This is a preprint of: "Lower bounds for the local cyclicity of centers using high order developments and parallelization", Luiz Fernando da Silva Gouveia, Joan Torregrosa, *J. Differential Equations*, vol. 271, 447–479, 2021.
DOI: [10.1016/j.jde.2020.08.027]

LOWER BOUNDS FOR THE LOCAL CYCLICITY OF CENTERS USING HIGH ORDER DEVELOPMENTS AND PARALLELIZATION

LUIZ F. S. GOUVEIA AND JOAN TORREGROSA

Abstract. We are interested in small-amplitude isolated periodic orbits, so-called limit cycles, surrounding only one equilibrium point, that we locate at the origin. We develop a parallelization technique to study higher order developments, with respect to the parameters, of the return map near the origin. This technique is useful to study lower bounds for the local cyclicity of centers. We denote by $M(n)$ the maximum number of limit cycles bifurcating from the origin via a degenerate Hopf bifurcation for a polynomial vector field of degree n. We get lower bounds for the local cyclicity of some known cubic centers and we prove that $M(4) \geq 20$, $M(5) \geq 33$, $M(7) \geq 61$, $M(8) \geq 76$, and $M(9) \geq 88$.

1. Introduction

Hilbert early last century presented a list of problems that almost all of them are solved. One problem that still is open is the second part of the 16th Hilbert's problem: It consists in determine the maximal number $H(n)$ of limit cycles, and their relative positions, of planar polynomial vector fields of degree n. In last years other related problems have been proposed. In 1977, Arnol'd ([2]) suggested a weakened version, focused on the study of the number of limit cycles bifurcating from the period annulus of Hamiltonian systems perturbing with polynomials of fixed degree n. We are interested here in another local version, that consists in providing the maximum number $M(n)$ of small-amplitude limit cycles bifurcating from an elementary equilibrium point of center or focus type, clearly $M(n) \leq H(n)$. In other words, $M(n)$ is an upper bound of the cyclicity of such equilibrium points. More details on related problems about the cyclicity of homoclinic and heteroclinic connections or of period annuli can be found in, for example, Roussarie's book ([28]).

There are very few general families for which this local number $M(n)$ is completely determined. In 1954 Bautin ([5]) proved that $M(2) = 3$ and Sibirskiĭ ([29]) and Blows and Lloyd ([6]) studied the special cubic family without quadratic terms. But the first complete proof that $M_h(3) = 5$ was obtained by Żołądek ([33]) in 1994. The first evidence that $M(3) \geq 11$ was presented also by Żołądek ([34]) in 1995, providing a cubic center with very high local cyclicity. This problem was revisited by himself in 2016 ([36]). The first and simplest proof that this lower bound is reached for the cubic family was given by Christopher in 2005 ([12]). Although we will describe more precisely this mechanism later, Christopher's idea is based in choosing a point on the center variety and, at this point, study the independence property of the first-order Taylor series with respect to the perturbation parameters, denoted by $L_k^{(1)}$, of the *Lyapunov constants*. In fact, if the point is (generically) chosen on a component of the center variety of codimension r, then there exist perturbations which can produce $r-1$ limit cycles. Apart from the fact that the center problem for polynomial vector fields of degree n is unsolved, the main problem is how to compute the codimension of each component of the center variety

2010 *Mathematics Subject Classification.* Primary 34C23, 34C25, 34C07.

Key words and phrases. small-amplitude limit cycle, polynomial vector field, center cyclicity, Lyapunov constants, higher-order developments and parallelization.

or, alternatively, how to find good center candidates to be perturbed for obtaining the highest (local) cyclicity value. This last procedure is the one mostly used to provide the best (highest) lower bounds for $M(n)$. It is somewhat curious that both centers appear listed in the same work of Żołądek ([35]). We notice that the idea of studying only linear developments, with respect to the parameters, near centers appears in some previous works of Chicone and Jacobs ([10]) and of Han ([23]).

In 2012, Giné ([19, 20]) conjectured that $M(n) = n^2 + 3n - 7$. This suggests a very high value for $M(n)$ for polynomial vector fields of low degree. In particular, for $n = 4$, it asserts that $M(4) \geq 21$. The local cyclicity problem in the quartic polynomial class was studied in [19] using only second-order Taylor developments of the Lyapunov constants. But, as we will describe in Section 5, orders 2 and 3 are not enough to prove this result. Then, this lower bound remains to be proved. For degrees $n = 5, 7, 8, 9$ the best lower bounds for $M(n)$ were obtained in [25] providing examples exhibiting 28, 54, 70, and 88 limit cycles, respectively. Next result improves all these values increasing the known lower bounds for $M(n)$ for the respective degrees. Up to our knowledge, the highest value for $n = 6$ is $M(6) \geq 48$ and it is proved in [4]. This suggests that the value for $M(n)$ should be increased (at least) by one. The new conjectured value appears recently in [21]. A final comment about the Giné's conjecture for $n = 5$ is that the value $M(5) \geq 33$ suggested is reached in our main result. The proof will follow perturbing a quintic center with homogeneous nonlinearities, appearing in [18], in the complete polynomial class of degree 5. The best result for general polynomial vector fields of degree n is $M(n) \geq n^2 - 2$. A result of 2004 due to Movasati ([27]).

Theorem 1.1. *The number of limit cycles bifurcating from a singular monodromic point for vector fields of degree four, five, seven, eight and nine is at least* $M(4) \geq 20$*,* $M(5) \geq 33$*,* $M(7) \geq 61$*,* $M(8) \geq 76$*, and* $M(9) \geq 88$*.*

A better description of the number of limit cycles in the cubic polynomial family can be obtained studying lower bounds for the local cyclicity of some cubic Darboux centers. We analyze here all those with codimension 12 and listed by Żołądek in [35] but having a real center. More concretely, we show that in almost all cases always 11 limit cycles bifurcate from each studied center. The proofs show why sometimes a very high-order study is necessary.

For proving our results first we extend, in Section 3, the parallelization technique introduced by Liang and Torregrosa in [25] to higher-order developments. Second, following the ideas in [12] for studying higher-order developments, we use a blow-up procedure to get a complete unfolding of the return map near a polynomial center perturbing with polynomials of the same degree. We remark that the parallelization technique drastically reduces the computation time. In particular, in the proof of Theorem 1.1 when only developments of order two are necessary to be used, instead of more than one month of computation time, we need less than one hour. We have used a cluster of computers with ninety processors simultaneously. All the computations have been made with the Computer Algebra System `Maple` [26].

This paper is structured as follows. In Section 2, we recall the necessary definitions and algorithms to get the coefficients of the return map, the so-called Lyapunov constants among other preliminary results as the *Poincaré–Miranda Theorem* (see [24]) and the *Gershgorin Theorem* (see [17]) about localization of eigenvalues of a matrix. In Section 3, we present and prove the parallelization setting. The study of the local cyclicity of cubic Darboux centers is developed in Section 4. Finally, in Sections 5 and 6 we prove our main result Theorem 1.1. All the computations are analytic but, in some cases, we have used numerical approximations that, with Computer Assisted Proofs techniques, have allowed us to obtain the analytical proof.

2. Degenerated Hopf bifurcation

In this section, we recall how to obtain the Lyapunov constants or focal values, that is, the coefficients of the return map near an equilibrium point with differential matrix of an elementary center type. As usual, it is not restrictive to assume that the equilibrium point is located at the origin. So, following [1], we can consider the polynomial system of degree n

$$\begin{cases} \dot{x} &= \lambda_0 x - y + \sum_{k=2}^{n} P_k(x,y), \\ \dot{y} &= x + \lambda_0 y + \sum_{k=2}^{n} Q_k(x,y), \end{cases} \tag{1}$$

being P_k and Q_k homogeneous polynomials of degree k in the variables x, y. Writing the above system in polar coordinates, $(r,\theta) = (r\cos\theta, r\sin\theta)$, we get

$$\frac{dr}{d\theta} = \lambda_0 r + \sum_{k=2}^{\infty} S_k(\theta) r^k, \tag{2}$$

being $S_k(\theta)$ trigonometric polynomials. Let $r(\theta,\rho)$ be the solution of system (2) such that $r(0,\rho) = \rho$. The stability of the origin is clearly determined, using Hartman–Grobman Theorem, when $\lambda_0 \neq 0$. When $\lambda_0 = 0$ the stability problem is known as the center-focus problem and there are some classical tools to distinguish when the origin is stable or unstable. So, for $\lambda_0 = 0$ and close to $\rho = 0$, we can develop this solution as a power series in ρ

$$r(\theta,\rho) = \rho + \sum_{k=2}^{\infty} r_k(\theta)\rho^k,$$

where $r_k(0) = 0$ for all $k \geq 2$. Then, the *Poincaré return map,* $\Pi(\rho)$, can be obtained evaluating the above expression at $\theta = 2\pi$, i.e.

$$\Pi(\rho) = r(2\pi,\rho) = \rho + \sum_{k=2}^{\infty} r_k(2\pi)\rho^k. \tag{3}$$

When $V_{\tilde{K}} = r_{\tilde{K}}(2\pi) \neq 0$ for some value $\tilde{K}$ we say that the origin of system (1) is a *weak-focus*, otherwise we say that the origin is a *center*. In this context, it is well known that the first nonzero value, when it exists, corresponds to an odd subscript $\tilde{K} = 2K+1$, see [1, 9, 16], and consequently the K-Lyapunov constant is defined as $L_K = V_{2K+1}$ when $L_1 = \cdots = L_{K-1} = 0$. Then, we say that the origin is a *weak-focus of order* K when $L_K \neq 0$ and $L_1 = \cdots = L_{K-1} = 0$. These constants are polynomials in the coefficients of P_k and Q_k defined in (1). For more details in the algebraic properties that they satisfy, we refer the reader to [13].

When the first Lyapunov constant is negative (positive), $L_1 \neq 0$, a small-amplitude stable (unstable) limit cycle bifurcates from the equilibrium when the trace parameter λ_0 moves from zero to positive (negative). Because the stability of the equilibrium point changes and the limit cycle appears by using the Poincaré–Bendixson Theorem. This phenomenon is known as the classical Hopf bifurcation. The results in this work are dealing with the K-degenerated Hopf bifurcation. That is, when K small-amplitude limit cycles bifurcate from a weak-focus of order K, for more details see for example [11]. According Roussarie in [28], at most K limit cycles can bifurcate from a weak-focus of order K under analytic perturbations. In the context of (1), the main difficulty is how can we ensure the existence of polynomial perturbations such that the K limit cycles bifurcate from a weak-focus of order K. To unify notation, we denote the trace parameter in (1) by $L_0 := \lambda_0$. The fact that the coefficients of the return map (3) corresponding to monomials of even degree in ρ do not play any role in the bifurcation phenomenon is due to the property that the Bautin ideal is generated only by the coefficients of odd degree, i.e. $\mathbf{B} = \langle L_1, \ldots, L_n, \ldots \rangle = \langle V_3, V_4, \ldots, V_{2n}, V_{2n+1}, \ldots \rangle$. See more details also in [28]. This property has been revisited recently in [14].

Another equivalent procedure to study the center-focus problem is to propose, for $\lambda_0 = 0$, a function $H(x,y) = x^2 + y^2 + O(||(x,y)||^3)$ such that, using (1),

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = h_4 r^4 + h_6 r^6 + \cdots + h_{2k} r^{2k} + \cdots ,$$

with $r^2 = x^2 + y^2$. Then, the first nonvanishing coefficient of the above derivative, which always has an even subscript, determines the stability of the origin of (1), being H a Lyapunov function. In fact, both coefficients, h_{2K+2} and L_{2K+1}, differ on a multiplicative nonzero constant when the previous vanish. Here the center property reads as $h_{2k} = 0$ for all k. In complex variables, via the change of variables $z = x + \mathrm{i}\, y$ and for $\lambda_0 = 0$, system (1) writes as

$$\dot{z} = R(z, \bar{z}) = \mathrm{i}\, z + \sum_{k=2}^{n} R_k(z, \bar{z}),$$

where $R_k(z,\bar{z})$ are homogeneous polynomials of degree k in $(z,\bar{z})$. Consequently, the above function $H = z\bar{z} + O_3(z,\bar{z})$ satisfies

$$\dot{H} = \frac{\partial H}{\partial z}\dot{z} + \frac{\partial H}{\partial \bar{z}}\dot{\bar{z}} = \sum_{k=2}^{\infty} g_k (z\bar{z})^k,$$

being the coefficient g_{k+1} the k-Lyapunov constant.

Following the Christopher's idea ([12]), to find good (or high) lower bounds for $M(n)$, instead of studying the local cyclicity of the origin of system (1), we consider perturbations of a fixed center. That is, taking P_c, Q_c, P, and Q as polynomials of degree n in (x,y) and studying the perturbed system

$$\begin{cases} \dot{x} &= P_c(x,y) + P(x,y,\lambda), \\ \dot{y} &= Q_c(x,y) + Q(x,y,\lambda), \end{cases} \tag{4}$$

with P, Q having only monomials of degree higher or equal than two. When $\lambda = 0$, $P(x,y,0) = Q(x,y,0) = 0$, and the unperturbed system must be written as (1), i.e. for $\lambda_0 = 0$, the origin should be an elementary center. Thus, we are interested in finding limit cycles of small-amplitude bifurcating from the origin of system (4).

In most cases, the explicit computation of the Lyapunov constants for system (1) is very hard and impossible to do by hand. So a Computer Algebra System[1] is necessary to be used. Moreover, sometimes more specific algorithms can be developed to decrease the computation time. Here, with the aim to reduce not only the computation time but also the total memory requirements, we extend, in the next section, the parallelization algorithm proposed by Liang and Torregrosa in [25] to higher-order developments but for systems of type (4).

Next two results, which are proved in [12], provide conditions to get the bifurcation of small-amplitude limit cycles near a polynomial center. The first uses only the linear developments of the Lyapunov constants. We notice that as we are perturbing centers,

[1] We have used `Maple` for all our computations ([26])

the Taylor approximations vanish at the origin ($\lambda = 0$) in the parameter space. So, the first-order Taylor approximation is linear. As we have commented before, similar versions of this result can be found in [10] and [23]. The second result uses higher-order Taylor approximations.

Theorem 2.1 ([12]). *Suppose that s is a point on the center variety and that the first k Lyapunov constants, $L_1, \ldots, L_k$, have independent linear parts (with respect to the expansion of L_i about s), then s lies on a component of the center variety of codimension at least k and there are bifurcations which produce k limit cycles locally from the center corresponding to the parameter value s. If, furthermore, we know that s lies on a component of the center variety of codimension k, then s is a smooth point of the variety, and the cyclicity of the center for the parameter value s is exactly k. In the latter case, k is also the cyclicity of a generic point on this component of the center variety.*

The scheme of the proof is as follows. Under the hypotheses of the above result, there exists a change of variables such that the first Lyapunov constants write as

$$L_i = u_i + O_2(u_1, \ldots, u_k, \ldots, u_m), i = 1, \ldots, k, \tag{5}$$

assuming that we have $m \geq k$ bifurcation parameters. Using the Implicit Function Theorem it is clear that we can write $L_i = v_i$, for $i = 1, \ldots, k$. Then the first coefficients of the return map (3) are independent. It is clear that, when the trace parameter $L_0 = \lambda_0 = 0$ in (1) we get only $k-1$ limit cycles. But adding the parameter $L_0 = \lambda_0$ we have an extra limit cycle by the classical Hopf bifurcation, obtaining in total k, as the above result ensures. In fact, this proves the existence of a variety, in the parameter space, of weak-foci of order k ($\{\lambda_0 = u_1 = u_2 = \cdots = u_{k-1} = 0\}$ and $u_k \neq 0$) that unfolds k hyperbolic limit cycles. Using the Weierstrass Preparation Theorem this is the maximal number near such curve.

