

LIMIT CYCLES OF DISCONTINUOUS PIECEWISE DIFFERENTIAL SYSTEMS FORMED BY LINEAR CENTERS IN \mathbb{R}^2 AND SEPARATED BY TWO CIRCLES

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ABSTRACT. We show that discontinuous planar piecewise differential systems formed by linear centers and separated by two concentric circles can have at most three limit cycles. Usually is a difficult problem to provide the exact upper bound that a class of differential systems can exhibit. Here we also provide examples of such systems with zero, one, two, or three limit cycles.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In the qualitative theory of the differential systems in \mathbb{R}^2 one of the main difficult objects to study are the limit cycles. Recall that a *limit cycle* is an isolated periodic solution in the set of all periodic solutions of the differential systems, see for instance the second part of the famous 16th Hilbert problem [5, 7, 9].

The study of piecewise linear discontinuous differential systems started with Andronov, Vitt and Khaikin in [1]. Due to the fact that these systems model many real phenomena and different modern devices, they have become a topic of great interest these last twenty years. For more details see for instance the books [2, 19] and the references therein.

In recent years many authors have been widely interested in solving the second part of the 16-th Hilbert’s problem for continuous and discontinuous piecewise linear differential systems in \mathbb{R}^2 , that is to determine an upper bound for the maximum number of limit cycles for these class of differential systems.

The easiest continuous piecewise linear differential systems are formed by two linear differential systems separated by a straight line. It is known that such systems have at most one limit cycle, see [4, 13, 17, 18]. But if both linear differential systems are linear centers, then it is known that the continuous piecewise linear differential systems have no limit cycles, see [11, 15].

Also the easiest discontinuous piecewise linear differential systems are formed by two linear differential systems separated by a straight line. It is known that this class of differential systems can have three limit cycles, but at this moment it is unknown if three is the maximum number of limit cycles that this class can

2010 *Mathematics Subject Classification.* 34C29, 34C25, 47H11.

Key words and phrases. Limit cycles, linear centers, continuous piecewise linear differential systems, discontinuous piecewise differential systems, first integrals.

exhibit, see for instance [8, 10, 14]. Other works on the limit cycles of discontinuous piecewise linear differential systems can be found in [6, 12].

A natural further step is to know which is the maximum number of limit cycles of continuous or discontinuous piecewise linear differential systems separated by other objects rather than a straight line and in particular if these linear systems are centers. This is the goal of this paper.

More precisely, in this paper we deal with discontinuous piecewise differential systems formed by two linear centers separated by two concentric circles. Without loss of generality we can assume that these circles are

$$\mathbb{S}_1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}, \quad \text{and} \quad \mathbb{S}_2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = s^2\},$$

with $s > 1$. We use the Filippov conventions for defining the discontinuous piecewise differential system on \mathbb{S}_1 and on \mathbb{S}_2 , see [3]. We denote by (C) the class of planar discontinuous piecewise differential systems formed by three linear centers and separated by two circles \mathbb{S}_1 and \mathbb{S}_2 .

We note that if we try to obtain continuous piecewise differential systems formed by three linear centers separated by the circles \mathbb{S}_1 and \mathbb{S}_2 , then we obtain a global linear center in the plane, and consequently we do not have limit cycles. Therefore the interesting piecewise differential systems formed by three linear centers separated by the circles \mathbb{S}_1 and \mathbb{S}_2 are the discontinuous ones, the ones that we shall study.

Here a *crossing periodic orbit* for a system in (C) is a periodic orbit which has exactly four points of discontinuity, that is, two points in the circle \mathbb{S}_1 and the other two points in the circle \mathbb{S}_2 . A *crossing limit cycle* for a system in (C) is a crossing periodic orbit isolated in the set of all crossing periodic orbits of the system.

We remark that we are not interested in studying the crossing limit cycles for the systems in (C) having only two intersection points with either the circle \mathbb{S}_1 or the circle \mathbb{S}_2 , because in [16] it was proved that such systems can have at most two crossing limit cycles intersecting a circle in two points, and that there are examples exhibiting two such crossing limit cycles.

The objective of this paper is to study the number of crossing limit cycles that systems in the class (C) can have intersecting in two points each of the circles \mathbb{S}_1 and \mathbb{S}_2 .

In what follows when we talk about a crossing limit cycle, it will be a crossing limit cycle that intersects in two points each of the circles \mathbb{S}_1 and \mathbb{S}_2 .

Our main result is the following.

Theorem 1. *The following statements hold.*

- (a) *Every system in (C) has at most three crossing limit cycles.*
- (b) *There are systems in (C) having three crossing limit cycles.*
- (c) *There are systems in (C) having exactly two crossing limit cycles.*
- (d) *There are systems in (C) having exactly one crossing limit cycle.*
- (e) *There are system in (C) without crossing limit cycles.*

The proof of Theorem 1 is given in the next section.

2. PROOF OF THEOREM 1

To prove Theorem 1 we first recall the following lemma, proved in [15] which provides a normal form for an arbitrary linear differential system having a center.

Lemma 2. *A linear differential system having a center can be written as*

$$(1) \quad \dot{x} = -bx - \frac{4b^2 + \omega^2}{4a}y + d, \quad \dot{y} = ax + by + c,$$

with $a > 0$ and $\omega > 0$.

We prove each statement of Theorem 1 separately.

Proof of statement (a) of Theorem 1. In the bounded region R_1 limited by the circle \mathbb{S}_1 we consider the arbitrary linear differential center (1), which has the first integral

$$H_1(x, y) = 4(ax + by)^2 + 8a(cx - dy) + \omega^2 y^2.$$

In the bounded region R_2 limited by the circles \mathbb{S}_1 and \mathbb{S}_2 we consider the arbitrary linear differential center

$$(2) \quad \dot{x} = -Bx - \frac{4B^2 + \Omega^2}{4A}y + D, \quad \dot{y} = Ax + By + C,$$

with $A > 0$ and $\Omega > 0$. This system has the first integral

$$H_2(x, y) = 4(Ax + By)^2 + 8A(Cx - Dy) + \Omega^2 y^2.$$

Finally in the region R_3 outside the circle \mathbb{S}_2 we consider the arbitrary linear differential center

$$(3) \quad \dot{x} = -\beta x - \frac{4\beta^2 + w^2}{4\alpha}y + \delta, \quad \dot{y} = \alpha x + \beta y + \gamma,$$

with $\alpha > 0$ and $w > 0$, that has first integral

$$H_3(x, y) = 4(\alpha x + \beta y)^2 + 8\alpha(\gamma x - \delta y) + w^2 y^2.$$

Take the scaling of the times $\sigma = at$ in R_1 , $\tau = At$ in R_2 and $\nu = \alpha t$ in R_3 . These three scalings change the velocity in which the orbits of system (1), (2) and (3) are traveled, but they do not change the orbits, and consequently they do not change the crossing limit cycles that the discontinuous piecewise linear differential systems may have. Note that after these scalings of the time we can always assume, without loss of generality, that $a = A = \alpha = 1$, with the dot in the systems denoting derivative with respect to the new times σ, τ and ν .

The proof of statement (a) will be done by contradiction. We thus assume that the discontinuous piecewise linear differential system formed by the three linear centers (1), (2) and (3) have four crossing periodic orbits that intersect the circle \mathbb{S}_1 in the points (x_i, y_i) for $i = 2, 3$ and the circle \mathbb{S}_2 in the points (x_i, y_i) for $i = 1, 4$,

so these points must satisfy the system

$$\begin{aligned}
 (4) \quad & f_1 = H_2(x_1, y_1) - H_2(x_2, y_2) = 0, \\
 & f_2 = H_1(x_2, y_2) - H_1(x_3, y_3) = 0, \\
 & f_3 = H_2(x_3, y_3) - H_2(x_4, y_4) = 0, \\
 & f_4 = H_3(x_4, y_4) - H_3(x_1, y_1) = 0, \\
 & f_5 = x_1^2 + y_1^2 - s^2 = 0, \quad f_6 = x_4^2 + y_4^2 - s^2 = 0, \\
 & f_7 = x_2^2 + y_2^2 - 1 = 0, \quad f_8 = x_3^2 + y_3^2 - 1 = 0,
 \end{aligned}$$

or equivalently

$$\begin{aligned}
 (5) \quad & f_1 = 4(By_1 + x_1)^2 - 4(By_2 + x_2)^2 + 8C(x_1 - x_2) - 8D(y_1 + y_2) + \Omega^2(y_1^2 - y_2^2) = 0, \\
 & f_2 = 4(by_2 + x_2)^2 - 4(by_3 + x_3)^2 + 8c(x_2 - x_3) - 8d(y_2 + y_3) + \omega^2(y_2^2 - y_3^2) = 0, \\
 & f_3 = 4(By_3 + x_3)^2 - 4(By_4 + x_4)^2 + 8C(x_3 - x_4) - 8D(y_3 + y_4) + \Omega^2(y_3^2 - y_4^2) = 0, \\
 & f_4 = 4(\beta y_4 + x_4)^2 - 4(\beta y_1 + x_1)^2 + 8\gamma(x_4 - x_1) - 8\delta(y_4 + y_1) + w^2(y_4^2 - y_1^2) = 0, \\
 & f_5 = 0, \quad f_6 = 0, \quad f_7 = 0, \quad f_8 = 0.
 \end{aligned}$$

Assume that the four solutions of this system are (p_1, p_2, p_3, p_4) , (q_1, q_2, q_3, q_4) , (t_1, t_2, t_3, t_4) and (w_1, w_2, w_3, w_4) with $p_i, q_i, t_i, w_i \in \mathbb{R}^2$, $p_i \neq p_j$, $q_i \neq q_j$, $t_i \neq t_j$, and $w_i \neq w_j$ for $i \neq j$ and $i, j = 1, 2, 3, 4$. We rewrite these points in the following way

$$(6) \quad p_i = (x_i, y_i), \quad q_i = (k_i, l_i), \quad t_i = (m_i, n_i), \quad w_i = (h_i, j_i),$$

for $i = 1, 2, 3, 4$.

