where ψ is given by (20). Let us also fix

$$q : \widetilde{B} \longrightarrow B \setminus S\,, \quad q' : \widetilde{B}' \longrightarrow B' \setminus S'$$

universal coverings of $B \setminus S$ and $B' \setminus S'$ respectively. In all the sequel we adopt the following conventions and notations:

(a) for $A \subset B$, resp. $A' \subset B'$, we write

$$\widetilde{A} := q^{-1}(A \setminus S)\,, \quad \text{resp.} \quad \widetilde{A'} := q'^{-1}(A' \setminus S')\,,$$

(b) we suppose that B' contains $\psi(B)$, we choose a lifting $\widetilde{\psi} : \widetilde{B} \to \widetilde{B}'$ of ψ and we denote the isomorphisms induced by $\widetilde{\psi}$ between the deck transformation groups of these coverings by

$$\widetilde{\psi}_* : \Gamma \xrightarrow{\sim} \Gamma' , \quad \widetilde{\psi}_*(f) := \widetilde{\psi} \circ f \circ \widetilde{\psi}^{-1} , \quad \Gamma := \mathrm{Aut}_q(\widetilde{B}) , \quad \Gamma' := \mathrm{Aut}_{q'}(\widetilde{B}') ,$$

(c) we fix fundamental systems of open neighborhoods of S in B, resp. S' in B',

$$(U_n)_{n\in\mathbb{N}} , \quad (U'_n)_{n\in\mathbb{N}} , \quad \text{with} \quad \psi(U_n) \subset U'_n ,$$

fulfilling Properties (1)-(3) of Theorem 11.2,

(d) $\widetilde{\mathcal{Q}}^{\mathcal{F}}_{U_n} := \widetilde{U}_n / \widetilde{\mathcal{F}}_{|\widetilde{U}_n}$ denotes the leaf space of the restriction to $\widetilde{U}_n$ of the foliation $\widetilde{\mathcal{F}} := q^*\mathcal{F}$, endowed with its structure of Riemann surface, cf. Remark 11.3,

(e) via $E_{\mathcal{F}} : B_{\mathcal{F}} \to B$ and $E_{\mathcal{G}} : B'_{\mathcal{G}} \to B'$, the reduction maps of $\mathcal{F}$ and $\mathcal{G}$, we perform the following identifications

$$B_{\mathcal{F}} \setminus \mathcal{S} \simeq B \setminus S , \quad B'_{\mathcal{G}} \setminus \mathcal{S}' \simeq B' \setminus S' ,$$

$$\mathcal{S} := E_{\mathcal{F}}^{-1}(S) , \quad \mathcal{S}' := E_{\mathcal{G}}^{-1}(S') ,$$

(f) we consider $\widetilde{B}$ and $\widetilde{B'}$ as universal coverings of $B_{\mathcal{F}} \setminus \mathcal{S}$ and $B'_{\mathcal{G}} \setminus \mathcal{S}'$ respectively and, with these identifications, for $A \subset B_{\mathcal{F}}$ or $A' \subset B'_{\mathcal{G}}$ we write:

$$\widetilde{A} := q^{-1}(A \setminus \mathcal{S}) , \quad \widetilde{A}' := q'^{-1}(A' \setminus \mathcal{S}') .$$

Definition 11.8. *The monodromy of $\mathcal{F}$ is the morphism between Γ and the group of automorphisms of inverse systems*

$$\mathfrak{M}^{\mathcal{F}}_S : \Gamma \to \mathrm{Aut}_{\overleftarrow{\mathrm{An}}}(\widetilde{\mathcal{Q}}^{\mathcal{F}}_\infty) \subset \mathrm{Aut}_{\overleftarrow{\mathrm{Top}}}(\widetilde{\mathcal{Q}}^{\mathcal{F}}_\infty) , \quad \widetilde{\mathcal{Q}}^{\mathcal{F}}_\infty := (\widetilde{\mathcal{Q}}^{\mathcal{F}}_{U_n})_{n\in\mathbb{N}} ,$$

given by the actions of Γ on the leaf spaces $(f, L) \mapsto f(L)$, $f \in \Gamma$, $L \in \widetilde{\mathcal{Q}}^{\mathcal{F}}_{U_n}$, with $\overleftarrow{\mathrm{An}}$ the category of pro-objects associated to the category of analytic spaces, and $\overleftarrow{\mathrm{Top}}$ the category of pro-objets associated to the category of topological spaces and continuous maps.