When the linear parts of the next Lyapunov constants are linear combination of the first k we can use the higher developments to obtain more limit cycles. This is the aim of the next result also proved by Christopher in [12, Theorem 3.1].

Theorem 2.2. *Suppose that we are in a point s where Theorem 2.1 applies. After a change of variables if necessary, we can assume that $L_0 = L_1 = \cdots = L_k = 0$ and the next Lyapunov constants $L_i = h_i(u) + O_{m+1}(u)$, for $i = k+1, \ldots, k+l$, where h_i are homogeneous polynomials of degree $m \geq 2$ and $u = (u_{k+1}, \ldots, u_{k+l})$. If there exists a line ℓ, in the parameter space, such that $h_i(\ell) = 0$, $i = k+1, \ldots, k+l-1$, the hypersurfaces $h_i = 0$ intersect transversally along ℓ for $i = k+1, \ldots, k+l-1$, and $h_{k+l}(\ell) \neq 0$, then there are perturbations of the center which produce $k+l$ limit cycles.*

The above result is not written exactly as in the original Christopher paper because we have adapted to include also the conclusion of Theorem 2.1. We have also included here an alternative proof.

Proof of Theorem 2.2. The first step uses Theorem 2.1 to assume that the first k Lyapunov constants together with the trace are the first $k+1$ coordinates $(u_0, u_1, \ldots, u_k)$, that will be taken as zero. The proof continues taking the blow-up change of variable $u_j = v_j u_{k+l}$, for $j = k+1, \ldots, k+l-1$. Then $v = (v_{k+1}, \ldots, v_{k+l-1})$ and we can write $h_i(u) = u_{k+l}^m \widehat{h}_i(v)$, for $i = k+1, \ldots, k+l$. Consequently,

$$L_i(u) = u_{k+l}^m \widetilde{L}_i(v) = u_{k+l}^m \left(\widehat{h}_i(v) + \sum_{j=1}^{\infty} g_{ij}(v) u_{k+l}^j \right).$$

The existence of a line ℓ as in the statement gets v^* such that $\widehat{h}_i(v^*) = 0$, for $i = k+1, \ldots, k+l-1$, $\widehat{h}_{k+l}(v^*) \neq 0$, and the determinant of the Jacobian matrix of

$(\widehat{h}_{k+1},\ldots,\widehat{h}_{k+l-1})$ with respect to v does not vanish at v^*, then the Implicit Function Theorem applies, in a small neighborhood of v^*, and the change of variables $w_i=\widetilde{L}_i(v)$ is well defined. The proof follows the same scheme explained in the comments before to state this result but changing $u_j=w_j u_{k+l}^m$. Because, now u_{k+l}^m is a common factor of the complete return and the polynomial provided by the Weierstrass Preparation Theorem has $\widehat{h}_{k+l}(v^*)\neq 0$ as the coefficient of maximal degree monomial and the other are the independent coefficients w_j, for $j=1,\ldots,k+l-1$. Finally, as above we can use the trace parameter to have a complete unfolding of $k+l$ limit cycles. □

We remark that, as in the previous result, this maximal number of limit cycles is obtained only near the weak-foci curve. Consequently, the previous results provide only lower bounds for the local cyclicity problem. Theorem 2.1 uses first order developments. After removing them (using the Implicit Function Theorem) Theorem 2.2 imposes that all the homogeneous principal parts at the origin of the remaining h_i have the same degree. Our proof of Theorem 2.2 suggests that we can study, restricting the parameter space if necessary, how the intersection of the algebraic varieties $\mathcal{S}_L=\{L_1(u)=L_2(u)=\cdots=L_m(u)=0\}$ is near $u=0$. This is equivalent to know if the function $f:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$

$$(u_1,\ldots,u_n)\mapsto(L_1,L_2,\ldots,L_n)$$

is locally surjective at the origin (see the interesting comment on this fact in [28, Page 71]). From the singularities classification theory, see [3], some properties used to compute the local multiplicity of a function in a point, $\mu_0[f]$, are useful to study the local intersection of system $\mathcal{S}_L$.

Proposition 2.3 ([3]). *Let $f:(\mathbb{R}^n,0)\to(\mathbb{R}^n,0)$ be a finite map germ. Defining $f=(f_1,\ldots,f_n)$ and $\mathcal{S}_f=\{f_1=f_2=\cdots=f_n=0\}$ the next properties hold,*

(i) if $f_i=h_i+H.O.T.$, where h_i is homogeneous of degree k_i, then $\mu_0[f]\geq\prod_{i=1}^n k_i$ and $\mu_0[f]=\prod_{i=1}^n k_i$ if and only if the system $h_i=0$, $i=1,\ldots,n$, has only the trivial solution in $\mathbb{C}^n$.

(ii) if $g_i=f_i+\sum_{j<i}A_j^i f_j$, then $\mu_0[f]=\mu_0[g]$ and $\mathcal{S}_f=\mathcal{S}_g$.

Inspired by the Gauss algorithm to triangularize a matrix, we can use the second property of the above result to convert the local intersection of the set $\mathcal{S}_L$ in a simpler one. For example, in (5) we have simplified with the first k linear terms. But, in general, we will use also to simplify higher-orders terms and the restriction of Theorem 2.1 on the same degree for all principal parts can be removed. Moreover, this simple mechanism helps also to reduce the total computation time.

Next, we present a simple example of the application using the aim of the above proposition. We assume that, up to an adequate linear change of variables in the parameter space, the Taylor approximations up degree three for the first four Lyapunov constants write as

$$\begin{aligned}
L_1(u)&=u_1+u_1^2+u_1u_3+u_3u_4+O_4(u),\\
L_2(u)&=u_2+u_2^2+u_2u_3+u_1^3+u_3u_4+u_2^2u_3+O_4(u),\\
L_3(u)&=u_1+u_2+2u_1^2+2u_1u_3-3u_3^2+3u_3u_4+u_3^3+O_4(u),\\
L_4(u)&=u_1-u_2+3u_1^2+u_2^2-u_1u_3+u_3^2+Au_3u_4+O_4(u),
\end{aligned}\tag{6}$$

with $u=(u_1,u_2,u_3,u_4)$ and A a free parameter. We notice that, as we have commented above, the perturbation of a center implies that $L_k(0)=0$, for $k=1,2,3,4$. Clearly, as the linear developments of $L_3^{(1)}=u_1+u_2$ and $L_4^{(1)}=u_1-u_2$ are linear combinations of $L_1^{(1)}=u_1$ and $L_2^{(1)}=u_2$, we have, up to this point and applying Theorem 2.1 considering also the trace parameter, only two limit cycles of small amplitude. To get more we need to look at the higher-orders in the local intersection of the algebraic varieties defined by all the considered Lyapunov constants, i.e. how the origin of $\mathcal{S}_L = \{L_1(u) = L_2(u) = L_3(u) = L_4(u) = 0\}$ looks like. Inspired by Proposition 2.3.(ii) and the fact that, near the origin, L_1 and L_2 act as new coordinates, we can change L_3 and L_4 by $L_3 - L_1 - L_2$ and $L_4 - L_1 + L_2$, respectively, in $\mathcal{S}_L$ and the local intersection at the origin remains unchanged taking

$$\begin{aligned} L_3(u) &= u_1^2 + u_1u_3 - u_2^2 - u_2u_3 - 3u_3^2 + u_3u_4 - u_1^3 - u_2^2u_3 + u_3^3 + O_4(u), \\ L_4(u) &= 2u_1^2 - 2u_1u_3 + 2u_2^2 + u_2u_3 + u_3^2 + Au_3u_4 + u_1^3 + u_2^2u_3 + O_4(u). \end{aligned}$$

By abusing notation we have called the new ones again as L_3 and L_4. In the next step, we remove the degree two terms having u_1 or u_2. Changing now L_3 and L_4 by $L_3 - (L_1^2 + L_1u_3 - L_2^2 - L_2u_3)$ and $L_4 - (2L_1^2 - 2L_1u_3 + 2L_2^2 + L_2u_3)$, respectively. Then, the new local expressions are

$$\begin{aligned} L_3(u) = {} & -3u_3^2 + u_3u_4 - 3u_1^3 - 3u_1^2u_3 - u_1u_3^2 - 2u_1u_3u_4 + 2u_2^3 + 2u_2^2u_3 \\ & + u_2u_3^2 + 2u_2u_3u_4 + u_3^3 + O_4(u), \\ L_4(u) = {} & u_3^2 + Au_3u_4 - 4u_1^3 - 2u_1^2u_3 + 2u_1u_3^2 - 4u_1u_3u_4 - 4u_2^3 - 4u_2^2u_3 \\ & - u_2u_3^2 - 4u_2u_3u_4 + u_3^2u_4 + O_4(u). \end{aligned}$$

Similarly we can also remove the degree three terms having again u_1 or u_2. Therefore, we get

$$\begin{aligned} L_3(u) &= -3u_3^2 + u_3u_4 + u_3^3 + O_4(u), \\ L_4(u) &= u_3^2 + Au_3u_4 + u_3^2u_4 + O_4(u). \end{aligned} \tag{7}$$

It is important to point out that the terms corresponding to linear independent parameters, u_1 and u_2 in this example, should be removed degree by degree. That is, first degree 1, then degree 2, and so on.

In this particular example, when $A = -1/3$ we need an extra step for removing from L_4 the degree two common part with L_3, changing L_4 by $L_4 + L_3/3$. Then, we finally obtain

$$\begin{aligned} L_3(u) &= -3u_3^2 + u_3u_4 + u_3^3 + O_4(u), \\ L_4(u) &= u_3^2(u_3 + 3u_4)/3 + O_4(u). \end{aligned} \tag{8}$$

We have computed the triangularized equivalent system $\mathcal{S}_L^T$ corresponding to (6) changing L_3 and L_4 by (7) or (8), depending on $A \neq -1/3$ or $A = -1/3$, respectively.

After the triangularization, by using the Implicit Function Theorem in a small neighborhood of the origin, we can change the first two parameters (u_1, u_2) by $(v_1, v_2) := (L_1, L_2)$. We notice that the local expressions of the degree three Taylor approximations of L_3 and L_4 are not modified, only the terms in $O_4(u)$ change. Then, assuming $v_1 = v_2 = 0$, (7) and (8) are valid writing $u = (u_3, u_4)$. At this point, when $A \neq -1/3$ we can use Theorem 2.2, with the degree two Taylor approximations $L_3^{(2)} = u_3(-3u_3 + u_4)$ and $L_4^{(2)} = u_3(u_3 + Au_4)$, to get 4 limit cycles of small amplitude taking $u_4 = 3u_3$ as the straight line ℓ because $L_4^{(2)}|_{\{u_4=3u_3\}} = (3A+1)u_3 \neq 0$, for small enough u_3. When $A = -1/3$ an extension of Theorem 2.2 is needed, because we must work with orders $(1, 1, 2, 3)$ instead of $(1, 1, 2, 2)$. But there are also 4 limit cycles of small amplitude, because the order 4 weak-foci curve passing through the origin in the parameter space also exists: $L_4^{(3)}|_{\{u_4=3u_3\}} = 10u_3^3 \neq 0$ for small enough u_3.

The versal unfolding follows as in the proof of Theorem 2.2 using an adequate weighted blow-up. In fact we have described the usual desingularization procedure to classify the local intersection of $\mathcal{S}_L$ at the origin as it is usual in algebraic geometry, see [8, 31].

As we will see in the proofs of the results of the next sections, sometimes the application of Theorem 2.2 is not so simple as the above example. It depends on how the curve ℓ can be explicitly obtained and if the varieties in $\mathcal{S}_L$ are transversal along it. In fact, the above example shows that, in some cases, the straight line ℓ in Theorem 2.2 can be changed by a line. The lines ℓ obtained from the intersection of the principal homogeneous parts of the varieties, defined by the polynomials h_i of degree m in Theorem 2.2, can be computed just with a blow-up type change of coordinates. That is, reordering subscripts if necessary, the line can be parameterized by $u_i = \lambda v_i$, for $i = k+1, \ldots, k+l-1$ and $u_{k+l} = \lambda$. Then the main difficulty is to find an explicit transversal solution of the system defined by $l-1$ equations with $l-1$ unknowns of degree m obtained dividing each h_i by λ^m. Moreover, at these solutions the last h_{k+l} must be nonzero. In the above example, the principal homogeneous parts have degree two (when $A \neq -1/3$) and there are two (possible) lines ℓ: $u_3 = 0$ and $u_4 = 3u_3$. We have used the second one because on the first one L_4 also vanishes up to third-order analysis. So higher-order terms must be computed and analyzed. In most cases, these special values could define a center curve in the parameter space. Although this intersection point can be obtained numerically, we use a computer assisted proof to prove analytically the existence of such point. This can be done using Poincaré–Miranda's Theorem together with the results of last section. For the transversality property, we can use the Gershgorin circles Theorem. For completeness, we have included both below.

Theorem 2.4 ([24], Poincaré–Miranda)**.** *Let c be a positive real number and $S = [-c, c]^n$ a n-dimensional cube. Consider $f = (f_1, \ldots, f_n) : S \to \mathbb{R}^n$ a continuous function such $f_i(S_i^-) < 0$ and $f_i(S_i^+) > 0$ for each $1 \leq i \leq n$, where $S_i^{\pm} = \{(x_1, \ldots, x_n) \in S : x_i = \pm c\}$. So, there exists $d \in S$ such that $f(d) = 0$.*

Theorem 2.5 ([17], Circles of Gershgorin)**.** *Let $A = (a_{i,j}) \in \mathbb{C}^{n \times n}$ and α_k its eigenvalues. Consider for each $i = 1, \ldots, n$*

$$D_i = \{z \in \mathbb{C} : |z - a_{i,i}| \leq r_i\},$$

where $r_i = \sum_{i \neq j} |a_{i,j}|$. So, for all k, each $\alpha_k \in D_i$ for some i.

The Poincaré–Miranda's Theorem was conjectured by Poincaré in the 19th century and proved by Miranda in last century. Note that this result is a generalization of the Bolzano's Theorem for higher dimensions. The reader can get more details on Gershgorin Circle Theorem in [22].

3. Parallelization

In [25], Liang and Torregrosa present a parallelization mechanism to compute the Taylor linear approximation of the Lyapunov constants near centers. In this section, we extend this result to compute also the terms of higher degree. We start recalling the linearization result for completeness.