Substituting the solutions (p_1, p_2, p_3, p_4) and (q_1, q_2, q_3, q_4) , with p_i, q_i given in (6) inside (5) we can determine the parameters d , δ , c and γ in function of the coordinates of the points p_i, q_i as follows

$$\begin{aligned}
 (7) \quad & d = \frac{(4b^2 + \omega^2)(y_2^2 - y_3^2) + 8x_2(by_2 + c) - 8x_3(by_3 + c) + 4x_2^2 - 4x_3^2}{8(y_2 - y_3)}, \\
 & \delta = \frac{(4\beta^2 + w^2)(y_1^2 - y_4^2) + 4x_1^2 + 8x_1(\gamma + \beta y_1) - 4x_4^2 - 8x_4(\gamma + \beta y_4)}{8(y_1 - y_4)}, \\
 & c = \frac{1}{8(l_2 - l_3)(x_2 - x_3) - 8(k_2 - k_3)(y_2 - y_3)} \left(4(l_2(b^2 y_3^2 + 2by_3(x_3 - k_2) + 2bk_2 y_2 \right. \\
 & \quad \left. - (by_2 + x_2)^2 + x_3^2) + b^2 l_2^2(y_2 - y_3) - b^2 l_3 y_3^2 + y_3((bl_3 + k_3)^2 - 2bl_3 x_3 - k_2^2) \right. \\
 & \quad \left. + l_3(-by_2(bl_3 + 2k_3)) + (by_2 + x_2)^2 - x_3^2) + 4y_2(k_2 - k_3)(k_2 + k_3) \right. \\
 & \quad \left. + \omega^2(l_2 - l_3)(y_2 - y_3)(l_2 + l_3 - y_2 - y_3) \right), \\
 & \gamma = \frac{1}{8(l_1 - l_4)(x_1 - x_4) - 8(k_1 - k_4)(y_1 - y_4)} \left(8\beta(k_1 l_1(y_1 - y_4) + k_4 l_4(y_4 - y_1) \right. \\
 & \quad \left. - (l_1 - l_4)(x_1 y_1 - x_4 y_4)) + 4(k_1 - k_4)(k_1 + k_4)(y_1 - y_4) \right. \\
 & \quad \left. + (l_1 - l_4)(w^2(y_1 - y_4)(l_1 + l_4 - y_1 - y_4) - 4x_1^2 + 4x_4^2) \right. \\
 & \quad \left. + 4\beta^2(l_1 - l_4)(y_1 - y_4)(l_1 + l_4 - y_1 - y_4) \right).
 \end{aligned}$$

We assume that the points (t_1, t_2, t_3, t_4) , (w_1, w_2, w_3, w_4) with t_i, w_i as in (6) satisfies system (5), then we can obtain the remaining parameters, namely, ω^2 and w^2 . In fact, from equation f_2 we obtain $\omega^2 = S/T$, where

$$\begin{aligned} S = & 4(-(k_2 - k_3)(y_3(-b^2n_2^2 - 2bm_2n_2 + (bn_3 + m_3)^2 + 2bx_3(n_2 - n_3) - m_2^2) + b^2n_2^2y_2 \\ & + b^2y_3^2(n_2 - n_3) - b^2n_2y_2^2 - b^2n_3^2y_2 + b^2n_3y_2^2 + 2bm_2n_2y_2 - 2bm_3n_3y_2 - 2bn_2x_2y_2 \\ & + 2bn_3x_2y_2 + k_2((n_2 - n_3)(x_2 - x_3) - (m_2 - m_3)(y_2 - y_3)) + k_3((n_2 - n_3)(x_2 - x_3) \\ & - (m_2 - m_3)(y_2 - y_3)) + y_2(m_2 - m_3)(m_2 + m_3) - n_2x_2^2 + n_2x_3^2 + n_3x_2^2 - n_3x_3^2) \\ & + l_2(m_2(b^2(y_3^2 - y_2^2) + 2b(k_2y_2 - k_2y_3 + n_2x_2 - n_2x_3 - x_2y_2 + x_3y_3) - x_2^2 + x_3^2) \\ & + m_3(b^2(y_2 - y_3)(y_2 + y_3) + 2b(-k_2y_2 + k_2y_3 - n_3x_2 + n_3x_3 + x_2y_2 - x_3y_3) \\ & + (x_2 - x_3)(x_2 + x_3)) + b(n_2 - n_3)(x_2 - x_3)(b(n_2 + n_3) - 2k_2) + m_2^2(x_2 - x_3) \\ & + m_3^2(x_3 - x_2)) + l_3(m_2(b^2(y_2 - y_3)(y_2 + y_3) + 2b(-k_3y_2 + k_3y_3 + n_2(x_3 - x_2) \\ & + x_2y_2 - x_3y_3) + (x_2 - x_3)(x_2 + x_3)) + m_3(b^2(y_3^2 - y_2^2) + 2b(k_3y_2 - k_3y_3 + n_3(x_2 - x_3) \\ & - x_2y_2 + x_3y_3) - x_2^2 + x_3^2) + b(n_2 - n_3)(x_3 - x_2)(b(n_2 + n_3) - 2k_3) + m_2^2(x_3 - x_2) \\ & + m_3^2(x_2 - x_3)) + l_2^2b^2((m_2 - m_3)(y_2 - y_3) + (n_3 - n_2)(x_2 - x_3)) + l_3^2b^2((n_2 - n_3)(x_2 - x_3) \\ & - (m_2 - m_3)(y_2 - y_3))), \end{aligned}$$

and

$$\begin{aligned} T = & (k_2 - k_3)(n_2 - n_3)(y_2 - y_3)(n_2 + n_3 - y_2 - y_3) + l_2^2((n_2 - n_3)(x_2 - x_3) \\ & - (m_2 - m_3)(y_2 - y_3)) + l_2((m_2 - m_3)(y_2 - y_3)(y_2 + y_3) + (n_3 - n_2)(n_2 + n_3)(x_2 - x_3)) \\ & + l_3^2((m_2 - m_3)(y_2 - y_3) + (n_3 - n_2)(x_2 - x_3)) + l_3((n_2 - n_3)(n_2 + n_3)(x_2 - x_3) \\ & - (m_2 - m_3)(y_2 - y_3)(y_2 + y_3)). \end{aligned}$$

Analogously from equation f_4 we obtain $w^2 = R/U$, where

$$\begin{aligned} R = & 4(-(k_1 - k_4)(k_1((n_1 - n_4)(x_1 - x_4) - (m_1 - m_4)(y_1 - y_4)) - k_4(m_1 - m_4)(y_1 - y_4) \\ & + k_4(n_1 - n_4)(x_1 - x_4) + 2\beta(m_1n_1(y_1 - y_4) + m_4n_4(y_4 - y_1) - (n_1 - n_4)(x_1y_1 - x_4y_4)) \\ & + (m_1 - m_4)(m_1 + m_4)(y_1 - y_4) - (n_1 - n_4)(x_1 - x_4)(x_1 + x_4) + \beta^2(n_1 - n_4)(y_1 - y_4) \\ & \times (n_1 + n_4 - y_1 - y_4)) + l_1(2\beta(k_1(m_1 - m_4)(y_1 - y_4) - k_1(n_1 - n_4)(x_1 - x_4) \\ & + m_1(n_1(x_1 - x_4) - x_1y_1 + x_4y_4) + m_4(n_4(x_4 - x_1) + x_1y_1 - x_4y_4)) + \beta^2((n_1 - n_4) \\ & \times (n_1 + n_4)(x_1 - x_4) - (m_1 - m_4)(y_1 - y_4)(y_1 + y_4)) + (m_1 - m_4)(x_1 - x_4) \\ & \times (m_1 + m_4 - x_1 - x_4)) + l_4(2\beta(-k_4(m_1 - m_4)(y_1 - y_4) + k_4(n_1 - n_4)(x_1 - x_4) \\ & + m_1(n_1(x_4 - x_1) + x_1y_1 - x_4y_4) + m_4(n_4(x_1 - x_4) - x_1y_1 + x_4y_4)) \\ & + \beta^2((m_1 - m_4)(y_1 - y_4)(y_1 + y_4) + (n_4 - n_1)(n_1 + n_4)(x_1 - x_4)) \\ & + (m_1 - m_4)(x_1 - x_4)(-m_1 - m_4 + x_1 + x_4)) + \beta^2l_1^2((m_1 - m_4)(y_1 - y_4) \\ & + (n_4 - n_1)(x_1 - x_4)) + \beta^2l_4^2((n_1 - n_4)(x_1 - x_4) - (m_1 - m_4)(y_1 - y_4))), \end{aligned}$$

and

$$\begin{aligned} U = & (k_1 - k_4)(n_1 - n_4)(y_1 - y_4)(n_1 + n_4 - y_1 - y_4) + l_1^2((n_1 - n_4)(x_1 - x_4) \\ & - (m_1 - m_4)(y_1 - y_4)) + l_1((m_1 - m_4)(y_1 - y_4)(y_1 + y_4) + (n_4 - n_1)(n_1 + n_4)(x_1 - x_4)) \\ & + l_4^2((m_1 - m_4)(y_1 - y_4) + (n_4 - n_1)(x_1 - x_4)) + l_4((n_1 - n_4)(n_1 + n_4)(x_1 - x_4) \\ & - (m_1 - m_4)(y_1 - y_4)(y_1 + y_4)). \end{aligned}$$