The monodromy $\mathfrak{M}^{\mathcal{G}}_{S'} : \Gamma' \to \mathrm{Aut}_{\overleftarrow{\mathrm{An}}}(\widetilde{\mathcal{Q}}^{\mathcal{G}}_\infty)$ of $\mathcal{G}$ is defined in the same way.

Remark 11.9. The lifting $\widetilde{\psi}$ of ψ fixed in (b) induces an isomorphism

$$h_{\widetilde{\psi}} : \widetilde{\mathcal{Q}}^{\mathcal{F}}_\infty \xrightarrow{\sim} \widetilde{\mathcal{Q}}^{\mathcal{G}}_\infty$$

in the category $\overleftarrow{\mathrm{Top}}$, given by the maps $\widetilde{\mathcal{Q}}^{\mathcal{F}}_{U_n} \to \widetilde{\mathcal{Q}}^{\mathcal{G}}_{U'_n}$ that associate to each leaf L of $\widetilde{\mathcal{F}}_{|\widetilde{U}_n}$ the leaf of $\widetilde{\mathcal{G}}_{|\widetilde{U}'_n}$ containing $\widetilde{\psi}(L)$. When $\mathcal{E}_{\mathcal{F}}$ contains an exceptional cut-component $h_{\widetilde{\psi}}$ may not be $\mathcal{N}$-analytic in the sense of [18, page 416] and we must extend to our context the notion of geometric conjugation of monodromy introduced in [18]. □

To any subset A in B meeting S we associate the following object and morphism in the category $\overleftarrow{\mathrm{Top}}$:

- $(\widetilde{A}, \infty) := (\widetilde{A} \cap U_n)_{n\in\mathbb{N}}$,

- $\tau_{\widetilde{A}} : (\widetilde{A}, \infty) \to \widetilde{\mathcal{Q}}^{\mathcal{F}}_{\infty}$ is induced by the family of maps $\tau^n_{\widetilde{A}} : \widetilde{A} \cap U_n \to \widetilde{\mathcal{Q}}^{\mathcal{F}}_{U_n}$ that associate to any point m in $\widetilde{A} \cap U_n$ the leaf containing m of the foliation $\widetilde{\mathcal{F}}_{|\widetilde{U}_n}$, $n \in \mathbb{N}$,
- $\Gamma_{\widetilde{A},\infty}$ is the group of automorphisms f of $(\widetilde{A}, \infty)$ such that $q \circ f = q$.

For $A' \subset B'$ meeting S', $(\widetilde{A'}, \infty)$, $\tau_{\widetilde{A'}}$ and $\Gamma_{\widetilde{A'},\infty}$ are similarly defined.

Definition 11.10. *A geometric $\mathcal{C}^0$-conjugation between $\mathfrak{M}^{\mathcal{F}}_S$ and $\mathfrak{M}^{\mathcal{G}}_{S'}$ is a pair $(\mathfrak{g}, h)$ formed by:*

- *an isomorphism of groups $\mathfrak{g} : \Gamma \xrightarrow{\sim} \Gamma'$ induced by the lifting*
$$\widetilde{g} : (\widetilde{B}, \infty) \xrightarrow{\sim} (\widetilde{B'}, \infty)\,, \quad q' \circ \widetilde{g} = g \circ q\,,$$
of a germ of homeomorphism along the separatices $g : (B, S) \xrightarrow{\sim} (B', S')$ preserving the orientations of B, B', S and S' (but not necessarily the foliations),
- *an isomorphism $h : \widetilde{\mathcal{Q}}^{\mathcal{F}}_{\infty} \xrightarrow{\sim} \widetilde{\mathcal{Q}}^{\mathcal{G}}_{\infty}$ in the category $\underleftarrow{\mathrm{Top}}$, such that $h_* \circ \mathfrak{M}^{\mathcal{F}}_S = \mathfrak{M}^{\mathcal{G}}_{S'} \circ \mathfrak{g}$, where h_* is the group morphism*
$$h_* : \mathrm{Aut}_{\underleftarrow{\mathrm{Top}}}(\widetilde{\mathcal{Q}}^{\mathcal{F}}_{\infty}) \to \mathrm{Aut}_{\underleftarrow{\mathrm{Top}}}(\widetilde{\mathcal{Q}}^{\mathcal{G}}_{\infty})\,, \quad \varphi \mapsto h \circ \varphi \circ h^{-1}\,.$$

We also say that $(g, \widetilde{g}, h)$ represents the conjugation $(\mathfrak{g}, h)$.