Theorem 3.1 ([25])**.** *Let $p(z, \bar{z})$ and $Q_j(z, \bar{z})$, $j = 1, \ldots, s$ be polynomials with monomials of degree higher or equal than two such that the origin of $\dot{z} = \mathrm{i}\, z + p(z, \bar{z})$ is a center. If $L_{k,j}^{(1)}$ denotes the first-order Taylor approximation, with respect to $\lambda_j \in \mathbb{R}$, of the k-Lyapunov constant of equation*

$$\dot{z} = \mathrm{i}\, z + p(z, \bar{z}) + \lambda_j Q_j(z, \bar{z}), \quad j = 1, \ldots, s,$$

then the first-order Taylor approximation of the k-Lyapunov constant, with respect to $\Lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$, *of equation*

$$\dot{z} = \mathrm{i}\, z + p(z, \bar{z}) + \sum_{j=1}^{N} \lambda_j Q_j(z, \bar{z}),$$

is $L_k^{(1)} = \sum\limits_{j=1}^{N} L_{k,j}^{(1)}$.

In the following Theorem 3.3, we show how to get the terms up to degree $\ell > 1$ of the Lyapunov constants. The main idea is to decompose the global computation problem into a collection of simpler problems. The advantage of the previous result is that each perturbation parameter λ_j appears in only one problem. We remark that the authors were interested only in the linear approximation. Now, because we study developments up to degree ℓ, we need to decompose the global problem in simpler problems having exactly ℓ parameters or monomials. But, as many parameters appear in more than one simple perturbation problem we need to correct (multiplying by a factor adequately chosen) the obtained coefficients of the developments of degree ℓ. We will see that the idea of parallelization only makes sense if the number of parameters is greater than the degree up to which we want to compute. To further clarify the main idea of parallelization, we present the next proposition where we perturb a quadratic center with four parameters and we calculate the first three Lyapunov constants up to degree two.

Proposition 3.2. *Consider the quadratic perturbed equation*

$$\dot{z} = \mathrm{i}\, z + z^2 + (a_{20} + \mathrm{i}\, b_{20}) z^2 + (a_{11} + \mathrm{i}\, b_{11}) z\bar{z}. \tag{9}$$

Then the Taylor developments up to degree two of the Lyapunov constants of the above system are

$$\begin{aligned}
L_1^{(2)} &= -2a_{20}b_{11} - 2b_{20}a_{11} - 2b_{11},\\
L_2^{(2)} &= 36a_{20}b_{11} + 12b_{20}a_{11} + 32a_{11}b_{11} + 8b_{11}^2 + 12b_{11},\\
L_3^{(2)} &= -540a_{20}b_{11} - 108b_{20}a_{11} - 582a_{11}b_{11} - 192b_{11}^2 - 108b_{11}.
\end{aligned}$$

Proof. We rename the four real parameters in the ordered list

$$\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (a_{20}, b_{20}, a_{11}, b_{11}).$$

As system (9) is a holomorphic quadratic center when $\lambda = 0$, all the Lyapunov constants vanish. Then, as we are interested in degree two developments and we have 4 parameters, we will consider the six combinatorial (without repetition) pairs

$$S = \{(\lambda_1, \lambda_2), (\lambda_1, \lambda_3), (\lambda_1, \lambda_4), (\lambda_2, \lambda_3), (\lambda_2, \lambda_4), (\lambda_3, \lambda_4)\},$$

and 6 respective differential equations, one for each element in S, such that in (9) the parameters that are not in the chosen pair are taken to be zero. For example, the corresponding differential equation to $S_3 = (\lambda_1, \lambda_4)$ is equation E_3: $\dot{z} = \mathrm{i}\, z + z^2 + \lambda_1 z^2 + \mathrm{i}\, \lambda_4 z\bar{z}$.

For each equation E_j corresponding to each pair S_j we compute, with the mechanism described in Section 2, the first three Lyapunov constants. We denote by $L_{k,j}$ the k-Lyapunov constant of equation E_j and by $L_{k,j}^{(2)}$ the corresponding Taylor approximations up to degree 2. Straightforward computations show $L_{k,j}^{(2)} = 0$, for $k = 1, 2, 3$ and $j = 1, 2$

and

$$\begin{aligned}
L_{1,3}^{(2)} &= -2\lambda_1\lambda_4 - 2\lambda_4, & L_{1,5}^{(2)} &= -2\lambda_4,\\
L_{2,3}^{(2)} &= 36\lambda_1\lambda_4 + 8\lambda_4^2 + 12\lambda_4, & L_{2,5}^{(2)} &= 8\lambda_4^2 + 12\lambda_4,\\
L_{3,3}^{(2)} &= -540\lambda_1\lambda_4 - 192\lambda_4^2 - 108\lambda_4, & L_{3,5}^{(2)} &= -192\lambda_4^2 - 108\lambda_4,\\
L_{1,4}^{(2)} &= -2\lambda_3\lambda_2, & L_{1,6}^{(2)} &= -2\lambda_4,\\
L_{2,4}^{(2)} &= 12\lambda_3\lambda_2, & L_{2,6}^{(2)} &= 32\lambda_3\lambda_4 + 8\lambda_4^2 + 12\lambda_4,\\
L_{3,4}^{(2)} &= -108\lambda_3\lambda_2, & L_{3,6}^{(2)} &= -582\lambda_3\lambda_4 - 192\lambda_4^2 - 108\lambda_4.
\end{aligned}$$

We notice that we can not compute $L_k^{(2)}$ by $\sum_{j=1}^{6} L_{k,j}^{(2)}$ as in the linear case, because of the repeated terms. The monomials having only one parameter, λ_l or λ_l^2, appear more than once in different E_j, while the monomials having two, $\lambda_l\lambda_m$, only once. So, for every fixed k, before considering the sum of all $L_{k,j}^{(2)}$ we need to correct the corresponding coefficient adequately. For example, the monomials λ_4 and λ_4^2 appear in S_3, S_5, and S_6, that is, exactly $3 = \binom{3}{1}$ times. Hence, these monomials will be divided by 3. But, as the monomial $\lambda_1\lambda_4$ only appears in S_3 the corresponding coefficient remains unchanged. So, we need to multiply by a correction factor which depends on the number of repetitions. As we have four perturbation parameters, we should divide each repeated term by 3 because is the number of times that they appear in S.

Denoting by $\hat{L}_{k,j}^{(2)}$ the corrected k-Lyapunov constant corresponding to the pair S_j, we can obtain $L_k^{(2)} = \sum_{j=1}^{6} \hat{L}_{k,j}^{(2)}$. The statement follows because, in our case, we have $\hat{L}_{k,j}^{(2)} = 0$, for $k = 1, 2, 3$ and $j = 1, 2$ and

$$\begin{aligned}
\hat{L}_{1,3}^{(2)} &= -2\lambda_1\lambda_4 - \frac{2}{3}\lambda_4, & \hat{L}_{1,5}^{(2)} &= -\frac{2}{3}\lambda_4,\\
\hat{L}_{2,3}^{(2)} &= 36\lambda_1\lambda_4 + \frac{8}{3}\lambda_4^2 + 4\lambda_4, & \hat{L}_{2,5}^{(2)} &= \frac{8}{3}\lambda_4^2 + 4\lambda_4,\\
\hat{L}_{3,3}^{(2)} &= -540\lambda_1\lambda_4 - 64\lambda_4^2 - 36\lambda_4, & \hat{L}_{3,5}^{(2)} &= -64\lambda_4^2 - 36\lambda_4,\\
\hat{L}_{1,4}^{(2)} &= -2\lambda_2\lambda_3, & \hat{L}_{1,6}^{(2)} &= -\frac{2}{3}\lambda_4,\\
\hat{L}_{2,4}^{(2)} &= 12\lambda_2\lambda_3, & \hat{L}_{2,6}^{(2)} &= 32\lambda_3\lambda_4 + \frac{8}{3}\lambda_4^2 + 4\lambda_4,\\
\hat{L}_{3,4}^{(2)} &= -108\lambda_2\lambda_3, & \hat{L}_{3,6}^{(2)} &= -582\lambda_3\lambda_4 - 64\lambda_4^2 - 36\lambda_4.
\end{aligned}$$

□

Now we can state the main result of this section, the computation in a parallelized form of the Taylor developments of the Lyapunov constants up to degree ℓ, assuming that the number of total parameters is $\ell \le N$.

Theorem 3.3. *Let $p(z,\bar z)$ and $Q_j(z,\bar z)$, $j = 1,\ldots,N$ be polynomials with monomials of degree higher or equal than two such that the origin of $\dot z = \mathrm{i}\, z + p(z,\bar z)$ is a center. For $\ell \le N$, we denote by $L_k^{(\ell)}$ the Taylor approximation of k-Lyapunov constant up to degree ℓ of equation*

$$\dot z = \mathrm{i}\, z + p(z,\bar z) + \sum_{j=1}^{N} \lambda_j Q_j(z,\bar z), \tag{10}$$

with $\Lambda = (\lambda_1, \ldots, \lambda_N) \in \mathbb{R}^N$*. Let* S *be the set of all combinations of the components of* Λ *taken* ℓ *at a time. That is,* $S = \{(\lambda_1, \ldots, \lambda_\ell), (\lambda_2, \ldots, \lambda_{\ell+1}), \ldots, (\lambda_{\ell-N}, \ldots, \lambda_N)\}$ *and having* $\binom{N}{\ell}$ *elements. For each element* $S_j \in S$*, we denote by* $\sigma(j, \zeta)$ *the subscript of the parameters in* S_j *at position* ζ*, i.e.* $S_j = (\lambda_{\sigma(j,1)}, \ldots, \lambda_{\sigma(j,\ell)})$*, and we denote by* $L_{k,j}^{(\ell)}$ *the Taylor approximation up to degree* ℓ *with respect to* Λ *of the* k*-Lyapunov constant of equation*

$$\dot{z} = \mathrm{i}\, z + p(z, \bar{z}) + \sum_{l=1}^{\ell} \lambda_{\sigma(j,l)} Q_{\sigma(j,l)}(z, \bar{z}). \tag{11}$$

Then

$$L_k^{(\ell)} = \sum_{l=1}^{N} \hat{L}_{k,j}^{(\ell)},$$

where $\hat{L}_{k,j}^{(\ell)} = \sum_p \frac{\mu_{k,j,p}}{\binom{N-s(p)}{\ell-s(p)}} \Lambda_j^p$*, for* $\Lambda_j^p = \lambda_{\sigma(j,1)}^{p_1} \lambda_{\sigma(j,2)}^{p_2} \cdots \lambda_{\sigma(j,\ell)}^{p_\ell}$ *and* $p = (p_1, \ldots, p_\ell)$ *writing* $L_{k,j}^{(\ell)} = \sum_p \mu_{k,j,p} \Lambda_j^p$ *with* $s(p) = \sum_{l=1}^{\ell} \operatorname{sgn}(p_l)$ *where* $\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x = 0. \end{cases}$

Proof. It is well known that the k-Lyapunov constant, L_k, of a differential equation (10) is a polynomial in the parameters $\Lambda = (\lambda_1, \ldots, \lambda_N)$, see [13]. Moreover, in our case each L_k vanishes when $\Lambda = 0$. Consequently, $L_k^{(\ell)}$ is the Taylor polynomial of L_k up to degree ℓ at $\Lambda = 0$ and $L_k^{(\ell)}(0) = 0$. We write it as

$$L_k^{(\ell)} = \sum_p \mu_{k,p} \Lambda^p, \tag{12}$$

where $\Lambda^p = \lambda_1^{p_1} \lambda_2^{p_2} \cdots \lambda_N^{p_N}$, $p = (p_1, \ldots, p_N)$, and $\sum_{l=1}^{N} p_l \leq \ell$. This last condition allows us to decompose the total sum (12) in partial sums of ℓ parameters in Λ. Each partial sum is in fact the Taylor polynomial of the k-Lyapunov constant up to degree ℓ of equation (11). As each monomial can appear more than once in each partial sum, we need to correct the corresponding coefficient with a multiplicative factor that controls how many times this monomial appears. This factor depends on the number of different λ_l appearing in each monomial besides the total number N and the degree ℓ, in fact, the number of times that it appears in the partial sums, that is the combinatorial number $\binom{N-s(p)}{\ell-s(p)}$. □

4. Applications to cubic centers

In this section, we study lower bounds for the local cyclicity of some Darboux cubic centers, in particular those called of codimension 12 in [35], but having a real center. We use the results and procedures described in Sections 2 and 3. Here we fix $n = 3$, the center point is located at the origin and rescaling variables and time if necessary, we consider the perturbed system (1), that writes in complex coordinates as

$$\dot{z} = (\mathrm{i} + \lambda_0) z + p(z, \bar{z}) + \sum_{k+l=2}^{3} (a_{kl} + \mathrm{i}\, b_{kl}) z^k \bar{z}^l. \tag{13}$$

Moreover, the corresponding unperturbed system $\dot{z} = \mathrm{i}\, z + p(z, \bar{z})$ has a cubic Darboux center and we have, in general and among the trace parameter λ_0, 14 real parameters

$$(a_{20}, a_{11}, a_{02}, \ldots, b_{20}, b_{11}, b_{02}, \ldots) \in \mathbb{R}^{14}.$$

In the proofs, we have denoted by $\lambda = (u_1, u_2, \ldots, u_{11}) \in \mathbb{R}^{11}$ the relevant parameters and there are three that have been chosen as zero to simplify the computations. This is because in each center it can be seen that there are three that do not play any role. Using Theorems 2.1 and 2.2, we first study the number of limit cycles appearing with these parameters, and then we add an extra one using the trace λ_0. In each result, we have detailed why, the unperturbed system, is a center and which is the labeled name following the notation in [35]. In some cases, we have described also what was known up to now. Most of the considered centers are 1-parameter families depending on a special parameter a. Following the ideas of Theorem 2.1, see also [12], most probably, the cyclicity results are generic. That is, are valid for almost every value a, but for simplicity, we have chosen only one value for it.

Proposition 4.1. *There exist cubic polynomial perturbations such that from the origin of system*

$$\begin{cases} \dot{x} &= -\dfrac{343}{576}x^3 - \dfrac{7}{72}x^2 - \dfrac{49}{36}xy - 2y, \\ \dot{y} &= \dfrac{343}{72}x^3 - \dfrac{343}{96}x^2y + \dfrac{49}{24}x^2 + \dfrac{259}{36}xy - \dfrac{49}{9}y^2 + x, \end{cases} \tag{14}$$

bifurcate at least 11 *limit cycles of small-amplitude.*

This family is labeled as CD_{11}^{12} in [35] and as $CD45$ in [36]. System (14) has

$$H = -\frac{81}{245}\frac{\left(\frac{5764801}{23887872}x^5 + \frac{588245}{331776}x^4 + \frac{132055}{20736}x^3 + \frac{20335}{2592}x^2 + \frac{1715}{648}xy + \frac{245}{72}x + \frac{35}{9}y + 1\right)^4}{\left(\frac{117649}{331776}x^4 + \frac{2401}{1152}x^3 + \frac{1715}{288}x^2 + \frac{49}{18}x + \frac{28}{9}y + 1\right)^5} \tag{15}$$

as a first integral. In both previous works, Melnikov theory or order 2 and 3, respectively, is used to prove the same statement. The first integral proposed in [35] depends on one parameter, named a, that it is fixed to zero in [36]. For this value the corresponding system has a real saddle equilibrium point at the origin but as the computations done by Żołądek are all in complex he comments that, generically, the result for real cyclicity is also valid. Recently, in [30] this special parameter value is fixed to $a = -3$ where the system has a real center at the point $(3/2, -11/4)$. The proof uses developments of seventh-order of the Lyapunov constants. The first integral (15) is obtained from the one proposed in [30] but moving the equilibrium point and doing an affine change of coordinates that writes the linear part simpler. This new expression reduces computation time. Our proof, which is different, shows also that a Taylor approximation up to degree 7 is the minimum necessary to unfold 11 limit cycles but we have used developments of the Lyapunov constants up to degree 10 for a better understanding of this local bifurcation phenomenon.