Substituting p_i, q_i, t_i, w_i as in (6) in equations f_2 and f_4 , we obtain the expressions for b and β . In order to simplify these expressions, we use the relations

$$x_i^2 = s_i^2 - y_i^2, \quad k_i^2 = s_i^2 - l_i^2, \quad m_i^2 = s_i^2 - n_i^2, \quad h_i^2 = s_i^2 - j_i^2$$

with $s_i = 1$ for $i = 2, 3$ and $s_i = r$ for $i = 1, 4$. Doing this we get

$$\begin{aligned} b = & (h_3^2 + j_3^2 - 1) \left(-(k_2 - k_3)(-k_2(m_2 - m_3)(y_2 - y_3) + k_2(n_2 - n_3)(x_2 - x_3) \right. \\ & - k_3(m_2 - m_3)(y_2 - y_3) + k_3(n_2 - n_3)(x_2 - x_3) + (m_2 - m_3)(m_2 + m_3)(y_2 - y_3) \\ & - (n_2 - n_3)(x_2 + x_3)(x_2 - x_3)) + l_2(m_2 - m_3)(x_2 - x_3)(m_2 + m_3 - x_2 - x_3) \\ & \left. - l_3(m_2 - m_3)(x_2 - x_3)(m_2 + m_3 - x_2 - x_3) \right), \\ \beta = & (h_1^2 + j_1^2 - r^2) \left(-(k_1 - k_4)(-k_1(m_1 - m_4)(y_1 - y_4) + k_1(n_1 - n_4)(x_1 - x_4) \right. \\ & - k_4(m_1 - m_4)(y_1 - y_4) + k_4(n_1 - n_4)(x_1 - x_4) + (m_1 - m_4)(m_1 + m_4)(y_1 - y_4) \\ & - (n_1 - n_4)(x_1 + x_4)(x_1 - x_4)) + l_1(m_1 - m_4)(x_1 - x_4)(m_1 + m_4 - x_1 - x_4) \\ & \left. - l_4(m_1 - m_4)(x_1 - x_4)(m_1 + m_4 - x_1 - x_4) \right). \end{aligned}$$

Since $h_3^2 + j_3^2 - 1 = 0$ and $h_1^2 + j_1^2 - r^2 = 0$ (because they are points on the circles of radius 1 and r , respectively), we obtain that $b = 0$ and $\beta = 0$. Therefore we get that the piecewise linear differential center is formed by the linear differential centers

$$\dot{x} = -y, \quad \dot{y} = x,$$

in the regions R_1 and R_3 . This is a contradiction because in the region R_1 inside the circle of radius 1 we have the first integral $H_1(x, y) = x^2 + y^2$. Then using this first integral it is not possible to generate crossing limit cycles. Therefore there is no discontinuous piecewise linear differential systems (1), (2) and (3) having more than three crossing limit cycles. \square

Proof of statement (b) of Theorem 1. We construct a system in (C) having three crossing limit cycles. In the bounded region R_1 limited by the circle \mathbb{S}_1 we consider the linear differential center

$$(8) \quad \dot{x} = 4x - 11.9153y + 2.59286, \quad \dot{y} = 8x - 4y - 7.70389,$$

with the first integral

$$\begin{aligned} H_1(x, y) = & -7.703888863058826x - 2.59286097618567y + 4x^2 - 4xy \\ & + 5.957636997088071y^2. \end{aligned}$$

In the bounded region R_2 limited by the circles \mathbb{S}_1 and \mathbb{S}_2 with $s = 2$ we consider the linear differential center

$$(9) \quad \dot{x} = -2.48171x - 3.50967y + 5.2645, \quad \dot{y} = 8x + 2.48171y - 12.2078,$$

with the first integral

$$\begin{aligned} H_2(x, y) = & -12.207843699316152x - 5.264498126903808y + 4x^2 \\ & + 2.481708216718382xy + 1.7548327089679396y^2. \end{aligned}$$

Finally in the region R_3 outside the circle \mathbb{S}_2 we consider the linear differential center

$$(10) \quad \dot{x} = -8x + 4y + 12.2222, \quad \dot{y} = 8x - 4y - 12.2222,$$

with the first integral

$$H_3(x, y) = -12.222157553827293x - 3.9748620974752176y + 4x^2 - 4xy + 6.143143553839666y^2.$$

The discontinuous piecewise differential system formed by the previous three linear differential centers has three crossing limit cycles (p_1, p_2, p_3, p_4) , (q_1, q_2, q_3, q_4) and (t_1, t_2, t_3, t_4) with p_i, q_i, t_i

$$p_i = (x_i, y_i) = (s_i \cos \rho_i, s_i \sin \rho_i), \quad q_i = (k_i, l_i) = (s_i \cos \nu_i, s_i \sin \nu_i),$$

$$t_i = (m_i, n_i) = (s_i \cos u_i, s_i \sin u_i),$$

where $\rho_i, \nu_i, u_i \in [0, 2\pi)$, $i = 1, 2, 3, 4$, $s_i = 1$ for $i = 2, 3$ and $s_i = 2$ for $i = 1, 4$. Moreover, $\rho_1 = \rho_2 = \pi/4$, $\rho_3 = \rho_4 = 1/10$, $\nu_1 = \nu_2 = \pi/2$, $\nu_3 = \nu_4 = -\pi/4$, $u_1 = 2$, $u_2 = 2.2778851991902505$, $u_3 = 4$ and $u_4 = 4.760510529471427$. See the three crossing limit cycles of this system in Figure 1. \square

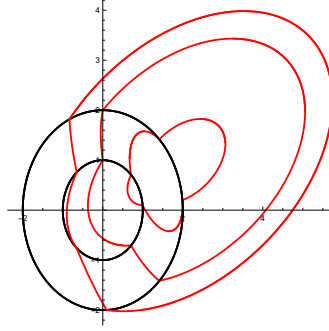


Figure 1: The three crossing limit cycles of the discontinuous piecewise differential systems formed by the linear centers (8), (9) and (10).

Proof of statement (c) of Theorem 1. We construct a system in (C) having exactly two crossing limit cycles. In the bounded region R_1 limited by the circle \mathbb{S}_1 we consider the linear differential center

$$(11) \quad \dot{x} = \frac{2}{5} - y, \quad \dot{y} = x - 1,$$

with the first integral

$$H_1(x, y) = 4x^2 + 8 \left(-x - \frac{2y}{5} \right) + 4y^2.$$

In the bounded region R_2 limited by the circles \mathbb{S}_1 and \mathbb{S}_2 with $s = 2$ we consider the linear differential center

$$(12) \quad \dot{x} = \frac{11x}{19} + \frac{18292y}{6859} - 4, \quad \dot{y} = -\frac{19x}{2} - \frac{11y}{19} - 10,$$

with the first integral

$$H_2(x, y) = 4 \left(-\frac{19x}{2} - \frac{11y}{19} \right)^2 - 76(4y - 10x) + 100y^2.$$

And in the region R_3 outside the circle S_2 we consider the linear differential center

$$(13) \quad \dot{x} = 16y + \frac{1}{5}, \quad \dot{y} = 1 - \frac{x}{4},$$

with the first integral

$$H_3(x, y) = \frac{x^2}{4} - 2\left(x - \frac{y}{5}\right) + 16y^2.$$

This discontinuous piecewise differential system formed by the previous linear centers (11), (12) and (13) has two crossing limit cycles, which are the unique two real solutions $(p_1^j, p_2^j, p_3^j, p_4^j)$ with $j = 1, 2$ satisfying (4). Since we want to use Groebner bases, we use the rational parametrization of the circle for obtain polynomial equations, so we write the points $p_i = (x_i, y_i)$ as

$$(14) \quad (x_i, y_i) = \left(\frac{R(1 - t_i^2)}{1 + t_i^2}, \frac{2Rt_i}{1 + t_i^2} \right),$$

with $t_i \neq t_j$, $i \neq j$ for $i = 1, 2, 3, 4$. Doing this system (4) reduces to the polynomial system

$$(15) \quad \begin{aligned} f_1 &= 116603t_1^4t_4^4 - 566048t_1^4t_4^3 - 169146t_1^4t_4^2 - 311904t_1^4t_4 + 1214043t_1^4 + 251256t_1^3t_4^4 \\ &\quad + 502512t_1^3t_4^3 + 251256t_1^3 + 59434t_1^2t_4^4 - 1132096t_1^2t_4^3 - 685836t_1^2t_4^2 - 623808t_1^2t_4 \\ &\quad + 2254314t_1^2 + 187720t_1t_4^4 + 375440t_1t_4^3 + 187720t_1 - 432117t_4^4 - 566048t_4^3 \\ &\quad - 1266586t_4^2 - 311904t_4 + 665323, \\ f_2 &= 2t_1t_2 + 5t_1 + 5t_2 - 2, \\ f_3 &= -116603t_2^4t_3^4 + 566048t_2^4t_3^3 + 169146t_2^4t_3^2 + 311904t_2^4t_3 - 1214043t_2^4 - 251256t_2^3t_3^4 \\ &\quad - 502512t_2^3t_3^3 - 251256t_2^3 - 59434t_2^2t_3^4 + 1132096t_2^2t_3^3 + 685836t_2^2t_3^2 + 623808t_2^2t_3 \\ &\quad - 2254314t_2^2 - 187720t_2t_3^4 - 375440t_2t_3^3 - 187720t_2 + 432117t_3^4 + 566048t_3^3 \\ &\quad + 1266586t_3^2 + 311904t_3 - 665323, \\ f_4 &= 2t_3^3t_4^3 + 305t_3^3t_4^2 + 2t_3^3t_4 - 10t_3^3 + 305t_3^2t_4^3 - 2t_3^2t_4^2 - 10t_3^2t_4 - 2t_3^2 + 2t_3t_4^3 - 10t_3t_4^2 \\ &\quad + 2t_3t_4 - 325t_3 - 10t_4^3 - 2t_4^2 - 325t_4 - 2, \\ f_5 &= 0, \quad f_6 = 0, \quad f_7 = 0, \quad f_8 = 0. \end{aligned}$$