Remark 11.11. The lifting $\widetilde{\psi}$ of ψ choosen in (b) induces a geometric $\mathcal{C}^0$-conjugation $(\widetilde{\psi}_*, h_{\widetilde{\psi}})$ between $\mathfrak{M}^{\mathcal{F}}_S$ and $\mathfrak{M}^{\mathcal{G}}_{S'}$, that is represented by $(\psi, \widetilde{\psi}, h_{\widetilde{\psi}})$, with $h_{\widetilde{\psi}}$ as in Remark 11.9. □

Now for each invariant component D of $\mathcal{S}$ we will denote by $\mathfrak{S}_D$ the set of singular points of $\mathcal{F}^\sharp$ belonging to D, except the nodal singularities which are attaching points of strict transforms of separatrices. Let us choose:

- a good fibration ρ_D transverse to D, cf. Definition 4.5; thus if $m \in D$ is not a singular point of $\mathcal{S}$ we can write without ambiguity:
$$\Delta_m = \rho_D^{-1}(m)\,; \tag{24}$$
- a collection $(Z_{D,s})_{s\in\mathfrak{S}_D}$ of holomorphically embedded compact discs in D, centered at the points s, without pairwise intersection.

We also require that $Z_{D,s} = D$ when D is the strict transform of an isolated separatrix.

Definition 11.12. *The following compact sets:*

(i) $Z_D := \overline{D \setminus \cup_{s\in\mathfrak{S}_D} Z_{D,s}}$, *with D an invariant component of $\mathcal{E}_{\mathcal{F}}$,*

(ii) $Z_s := Z_{D_1,s} \cup Z_{D_2,s}$, *with s the intersection point of two invariant components of $\mathcal{S}$,*

will be called elementary pieces of $\mathcal{S}$.

We perform for $\mathcal{G}$ similar choices of good fibrations $\rho_{D'}$ and embedded discs defining similarly elementary pieces $Z'_{s'}$ and $Z'_{D'}$ of $\mathcal{S}'$.

Theorem A of [17] gives immediately:

Lemma 11.13. *For any geometric $\mathcal{C}^0$-conjugation $(\mathfrak{g}, h)$ between $\mathfrak{M}^{\mathcal{F}}_S$ and $\mathfrak{M}^{\mathcal{G}}_{S'}$ there exists a representation $(g, \widetilde{g}, h)$ and a choice of elementary pieces of $\mathcal{S}$ and $\mathcal{S}'$ such that:*

(1) g lifts through the reduction maps $E_{\mathcal{F}}$ and $E_{\mathcal{G}}$ to a germ of homeomorphism $g^\sharp : (B_{\mathcal{F}}, \mathcal{S}) \to (B'_{\mathcal{G}}, \mathcal{S}')$, i.e. $E_{\mathcal{G}} \circ g^\sharp = g \circ E_{\mathcal{F}}$,

and, for any elementary piece Z_D, $D \in \mathrm{Comp}(\mathcal{E}_{\mathcal{F}})$ invariant, and Z_s, $s \in \mathrm{Sing}(\mathcal{S})$, it satisfies:

(2) $g^\sharp(Z_D) = Z'_{g^\sharp(D)}$ and $g^\sharp$ is compatible with the transverse fibrations over Z_D, i.e. $(\rho_{g^\sharp(D)} \circ g^\sharp)_{|\rho_D^{-1}(Z_D)} = (g^\sharp \circ \rho_D)_{|\rho_D^{-1}(Z_D)}$,

(3) $g^\sharp(Z_s) = Z'_{g^\sharp(s)}$, $g^\sharp$ is holomorphic on a neigborhood of Z_s and is compatible with the good fibration over ∂Z_s.