Proof of Proposition 4.1. Clearly system (14) has a center at the origin because (15) is a first integral well defined at the origin and the corresponding level curves in a neighborhood of the origin are ovals. In fact $H = -81/245 + x^2 + 2y^2 + \cdots$.

The first step is the computation of the Taylor approximations of the Lyapunov constants, L_k, corresponding to a cubic perturbation of system (14) having only quadratic and cubic terms. We will study which are the principal parts, near the origin in the parameters space, of each Lyapunov constant when the previous vanish.

The second step is the study of linear parts of L_k. Unfortunately, we get only 9 linearly independent. More concretely, the 10th and the 11th are linearly dependent with respect to the first nine. Consequently, Theorem 2.1, adding the trace parameter, provides an unfolding with 9 limit cycles. To obtain the remaining two as it is stated, we need to look at higher-order developments. After a linear change of variables and doing the necessary transformations following the scheme detailed in Section 2, we get

$$\begin{aligned} L_k &= u_k + O_2(\lambda), \text{ for } k = 1,\dots,9,\\ L_{10} &= O_2(\lambda),\\ L_{11} &= O_2(\lambda). \end{aligned}$$

Here $\lambda = (u_1, u_2, \dots, u_{11}) \in \mathbb{R}^{11}$ denotes the relevant parameters, the other are zero, and $O_k(\lambda)$ contains all monomials of degree at least k in λ.

In the third step, we study the higher-order developments, in particular, the ones corresponding to L_{10} and L_{11}. The Implicit Function Theorem ensures that there exists an analytical local change of variables in the parameter space, well defined in a neighborhood of the origin, such that $L_k = v_k$, for $k = 1,\dots,9$. With this change, doing an affine change of variables, and following the scheme detailed in Section 2 we can write the last two Lyapunov constants depending only on two parameters (v_{10}, v_{11}), except positive multiplicative constants, as

$$\begin{aligned} L_{10} &= v_{10}^3 + O_4(v_{10}, v_{11}),\\ L_{11} &= -v_{10}^3 + O_4(v_{10}, v_{11}). \end{aligned}$$

Here, taking $v_{11} = 0$, it is clear that we have a curve, in the parameters space, of weak-foci of order 10, because $L_{10} \neq 0$, that unfolds (using the linearity of $v_1,\dots,v_9$ and the trace parameter) 10 limit cycles. This is also true taking $v_{10} = 0$, but with a higher development, because $L_{10} \neq 0$. In fact, except a positive multiplicative constant, $L_{10} = v_{11}^6 + O_7(v_{11})$. It is evident that, only with a third-order development, we can not unfold the eleventh limit cycle.

The next step is the study, if they exist, of the different real branches near the origin that has the algebraic curve $L_{10} = 0$ and if there exists one such that $L_{11} \neq 0$. This is done computing more terms in the development of L_{10} and L_{11}. We can follow [8] to use the Newton–Puiseux algorithm with different weights for the variables (v_{10}, v_{11}). With weights $(2,1)$ the principal parts are

$$\begin{aligned} L_{10} &= v_{10}^3 + \alpha_{22}v_{10}^2v_{11}^2 + \alpha_{14}v_{10}v_{11}^4 + \alpha_{06}v_{11}^6 + O_7^{(2,1)}(v_{10}, v_{11}),\\ L_{11} &= -(v_{10}^3 + \alpha_{22}v_{10}^2v_{11}^2 + \alpha_{14}v_{10}v_{11}^4 + \alpha_{06}v_{11}^6) + O_7^{(2,1)}(v_{10}, v_{11}), \end{aligned}$$

where $O_m^{(2,1)}$ denotes the monomials with degree higher or equal than m with respect to the weight $(2,1)$. With this weight, it is easy to see the different simple branches of the intersection at the origin. In fact, the principal part of L_{10}, with this specific choice of weights, decomposes as a product of two factors, one simple and one double,

$$L_{10}(v_{10}, v_{11}) = (v_{10} - a_2v_{11}^2)\left(v_{10} - b_2v_{11}^2\right)^2 + O_7^{(2,1)}(v_{10}, v_{11}), \tag{16}$$

with

$$a_2 = -\frac{660160595890746}{37506906889} \text{ and } b_2 = -\frac{487045680336990}{37506906889}.$$

The study of the different branches, $v_{10}^{[j]}(v_{11})$, of the curve $L_{10}(v_{10}, v_{11}) = 0$ near the origin is done using the weighted blow-up $v_{10} = v_{11}^2 w_{10}$. Then the function (16), collecting in v_{11}, can be written as polynomials in v_{11} of degree m with coefficients polynomials in w_{10} of degree $[m/2]$, for $m \geq 6$, where $[\cdot]$ denotes the integer part function. Consequently, dividing by v_{11}^6, (16) writes as

$$\widetilde{L}_{10}(w_{10}, v_{11}) = (w_{10} - a_2)\,(w_{10} - b_2)^2 + \sum_{m\geq 1}^{\infty} W_m(w_{10})v_{11}^m,$$

with W_m polynomials of degree $[(m+6)/2]$. An equivalent expression, $\widetilde{L}_{11}(w_{10},v_{11})$ can be obtained for L_{11} but with different functions $W_m(w_{10})$.

Now, we can write $\widetilde{L}_{10}=\widetilde{w}_{10}$ in a neighborhood of $(w_{10},v_{11})=(a_2,0)$ using the Implicit Function Theorem, because $\widetilde{L}_{10}(a_2,0)=0$ and the partial derivative

$$\left.\frac{\partial L_{10}}{\partial w_{10}}\right|_{(a_2,0)}=(a_2-b_2)^2\neq 0.$$

Clearly, when $\widetilde{w}_{10}=0$, there exists an analytic branch, $w_{10}=\omega^{[1]}(v_{11})=a_2+O_1(v_{11})$, such that $\widetilde{L}_{10}$ and also L_{10} vanish on it. Consequently, $v_{10}=v_{11}^2\omega^{[1]}(v_{11})=a_2v_{11}^2+O_3(v_{11})$. With this change of variables, after multiplying by v_{11}^6, we write (16) as $L_{10}=v_{11}^6\widetilde{w}_{10}$. Additionally, except a positive multiplicative constant, we get

$$L_{11}|_{L_{10}=0}=L_{11}|_{\widetilde{w}_{10}=0}=-v_{11}^7+O_8(v_{11})\neq 0,$$

for $v_{11}\neq 0$ small enough, and $L_{11}=-v_{11}^7(1+O_1(\widetilde{w}_{10},v_{11}))$. We notice that the neighborhood of $(a_2,0)$ for the variables (w_{10},v_{10}) is transformed to a neighborhood of the origin for the variables $(\widetilde{w}_{10},v_{11})$.

This proves that there exists a curve, in the parameters space, for v_{11} small enough, of weak-foci of order 11 that is born at the origin. The cubic perturbation mechanism described in Section 2 following the scheme of Roussarie, see [28], proves that only 11 limit cycles can bifurcate from the origin of system (14). This is because, from the Weierstrass Preparation Theorem, see [31], the Poincaré return map is a polynomial with coefficients the Lyapunov constants, that write, after all the changes of variables, as $L_k=u_k$, for $k=1,\ldots,9$, $L_{10}=v_{11}^6\widetilde{w}_{10}$ and $L_{11}=-v_{11}^7(1+O_1(\widetilde{w}_{10},v_{11}))$. Finally, using the weighted blow-up $\{u_1=z^7z_1,u_2=z^7z_2,\ldots,u_9=z^7z_9,\widetilde{w}_{10}=zz_{10},u_{11}=z\}$, after dividing by z^7, it is clear that we have constructed a versal unfolding, obtaining the maximal number of limit cycles. As v_{11} can vanish this upper bound is in fact a lower bound for the cyclicity of the center as we wanted to prove. □

As we have explained in Section 2, the above proof is a kind of generalization of the result provided by Christopher in [12], see Theorem 2.2. After the simplification due to the linear developments of the first 9 Lyapunov constants, the next two has not the same order at the origin. It is also clear that the transversal straight line ℓ is now an analytic curve.

We notice that, as we have proved that the curve $L_{10}=0$ has a real branch associated to the simple factor, using the Weierstrass Preparation and Division Theorems, [31], we can write (16), except for a nonvanishing multiplicative function, as

$$(v_{10}-v_{11}^2\omega^{[1]}(v_{11}))(v_{10}^2+v_{11}^2\phi_1(v_{11})v_{10}+v_{11}^4\phi_0(v_{11})), \tag{17}$$

where ϕ_0 and ϕ_1 are analytic functions that vanish at zero. As we have computed the Taylor approximations of the Lyapunov constants up to degree 10, using the Puiseux series (see [8]), we can compute four extra terms of this analytic simple branch $v_{10}=v_{11}^2\omega^{[1]}(v_{11})=\sum_{m\geq 2}^{\infty}a_mv_{11}^m$. In this case, the Puiseux series has only natural exponents then it is, in fact, a Taylor series. Straightforward computations show that

$$
\begin{aligned}
a_3 &= \frac{16104570945819692121638226351}{25209283713691672597112320},\\
a_4 &= \frac{386258251571578220793485476718239267056083732367}{123628442038275561958744770186426115741450240},\\
a_5 &= -\frac{7344527305232752838312438300220617202784745335855878366073914837}{10802169913777097568319537676224431667868634439810816147456000},\\
a_6 &= -\frac{1040667719410212727048282608984\cdots 195082403446045341049879843107}{2913611107447792035945464752572\cdots 844814820535324990043586560000}.
\end{aligned}
$$

Using the above coefficients and the developments up to degree 10 of L_{10} we can compute the first terms of the Taylor series of the functions ϕ_0 and ϕ_1 in (17). It can be checked that they provide the double factor appearing in (16) and that the discriminant, with respect to v_{10}, writes as $\phi_1^2(v_{11}) - 4\phi_0(v_{11}) = Av_{11}^8 + \cdots$ with $A > 0$. This proves that there exist another two real branches, tangent to the double factor in (16), $v_{10} = v_{11}^2\omega^{[j]}(v_{11}) = \sum_{m\geq 2}^{\infty} b_m^{[j]} v_{11}^m$, for $j = 2, 3$. In the above proof we have got the first coefficient, that coincides for both branches, $b_2^{[2]} = b_2^{[3]} = b_2$. In fact, the second also coincides,

$$
b_3^{[2]} = b_3^{[3]} = b_3 = -\frac{30834092507446246450289832}{9847376450660809608247},
$$

because the discriminant starts with terms of degree 8. The next coefficients $b_4^{[2]}$ and $b_4^{[3]}$, which are different, are the zeros of the quadratic equation $\beta^2 + \beta_1\beta + \beta_0 = 0$, where

$$
\beta_1 = \frac{374029705710452551852715772722429520}{369344631635866288497876795513583},
$$

$$
\beta_0 = \frac{8566777417085455703175077527180594435196203104777368650266555098833382208}{33421786944967271392447900695765062189771656497217124612800686982805}.
$$

We observe that we have needed approximations up to degree 10 to distinguish the other two real branches. Over them, we have checked that L_{11} vanishes up to this level of approximation.

Proposition 4.2. *There exist cubic polynomial perturbations such that from the origin of system*

$$
\left\{
\begin{array}{rcl}
\dot{x} &=& \dfrac{5}{32}\,x^3 - \dfrac{15}{64}\,x^2y - \dfrac{5}{32}\,xy^2 - y,\\
\dot{y} &=& \dfrac{15}{64}\,x^3 + \dfrac{35}{32}\,x^2y - \dfrac{15}{16}\,xy^2 - \dfrac{15}{32}\,y^3 + x,
\end{array}
\right.
\tag{18}
$$

bifurcate at least 11 *limit cycles of small-amplitude.*

This is the system CD_{12}^{12} in [35]. Christopher studies in [12] the local cyclicity for this family fixing the free parameter to $a = 2$. He also needs developments up to degree two, but with a high computational effort using Grobner Basis. The above system is obtained taking $a = 3/5$, but doing an adequate affine change of coordinates. The computations are simpler because the linear part at the equilibrium, which has been moved to the origin, is in the normal form of an elementary center type point.

Proof of Proposition 4.2. System (18) has a center at the origin because it has the rational first integral

$$
H = -\frac{36}{5}\,\frac{(625x^4 + 1920x^2 + 2560xy + 4096)^3}{(78125x^6 + 360000x^4 + 480000x^3y + 1044480x^2 + 737280xy + 491520y^2 + 786432)^2},
$$

which is well defined at the origin and with Taylor series $-4/5 + x^2 + y^2 + \cdots$.

Computing the Lyapunov constants, following the scheme explained in Section 3 up to degree 1, corresponding to the perturbed system (13), we have that the first five linear terms have rank five with respect to the parameters. Doing a linear change of coordinates we have that they write $L_k^{(1)} = u_k$, for $k = 1, \ldots, 5$. Using the triangularization procedure describe before Proposition 2.3 we can simplify the next Lyapunov constants to get $L_k^{(1)} = 0$, for $k = 6, \ldots, 11$. Consequently, up to first-order only five limit cycles bifurcate from the origin.