Doing the Groebner basis using Mathematica we can eliminate the variables t_2, t_4 using the equations (f_2, f_3) and (f_1, f_4) , respectively. Moreover we get two polynomials: p_1 of degree 8 and p_2 of degree 24 with respect to variables t_1 and t_3 . Solving the system obtained for these two polynomials we obtain the following six real solutions

$$\begin{aligned} (t_1^1, t_3^1) &= (3.113683889, -3.345831873), \quad (t_1^2, t_3^2) = (-1.208512958, 3.444180431), \\ (t_1^3, t_3^3) &= (-0.07918973421, -2.003557002), \quad (t_1^4, t_3^4) = (0.7258741121, -0.9412890517), \\ (t_1^5, t_3^5) &= (-0.2525471397, 0.8580640103), \quad (t_1^6, t_3^6) = (0.4948650262, 0.5345302181). \end{aligned}$$

Therefore (15) has the next six solutions

$$\begin{aligned}
(t_1^1, t_2^1, t_3^1, t_4^1) &= (3.113684, -1.208513, -3.345832, 3.444180), \\
(t_1^2, t_2^2, t_3^2, t_4^2) &= (-1.208513, 3.113684, 3.444180, -3.345832), \\
(t_1^3, t_2^3, t_3^3, t_4^3) &= (0.7258741, -0.2525471, -0.941289, 0.8580640), \\
(t_1^4, t_2^4, t_3^4, t_4^4) &= (-0.2525472, 0.7258742, 0.8580642, -0.941289), \\
(t_1^5, t_2^5, t_3^5, t_4^5) &= (-0.07919058, 0.4948659, -2.003559, 0.5345301), \\
(t_1^6, t_2^6, t_3^6, t_4^6) &= (0.4948641, -0.07918876, 0.5345306, -2.003555).
\end{aligned}$$

The solutions $(t_1^j, t_2^j, t_3^j, t_4^j)$ for $j = 2, 4, 6$, define the same solutions as for $j = 1, 3, 4$. Therefore the solutions of system (4) in the variables (x_i, y_i) are

$$\begin{aligned}
(p_1^1, p_2^1, p_3^1, p_4^1) &= (0.880088, -0.474811, 0.309849, 0.950786, 0.303784, 1.97679, \\
&\quad 0.120863, -1.99634), \\
(p_1^2, p_2^2, p_3^2, p_4^2) &= (-0.187158, -0.98233, -0.812997, 0.582268, -1.68902, 1.07109, \\
&\quad -1.67199, -1.09748), \\
(p_1^3, p_2^3, p_3^3, p_4^3) &= (0.987536, -0.157392, 0.606566, 0.795033, -1.20227, -1.59829, \\
&\quad 1.11109, 1.66297).
\end{aligned}$$

The third solution does not define a crossing limit cycle. See the two crossing limit cycles in Figure 2. \square

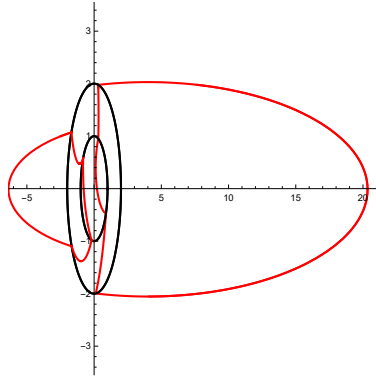


Figure 2: The two crossing limit cycles of the discontinuous piecewise differential systems formed by the linear centers (11), (12) and (13).

Proof of statement (d) of Theorem 1. We construct a system in (C) having exactly one crossing limit cycle. In the bounded region R_1 limited by the circle S_1 we consider the linear differential center

$$(16) \quad \dot{x} = -\frac{x}{3} + \frac{2849y}{1917} + \frac{34}{25}, \quad \dot{y} = -\frac{213x}{77} + \frac{y}{3} + 1,$$

with the first integral

$$H_1(x, y) = 4 \left(\frac{y}{3} - \frac{213x}{77} \right)^2 - \frac{1704}{77} \left(x - \frac{34y}{25} \right) + 16y^2.$$

In the bounded region R_2 limited by the circles \mathbb{S}_1 and \mathbb{S}_2 with $s = 2$ we consider the linear differential center

$$(17) \quad \dot{x} = \frac{11x}{19} + \frac{18292y}{6859} - 4, \quad \dot{y} = -\frac{19x}{2} - \frac{11y}{19} - 10,$$

with the first integral

$$H_2(x, y) = 4 \left(-\frac{19x}{2} - \frac{11y}{19} \right)^2 - 76(4y - 10x) + 100y^2.$$

Finally in the region R_3 outside the circle \mathbb{S}_2 we consider the linear differential center

$$(18) \quad \dot{x} = \frac{23x}{22} - \frac{9117y}{5324} + \frac{2}{5}, \quad \dot{y} = \frac{11x}{9} - \frac{23y}{22} - \frac{131}{88},$$

with the first integral

$$H_3(x, y) = 4 \left(\frac{11x}{9} - \frac{23y}{22} \right)^2 + \frac{88}{9} \left(-\frac{131x}{88} - \frac{2y}{5} \right) + 4y^2.$$

This discontinuous piecewise differential system formed by the linear centers (16), (17) and (18) has exactly one crossing limit cycle, coming from the unique real solution (p_1, p_2, p_3, p_4) with $p_i = (x_i, y_i)$, $p_i \neq p_j$ for $i \neq j$ and $i, j = 1, 2, 3, 4$ that satisfies system (4). Again since we want to use Groebner bases we use the rational parametrization (14) and system (4) reduces to the polynomial system

$$(19) \quad \begin{aligned} f_1 &= 116603t_1^4t_4^4 - 566048t_1^4t_4^3 - 169146t_1^4t_4^2 - 311904t_1^4t_4 + 1214043t_1^4 + 251256t_1^3t_4^4 \\ &\quad + 502512t_1^3t_4^3 + 251256t_1^3t_4^2 + 59434t_1^2t_4^4 - 1132096t_1^2t_4^3 - 685836t_1^2t_4^2 - 623808t_1^2t_4 \\ &\quad + 2254314t_1^2 + 187720t_1t_4^4 + 375440t_1t_4^3 + 187720t_1 - 432117t_4^4 - 566048t_4^3 \\ &\quad - 1266586t_4^2 - 311904t_4 + 665323, \\ f_2 &= 6248781t_1^3t_2^3 - 8413925t_1^3t_2^2 + 3788631t_1^3t_2 - 3690225t_1^3 - 8413925t_1^2t_2^3 - 8708931t_1^2t_2^2 \\ &\quad - 3690225t_1^2t_2 - 6248781t_1^2 + 3788631t_1t_2^3 - 3690225t_1t_2^2 + 1328481t_1t_2 + 1033475t_1 \\ &\quad - 3690225t_2^3 - 6248781t_2^2 + 1033475t_2 - 3788631, \\ f_3 &= -116603t_2^4t_3^4 + 566048t_2^4t_3^3 + 169146t_2^4t_3^2 + 311904t_2^4t_3 - 1214043t_2^4 - 251256t_2^3t_3^4 \\ &\quad - 502512t_2^3t_3^3 - 251256t_2^3t_3^2 - 59434t_2^2t_3^4 + 1132096t_2^2t_3^3 + 685836t_2^2t_3^2 + 623808t_2^2t_3 \\ &\quad - 2254314t_2^2 - 187720t_2t_3^4 - 375440t_2t_3^3 - 187720t_2 + 432117t_3^4 + 566048t_3^3 \\ &\quad + 1266586t_3^2 + 311904t_3 - 665323, \\ f_4 &= 810216t_3^3t_4^3 - 243515t_3^3t_4^2 - 1193544t_3^3t_4 - 713295t_3^3 - 243515t_3^2t_4^3 - 2813976t_3^2t_4^2 \\ &\quad - 713295t_3^2t_4 - 810216t_3^2 - 1193544t_3t_4^3 - 713295t_3t_4^2 - 3197304t_3t_4 - 1183075t_3 \\ &\quad - 713295t_4^3 - 810216t_4^2 - 1183075t_4 + 1193544, \\ f_5 &= 0, \quad f_6 = 0, \quad f_7 = 0, \quad f_8 = 0. \end{aligned}$$

Doing the Groebner basis using Mathematica we can eliminate the variables t_2, t_4 using the equations (f_2, f_3) and (f_1, f_4) , respectively. Moreover, we get two polynomials p_3 and p_4 both of degree 24 with respect to variables t_1 and t_3 . Solving the

system formed by these two polynomials we obtain its two unique real solutions

$$(t_1^1, t_3^1) = (-0.2761369370, -5.638072382), (t_1^2, t_2^2) = (-1.456914010, -0.9512182090).$$

Therefore system (19) has the two solutions

$$(t_1^1, t_2^1, t_3^1, t_4^1) = (-1.4576899, -0.27613694, -0.95176617, -5.6380724),$$

$$(t_1^2, t_2^2, t_3^2, t_4^2) = (-0.27613694, -1.4576899, -5.6380724, -0.95176617),$$

From these two solutions we obtain the solutions of system (4) in the variables (x_i, y_i) , i.e.

$$(p_1^1, p_2^1, p_3^1, p_4^1) = (-0.359971, -0.932963, 0.858302, -0.513146, 0.0987913, -1.99756, \\ -1.878, -0.687824),$$

$$(p_1^2, p_2^2, p_3^2, p_4^2) = (0.858302, -0.513146, -0.359971, -0.932963, -1.878, -0.687824, \\ 0.0987913, -1.99756).$$

But the both solution define the same crossing limit cycle. See this crossing limit cycle of the piecewise differential system in Figure 3. \square

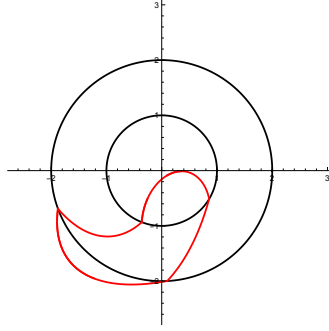


Figure 3: The crossing limit cycle of the discontinuous piecewise differential system formed by the linear centers (16), (17) and (18).