Definition 11.14. *Let $V \subset (B \setminus S)$ and $V' \subset (B' \setminus S')$ be subsets whose closures meet respectively S and S'. A* $\mathcal{C}^0$-realization of $(\mathfrak{g}, h)$ over V and V' *is the data $(\chi, \widetilde{\chi})$ of a homeomorphism $\chi : V \to V'$ and a lifting of it (in the category $\underleftarrow{\mathrm{Top}}$)*

$$\widetilde{\chi} : (\widetilde{V}, \infty) \xrightarrow{\sim} (\widetilde{V'}, \infty)\,, \quad q' \circ \widetilde{\chi} = \chi \circ q\,,$$

such that the following diagrams commute:

$$\begin{array}{ccc} (\widetilde{V}, \infty) & \xrightarrow{\widetilde{\chi}} & (\widetilde{V'}, \infty) \\ \downarrow \tau_{\widetilde{V}} & & \downarrow \tau_{\widetilde{V'}} \\ \widetilde{\mathcal{Q}}^{\mathcal{F}}_\infty & \xrightarrow{h} & \widetilde{\mathcal{Q}}^{\mathcal{G}}_\infty \end{array} \qquad\qquad \begin{array}{ccc} \Gamma & \xrightarrow{\iota} & \Gamma_{\widetilde{V},\infty} \\ \downarrow \mathfrak{g} & & \downarrow \widetilde{\chi}_* \\ \Gamma' & \xrightarrow{\iota'} & \Gamma'_{\widetilde{V'},\infty} \end{array}$$

When χ can be lifted via $E_{\mathcal{F}}$ and $E_{\mathcal{G}}$ to an open neighborhood of a subset Z of $\mathcal{E}_{\mathcal{F}}$, we will say that $(\chi, \widetilde{\chi})$ is a $\mathcal{C}^0$-realization of $(\mathfrak{g}, h)$ along Z.

Remark 11.15. If V is an open set then the commutativity of the first diagram implies that χ conjugates the foliations $\mathcal{F}_{|V}$ and $\mathcal{G}_{|V'}$. □

We now fix a geometric $\mathcal{C}^0$-conjugation $(\mathfrak{g}, h)$ between the monodromies $\mathfrak{M}^{\mathcal{F}}_S$ and $\mathfrak{M}^{\mathcal{G}}_{S'}$ and a representation $(g, \widetilde{g}, h)$ of it fulfilling Assertion (1) of Lemma 11.13. Let D be an invariant component of $\mathcal{S}$ and let $m_0 \in D \setminus \mathrm{Sing}(\mathcal{F}^\sharp)$. We will use the notations:

$$D' := g^\sharp(D)\,,\ m'_0 := g^\sharp(m_0)\,, \quad D^* := D \setminus \mathrm{Sing}(\mathcal{F}^\sharp),\ D'^* := D' \setminus \mathrm{Sing}(\mathcal{G}^\sharp)\,.$$

Let us fix now (Δ, m_0), resp. (Δ', m'_0), a germ at m_0, resp. m'_0, of an holomorphically embedded disc transversal to the foliation. We also denote by

$$\mathcal{H}^{\mathcal{F}^\sharp}_D : \pi_1(D^*, m_0) \to \mathrm{Diff}(\Delta, m_0)\,, \qquad \mathcal{H}^{\mathcal{G}^\sharp}_{D'} : \pi_1(D'^*, m'_0) \to \mathrm{Diff}(\Delta', m'_0)\,,$$

the holonomy representations of the foliation $\mathcal{F}^\sharp$, resp. $\mathcal{G}^\sharp$, associated to the leaves D^* and D'^*.