The second step is the computation of the second-order terms. Using again Proposition 2.3 to eliminate the parameters u_k, $k = 1, \ldots, 5$, we obtain 6 homogeneous polynomials of degree 2 for the second-order terms of L_k for $k = 6, \ldots, 11$.

$$
\begin{aligned}
L_6^{(2)} = {} & \frac{65234375}{2491416576}a_{11}^2 - \frac{5695234375}{226718908416}a_{02}a_{11} + \frac{187109375}{32388415488}a_{11}a_{20} + \frac{1842265625}{75572969472}a_{11}b_{02} \\
& - \frac{250390625}{5813305344}a_{11}b_{11} + \frac{23046875}{1937768448}a_{11}b_{20} + \frac{390625}{899678208}b_{11}^2 - \frac{1466328125}{75572969472}a_{02}b_{11} \\
& - \frac{250390625}{75572969472}a_{20}b_{11} - \frac{950546875}{75572969472}b_{02}b_{11} + \frac{266796875}{25190989824}b_{11}b_{20} - \frac{141015625}{18893242368}a_{02}^2 \\
& + \frac{21171875}{4359979008}a_{02}a_{20} - \frac{20703125}{25190989824}a_{02}b_{02} + \frac{708828125}{75572969472}a_{02}b_{20} + \frac{390625}{1349517312}a_{20}^2 \\
& - \frac{204921875}{75572969472}a_{20}b_{02} + \frac{23046875}{25190989824}a_{20}b_{20} + \frac{241796875}{75572969472}b_{02}^2 + \frac{47421875}{37786484736}b_{02}b_{20} - \frac{11328125}{3598712832}b_{20}^2, \\
L_7^{(2)} = {} & -\frac{804296875}{53150220288}a_{11}^2 + \frac{62870546875}{4836670046208}a_{02}a_{11} - \frac{229296875}{76772540416}a_{11}a_{20} - \frac{488059296875}{43530030415872}a_{11}b_{02} \\
& + \frac{221214453125}{14510010138624}a_{11}b_{11} - \frac{240866796875}{43530030415872}b_{20}a_{11} - \frac{94140625}{345476431872}b_{11}^2 + \frac{406132109375}{43530030415872}a_{02}b_{11} \\
& + \frac{8782421875}{3348463878144}a_{20}b_{11} + \frac{3529140625}{537407782912}b_{02}b_{11} - \frac{26304296875}{4836670046208}b_{11}b_{20} + \frac{9333203125}{3627502534656}a_{02}^2 \\
& - \frac{1203359375}{329772957696}a_{02}a_{20} + \frac{11367578125}{43530030415872}a_{02}b_{02} - \frac{163425546875}{43530030415872}a_{02}b_{20} - \frac{12109375}{86369107968}a_{20}^2 \\
& + \frac{39773359375}{14510010138624}a_{20}b_{02} - \frac{5510546875}{4836670046208}a_{20}b_{20} + \frac{126171875}{76772540416}b_{20}^2 - \frac{10987578125}{7255005069312}b_{02}b_{20} - \frac{8826953125}{14510010138624}b_{02}^2, \\
L_8^{(2)} = {} & \frac{205818359375}{68341519613952}a_{11}^2 - \frac{27010282421875}{15786891030822912}a_{02}a_{11} + \frac{566685546875}{2255270147260416}a_{11}a_{20} \\
& + \frac{6561098046875}{15786891030822912}a_{11}b_{02} + \frac{49375697265625}{15786891030822912}a_{11}b_{11} + \frac{2022599609375}{15786891030822912}b_{20}a_{11} \\
& + \frac{471892578125}{5920084136558592}b_{11}^2 - \frac{6456421484375}{15786891030822912}a_{02}b_{11} - \frac{4613353515625}{5262297010274304}a_{20}b_{11} \\
& - \frac{47627121484375}{47360673092468736}b_{02}b_{11} + \frac{33662533203125}{47360673092468736}b_{11}b_{20} + \frac{733501953125}{1315574252568576}a_{02}^2 \\
& + \frac{1488928515625}{1315574252568576}a_{02}a_{20} + \frac{3438419921875}{15786891030822912}a_{02}b_{02} - \frac{5852162890625}{15786891030822912}a_{02}b_{20} \\
& - \frac{5248046875}{140954384203776}a_{20}^2 - \frac{121922265625}{103182294319104}a_{20}b_{02} + \frac{30685546875}{53154515255296}a_{20}b_{20} \\
& - \frac{3643357421875}{15786891030822912}b_{20}^2 + \frac{509573828125}{607188116570112}b_{02}b_{20} - \frac{3731908203125}{5262297010274304}b_{02}^2, \\
L_9^{(2)} = {} & \frac{4251095908203125}{2742408499068665856}a_{11}^2 - \frac{1670155486328125}{914136166356221952}a_{02}a_{11} + \frac{120403642578125}{210954499928358912}a_{11}a_{20} \\
& + \frac{1678778216796875}{747929590655090688}a_{11}b_{02} - \frac{43118063095703125}{8227225497205997568}a_{11}b_{11} + \frac{9782838056640625}{8227225497205997568}b_{20}a_{11} \\
& + \frac{144507529296875}{9598429746740330496}b_{11}^2 - \frac{9930148478515625}{5235507134585634816}a_{02}b_{11} - \frac{240089833984375}{2742408499068665856}a_{20}b_{11} \\
& - \frac{757186115234375}{914136166356221952}b_{02}b_{11} + \frac{14849983759765625}{19196859493480660992}b_{11}b_{20} - \frac{377087236328125}{436292261215469568}a_{02}^2 \\
& + \frac{143162263671875}{685602124767166464}a_{02}a_{20} - \frac{1169718330078125}{44300444984955371520}a_{02}b_{02} + \frac{62189059427734375}{57590578480441982976}a_{02}b_{20} \\
& + \frac{2179873046875}{28566755198631936}a_{20}^2 + \frac{9000880859375}{210954499928358912}a_{20}b_{02} - \frac{210343720703125}{2742408499068665856}a_{20}b_{20} \\
& - \frac{1433194228515625}{6398953164493553664}b_{20}^2 - \frac{1701270740234375}{9598429746740330496}b_{02}b_{20} + \frac{1006989326171875}{1745169044861878272}b_{02}^2,
\end{aligned}
$$

$$
\begin{aligned}
L_{10}^{(2)} = & -\frac{16403503182763671875}{11057391068244860731392}a_{11}^2 + \frac{6019489875693359375}{4553043381042001477632}a_{02}a_{11} - \frac{4110241175146484375}{11057391068244860731392}a_{11}a_{20} \\
& - \frac{226410773740234375}{180423630484181876736}a_{11}b_{02} + \frac{1537815485986328125}{781835732098121465856}a_{11}b_{11} - \frac{1853450635595703125}{2866731017693112041472}b_{20}a_{11} \\
& - \frac{543782850048828125}{19350434369428506279936}b_{11}^2 + \frac{27828407758955078125}{25800579159238008373248}a_{02}b_{11} + \frac{719020527197265625}{2866731017693112041472}a_{20}b_{11} \\
& + \frac{50925810357353515625}{77401737477714025119744}b_{02}b_{11} - \frac{43579580815966796875}{77401737477714025119744}b_{11}b_{20} + \frac{1929508851220703125}{6450144789809502093312}a_{02}^2 \\
& - \frac{7638986422255859375}{19350434369428506279936}a_{02}a_{20} + \frac{54847838720703125}{505893709004666830848}a_{02}b_{02} - \frac{103268798154296875}{216811589573428641792}a_{02}b_{20} \\
& - \frac{17619605615234375}{460724627843535863808}a_{20}^2 + \frac{6869360360732421875}{25800579159238008373248}a_{20}b_{02} - \frac{279467075634765625}{2866731017693112041472}a_{20}b_{20} \\
& + \frac{4340212115087890625}{25800579159238008373248}b_{20}^2 - \frac{15956755576171875}{159262834316284002304}b_{02}b_{20} - \frac{3536245476904296875}{25800579159238008373248}b_{02}^2, \\
L_{11}^{(2)} = & \frac{42718448752197265625}{95184091534832953196544}a_{11}^2 - \frac{171788232009716796875}{602832579720608703578112}a_{02}a_{11} + \frac{6210508852783203125}{95184091534832953196544}a_{11}a_{20} \\
& + \frac{5259382308953564453125}{37978452522398348325421056}a_{11}b_{02} + \frac{2996301505219970703125}{12659484174132782775140352}a_{11}b_{11} + \frac{2217310253376220703125}{37978452522398348325421056}b_{20}a_{11} \\
& + \frac{79495114489990234375}{6329742087066391387570176}b_{11}^2 - \frac{4526167344325244140625}{37978452522398348325421056}a_{02}b_{11} - \frac{1501926097403076171875}{12659484174132782775140352}a_{20}b_{11} \\
& - \frac{2113627250105517578125}{12659484174132782775140352}b_{02}b_{11} + \frac{8402376020751953125}{67697776332260870455296}b_{11}b_{20} + \frac{170019216537353515625}{3164871043533195693785088}a_{02}^2 \\
& + \frac{77972993629345703125}{452124434790456527683584}a_{02}a_{20} + \frac{378160348073486328125}{37978452522398348325421056}a_{02}b_{02} - \frac{344733712266748046875}{37978452522398348325421056}a_{02}b_{20} \\
& + \frac{58214882568359375}{32294602485032609120256}a_{20}^2 - \frac{2056445968159619140625}{12659484174132782775140352}a_{20}b_{02} + \frac{56233442866943359375}{744675539654869575008256}a_{20}b_{20} \\
& - \frac{371336987060546875}{9568771106676328628224}b_{20}^2 + \frac{666554572872607421875}{6329742087066391387570176}b_{02}b_{20} - \frac{930355010247314453125}{12659484174132782775140352}b_{02}^2.
\end{aligned}
$$

We consider now the system $\{L_6^{(2)} = \cdots = L_{10}^{(2)} = 0\}$. Doing, for example, the blow-up $a_{02} = zv_1, a_{11} = zv_2, a_{20} = zv_3, b_{02} = z, b_{11} = zv_4, b_{20} = zv_5$, we can solve a system of five equations of degree 2 with respect to 5 variables. Using a computer algebra system we get that $v_k = p_k(\alpha)/q(\alpha)$ with p_k and q polynomials with rational coefficients of degree 27 and α being a solution of a given polynomial, $Q(\alpha)$, also with rational coefficients, of degree 28. This polynomial has 20 simple real solutions,

$$
\begin{aligned}
\{ & -1.460571830, -0.6718444255, -0.6670390163, -0.5158935998, -0.4874999611, \\
& -0.3970369469, -0.3874233159, -0.2401990480, -0.02992848475, 0.02384205186, \\
& 0.03979267840, 0.08015376288, 0.2087598950, 0.2131172755, 0.2232320471, \\
& 0.2463926997, 0.2995004189, 0.3312032992, 0.3788127882, 1.397031032\}.
\end{aligned}
$$

The next step is to check that the Jacobian of the five equations with respect to $\{v_1, \ldots, v_5\}$ is different from zero. The size of the polynomials does not make possible to compute directly the determinant of the Jacobian matrix in terms of α. Then, we do a Gauss elimination to get a triangular matrix. The diagonal elements are now of the form $J_{11} = q_1(\alpha)/q(\alpha)$ and $J_{kk} = q_k(\alpha)/(q_{k-1}(\alpha)q(\alpha))$, for $k = 2, \ldots, 4$, where all polynomials have also rational coefficients and degree 27 in α, in particular $q_1 = p_1$. Consequently, the determinant is $q_5(\alpha)/q^5(\alpha)$. Moreover, it can be checked also that $L_{11}^{(2)} = p_6(\alpha)/q^2(\alpha)$. The last step is the computation of the resultants of all polynomials p_k, q_k with Q with respect to α, checking that all are different from zero. So all the variables v_k, among of the determinant and the value of $L_{11}^{(2)}$ are nonzero real numbers.

It is clear that with the proposed blow-up, after dividing by z^2 we can apply the Implicit Function Theorem to find an analytic curve $v_k = V_k(z)$ where the varieties $\{L_6^{(2)}, \ldots, L_{10}^{(2)}\}$ intersect transversally and $L_{11}^{(2)}$ is nonzero. Then 6 extra limit cycles appear and the statement follows.

The scheme of the proof is the same as the proof of Proposition 4.1. But, in fact, we could apply Theorem 2.2. But the work to find the line ℓ is the same.

We notice that all the computations in this proof have been made with `Maple` on a personal computer in a few minutes. We have not shown the polynomials because of the size of them. The coefficients are rational numbers with numerators and denominators having more than 200 digits. □

Remark 4.3. *An alternative proof of the above result can be done computing numerically an approximation of a solution*

$$\begin{aligned} v_1 &= 0.1924453833548429, \quad v_2 = 0.2384205185830677, \\ v_3 &= 0.1490077870024313, \quad v_4 = 0.7626068770346651, \\ v_5 &= 1.5437258992144801, \end{aligned}$$

with enough digits to ensure that $L_6^{(2)}/z^2 \approx 6.80504529555082 \cdot 10^{-9}$ *and the Jacobian,* $-4.45823335837756 \cdot 10^{-10}$, *are nonzero real numbers. This can be obtained using a computer assisted proof with the Poincaré–Miranda Theorem, as we will do in some of the following results.*

The next proposition shows the difficulties to get more than 10 limit cycles. Our computations do not provide a better result for the cyclicity of the next cubic polynomial system. The unperturbed system is labeled as CD_{29}^{12} in [35] but we have not considered it directly. To simplify the computations, we have made an affine change of coordinates.

Proposition 4.4. *There exist cubic polynomial perturbations such that the origin of system*

$$\begin{cases} \dot{x} &= 8x^3 - 40x^2y + 2x^2 - 30xy - 5y, \\ \dot{y} &= \dfrac{24}{5}x^3 + 24x^2y - 80xy^2 + 4x^2 + 10xy - 10y^2 + x, \end{cases} \tag{19}$$

has cyclicity at least 10.

Proof. System (19) has a center at the origin because it has the rational first integral

$$\frac{(64x^3 + 72x^2 + 120xy + 30x + 30y + 5)^4}{(128x^4 + 192x^3 + 320x^2y + 128x^2 + 240xy + 40x + 40y + 5)^3},$$

which is well defined at the origin and the level curves are topologically circumferences.

Doing a first-order analysis of the Lyapunov constants, only the first 6 are linearly independent. Then, as the previous studies, we can write $L_k^{(1)} = u_k$, for $k = 1, \ldots, 6$. Applying the simplification algorithm described previously, using Proposition 2.3, we get

$$\begin{aligned} L_7^{(2)} &= A_7 u_7 u_8, \\ L_8^{(3)} &= A_8 u_7 u_9 u_{10}, \\ L_9^{(3)} &= u_7 u_9 (A_9 u_7 + B_9 u_{10}), \\ L_{10}^{(3)} &= u_7 u_9 (A_{10} u_7 + B_{10} u_{10}), \\ L_{11}^{(3)} &= u_7 u_9 (A_{10} u_7 + B_{10} u_{10}), \end{aligned}$$

where A_k, B_k are nonvanishing rational numbers. We notice that $L_{11}^{(3)} = L_{10}^{(3)}$. To study how is the local intersection of the varieties L_k, for $k = 7, \ldots, 11$ we need to do an adequate weighted blow-up using a privileged parameter. Here we have chosen $u_7 = z$, $u_8 = z^2 z_8$, $u_9 = z z_9$, $u_{10} = z z_{10}$, $u_{11} = z z_{11}$, the other three parameters are taken as zero. The Taylor development with respect to z, dividing by a nonzero rational number, is

$$
\begin{aligned}
L_7 &= z^3(z_8 + p_2(z_9, z_{10})) + \sum_{j\geq 4} W_{7j}(z_8, z_9, z_{10}, z_{11}) z^j, \\
L_8 &= z^3 z_9 z_{10} + \sum_{j\geq 4} W_{8j}(z_8, z_9, z_{10}, z_{11}) z^j, \\
L_9 &= z^3 z_9 p_1(z_9) + \sum_{j\geq 4} W_{9j}(z_8, z_9, z_{10}, z_{11}) z^j, \\
L_{10} &= z^3 z_9^2 + \sum_{j\geq 4} W_{10j}(z_8, z_9, z_{10}, z_{11}) z^j,
\end{aligned}
$$

where p_1 and p_2 are polynomials of degree 1 and 2, respectively, $p_1(0) \neq 0$, and $W_{k,j}$ are also polynomials.