Proof of statement (e) of Theorem 1. To prove statement (e) it is sufficient consider the discontinuous piecewise linear differential system having the linear center $\dot{x} = -y, \dot{y} = x$ in the region R_2 limited by the circles \mathbb{S}_1 and \mathbb{S}_2 with first integral $H(x, y) = x^2 + y^2$, and two arbitrary different centers in the region R_1 limited by the circle \mathbb{S}_1 and in the region R_3 outside the circle \mathbb{S}_2 . Then it is impossible to connect orbits of the region R_1 with orbits of the region R_3 , so they do not exist crossing limit cycle for this discontinuous piecewise differential system. \square

3. APPENDIX

The polynomials p_1, p_2, p_3 and p_4 of the proofs of statements (c) and (d),

$$p_1 = a_1 t_1^4 t_3^4 + a_2 t_1^4 t_3^3 + a_3 t_1^4 t_3^2 + a_4 t_1^4 t_3 + a_5 t_1^4 + a_6 t_1^3 t_3^4 + a_7 t_1^3 t_3^3 + a_8 t_1^3 + a_9 t_1^2 t_3^4 \\ + a_{10} t_1^2 t_3^3 + a_{11} t_1^2 t_3^2 + a_{12} t_1^2 t_3 + a_{13} t_1^2 + a_{14} t_1 t_3^4 + a_{15} t_1 t_3^3 + a_{16} t_1 + a_{17} t_3^4 \\ + a_{18} t_3^3 + a_{19} t_3^2 + a_{20} t_3 + a_{21}$$

where the coefficients a_i are

$a_1 = 54607$	$a_2 = -16415392$	$a_3 = -11558994$	$a_4 = -9045216$
$a_5 = 31880367$	$a_6 = -10198280$	$a_7 = -20396560$	$a_8 = -10198280$
$a_9 = -4653694$	$a_{10} = -32830784$	$a_{11} = -32643804$	$a_{12} = -18090432$
$a_{13} = 58997826$	$a_{14} = -20969016$	$a_{15} = -41938032$	$a_{16} = -20969016$
$a_{17} = -7078753$	$a_{18} = -16415392$	$a_{19} = -25825714$	$a_{20} = -9045216$
$a_{21} = 24747007$			

$$\begin{aligned}
p_2 = & b_1 t_3^{12} t_1^{12} + b_2 t_3^{11} t_1^{12} + b_3 t_3^{10} t_1^{12} + b_4 t_3^9 t_1^{12} + b_5 t_3^8 t_1^{12} + b_6 t_3^7 t_1^{12} + b_7 t_3^6 t_1^{12} + b_8 t_3^5 t_1^{12} \\
& + b_9 t_3^4 t_1^{12} + b_{10} t_3^3 t_1^{12} + b_{11} t_3^2 t_1^{12} + b_{12} t_3 t_1^{12} + b_{13} t_1^{12} + b_{14} t_3^{12} t_1^{11} + b_{15} t_3^{11} t_1^{11} \\
& + b_{16} t_3^{10} t_1^{11} + b_{17} t_3^9 t_1^{11} + b_{18} t_3^8 t_1^{11} + b_{19} t_3^7 t_1^{11} + b_{20} t_3^6 t_1^{11} + b_{21} t_3^5 t_1^{11} + b_{22} t_3^4 t_1^{11} \\
& + b_{23} t_3^3 t_1^{11} + b_{24} t_3^2 t_1^{11} + b_{25} t_3 t_1^{11} + b_{26} t_1^{11} + b_{27} t_3^{12} t_1^{10} + b_{28} t_3^{11} t_1^{10} + b_{29} t_3^{10} t_1^{10} \\
& + b_{30} t_3^9 t_1^{10} + b_{31} t_3^8 t_1^{10} + b_{32} t_3^7 t_1^{10} + b_{33} t_3^6 t_1^{10} + b_{34} t_3^5 t_1^{10} + b_{35} t_3^4 t_1^{10} + b_{36} t_3^3 t_1^{10} \\
& + b_{37} t_3^2 t_1^{10} + b_{38} t_3 t_1^{10} + b_{39} t_1^{10} + b_{40} t_3^{12} t_1^9 + b_{41} t_3^{11} t_1^9 + b_{42} t_3^{10} t_1^9 + b_{43} t_3^9 t_1^9 \\
& + b_{44} t_3^8 t_1^9 + b_{45} t_3^7 t_1^9 + b_{46} t_3^6 t_1^9 + b_{47} t_3^5 t_1^9 + b_{48} t_3^4 t_1^9 + b_{49} t_3^3 t_1^9 + b_{50} t_3^2 t_1^9 + b_{51} t_3 t_1^9 \\
& + b_{52} t_1^9 + b_{53} t_3^{12} t_1^8 + b_{54} t_3^{11} t_1^8 + b_{55} t_3^{10} t_1^8 + b_{56} t_3^9 t_1^8 + b_{57} t_3^8 t_1^8 + b_{58} t_3^7 t_1^8 + b_{59} t_3^6 t_1^8 \\
& + b_{60} t_3^5 t_1^8 + b_{61} t_3^4 t_1^8 + b_{62} t_3^3 t_1^8 + b_{63} t_3^2 t_1^8 + b_{64} t_3 t_1^8 + b_{65} t_1^8 + b_{66} t_3^{12} t_1^7 + b_{67} t_3^{11} t_1^7 \\
& + b_{68} t_3^{10} t_1^7 + b_{69} t_3^9 t_1^7 + b_{70} t_3^8 t_1^7 + b_{71} t_3^7 t_1^7 + b_{72} t_3^6 t_1^7 + b_{73} t_3^5 t_1^7 + b_{74} t_3^4 t_1^7 + b_{75} t_3^3 t_1^7 \\
& + b_{76} t_3^2 t_1^7 + b_{77} t_3 t_1^7 + b_{78} t_1^7 + b_{79} t_3^{12} t_1^6 + b_{80} t_3^{11} t_1^6 + b_{81} t_3^{10} t_1^6 + b_{82} t_3^9 t_1^6 + b_{83} t_3^8 t_1^6 \\
& + b_{84} t_3^7 t_1^6 + b_{85} t_3^6 t_1^6 + b_{86} t_3^5 t_1^6 + b_{87} t_3^4 t_1^6 + b_{88} t_3^3 t_1^6 + b_{89} t_3^2 t_1^6 + b_{90} t_3 t_1^6 + b_{91} t_1^6 \\
& + b_{92} t_3^{12} t_1^5 + b_{93} t_3^{11} t_1^5 + b_{94} t_3^{10} t_1^5 + b_{95} t_3^9 t_1^5 + b_{96} t_3^8 t_1^5 + b_{97} t_3^7 t_1^5 + b_{98} t_3^6 t_1^5 + b_{99} t_3^5 t_1^5 \\
& + b_{100} t_3^4 t_1^5 + b_{101} t_3^3 t_1^5 + b_{102} t_3^2 t_1^5 + b_{103} t_3 t_1^5 + b_{104} t_1^5 + b_{105} t_3^{12} t_1^4 + b_{106} t_3^{11} t_1^4 \\
& + b_{107} t_3^{10} t_1^4 + b_{108} t_3^9 t_1^4 + b_{109} t_3^8 t_1^4 + b_{110} t_3^7 t_1^4 + b_{111} t_3^6 t_1^4 + b_{112} t_3^5 t_1^4 + b_{113} t_3^4 t_1^4 \\
& + b_{114} t_3^3 t_1^4 + b_{115} t_3^2 t_1^4 + b_{116} t_3 t_1^4 + b_{117} t_1^4 + b_{118} t_3^{12} t_1^3 + b_{119} t_3^{11} t_1^3 + b_{120} t_3^{10} t_1^3 \\
& + b_{121} t_3^9 t_1^3 + b_{122} t_3^8 t_1^3 + b_{123} t_3^7 t_1^3 + b_{124} t_3^6 t_1^3 + b_{125} t_3^5 t_1^3 + b_{126} t_3^4 t_1^3 + b_{127} t_3^3 t_1^3 \\
& + b_{128} t_3^2 t_1^3 + b_{129} t_3 t_1^3 + b_{130} t_1^3 + b_{131} t_3^{12} t_1^2 + b_{132} t_3^{11} t_1^2 + b_{133} t_3^{10} t_1^2 + b_{134} t_3^9 t_1^2 \\
& + b_{135} t_3^8 t_1^2 + b_{136} t_3^7 t_1^2 + b_{137} t_3^6 t_1^2 + b_{138} t_3^5 t_1^2 + b_{139} t_3^4 t_1^2 + b_{140} t_3^3 t_1^2 + b_{141} t_3^2 t_1^2 + b_{142} t_3 t_1^2 \\
& + b_{143} t_1^2 + b_{144} t_3^{12} t_1 + b_{145} t_3^{11} t_1 + b_{146} t_3^{10} t_1 + b_{147} t_3^9 t_1 + b_{148} t_3^8 t_1 + b_{149} t_3^7 t_1 \\
& + b_{150} t_3^6 t_1 + b_{151} t_3^5 t_1 + b_{152} t_3^4 t_1 + b_{153} t_3^3 t_1 + b_{154} t_3^2 t_1 + b_{155} t_3 t_1 + b_{156} t_1 \\
& + b_{157} t_3^{12} + b_{158} t_3^{11} + b_{159} t_3^{10} + b_{160} t_3^9 + b_{161} t_3^8 + b_{162} t_3^7 + b_{163} t_3^6 \\
& + b_{164} t_3^5 + b_{165} t_3^4 + b_{166} t_3^3 + b_{167} t_3^2 + b_{168} t_3 + b_{169},
\end{aligned}$$