Theorem 11.16. *Let $\chi : (\Delta, m_0) \to (\Delta', m'_0)$ be a germ of homeomorphism and $\widetilde{\chi} : (\widetilde{\Delta}, \infty) \to (\widetilde{\Delta}', \infty)$ be a lifting of it, such that $(\chi, \widetilde{\chi})$ is a realization over Δ and Δ' of the geometric $\mathcal{C}^0$-conjugation $(\mathfrak{g}, h)$ between the monodromies $\mathfrak{M}^{\mathcal{F}}_S$ and $\mathfrak{M}^{\mathcal{G}}_{S'}$. Then χ and the group isomorphism $g^\sharp_*$ from $\pi_1(D^*, m_0)$ to $\pi_1(D'^*, m'_0)$ induced by $g^\sharp$ define a conjugation between $\mathcal{H}^{\mathcal{F}^\sharp}_D$ and $\mathcal{H}^{\mathcal{G}^\sharp}_{D'}$, i.e.*

$$\mathcal{H}^{\mathcal{G}^\sharp}_{D'}(g^\sharp_*(\dot\gamma)) = \chi \circ \mathcal{H}^{\mathcal{F}^\sharp}_D(\dot\gamma) \circ \chi^{-1}\,, \quad \dot\gamma \in \pi_1(D^*,\, m_0)\,.$$

Proof. The proof of [18, Theorem 4.3.1] remains literally valid for this $\mathcal{C}^0$-version. □

Let us assume now that the previously fixed representation $(g, \widetilde{g}, h)$ of $(\mathfrak{g}, h)$ satisfies all the properties (1)-(3) of Lemma 11.13 and that the point m_0 is in the boundary ∂Z of an elementary piece Z such that either $Z = Z_D$ with D an invariant component of $\mathcal{E}_{\mathcal{F}}$, or $Z = Z_s$ with s a non-nodal singularity. We can suppose that $\Delta = \Delta_{m_0}$ and $\Delta' = \Delta'_{m'_0}$, cf. Notation (24).

Lemma 11.17. *With the notations introduced above and under the hypothesis of Theorem 11.16, let us also suppose that the following additional assumptions are satisfied:*

- *$\widetilde{\chi}$ and $\widetilde{g}$ induce the same map from $\pi_0(\widetilde{\Delta})$ to $\pi_0(\widetilde{\Delta}')$,*
- *χ is holomorphic,*
- *if $Z = Z_s$, the Camacho-Sad indices of $\mathcal{F}^\sharp$ at s along D and of $\mathcal{G}^\sharp$ at $g^\sharp(s)$ along $g^\sharp(D)$ are equal.*

Then there exists a $\mathcal{C}^0$-realization $(\Phi, \widetilde{\Phi})$,

$$\Phi : V \to V', \quad \widetilde{\Phi} : (\widetilde{V}, \infty) \to (\widetilde{V'}, \infty),$$

of $(\mathfrak{g}, h)$ along Z, such that:

1. *the restriction of Φ to $\mathcal{E}_{\mathcal{F}} \cap V$ is equal to $g^\sharp$, its restriction to (Δ, m_0) is equal to χ and that of $\widetilde{\Phi}$ to $(\widetilde{\Delta}, \infty)$ is equal to $\widetilde{\chi}$,*
2. *on a neighborhood of $\partial Z \cap \mathcal{E}_{\mathcal{F}}$, Φ is compatible[10] with the good fibrations previously choosen; moreover the restriction of Φ to each fiber Δ_m, $m \in \partial Z \cap \mathcal{E}_{\mathcal{F}}$, is holomorphic, cf. Notation (24);*
3. *for every $m \in \partial Z \cap \mathcal{E}_{\mathcal{F}}$, the restrictions $\widetilde{\Phi}_{|\widetilde{\Delta}_m}$ and $\widetilde{g}_{|\widetilde{\Delta}_m}$ induce the same map $\pi_0(\widetilde{\Delta}_m) \to \pi_0(\widetilde{\Delta}'_{g^\sharp(m)})$.*

Proof. It is literally the same proof as that of [18, Lemma 8.3.2] but using Theorem 11.16 above instead of [18, Theorem 4.3.1]. □

11.3. **Step (iii): Conjugation at non-exceptional cut-components.** We keep the notations and conventions (a)-(f) introduced at the beginning of Section 11.2. We will also use the notation Δ_m, resp. $\Delta'_{m'}$, introduced at (24) for the fibers of the good fibrations at regular points of $\mathcal{S}$, resp. $\mathcal{S}'$.