The statement follows as in the previous studies because there exists a transversal intersection of the varieties $\{z_8 + p_2(z_9, z_{10}) = 0, z_9 z_{10} = 0, z_9 p_1(z_9) = 0\}$ with $z_9 \neq 0$. Then, clearly, L_{10} is nonvanishing for z small enough. $\square$

Remark 4.5. *We remark that we have tried to improve the above result unsuccessfully. We have used different weighted blow-ups, higher-order developments, and the study of the Newton polyhedron.*

Proposition 4.6. *There exist cubic polynomial perturbations such that from the origin of system*

$$
\left\{
\begin{array}{rcl}
\dot{x} &=& -\dfrac{338}{441}x^3 + \dfrac{4394}{9261}x^2 y - \dfrac{338}{147}xy^2 + \dfrac{338}{147}y^3 - \dfrac{26}{21}x^2 - \dfrac{793}{441}xy + \dfrac{65}{21}y^2 + y, \\
\dot{y} &=& \dfrac{338}{3087}x^3 - \dfrac{338}{343}x^2 y - \dfrac{338}{1323}xy^2 - \dfrac{338}{1029}y^3 - \dfrac{65}{147}x^2 - \dfrac{247}{147}xy - \dfrac{26}{147}y^2 - x,
\end{array}
\right. \tag{20}
$$

bifurcate at least 11 *limit cycles of small-amplitude.*

Proof. System (20) has a center at the origin because it has the rational first integral

$$
H = \frac{324135}{86528} \frac{\left(\frac{2704\,x^2}{3087} - \frac{5408\,xy}{3087} + \frac{2704\,y^2}{1323} - \frac{208\,x}{147} + \frac{416\,y}{147} + 1\right)\left(\frac{104\,x}{441} + \frac{104\,y}{147} + 1\right)^3}{\left(-\frac{2704\,x^2}{15435} + \frac{2704\,y^2}{1715} - \frac{52\,x}{147} + \frac{52\,y}{21} + 1\right)^2}
$$

which is well defined at the origin, $H = 324135/86528 + x^2 + y^2 + \cdots$.

We consider the perturbed system (13) being (20) the unperturbed center. Following the scheme explained in Section 3, the developments up to degree 1 of the first 8 Lyapunov constants are linearly independent. Then, after a linear change of the perturbation parameters, we have that they write as $L_k^{(1)} = u_k$, for $k = 1, \ldots, 8$. Using the properties detailed in Proposition 2.3 we can simplify the next Lyapunov constants to get $L_k^{(1)} = 0$, for $k = 9, \ldots, 11$. At this point we assume, to simplify computations, that $b_{03} = b_{12} = b_{30} = 0$. Then, we can write

$$
\begin{aligned}
L_9^{(2)} &= -A u_9 u_{10}, \\
L_{10}^{(2)} &= B u_9 u_{10}, \\
L_{11}^{(2)} &= -C u_9 u_{10},
\end{aligned}
$$

where A, B, and C are rational numbers having between 48 and 71 digits in the numerators and between 68 and 86 digits in the denominators. So, we see clearly here that we have at least 9 limit cycles using the trace parameter together with u_k, for $k = 1, \ldots, 8$. In fact, at most 9 with only order 2 developments.

Hence, if we want more limit cycles, we should compute up to degree 4 developments. Because, after using again the simplification with L_9 and Proposition 2.3, $L_{10}^{(3)} = L_{11}^{(3)} = 0$.

Doing the blow-up $u_9 = z, u_{10} = zz_{10}$, and $u_{11} = zz_{11}$, we can divide L_9 by z^2 and we can use the Implicit Function Theorem to write z_{10} as a function of z and z_{11}. Then, $L_{10}^{(4)} = z^4 p_4(z_{11})$ and $L_{11}^{(4)} = z^4 q_4(z_{11})$ where p_4 and q_4 are polynomial with rational coefficients of degree 4 having both 4 different real roots. Moreover, z^4 is a common factor of the complete L_{10} and L_{11} and the resultant of p_4 and q_4 with respect to z_{11} is different from zero. Therefore, applying also the Implicit Function Theorem near the simple zeros of p_4, there exists values of the parameters, for small enough z, such that $L_{10} = 0$ and $L_{11} \neq 0$.

The statement follows because we have proved the existence of an analytic curve of weak-foci of order 11 such that, along it and as in the previous proofs, 11 limit cycles of small-amplitude bifurcate from the origin. $\square$

Proposition 4.7. *There exist cubic polynomial perturbations such that from the origin of system*

$$\begin{cases} \dot{x} &= \frac{2809}{32946}x^3 - \frac{22472}{49419}yx^2 + \frac{5618}{16473}xy^2 + \frac{69695}{98838}x^2 - \frac{66992}{16473}xy + \frac{636}{289}y^2 + x - \frac{2112}{289}y, \\ \dot{y} &= \frac{22472}{247095}xy^2 - \frac{2809}{16473}y^3 + \frac{86549}{197676}xy - \frac{20988}{27455}y^2 + \frac{151447}{247095}x - y, \end{cases} \tag{21}$$

bifurcate at least 11 *limit cycles of small-amplitude.*

This is the case labeled CD_{31}^{12} in [35]. Christopher in [12] provides with it the first analytic proof that 11 limit cycles of small-amplitude exist for a cubic polynomial vector field. Here we add it for completeness.

Proof of Proposition 4.7. The corresponding first integral of (21), which is well defined at the origin, is

$$H = \frac{\left(xy^2 + \dfrac{280\,xy}{53} + \dfrac{342\,y^2}{53} + \dfrac{22409\,x}{2809} + \dfrac{95760\,y}{2809} + \dfrac{7812755}{148877}\right)^5}{\left(x + \dfrac{342}{53}\right)^3 F_6^2(x,y)},$$

where

$$\begin{aligned} F_6(x,y) = xy^5 &+ \frac{700}{53}xy^4 + \frac{342}{53}y^5 + \frac{406045}{5618}xy^3 + \frac{239400}{2809}y^4 + \frac{30389450}{148877}xy^2 \\ &+ \frac{139611775}{297754}y^3 + \frac{18788141215}{63123848}xy + \frac{10549512750}{7890481}y^2 \\ &+ \frac{150246782525}{836390986}x + \frac{826646189040}{418195493}y + \frac{26977377387858}{22164361129}. \end{aligned}$$

The proof of the above proposition follows computing the linear terms of the Lyapunov constants and then using Theorem 2.1 to provide the complete unfolding of 11 limit cycles. $\square$

5. Order one studies to get lower bounds for $M(8)$ and $M(9)$

This section is devoted to proving the statement of Theorem 1.1 corresponding to local cyclicity of polynomial vector fields of degrees 8 and 9, using only linear developments. The proofs follow from Theorem 2.1 just computing the Taylor approximations up to degree

1 and computing how many are linearly independent. In both results, the unperturbed systems are centers of degrees 7 and 8 having a straight line of equilibrium points, that, for simplicity, we have fixed to $\{1 - x - y = 0\}$.

We notice that the parallelization procedure described in Section 3 is indispensable to get the results. The total computation time, in both cases, is less than one hour.

Proposition 5.1. *Consider the perturbed system* (4) *of degree* $n = 8$ *with the center* $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$ *given by*

$$\begin{cases} \dot{x} &= (1-x-y)(-\frac{2527}{3}x^6y - \frac{2968}{3}x^5y^2 - \frac{4186}{3}x^4y^3 - \frac{2800}{3}x^3y^4 - 553x^2y^5 \\ & \quad +56xy^6 + \frac{184}{3}x^3y + \frac{88}{3}x^2y^2 + 48xy^3 - y), \\ \dot{y} &= (1-x-y)(672x^7 + 1484x^6y + \frac{2219}{3}x^5y^2 + \frac{5684}{3}x^4y^3 - \frac{742}{3}x^3y^4 + \frac{1148}{3}x^2y^5 \\ & \quad -315y^6 - 28y^7 - 58x^4 - 44x^3y - \frac{104}{3}x^2y^2 - \frac{44}{3}xy^3 + 10y^4 + x). \end{cases}$$

There are perturbation parameters λ *such that at least* 76 *limit cycles of small-amplitude bifurcate from the origin.*

The above system, without the straight line of equilibrium points, is a center because it has the first integral $H(x(x^2 + y^2), y(x^2 + y^2))$ where

$$H(x, y) = \frac{(42x - 7y - 1)^3 f_3(x, y)}{(448x^2 + 336xy + 63y^2 - 44x - 12y + 1)^3 (1183x^2 - 68x + 1)} \tag{22}$$

and $f_3(x, y) = 10752x^3 + 29568x^2y + 17640xy^2 + 3024y^3 - 1600x^2 - 2760xy - 576y^2 + 74x + 57y - 1$. The rational first integral (22) corresponds to the cubic polynomial center provided by Bondar and Sadovskiĭ in [7]. They prove that the cubic perturbations provide also 11 limit cycles using only up to degree 1 Taylor approximations as for system (21). Although we have also verified it, we have not included this result here.

Proposition 5.2. *Consider the perturbed system* (4) *of degree* $n = 9$ *with the center* $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$ *given by*

$$\begin{cases} \dot{x} &= (1-x-y)(\frac{54}{175}x^8 + \frac{18}{35}x^7y - \frac{54}{175}x^6y^2 + \frac{894}{175}x^5y^3 - 2x^4y^4 + \frac{66}{25}x^3y^5 \\ & \quad -\frac{26}{35}x^2y^6 - \frac{342}{175}xy^7 + \frac{16}{25}y^8 - y), \\ \dot{y} &= (1-x-y)(-\frac{198}{175}x^7y - \frac{1254}{175}x^6y^2 - \frac{586}{175}x^5y^3 - \frac{258}{35}x^4y^4 - \frac{22}{5}x^3y^5 \\ & \quad +\frac{18}{25}x^2y^6 - \frac{382}{175}xy^7 + \frac{162}{175}y^8 + x). \end{cases}$$

There are perturbation parameters λ *such that at least* 88 *limit cycles of small-amplitude bifurcate from the origin.*

The proof that the above system, without the straight line of equilibrium points, is a center follows from an idea of Giné in [19]. We consider the center with homogeneous quartic nonlinearities given in [19, System (6), Pag. 8857] taking $c = 4/5$ and $s = 3/5$. The change of variables $(x, y) = r^{3/7}(\cos\theta, \sin\theta)$ in such a quartic system gets a system of degree 8 having also a center at the origin.

We notice that for other degrees, $n = 3, \ldots, 7$, adding a straight line of equilibria to a center of degree $n - 1$, we have not obtained higher lower bounds for the local cyclicity than the ones obtained previously, nor better than the ones given in the results of the next section using higher-order Taylor series. For example, the best cubic system of Section 4 adding such curve provides a quartic system with only 19 limit cycles up to first and second-order studies. For degree 6, we have not found any system to improve the highest value found in [4], $M(6) \geq 48$, which improves in 8 the best known value for $M(6)$ in [25]. We remark that the number of total parameters for $n = 6$ is 50.

6. Higher order studies to get lower bounds for $M(4)$, $M(5)$, and $M(7)$

This section is devoted to proving the statement of Theorem 1.1 for the local cyclicity of polynomial vector fields of degrees 4, 5, and 7. In the proofs, we will use higher-order Taylor developments.

Proposition 6.1. *Consider the perturbed system* (4) *of degree* $n = 4$ *with the center* $(\dot{x}, \dot{y}) = (P_c(x,y), Q_c(x,y))$,

$$\begin{cases} \dot{x} &= \dfrac{1}{25}(32x^4 - 168x^2y^2 + 32xy^3 + 24y^4) - y, \\ \dot{y} &= \dfrac{1}{25}(-32x^3y - 192x^2y^2 - 24xy^3 + 64y^4) + x. \end{cases} \tag{23}$$

There are perturbation parameters λ *such that at least* 20 *limit cycles of small-amplitude bifurcate from the origin.*

Proof. The system in the statement is presented in [19, 20]. In [19] it is proved that the origin is a center because it has the polynomial inverse integrating factor

$$\begin{aligned} V(x,y) =& (8xy - 4y^2 + 10y - 5)(64x^3 - 96x^2y + 48xy^2 - 8y^3 - 25)\times \\ & (64x^2y^2 - 64xy^3 + 16y^4 - 80xy^2 + 40y^3 + 40xy + 80y^2 + 50y + 25) \end{aligned}$$

We need to compute the Taylor developments up to degree 4 of the Lyapunov constants. Because we will see that up to degree 3 is not enough to prove the statement. As we have detailed in the proofs of Propositions 4.1 and 4.2, simplifying as we have described in Section 2 and up to multiplicative nonzero constants, we have the next triangularized expressions up to degree two:

$$\begin{aligned} &L_k = u_k + O_2(\lambda), \text{ for } k = 1, \ldots, 16, && L_{19} = O_3(\lambda), \\ &L_{17} = -u_{17}u_{19} + u_{18}^2 + O_3(\lambda), && L_{20} = O_3(\lambda), \\ &L_{18} = u_{17}u_{18} + O_3(\lambda), && L_{21} = u_{20}^2 + O_3(\lambda). \end{aligned} \tag{24}$$

Here λ denotes the perturbation parameters and $O_k(\lambda)$ are the monomials of degree at least k in λ. Clearly, using only up to degree two of Taylor approximations there are two Lyapunov constants that vanish. So, higher degrees must be computed.

Using the Implicit Function Theorem and after using adequately the expressions of $L_1, \ldots, L_{16}$, we can consider only L_k, for $k = 17, \ldots, 21$ depending on $(u_{17}, \ldots, u_{24})$.