where the coefficients b_i are

$b_1 = 303991592446001788228938487$	$b_2 = 1408045233262894200409887264$
$b_3 = -610348310045438609985979526$	$b_4 = 150794822399721357288599264$
$b_5 = 3175292247340716691173055833$	$b_6 = -320740208014163780149933504$
$b_7 = -1704899763947849707377573908$	$b_8 = -405573064211186754732584256$
$b_9 = 62276987629682561558117113$	$b_{10} = -175671366937609682473351264$
$b_{11} = -909165874719067126843100166$	$b_{12} = 8410996908736764862587488$
$b_{13} = 67790690035723395857234967$	$b_{14} = 769919768775604394983693704$
$b_{15} = 608614820298158930816555520$	$b_{16} = 1006338413132290284840738864$
$b_{17} = 1431321555103103515287098880$	$b_{18} = 1106920085504028821353449720$
$b_{19} = 1051042237498193081988510720$	$b_{20} = 1785102108481388009724969120$
$b_{21} = 274182185151143817728302080$	$b_{22} = 598987412480135128015248120$
$b_{23} = 77449776729327796556689920$	$b_{24} = -141824611475603542032144336$
$b_{25} = 31603094271432476346355200$	$b_{26} = 173788643378306408181172104$
$b_{27} = 1588372403382159061063149654$	$b_{28} = 8092975591745815633602116544$
$b_{29} = -3248693041709167069185540444$	$b_{30} = 211406878520421069494121024$
$b_{31} = 19938833981071197564188426058$	$b_{32} = -2121423179067511352527553664$
$b_{33} = -10009180546277357760027249288$	$b_{34} = -2156318684700125763185882496$
$b_{35} = 1048296922418733508883452938$	$b_{36} = -905812191897963002200866624$
$b_{37} = -4698504379735191229617889884$	$b_{38} = 62646607676949750357457728$
$b_{39} = 470711522393073093178291734$	$b_{40} = 3275446737045079891833146776$
$b_{41} = 2798389897743457301579527680$	$b_{42} = 3430073858498420097704336016$
$b_{43} = 6384456051429072119840709120$	$b_{44} = 3437215685906732064733269480$
$b_{45} = 4256413077567038680571489280$	$b_{46} = 7086432988050350508983168480$
$b_{47} = 655959555828558576937681920$	$b_{48} = 1805773207367510400930991080$
$b_{49} = 885545955003083626094080$	$b_{50} = -1396578676010723603677244784$
$b_{51} = 102941964007868368998720000$	$b_{52} = 601492540218724645512852376$
$b_{53} = 1863185952562665535144523113$	$b_{54} = 18368537004625498191672760800$
$b_{55} = -10118633311500150875666933562$	$b_{56} = -3645531639364227160097798880$
$b_{57} = 47742911997329850459275889735$	$b_{58} = -7908940207973105675684819520$
$b_{59} = -29927104493544736358479363180$	$b_{60} = -5586703187951216642169113280$
$b_{61} = 1322636002702229535320877735$	$b_{62} = -2216882120199644964482927520$
$b_{63} = -10708672793085961168931909562$	$b_{64} = 104935955560450798251925920$
$b_{65} = 823337441741604282540503113$	$b_{66} = 4610181176067349319811463504$
$b_{67} = 4934488104553140293973765120$	$b_{68} = 673222061320160933366852064$
$b_{69} = 10443232592303538229139911680$	$b_{70} = -1826353774138329805781401680$
$b_{71} = 5118958520150359554781009920$	$b_{72} = 6459378942855282670710355520$
$b_{73} = -1347774333705502370068101120$	$b_{74} = -1787454240730479998434739280$
$b_{75} = -951934381413631957788410880$	$b_{76} = -5963483656776835980091631136$
$b_{77} = 6053984691832031894553600$	$b_{78} = 172744186200068108390253904$
$b_{79} = -1621708105402125393637763148$	$b_{80} = 20878158753803570951808551040$
$b_{81} = -17588058999912760831475813128$	$b_{82} = -12899199805848832415420900480$
$b_{83} = 58653267579916511026612363020$	$b_{84} = -15440556721247650813313611520$
$b_{85} = -50934444886457453503292294000$	$b_{86} = -7665732406157195645450753280$
$b_{87} = -1867587215720614217259928180$	$b_{88} = -2791367357827462082945909120$
$b_{89} = -12614307807012810344397195528$	$b_{90} = 51148822457498709060684160$
$b_{91} = 374525371112801525600025652$	$b_{92} = 2312913197949713418835292976$
$b_{93} = 4148805890329529288800711680$	$b_{94} = -6389500990001080093169282784$
$b_{95} = 7559997156913701546733317120$	$b_{96} = -10609352208188163894569321520$
$b_{97} = 728772418724086020836321280$	$b_{98} = 43899111766812611810796480$
$b_{99} = -4884992862282772005446830080$	$b_{100} = -6208792316160131100545359920$
$b_{101} = -2460343804730642330472913920$	$b_{102} = -9161706662669674635942223584$
$b_{103} = -257769790307956562122368000$	$b_{104} = -1002077214505360541021321424$

$b_{105} = -3971430621041392996112889447$	$b_{106} = 12374289928344354025078994400$
$b_{107} = -12344000458594114924782416922$	$b_{108} = -16330806431518499543110022880$
$b_{109} = 46437110133303585391251181335$	$b_{110} = -14124913545548867479214291520$
$b_{111} = -42477362711277341585174494380$	$b_{112} = -3946994294230016748826649280$
$b_{113} = -33305509685054214583510665$	$b_{114} = -1059408107241328466289647520$
$b_{115} = -4873852326922304202721184922$	$b_{116} = 97755453094933815991727520$
$b_{117} = 378892493096299322127122553$	$b_{118} = 200730767420231523311578024$
$b_{119} = 1644632956482672247915461120$	$b_{120} = -5190506023107083703857888016$
$b_{121} = 2171512967117559308974241280$	$b_{122} = -5226483846546228429709648680$
$b_{123} = -2128889600225717310326983680$	$b_{124} = 3195078713173340252023634720$
$b_{125} = -4473735425890521681580139520$	$b_{126} = -528169211751130452219818280$
$b_{127} = -2097914965047015435859668480$	$b_{128} = -4390595229150387720190227216$
$b_{129} = -279949150017098125665292800$	$b_{130} = -832100248207003813406591576$
$b_{131} = -1962737440728361463430634346$	$b_{132} = 3571645054628274995872119744$
$b_{133} = -1135972992384322472318023644$	$b_{134} = -8698405461151187684087549376$
$b_{135} = 23946171151847913614553237258$	$b_{136} = -4881243111994631139488859264$
$b_{137} = -16439941847018161873601156488$	$b_{138} = 1104550916440195157556415104$
$b_{139} = 2661394009267641937515912138$	$b_{140} = 862790136065306374490947776$
$b_{141} = 1979063068004556038527706916$	$b_{142} = 199041204126846535197342528$
$b_{143} = 760913305201756890114011734$	$b_{144} = -59789326956919740039496520$
$b_{145} = 240540093260984882325542400$	$b_{146} = -865085042503842495052102320$
$b_{147} = 101613906181427386827225600$	$b_{148} = 442302423721227403531571400$
$b_{149} = -944074338868289374965273600$	$b_{150} = 3264005580186614462109154400$
$b_{151} = -1318294267990666805410790400$	$b_{152} = 1902466505411470364808179400$
$b_{153} = -600610360955680350244569600$	$b_{154} = -251184784074299007561286320$
$b_{155} = -87464244753745424300736000$	$b_{156} = -137243848567305068804488520$
$b_{157} = -286675820524507063672675993$	$b_{158} = 391907233944915164121144864$
$b_{159} = 531122523280024395014816794$	$b_{160} = -1560655139996593684095106336$
$b_{161} = 3760122027727895557551059433$	$b_{162} = -88629050773656370280979904$
$b_{163} = -5171315387161144294026388308$	$b_{164} = 1221770371975293894648126144$
$b_{165} = -932340071814670759258887287$	$b_{166} = 617572180718464402590098336$
$b_{167} = 588807353130457320567680154$	$b_{168} = 101734228697173615509243488$
$b_{169} = 241273820025099567960612487$	