Using Theorem 11.5, in each non-exceptional cut-component $\mathcal{C} \subset \mathcal{E}_{\mathcal{F}}$, let us choose a non-nodal singular point $s_{\mathcal{C}} \in \mathrm{Sing}(\mathcal{F}^\sharp)$ where the strict transform $X_{\mathcal{C}}$ of an isolated separatrix meets $\mathcal{C}$. Let us consider the following filtration of $\mathcal{S}$:

$$C_0 \subset C_1 \cdots \subset C_k \subset \mathcal{S}$$

defined by:

- C_0 is the union of all elementary pieces $Z_{s_{\mathcal{C}}}$ for any cut-component $\mathcal{C}$ of $\mathcal{E}_{\mathcal{F}}$,
- each $\mathcal{Z}_j := \overline{C_j \setminus C_{j-1}}$, $j = 1, \ldots, k$, is the union of all (disjoint) elementary pieces meeting C_{j-1}, but not contained in C_{j-1}.

[10] i.e. $\rho' \circ \Phi(m) = \Phi \circ \rho(m)$ if $\rho(m) \in \partial Z$, ρ and ρ' denoting the good fibrations of the corresponding components of the exceptional divisors.

- $\mathcal{S} \setminus C_k$ is the union of the dicritical components, the strict transforms of the components of $\mathcal{S}$ meeting a dicritical component and the elementary pieces associated to nodal corners.

Lemma 11.18 ([18], §8.4)**.** *There exists a representation $(g, \widetilde{g}, h)$ of $(\mathfrak{g}, h)$ satisfying properties (1)-(3) of Lemma 11.13, such that moreover the restrictions of $\widetilde{g}$ to Δ and the lifting $\widetilde{\psi}$ of ψ chosen in (b) induce the same map from $\pi_0(\widetilde{\Delta})$ to $\pi_0(\widetilde{\Delta}')$.*

Let us choose $(g, \widetilde{g}, h)$ given by this lemma. Notice that according to Lemma 11.6, the points $s'_{\mathcal{C}'} = g^\sharp(s_{\mathcal{C}})$ are not nodal singular points of $\mathcal{G}^\sharp$ and we can consider the filtration of $\mathcal{S}'$

$$C'_0 \subset C'_1 \cdots \subset C'_k \subset \mathcal{S}' \,, \quad C'_j := g^\sharp(C_j)\,.$$

According to Lemma 11.13, C'_0 is a union of elementary pieces of $\mathcal{S}'$, and for $j = 1, \ldots, k$,

$$\mathcal{Z}'_j := \overline{C'_j \setminus C'_{j-1}} = g^\sharp(\mathcal{Z}_j)$$

is a union of disjoint elementary pieces not contained in C'_{j-1} but meeting C'_{j-1}. By induction we will now define a realization $(\Phi_j, \widetilde{\Phi}_j)$ along every $\mathcal{Z}_j$ of the conjugation $(\widetilde{\psi}_*, h_{\widetilde{\psi}})$, $j = 0, \ldots, k$.

- $(\Phi_0, \widetilde{\Phi}_0)$ is defined along any connected component $Z_{s_{\mathcal{C}}}$ of C_0 as the realization of $(\widetilde{\psi}_*, h_{\widetilde{\psi}})$ obtained by: first choosing a point $m_{\mathcal{C}}$ in the boundary of $X_{\mathcal{C}}$, then modifying ψ near $m_{\mathcal{C}}$ by performing a foliated isotopy such that $\psi(\Delta_{m_{\mathcal{C}}}) = \Delta'_{\psi(m_{\mathcal{C}})}$ and finally applying the extension Lemma 11.17 with $Z = Z_{s_{\mathcal{C}}}$, $\Delta = \Delta_{m_{\mathcal{C}}}$, $\Delta' = g^\sharp(\Delta)$, $\chi = \psi_{|\Delta}$ and $\widetilde{\chi} = \widetilde{\psi}_{|\widetilde{\Delta}}$;
- $(\Phi_j, \widetilde{\Phi}_j)$ is defined along each connected component $\mathcal{Z}$ of $\mathcal{Z}_j$ as the realization of $(\widetilde{\psi}_*, h_{\widetilde{\psi}})$ obtained by: first choosing a point $m_{\mathcal{Z}}$ in the (unique) component of $\partial\mathcal{Z}$ contained in C_{j-1}, then applying Lemma 11.17 with $Z = \mathcal{Z}$, $\Delta = \Delta_{m_{\mathcal{Z}}}$, $\Delta' = g^\sharp(\Delta)$, and χ, $\widetilde{\chi}$ being respectively the restrictions of Φ_{j-1} and $\widetilde{\Phi}_{j-1}$ to Δ and $\widetilde{\Delta}$.