With the Taylor approximation up to degree three, an adequate blow-up is

$$\begin{aligned} &u_{17} = z, && u_{18} = z^2w_1, && u_{19} = z^2w_2, && u_{20} = zw_3, \\ &u_{21} = zw_4, && u_{22} = zw_5, && u_{23} = zw_6, && u_{24} = zw_7. \end{aligned}$$

Then, after dividing by z^3, we have

$$\begin{aligned} \widetilde{L}_{17}^{(3)} &= w_2 + \frac{1}{3402000}w_3^2 + \frac{79}{4838400}w_6^2 + \frac{1}{241920}w_3w_6, \\ \widetilde{L}_{18}^{(3)} &= w_1 - \frac{387}{39040}w_6w_7 + \frac{9}{2440}w_4w_6 + \frac{1}{1220}w_5w_6 - \frac{1}{1600}w_3w_7 \\ &\quad - \frac{227971304366744606815875047421098088169074843984539306177 69}{22599395017588000741825603692446552868781675590536155251513600}w_6 \\ &\quad - \frac{99996445400250450505317073169188260741337096265876 11529}{217079229805212609584646551861359050353307590841471778158750}w_3, \\ \widetilde{L}_{19}^{(3)} &= \frac{54197\cdots00000}{866211\cdots28447}w_3 + \frac{362945\cdots0000}{28873\cdots76149}w_7, \\ \widetilde{L}_{20}^{(3)} &= 0. \end{aligned}$$

From the above expressions is clear that only up to degree three is not enough to prove the statement. But, as the above nonvanishing three terms have rank 3 with respect to w_1, w_2, and w_3, we can use also the Implicit Function Theorem to solve $\widetilde{L}_k^{(3)} = z_k$, with respect to w_1, w_2, w_3 and compute the Taylor approximation up to degree four we get

$$L_{17}^{(3)} = z^3 z_{17}, \quad L_{18}^{(3)} = z^3 z_{18}, \quad L_{19}^{(3)} = z^3 z_{19}, \quad L_{20}^{(4)} = z^4 w_6^2.$$

From which it follows that a curve of weak-foci of order 20 exists. Moreover, it unfolds 20 limit cycles of small-amplitude at it is indicated in the statement. The proof finishes as the proofs of Propositions 4.1 and 4.2. □

Remark 6.2. *In* [19]*, it is studied the homogeneous nonlinearities perturbation of degree* 4 *of* (23)*. It is proved, with approximations up to degree two, that* 7 *limit cycles exist in this special subclass. This proves that there exists a curve of weak-foci of order* 21*. This can be seen intuited from* (24) *but it is necessary to prove, using only homogeneous degree four perturbation terms, that L_{21} is nonvanishing. But it is not possible to find a complete unfolding of the* 21 *limit cycles with degree two because there are no free parameters, in fact $L_{19}^{(2)} = L_{20}^{(2)} = 0$. In the proof, we have shown that with degree three we get only* 19 *limit cycles and with degree four we get* 20*. The problem about the existence of a complete unfolding of* 21 *remains open. The computations to go further in the order are very hard.*

Proposition 6.3. *Consider the perturbed system* (4) *of degree $n = 5$ with the center $(\dot{x}, \dot{y}) = (P_c(x, y), Q_c(x, y))$,*

$$\begin{cases} \dot{x} &= \dfrac{1}{25}(42x^5 - 12x^4y - 476x^3y^2 - 68x^2y^3 + 266xy^4 + 56y^5) - y, \\ \dot{y} &= \dfrac{1}{25}(-8x^5 - 26x^4y - 28x^3y^2 - 4x^2y^3 - 132xy^4 + 6y^5) + x. \end{cases} \tag{25}$$

There are perturbation parameters λ such that at least 33 *limit cycles of small-amplitude bifurcate from the origin.*

Proof. As the previous result, system (25) appears also in [19, 20]. The system has a center because

$$\begin{aligned} V(x, y) =&(64x^8 + 1600x^7y + 13456x^6y^2 + 32000x^5y^3 - 99616x^4y^4 - 380800x^3y^5 \\ &+ 320656x^2y^6 + 548800xy^7 + 153664y^8 + 20000x^4 + 85000x^3y \\ &- 10000x^2y^2 - 155000xy^3 - 100000y^4 + 15625)\times \\ &(-64x^4 + 192x^3y - 16x^2y^2 - 192xy^3 - 64y^4 + 25)^{\frac{1}{4}} \end{aligned}$$

is an inverse integrating factor.

Computing the Lyapunov constants up to degree 2, we can check that the first 17 linear parts are linearly independent. Then, up to a linear change of coordinates in the parameter space we can write $L_k^{(1)} = u_k$, for $k = 1, \ldots, 17$, and $L_k^{(1)} = 0$, for $k = 18, \ldots, 33$. Consequently, using the same scheme as in the proof of Proposition 4.2, simplifying also with triangularization scheme detailed after Proposition 2.3, and using the Implicit Function Theorem we can restrict our study to see the intersection of 16 homogeneous polynomials of degree 2,

$$L_k^{(2)} = \mathcal{L}_k(\hat{\lambda}), \text{ for } k = 18, \ldots, 33, \tag{26}$$

with $\hat{\lambda} = (u_{18}, u_{19}, \ldots, u_{33})$ choosing three perturbation parameters as zero. We recall that for degree 5 perturbations we have 36 parameters, but here only 33 will be relevant. There are three that they do not play any role.

The next step is to consider the blow-up $u_k = zz_k$, for $k = 18, \ldots, 32$ and $u_{33} = z$ in (26), writing $\mathcal{L}_k(\hat{\lambda}) = z^2 \mathcal{L}_k(\hat{z})$ with $\hat{z} = (z_{18}, z_{19}, \ldots, z_{32})$. Then, we need to show that this system of 15 equations of degree 2 with respect to 15 variables has at least a transversal intersection real point, $\hat{z}^*$. Moreover, we should check that $\mathcal{L}_{33}(\hat{z}^*)$ is nonvanishing. The proof finishes applying Theorem 2.2 to provide the complete unfolding of 33 limit cycles of small-amplitude. The main difference with respect to the proof of Proposition 4.2 is that here we can not obtain the explicit solution in terms of polynomials in one privileged variable. Because of the high number of variables and the size of the coefficients of the polynomials $\mathcal{L}_k$.

Numerically, we can get an approximate solution

$$
\begin{array}{lll}
z_{18}^* \approx 0.414467055443, & z_{19}^* \approx 0.977703106281, & z_{20}^* \approx 0.831273897080, \\
z_{21}^* \approx 10.87232453671, & z_{22}^* \approx 0.089114602089, & z_{23}^* \approx 5.803007782422, \\
z_{24}^* \approx -13.46886905316, & z_{25}^* \approx -2.653100632593, & z_{26}^* \approx 0.071920750628, \\
z_{27}^* \approx 1.279070836650, & z_{28}^* \approx -0.963042490919, & z_{29}^* \approx -7.708796674748, \\
z_{30}^* \approx -0.27853535522, & z_{31}^* \approx -7.245147157590, & z_{32}^* \approx 2.513953010283.
\end{array}
\tag{27}
$$

Then, $\mathcal{L}_{33}(\hat{z}^*) = -5.28936073528$ and the Jacobian matrix of $(\mathcal{L}_{18}, \ldots, \mathcal{L}_{32})$ with respect to $\hat{z}$ at $\hat{z}^*$ is $1.2572040284 \cdot 10^{14}$. We have solved numerically with different number of digits (up to 1000 digits) to ensure the convergence of the numerical solution. The above numerical approximation is shown with only 12 digits.

Then we can use the Poincaré–Miranda Theorem, to prove analytically the existence of the point $\hat{z}^*$. This will be doing with an interval analysis for applying Theorem 2.4. Finally, we need to check that the Jacobian and $\mathcal{L}_{33}$ are nonvanishing at $\hat{z}^*$, also with another accurate interval analysis. This can be done with a *Computer Assisted Proof* mechanism using the technical lemmas of Section 8.

The first step is to convert the approximate solution (27) to the adequate rational expression:

$$
\begin{array}{ll}
z_{18}^* \approx \dfrac{8289341108806467679}{20000000000000000000}, & z_{19}^* \approx \dfrac{97770310628309363091}{100000000000000000000}, \\
z_{20}^* \approx \dfrac{83127389707517527887}{100000000000000000000}, & z_{21}^* \approx \dfrac{10872324536703550641}{1000000000000000000}, \\
z_{22}^* \approx \dfrac{22278650522188043761}{250000000000000000000}, & z_{23}^* \approx \dfrac{14507519456029022899}{2500000000000000000}, \\
z_{24}^* \approx -\dfrac{6734434526579186771}{500000000000000000}, & z_{25}^* \approx -\dfrac{26531006325905618217}{10000000000000000000}, \\
z_{26}^* \approx \dfrac{35960375313628249487}{500000000000000000000}, & z_{27}^* \approx \dfrac{12790708366491310147}{10000000000000000000}, \\
z_{28}^* \approx -\dfrac{240760622728448541}{250000000000000000}, & z_{29}^* \approx -\dfrac{38543983373719519357}{5000000000000000000}, \\
z_{30}^* \approx -\dfrac{1740845970112931499}{6250000000000000000}, & z_{31}^* \approx -\dfrac{14490294315170358141}{2000000000000000000}, \\
z_{32}^* \approx \dfrac{25139530102793502293}{10000000000000000000}. &
\end{array}
$$

Then, we consider an affine change of parameters such that the linear part of $f = (\mathcal{L}_{18}, \ldots, \mathcal{L}_{32})$ will be the new variables. Then, the Jacobian matrix at this point will be near the identity. Next, we apply Theorem 2.4 with $n = 15$, $c = 10^{-9}$ to $f = (\mathcal{L}_{18}, \ldots, \mathcal{L}_{32})$. The conditions about the sign of the components of f on the faces $S_i^{\pm}$ are obtained from Lemmas 8.1 and 8.2. The existence of $\hat{z}^*$ is guaranteed because

$f_i(S_i^-) \subset [-2.01 \cdot 10^{-9}, -1.99 \cdot 10^{-9}]$ and $f_i(S_i^-) \subset [1.99 \cdot 10^{-9}, 2.01 \cdot 10^{-9}]$. We notice that the numerator and denominators of $\mathcal{L}_i$, for $i = 18, \ldots, 32$ are integer numbers having between 138 to 152 digits each. Moreover, $\mathcal{L}_{33} \in [-5.482536, -5.0966185]$ and its numerator and denominator have more than 120 digits each, so we have $\mathcal{L}_{33} < 0$. Finally, we must show that the determinant of the Jacobian matrix, $Jf(\hat{z})$, of f with respect to $\hat{z}$ does not vanish at $\hat{z}^*$. This determinant, as it is a 15×15 matrix, needs a very high computational cost. Alternatively, we can use the Gershgorin result, Theorem 2.5, to show that its eigenvalues are in a ball centered at 1 with radius 10^{-3}. That is $|\lambda - 1| < 10^{-3}$. Calling $J_{i,j}$ the i,j-element of $Jf(\hat{z}^*)$, and using again Lemmas 8.1 and 8.2, we can check that $J_{i,i} \in [0.999919, 1.000009]$ and, for $i \neq j$, $J_{i,j} \in [0.000018, 0.00212]$ or $J_{i,j} \in [-0.0002926, -5.7414588 \times 10^{-7}]$ when $J_{i,j}$ are positive or negative, respectively. Therefore, all eigenvalues are nonzero and, consequently, also the determinant of $Jf(\hat{z}^*)$. Then, the result follows. □

Proposition 6.4. *Consider the perturbed system* (4) *of degree $n = 7$ with the center $(\dot{x}, \dot{y}) = (P_c(x,y), Q_c(x,y))$,*

$$\begin{cases} \dot{x} &= (1-x)\Big(-y + \dfrac{8}{45}(2x-y)(24x^5 - 12x^4y - 32x^3y^2 + 12x^2y^3 - 42xy^4 - 5y^5)\Big), \\ \dot{y} &= (1-x)\Big(2x - \dfrac{16}{45}y(2x-y)(28x^4 + 66x^3y + 6x^2y^2 + 19xy^3 + 6y^4)\Big). \end{cases} \tag{28}$$

There are perturbation parameters λ such that at least 61 *limit cycles of small-amplitude bifurcate from the origin.*

Proof of Proposition 6.4. A similar system as the unperturbed one, without the straight line of equilibria $1 - x = 0$, appears in [19] in the study of local cyclicity for homogeneous nonlinearities perturbation. Giné proposes to start with a quartic system as (23) having a center at the origin because it has an integrating factor. Then, we should change to polar coordinates and transform variable r to a new radial variable $R^{3/5}$. With these changes the new system has degree 6 and we get system (28) after adding a straight line of equilibria. So, it has a center at the origin. Moreover, it has the next inverse integrating factor

$$\begin{aligned} V(x,y) =& (2x^2+y^2)^{-2/3}(128x^5 - 192x^4y + 160x^3y^2 - 112x^2y^3 + 48xy^4 - 8y^5 - 9)\times \\ & (2048x^7y^3 - 3072x^6y^4 + 3584x^5y^5 - 3328x^4y^6 + 2048x^3y^7 - 1024x^2y^8 \\ & + 384xy^9 - 64y^{10} + 864x^3y^2 + 432xy^4 - 27). \end{aligned}$$

Using only linear developments of the Lyapunov constants, see Theorem 2.1, we get only 58 limit cycles of small-amplitude because there exists a linear change of variables in the parameter space such that $L_k^{(1)} = u_k$, for $k = 1, \ldots, 58$ and $L_k^{(1)} = 0$, for $k = 59, 60, 61$.

Computing up to degree two of Taylor approximations of the Lyapunov constants and doing the simplifications as in the previous proofs, we can remove u_k for $k = 1, \ldots, 58$ from $L_k^{(2)}$, for $k = 59, 60, 61$. Vanishing the nonrelevant parameters and doing the adequate simple blow-up $u_{59} = z\, z_1$, $u_{60} = z\, z_2$, and $u_{61} = z$, we get

$$L_{59}^{(2)} = A_{59}\, z^2 \mathcal{L}_1(z_1, z_2), \quad L_{60}^{(2)} = A_{60}\, z^2 \mathcal{L}_2(z_1, z_2), \quad L_{61}^{(2)} = A_{61}\, z^2 \mathcal{L}_3(z_1, z_2),$$

where $\mathcal{L}_k$ are polynomials of degree 2 and A_k rational nonvanishing numbers. These polynomials have rational coefficients with numerators and denominators of around 1900 digits each. Approximately, they write as

$$\mathcal{L}_1 \approx z_1^2 + 77.576637 z_1 z_2 + 22.493284 z_2^2 + 107.76288 z_1 + 1038.9032 z_2 + 1265.0912,$$

$$\mathcal{L}_2 \approx z_1^2 - 2.6270001 z_1 z_2 + 0.27770877 z_2^2 + 25.446941 z_1 - 35.950489 z_2 + 160.06265,$$

$$\mathcal{L}_3 \approx z_1^2 + 3.4484543 z_1 z_2 + 2.9923181 z_2^2 + 32.128183 z_1 + 44.721607 z_2 + 248.24137.$$

The last step is to show that there exists at least a transversal real solution, $z^* = (z_1^*, z_2^*)$, of $\{\mathcal{L}_1 = 0, \mathcal{L}_2 = 0\}$ such that $\mathcal{L}_3(z^*)$ is nonvanishing. Then, Theorem 2.2 applies and the proof follows.

With an algebraic manipulator we can find the solution of $\mathcal{S}_{\mathcal{L}} = \{\mathcal{L}_1 = 0, \mathcal{L}_2 = 0\}$. It writes as $(z_1^*, z_2^*) = (p_3(\alpha), \alpha)$ where $p_3(\alpha)$ is a polynomial of degree 3 with rational coefficients and α is a real root of a given polynomial of degree 4, $p_4(\alpha)$. The polynomials p_3 and p_4 have rational coefficients with numerators and denominators of around 6000 and 4000 digits each, respectively. Approximately they write as

$$p_4(\alpha) \approx \alpha^4 + 0.87600811\,\alpha^3 - 1.97816765\,\alpha^2 - 3.22558688\,\alpha - 1.29759793,$$
$$p_3(\alpha) \approx 3.43467644\,\alpha^3 - 0.51633002\,\alpha^2 - 6.541427002\,\alpha - 17.7666993.$$

As the polynomial p_4 has only 2 real roots, the system $\mathcal{S}_{\mathcal{L}}$ has only two real solutions

$$(z_{1a}^*, z_{2a}^*) \approx (-14.693346428044632240, -0.85248929481003427092),$$
$$(z_{1b}^*, z_{2b}^*) \approx (-13.701549420630826548, 1.6907716352120896856).$$

As a function of α, we can find explicitly the values $\det \mathrm{J}(\mathcal{L}_1, \mathcal{L}_2)_{(z_1^*, z_2^*)}$ and $\mathcal{L}_3(z_1^*, z_2^*)$. They are also polynomials of degree 3 in α with rational coefficients that approximately write as

$$\det \mathrm{J}(\mathcal{L}_1, \mathcal{L}_2)_{(z_1^*, z_2^*)} \approx 611.007749\,\alpha^3 + 530.704211\,\alpha^2 - 846.328099\,\alpha - 1100.76580,$$
$$\mathcal{L}_3(z_1^*, z_2^*) \approx 1.53111109\,\alpha^3 + 0.80160181\,\alpha^2 - 3.66450101\,\alpha - 3.44764247.$$

It can be seen that all the polynomials of degree 3, p_3, $\det \mathrm{J}(\mathcal{L}_1, \mathcal{L}_2)_{(z_1^*, z_2^*)}$, and $\mathcal{L}_3(z_1^*, z_2^*)$ have no common zeros with p_4 because their respective resultants, with respect to α, are nonzero rational numbers. This proves the transversality and that the last Lyapunov constant, L_{61}, is nonvanishing.