$$\begin{aligned}
p_3 = & c_1 t_1^{12} t_3^{12} + c_2 t_1^{12} t_3^{11} + c_3 t_3^{10} t_1^{12} + c_4 t_3^9 t_1^{12} + c_5 t_3^8 t_1^{12} + c_6 t_3^7 t_1^{12} + c_7 t_3^6 t_1^{12} + c_8 t_3^5 t_1^{12} \\
& + c_9 t_3^4 t_1^{12} + c_{10} t_3^3 t_1^{12} + c_{11} t_3^2 t_1^{12} + c_{12} t_3 t_1^{12} + c_{13} t_1^{12} + c_{14} t_3^{12} t_1^{11} + c_{15} t_1^{11} t_3^{11} \\
& + c_{16} t_3^{10} t_1^{11} + c_{17} t_3^9 t_1^{11} + c_{18} t_3^8 t_1^{11} + c_{19} t_3^7 t_1^{11} + c_{20} t_3^6 t_1^{11} + c_{21} t_3^5 t_1^{11} + c_{22} t_3^4 t_1^{11} \\
& + c_{23} t_3^3 t_1^{11} + c_{24} t_3^2 t_1^{11} + c_{25} t_3 t_1^{11} + c_{26} t_1^{11} + c_{27} t_3^{12} t_1^{10} + c_{28} t_3^{11} t_1^{10} + c_{29} t_1^{10} t_3^{10} \\
& + c_{30} t_3^9 t_1^{10} + c_{31} t_3^8 t_1^{10} + c_{32} t_3^7 t_1^{10} + c_{33} t_3^6 t_1^{10} + c_{34} t_3^5 t_1^{10} + c_{35} t_3^4 t_1^{10} + c_{36} t_3^3 t_1^{10} \\
& + c_{37} t_3^2 t_1^{10} + c_{38} t_3 t_1^{10} + c_{39} t_1^{10} + c_{40} t_3^{12} t_1^9 + c_{41} t_3^{11} t_1^9 + c_{42} t_3^{10} t_1^9 + c_{43} t_3^9 t_1^9 \\
& + c_{44} t_3^8 t_1^9 + c_{45} t_3^7 t_1^9 + c_{46} t_3^6 t_1^9 + c_{47} t_3^5 t_1^9 + c_{48} t_3^4 t_1^9 + c_{49} t_3^3 t_1^9 + c_{50} t_3^2 t_1^9 + c_{51} t_3 t_1^9 \\
& + c_{52} t_1^9 + c_{53} t_3^{12} t_1^8 + c_{54} t_3^{11} t_1^8 + c_{55} t_3^{10} t_1^8 + c_{56} t_3^9 t_1^8 + c_{57} t_3^8 t_1^8 + c_{58} t_3^7 t_1^8 + c_{59} t_3^6 t_1^8 \\
& + c_{60} t_3^5 t_1^8 + c_{61} t_3^4 t_1^8 + c_{62} t_3^3 t_1^8 + c_{63} t_3^2 t_1^8 + c_{64} t_3 t_1^8 + c_{65} t_1^8 + c_{66} t_3^{12} t_1^7 + c_{67} t_3^{11} t_1^7
\end{aligned}$$

$$\begin{aligned}
& + c_{68}t_3^{10}t_1^7 + c_{69}t_3^9t_1^7 + c_{70}t_3^8t_1^7 + c_{71}t_3^7t_1^7 + c_{72}t_3^6t_1^7 + c_{73}t_3^5t_1^7 + c_{74}t_3^4t_1^7 + c_{75}t_3^3t_1^7 \\
& + c_{76}t_3^2t_1^7 + c_{77}t_3t_1^7 + c_{78}t_1^7 + c_{79}t_3^{12}t_1^6 + c_{80}t_3^{11}t_1^6 + c_{81}t_3^{10}t_1^6 + c_{82}t_3^9t_1^6 + c_{83}t_3^8t_1^6 \\
& + c_{84}t_3^7t_1^6 + c_{85}t_3^6t_1^6 + c_{86}t_3^5t_1^6 + c_{87}t_3^4t_1^6 + c_{88}t_3^3t_1^6 + c_{89}t_3^2t_1^6 + c_{90}t_3t_1^6 + c_{91}t_1^6 \\
& + c_{92}t_3^{12}t_1^5 + c_{93}t_3^{11}t_1^5 + c_{94}t_3^{10}t_1^5 + c_{95}t_3^9t_1^5 + c_{96}t_3^8t_1^5 + c_{97}t_3^7t_1^5 + c_{98}t_3^6t_1^5 + c_{99}t_3^5t_1^5 \\
& + c_{100}t_3^4t_1^5 + c_{101}t_3^3t_1^5 + c_{102}t_3^2t_1^5 + c_{103}t_3t_1^5 + c_{104}t_1^5 + c_{105}t_3^{12}t_1^4 + c_{106}t_3^{11}t_1^4 \\
& + c_{107}t_3^{10}t_1^4 + c_{108}t_3^9t_1^4 + c_{109}t_3^8t_1^4 + c_{110}t_3^7t_1^4 + c_{111}t_3^6t_1^4 + c_{112}t_3^5t_1^4 + c_{113}t_3^4t_1^4 \\
& + c_{114}t_3^3t_1^4 + c_{115}t_3^2t_1^4 + c_{116}t_3t_1^4 + c_{117}t_1^4 + c_{118}t_3^{12}t_1^3 + c_{119}t_3^{11}t_1^3 + c_{120}t_3^{10}t_1^3 \\
& + c_{121}t_3^9t_1^3 + c_{122}t_3^8t_1^3 + c_{123}t_3^7t_1^3 + c_{124}t_3^6t_1^3 + c_{125}t_3^5t_1^3 + c_{126}t_3^4t_1^3 + c_{127}t_3^3t_1^3 \\
& + c_{128}t_3^2t_1^3 + c_{129}t_3t_1^3 + c_{130}t_1^3 + c_{131}t_3^{12}t_1^2 + c_{132}t_3^{11}t_1^2 + c_{133}t_3^{10}t_1^2 + c_{134}t_3^9t_1^2 \\
& + c_{135}t_3^8t_1^2 + c_{136}t_3^7t_1^2 + c_{137}t_3^6t_1^2 + c_{138}t_3^5t_1^2 + c_{139}t_3^4t_1^2 + c_{140}t_3^3t_1^2 + c_{141}t_3^2t_1^2 \\
& + c_{142}t_3t_1^2 + c_{143}t_1^2 + c_{144}t_3^{12}t_1 + c_{145}t_3^{11}t_1 + c_{146}t_3^{10}t_1 + c_{147}t_3^9t_1 + c_{148}t_3^8t_1 \\
& + c_{149}t_3^7t_1 + c_{150}t_3^6t_1 + c_{151}t_3^5t_1 + c_{152}t_3^4t_1 + c_{153}t_3^3t_1 + c_{154}t_3^2t_1 + c_{155}t_3t_1 \\
& + c_{156}t_1 + c_{157}t_3^{12} + c_{158}t_3^{11} + c_{159}t_3^{10} + c_{160}t_3^9 + c_{161}t_3^8 + c_{162}t_3^7 + c_{163}t_3^6 \\
& + c_{164}t_3^5 + c_{165}t_3^4 + c_{166}t_3^3 + c_{167}t_3^2 + c_{168}t_3 + c_{169},
\end{aligned}$$