According to Remark 11.15, Φ_j conjugates the germ of $\mathcal{F}^\sharp$ along $\mathcal{Z}_j$ to the germ of $\mathcal{G}^\sharp$ along Z'_j. To achieve this step and to obtain a conjugation Φ' between $\mathcal{F}^\sharp$ and $\mathcal{G}^\sharp$ on a neighborhood of C_k it suffices to note that for $j = 1, \ldots, k$ and for each connected component Θ of $\mathcal{Z}_{j-1} \cap \mathcal{Z}_j$, Φ_{j-1} and Φ_j are necessary equal when we restrict them to the real hypersurfaces $\rho_D^{-1}(\Theta)$, D being the component of $\mathcal{S}$ containing Θ. This fact results from the property (2) in Lemma 11.17 and the uniqueness of the extension of a conjugation between the holonomies to a conjugation of the corresponding foliations preserving transversal fibrations.

11.4. **Step (iv): Construction along the exceptional cut-components.** Let $\mathcal{C}$ be an exceptional cut-component of $\mathcal{E}_{\mathcal{F}}$ and let us keep the notations introduced in the proof of Lemma 11.6: $\mathcal{C} = D_1 \cup \cdots \cup D_\ell$ has two possible configurations (ci) and (cii), see Figures 4 and 5.

In the first case (ci), D_ℓ contains only one singular point of the foliation, hence there is a holomorphic first integral defined on a neighborhood of $\mathcal{C}$ and specifically the foliation is linearizable at each singular point. Thus by equality of Camacho-Sad indices given by Lemma 11.6 the foliations considered are locally holomorphically conjugated at the singular points corresponding by (21). The equalities (22) of self-intersections of the components of $\mathcal{C}$ and $\mathcal{C}'$ allow us to glue these conjugacies and to obtain a homeomorphism defined on a neighborhood of $\mathcal{C}$. We leave the details of this construction to the reader.

The situation in case (cii) is similar: we have again equality of Camacho-Sad indices and therefore local conjugacies, and then equality of self-intersections allowing to glue and to obtain a global $\mathcal{C}^0$-conjugation.

11.5. **Step (v): Extension and gluing.** On the elementary pieces Z_s corresponding to a nodal corner s, we perform the gluing of the homeomorphisms already constructed by the process described in [18, §8.5]. It remains to extend the obtained homeomorphisms to the dicritical components. Notice that in all the above constructions the homeomorphisms can be built by respecting the dicritical components meeting their domains of definition. Finally we arrive at the following situation described in [19, page 147]:

- we identify tubular neighborhoods of the dicritical components $D \subset \mathcal{E}_{\mathcal{F}}$ and $D' \subset \mathcal{E}_{\mathcal{G}}$ corresponding by (21), with the same tubular neighborhood of the zero section of the normal bundle of D; it is possible because D and D' have same negative self-intersection;
- the corresponding foliations are identified with the natural normal fiber bundle;
- we have to extend to the whole D a continuous map f from a union $\mathcal{K}$ of disjoint closed discs to the group $\mathrm{Aut}_0(\mathbb{C}, 0)$ of germs of homeomorphisms of $(\mathbb{C}, 0)$. This extension can be easily made by extending f to a union $\mathcal{K}'$ of bigger discs being a constant automorphism on $\partial\mathcal{K}'$.

This ends the proof of Theorem 11.4.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona, E-08193 Bellaterra (Barcelona), Spain
E-mail address: davidmp@mat.uab.es

Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118, Route de Narbonne, F-31062 Toulouse Cedex 9, France
E-mail address: jean-francois.mattei@math.univ-toulouse.fr

Sorbonne Université, Université Paris Diderot, CNRS, Institut de Mathématiques de Jussieu-Paris Rive Gauche, IMJ-PRG, F-75005, Paris, France
E-mail address: eliane.salem@imj-prg.fr