In Figure 1, we have drawn the zero level curves of the polynomials $\mathcal{L}_k$, for $k = 1, 2, 3$, in a neighborhood of the intersection points. Graphically, the transversality is also clear. $\square$

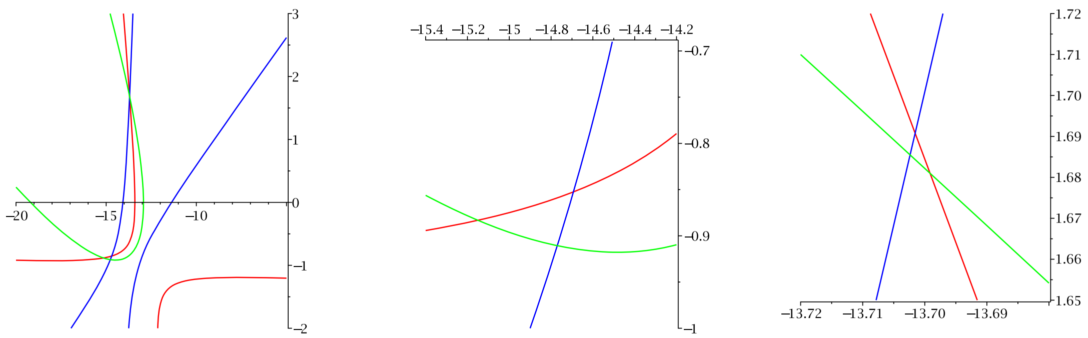

FIGURE 1. The level curves $\mathcal{L}_1 = 0$, $\mathcal{L}_2 = 0$, and $\mathcal{L}_3 = 0$ in red, green, and blue, respectively. The middle and right pictures are the corresponding zooms near the intersection points

7. Final comments

Taking a look at all analyzed systems, it is clear that we need new good examples to get higher lower bounds for the local cyclicity. The main difficulty is to know how to get them to ensure that only with developments of first-order it is enough to get the value originally conjectured by Giné ([19, 20]) and recently updated in [21]. In the language of

Żołądek, see [32], this is equivalent to find systems with maximal codimension. It should also be noted that the solution of the center problem is still open, even for polynomial vector fields of degree 3. It is also clear that with this mechanism we will never provide upper bounds.

We notice that the importance of Christopher work in [12] is that he pointed out that the computation of high-degree Taylor developments of the Lyapunov constants near a fixed center can be done without knowing their explicit and complete expressions. In no case have we calculated, because of the difficulties for their sizes, the constants first and then we have computed the Taylor developments. This fact has been crucial to perform all the computations made in this work and allow us to go further in determining the best lower bounds for $M(n)$ for lower degrees n. In particular, to design our parallelization algorithm.

For studying this local problem, the parallelization mechanism has been really a good tool. It has two computational advantages, the first is the decreasing of the total computation time, the second is the decreasing of memory necessities. Because partial computations require less time and less memory. Among these advantages, the difficulties now do not depend on the computation mechanisms. They are the size of the objects of higher developments, the knowledge of the local intersection of the varieties, and the high number of variables.

Finally, numerical computations are also not easy. Because as the degrees are quite high, to have small approximation errors we need to work with very high precision.

8. Accurate interval analysis

Next two technical results will help us to find upper and lower bounds for a polynomial of n variables in a n-dimensional cube. Their proofs can be found in [15].

Lemma 8.1 ([15]). *Consider $h>0$, $p>0$, q real numbers such that $p\in[\underline{p},\overline{p}]$, with $\underline{p}\overline{p}>0$, and $q\in[\underline{q},\overline{q}]$, with $\underline{q}\overline{q}>0$.*

(i) Then, $\sigma^l(q,p)\le qp\le\sigma^r(q,p)$, where

$$\sigma^l(q,p)=\begin{cases} q\underline{p}, & \text{if } q>0,\\ q\overline{p}, & \text{if } q<0,\end{cases}$$
$$\sigma^r(q,p)=\begin{cases} q\overline{p}, & \text{if } q>0,\\ q\underline{p}, & \text{if } q<0.\end{cases}$$

(ii) If $u_j\in[-h,h]$, for $j=1,\ldots,n$, and denoting $u^i=u_1^{i_1}\cdots u_n^{i_n}$, for $i=(i_1,\ldots,i_n)\neq 0$, we have $\mathcal{X}^l(q,u^i)\le qu^i\le\mathcal{X}^r(q,u^i)$, where

$$\mathcal{X}^l(q,u^i)=\begin{cases} 0, & \text{if } q>0 \text{ and } i_k \text{ even for all } k=1,\ldots,n,\\ -\overline{q}h^{i_1+\cdots+i_n}, & \text{if } q>0 \text{ and } i_k \text{ odd for some } k=1,\ldots,n,\\ \underline{q}h^{i_1+\cdots+i_n}, & \text{if } q<0,\end{cases}$$

and

$$\mathcal{X}^r(q,u^i)=\begin{cases} -\overline{q}h^{i_1+\cdots+i_n}, & \text{if } q>0 \text{ and } i_k \text{ even for all } k=1,\ldots,n,\\ 0, & \text{if } q<0 \text{ and } i_k \text{ odd for some } k=1,\ldots,n,\\ \underline{q}h^{i_1+\cdots+i_n}, & \text{if } q<0.\end{cases}$$

Furthermore, $\mathcal{X}^l(q,1)=\underline{q}$ and $\mathcal{X}^r(q,1)=\overline{q}$.

Lemma 8.2 ([15]). *Let $h > 0$ and p_j be positive nonrational numbers such that $p_j \in [\underline{p_j}, \overline{p_j}]$ with $\underline{p_j}, \overline{p_j}$ rational numbers satisfying $\underline{p_j}, \overline{p_j} > 0$, for $j = 1, \dots, m$. Consider the polynomial*

$$\mathcal{U}(u_1, \dots, u_n) = \sum_{i_1+\cdots+i_n=0}^{M} \left(\sum_{j=1}^{m} U_{j,i}\, p_j \right) u^i$$

with $u^i = u_1^{i_1} \cdots u_n^{i_n}$, for $i = (i_1, \dots, i_n)$, and $U_{j,i}$ rational numbers. Then

$$U_i^l \leq \sum_{j=1}^{m} U_{j,i}\, p_j \leq U_i^r,$$

with $U_i^l = \sum_{j=1}^{m} U_{j,i} \sigma^l(U_{j,i}, p_j)$ and $U_i^r = \sum_{j=1}^{m} U_{j,i} \sigma^r(U_{j,i}, p_j)$. Moreover, if $u_j \in [-h, h]$, for $j = 1, \dots, n$, and $U_i^l > U_i^r$ then

$$\underline{\mathcal{U}} = \sum_{i_1+\cdots+i_n=0}^{M} \mathcal{X}^l(U_i^l, u^i) \leq \mathcal{U}(u_1, \dots, u_n) \leq \sum_{i_1+\cdots+i_n=0}^{M} \mathcal{X}^r(U_i^r, u^i) = \overline{\mathcal{U}}.$$

9. Acknowledgments

We want to thank Armengol Gasull and Francesc Mañosas for the interesting discussions that have allowed us to understand and develop the results of this work. We also thank Fernando Sanz and Maria Aparecida Soares Ruas for guiding us in the interesting geometry of the local intersection of varieties. Finally, we thank the referees for their indications to improve the clarity of the exposition of our results.

This work has been realized thanks to the Catalan AGAUR 2017SGR1617 grant, the Spanish Ministerio de Ciencia, Innovación y Universidades - Agencia estatal de investigación MTM2016-77278-P (FEDER) grant, the European Community H2020-MSCA-RISE-2017-777911 grant and the Brazilian CNPq 200484/2015-0 and FAPESP 2020/04717-0 grants.

References

[1] A. A. Andronov, E. A. Leontovich, I. I. Gordon, and A. G. Maĭer. *Theory of bifurcations of dynamic systems on a plane.* Halsted Press [A division of John Wiley & Sons], New York-Toronto, Ont.; Israel Program for Scientific Translations, Jerusalem-London, 1973. Translated from the Russian.

[2] V. I. Arnol'd. Loss of stability of self-oscillations close to resonance and versal deformations of equivariant vector fields. *Functional Analysis and Its Applications*, 11(2):85–92, Apr 1977.

[3] V. I. Arnol'd, S. M. Guseĭn-Zade, and A. N. Varchenko. *Singularities of differentiable maps. Vol. I*, volume 82 of *Monographs in Mathematics.* Birkhäuser Boston, Inc., Boston, MA, 1985. The classification of critical points, caustics and wave fronts, Translated from the Russian by Ian Porteous and Mark Reynolds.

[4] J. L. R. Bastos, C. A. Buzzi, and J. Torregrosa. Factor reversible systems with applications to planar centers. Preprint, May 2020.

[5] N. N. Bautin. On the number of limit cycles which appear with the variation of coefficients from an equilibrium position of focus or center type. *American Math. Soc. Translation*, 1954(100):19, 1954.

[6] T. Blows and N. Lloyd. The number of limit cycles of certain polynomial differential equations. *Proc. Roy. Soc. Edinburgh Sect. A*, 98(3-4):215–239, 1984.

[7] Y. L. Bondar and A. P. Sadovskiĭ. On a theorem of Zoladek. *Differ. Uravn.*, 44(2):263–265, 287, 2008.

[8] E. Casas-Alvero. *Singularities of plane curves*, volume 276 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 2000.

[9] C. Chicone. *Ordinary differential equations with applications*, volume 34 of *Texts in Applied Mathematics.* Springer, New York, second edition, 2006.

[10] C. Chicone and M. Jacobs. Bifurcation of critical periods for plane vector fields. *Trans. Amer. Math. Soc.*, 312(2):433–486, 1989.

[11] S.-N. Chow, C. Li, and D. Wang. *Normal forms and bifurcation of planar vector fields*. Cambridge University Press, Cambridge, 2008. Reprint of the 1994 original.
[12] C. Christopher. Estimating limit cycle bifurcations from centers. In *Differential equations with symbolic computation*, Trends Math., pages 23–35. Birkhäuser, Basel, 2005.
[13] A. Cima, A. Gasull, V. Mañosa, and F. Mañosas. Algebraic properties of the Liapunov and period constants. *Rocky Mountain J. Math.*, 27(2):471–501, 1997.
[14] A. Cima, A. Gasull, and F. Mañosas. A note on the Lyapunov and period constants. *Qual. Theory Dyn. Syst.*, 19(33):1–13, 2020.
[15] L. P. da Cruz, D. D. Novaes, and J. Torregrosa. New lower bound for the Hilbert number in piecewise quadratic differential systems. *J. Differential Equations*, 266(7):4170–4203, 2019.
[16] F. Dumortier, J. Llibre, and J. C. Artés. *Qualitative theory of planar differential systems*. Universitext. Springer-Verlag, Berlin, 2006.
[17] S. Gerschgorin. Über die abgrenzung der eigenwerte einer matrix. *Izv. Akad. Nauk SSSR Otd. Fiz.-Mat.*, page 749–754, 1931.
[18] J. Giné. The center problem for a linear center perturbed by homogeneous polynomials. *Acta Math. Sin. (Engl. Ser.)*, 22(6):1613–1620, 2006.
[19] J. Giné. Higher order limit cycle bifurcations from non-degenerate centers. *Appl. Math. Comput.*, 218(17):8853–8860, 2012.
[20] J. Giné. Limit cycle bifurcations from a non-degenerate center. *Appl. Math. Comput.*, 218(9):4703–4709, 2012.
[21] J. Giné, L. F. Gouveia, and J. Torregrosa. Lower bounds for the local cyclicity for families of centers. Preprint, March 2020.
[22] G. H. Golub and C. F. Van Loan. *Matrix Computations (3rd Ed.)*. Johns Hopkins University Press, Baltimore, MD, USA, 1996.
[23] M. Han. Liapunov constants and Hopf cyclicity of Liénard systems. *Ann. Differential Equations*, 15(2):113–126, 1999.
[24] W. Kulpa. The Poincaré–Miranda Theorem. *The American Mathematical Monthly*, 104(6):545–550, 1997.
[25] H. Liang and J. Torregrosa. Parallelization of the Lyapunov constants and cyclicity for centers of planar polynomial vector fields. *J. Differential Equations*, 259(11):6494–6509, 2015.
[26] Maplesoft, a division of Waterloo Maple Inc. Maple 2018. http://www.maplesoft.com/, Waterloo, Ontario.
[27] H. Movasati. Center conditions: rigidity of logarithmic differential equations. *J. Differential Equations*, 197(1):197–217, 2004.
[28] R. Roussarie. *Bifurcation of planar vector fields and Hilbert's sixteenth problem*, volume 164 of *Progress in Mathematics*. Birkhäuser Verlag, Basel, 1998.
[29] K. S. Sibirskiĭ. On the number of limit cycles arising from a singular point of focus or center type. *Dokl. Akad. Nauk SSSR*, 161:304–307, 1965.
[30] Y. Tian and P. Yu. Bifurcation of small limit cycles in cubic integrable systems using higher-order analysis. *J. Differential Equations*, 264(9):5950–5976, 2018.
[31] C. T. C. Wall. *Singular points of plane curves*, volume 63 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2004.
[32] H. Żołądek. The classification of reversible cubic systems with center. *Topol. Methods Nonlinear Anal.*, 4(1):79–136, 1994.
[33] H. Żołądek. On certain generalization of the Bautin's Theorem. *Nonlinearity*, 7(1):233–279, 1994.
[34] H. Żołądek. Eleven small limit cycles in a cubic vector field. *Nonlinearity*, 8(5):843–860, 1995.
[35] H. Żołądek. Remarks on: "The classification of reversible cubic systems with center" [Topol. Methods Nonlinear Anal. **4** (1994), no. 1, 79–136]. *Topol. Methods Nonlinear Anal.*, 8(2):335–342 (1997), 1996.
[36] H. Żołądek. The CD45 case revisited. In *Mathematical Sciences with Multidisciplinary Applications*, pages 595–625, Cham, 2016. Springer International Publishing.

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Departamento de Matemática, Universidade Estadual Paulista, 15054-000 São José do Rio Preto, Brazil
Email address: fernandosg@mat.uab.cat, fernando.gouveia@unesp.br

Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain
Email address: torre@mat.uab.cat