where the coefficients c_i are

$c_1 = 492827014259845915299608227319536637731419$	$c_2 = -2438080869483823805524125927423327101760864$
$c_3 = -583967369729745691407332725982065481194222$	$c_4 = -14030753213599793266190916869868749229728928$
$c_5 = -2112398880888992440649575226736629098193739$	$c_6 = -20704537678061895220347710778183777494702272$
$c_7 = 16314059677623217238964322331384947587760444$	$c_8 = -2952515995403178561164076745475852546919488$
$c_9 = 31162658776998996833900763726327531493653461$	$c_{10} = 2995100914173664161345254044563304380156448$
$c_{11} = 8694194157488590194646535078660583863610258$	$c_{12} = 252497957065961189248293575799739160341728$
$c_{13} = 1082715163783039276166468232443383736985339$	$c_{14} = 3492602981916068139510311697321894896336160$
$c_{15} = -1086772216910382971847827935580184122246400$	$c_{16} = 19104079234490622993495661251498388385943360$
$c_{17} = -6479921110772705738680288056214355877292800$	$c_{18} = 47513651424426686589606400322923368008940000$
$c_{19} = -8645153532208056857462556794504613480307200$	$c_{20} = 71743869842303927706128932848762131389449600$
$c_{21} = -490954940952138178301360594813097836428800$	$c_{22} = 61443705627491408447388292352607012026700000$
$c_{23} = 4467727356180986120302936646931141870124800$	$c_{24} = 23145629144821074004398951377060529369111360$
$c_{25} = 1706677658787390207974200567873797981292800$	$c_{26} = 1543618187781461527518541104468774212528160$
$c_{27} = -2323816561894020815291862412380876186228962$	$c_{28} = -15221232929930773458456472178578407675326784$
$c_{29} = -34262923829620036787064124222157609493388588$	$c_{30} = -84967373635139092702075928473474253572696768$
$c_{31} = -84763593152089934929362358545649791350450174$	$c_{32} = -129428210040115971570675908069145369017890432$
$c_{33} = -3289163113523431857186036315376701628705256$	$c_{34} = -31343870431795582756934128868875937872224128$
$c_{35} = 113595081364525278063153886360041182910873026$	$c_{36} = 6148329632500583640777991728611485795069888$
$c_{37} = 31478112592247655368997669319792092455630292$	$c_{38} = -1689390982001220570825035447768473853226432$
$c_{39} = 7427447651187831461330285031989406759342558$	$c_{40} = 13383012203227907917824621303121452747105648$
$c_{41} = -142155928485548360829961693156575825875200$	$c_{42} = 74294378073956355629576991136310304510349088$
$c_{43} = -13568022171936691683662440860954055825350400$	$c_{44} = 181447380215363356673809177074365638872101520$
$c_{45} = -10409412470586033575120823305984180370649600$	$c_{46} = 263823724112907293601299014675982880399028160$
$c_{47} = 24084297326297726275359823584854414717401600$	$c_{48} = 210938020177403305017237463430401479109381520$
$c_{49} = 32232539800884528345582162109628026374726400$	$c_{50} = 65020571727123825284877363267494066440973088$
$c_{51} = 9885292891081911817933994386586911286150400$	$c_{52} = -2629738682007095093117892728101319378238352$
$c_{53} = -8585634321747039597044812015468889237484059$	$c_{54} = -38138639261621573185878772093209972833203360$
$c_{55} = -99459097371239746361552224007534640828174994$	$c_{56} = -208662958402028599241398852262553107190714720$
$c_{57} = -250110400612387426498394945973529777602342645$	$c_{58} = -325004178824310003530726853593163716251753280$
$c_{59} = -58308874505884115504607589998842526587829820$	$c_{60} = -99112247874103864818271271760567769256669120$
$c_{61} = 244442364943242843598877630775076244637925355$	$c_{62} = -6383670908668405183963835675976378354920480$
$c_{63} = 77409215265931032806019933596610819345940206$	$c_{64} = -10500086996941854444601098016745126992493280$
$c_{65} = 26917647511261211074128350933511426131836741$	$c_{66} = 19094043267862933710085655825380950597913792$
$c_{67} = 2772610408987514284138792210269974004083200$	$c_{68} = 107352615889938001273408807761836539829959552$
$c_{69} = -435715139505203306977082946120434998233600$	$c_{70} = 255794639028007854636979038401682749241718080$
$c_{71} = 25051281408735678347817878192373970590233600$	$c_{72} = 356024253763926654472363257956615418049552640$
$c_{73} = 83647205771681762950064507358973292712934400$	$c_{74} = 257807992454289979345818756939808912045238080$
$c_{75} = 76534654714198106208073087304993063641817600$	$c_{76} = 48522959886319009715933789759605571346375552$
$c_{77} = 21147055899744739196942333294784150521433600$	$c_{78} = -20796845209977102231177595688815082658982208$

$c_{79} = -9849593356850675629057265912349452548668636$	$c_{80} = -49076228308296251339217265678542819154922880$
$c_{81} = -118477440137686412135260416080073153484440936$	$c_{82} = -265467476778339568055632356837203309190789760$
$c_{83} = -296055099828644208169001933424649463930059620$	$c_{84} = -419598797741386963273170324312105825561834240$
$c_{85} = -46194302401357875971815524917860928049905840$	$c_{86} = -141858835155420258253205093148770087801544960$
$c_{87} = 335802656231088162646087412919754618878724380$	$c_{88} = -27027475103068849017876051450881071906131840$
$c_{89} = 116934505413169408583611402486349874317120664$	$c_{90} = -20041261590291352793028539025577567630554240$
$c_{91} = 46395112003948398478782053411467226382777764$	$c_{92} = 13074535060747681538889606780253951477429088$
$c_{93} = 6376784916240002535742343118761989697036800$	$c_{94} = 74601482244135768710184485014226207152977728$
$c_{95} = 17109988613179601769992721432844435088473600$	$c_{96} = 171662240237361494428105915501401410658345120$
$c_{97} = 54418430666371151454317072290336175287526400$	$c_{98} = 224471342258156540611335495031295585356184960$
$c_{99} = 104429153609590278533044694929123115854105600$	$c_{100} = 136207429899252139163881906944016285022825120$
$c_{101} = 82159045091585497104397343124805497714342400$	$c_{102} = -8910406882712120111403793589855870289838272$
$c_{103} = 21415118451426770791419342171936111756326400$	$c_{104} = -30781787577781125920761242770005724939274912$
$c_{105} = -5082172265702464585247355603705187123392459$	$c_{106} = -34122512559156973906757888417258486428403360$
$c_{107} = -71107324421539628081805956301982089608425394$	$c_{108} = -184021892951246111748373441843811156181114720$
$c_{109} = -177778722876474358884925292729102608494168645$	$c_{110} = -293178985563079241017830172614709547461353280$
$c_{111} = 5630357503192470717671281385280068729202180$	$c_{112} = -100671692955629954960193684385262770495069120$
$c_{113} = 263758420017792989370778900707430135626099355$	$c_{114} = -26494512564605919430001480565518515188520480$
$c_{115} = 92812247919220694808917872229478948389689806$	$c_{116} = -17851229058440404249724176045928558402093280$
$c_{117} = 39460490256819789181883049067509939501928341$	$c_{118} = 4321739489821064172804252764526646351858592$
$c_{119} = 4183362550647183634987602511988674689555200$	$c_{120} = 25116824553329108660182684756403623031756352$
$c_{121} = 13654639510991575419668368541305731256710400$	$c_{122} = 54948306158767587465505621794879169295242080$
$c_{123} = 35271134885231417852010000752136784723289600$	$c_{124} = 65160775836306878710834209738570954892576640$
$c_{125} = 56708653914086864208841965094809501621158400$	$c_{126} = 26310914036538686895342134341455570473962080$
$c_{127} = 41305648463209858365389389538488200651513600$	$c_{128} = -22720960628492070743883780548143283596467648$
$c_{129} = 10396852474010020223876659166498427186489600$	$c_{130} = -18024319923983421906518894954030091793197408$
$c_{131} = -1125002969761079454640643743261266930181922$	$c_{132} = -12093217807160562080241462843376330958782784$
$c_{133} = -21265579686952485525535597460059392318450348$	$c_{134} = -65640840281491212382211081078066041067608768$
$c_{135} = -53834367995688054524441882263226165499440574$	$c_{136} = -104664469067029119807763510989132303122978432$
$c_{137} = 16079375481294603024166020357060126186891544$	$c_{138} = -33211615754855752016170992494961780532576128$
$c_{139} = 105734685200422480615509581760981255520282626$	$c_{140} = -10212326132139845778048583857976930772738112$
$c_{141} = 34774765944655369305599473438282827293288532$	$c_{142} = -7617078681372297909152188388464117654314432$
$c_{143} = 1602341156649107222885282323531878871069598$	$c_{144} = 546974052057527699345306472081444121573808$
$c_{145} = 913975111339860772366185361073050700230400$	$c_{146} = 3254472818263915389281563228566178584332448$
$c_{147} = 3197036819470756287680716967080561118060800$	$c_{148} = 65890098049730444595065681967128584515453520$
$c_{149} = 7668244565973921463169073165654146916339200$	$c_{150} = 6556544600134649349878176201217168576365760$
$c_{151} = 11351453808928484172200593344992165926156800$	$c_{152} = -140804311839602487680472717135922086306480$
$c_{153} = 7916445641118925243785908255978882074387200$	$c_{154} = -6585722983165385395068723479751240032115552$
$c_{155} = 1950174690033467019439856470633352646739200$	$c_{156} = -3769885109957790462639499772041999422138192$
$c_{157} = -68684423999952508585735482541688566934421$	$c_{158} = -1691446355125215979754414262605771425664864$
$c_{159} = -2527061143852823572625761565219155128485262$	$c_{160} = -9360086128095368883956155412510591229536928$
$c_{161} = -6630305236928942947664202400225894960245339$	$c_{162} = -14803774783080914882705006384326456494894272$
$c_{163} = 4055345117805025211365989277907354279147644$	$c_{164} = -3677864516934594686681181621301413907687488$
$c_{165} = 16590212427743568372365743735873656177201861$	$c_{166} = -1208389043607450702526581383314927720515552$
$c_{167} = 4600219955329106189093067501246646716799218$	$c_{168} = -1248913155348573767928782374936212903850272$
$c_{169} = 2438796771237462754231229636119130017439499$	

$$\begin{aligned}
p_4 = & d_1 t_3^{12} t_1^{12} + d_2 t_3^{11} t_1^{12} + d_3 t_3^{10} t_1^{12} + d_4 t_3^9 t_1^{12} + d_5 t_3^8 t_1^{12} + d_6 t_3^7 t_1^{12} + d_7 t_3^6 t_1^{12} + d_8 t_3^5 t_1^{12} + d_9 t_3^4 t_1^{12} + d_{10} t_3^3 t_1^{12} \\
& + d_{11} t_3^2 t_1^{12} + d_{12} t_3 t_1^{12} + d_{13} t_1^{12} + d_{14} t_3^{12} t_1^{11} + d_{15} t_3^{11} t_1^{11} + d_{16} t_3^{10} t_1^{11} + d_{17} t_3^9 t_1^{11} + d_{18} t_3^8 t_1^{11} + d_{19} t_3^7 t_1^{11} \\
& + d_{20} t_3^6 t_1^{11} + d_{21} t_3^5 t_1^{11} + d_{22} t_3^4 t_1^{11} + d_{23} t_3^3 t_1^{11} + d_{24} t_3^2 t_1^{11} + d_{25} t_3 t_1^{11} + d_{26} t_1^{11} + d_{27} t_3^{12} t_1^{10} + d_{28} t_3^{11} t_1^{10} \\
& + d_{29} t_3^{10} t_1^{10} + d_{30} t_3^9 t_1^{10} + d_{31} t_3^8 t_1^{10} + d_{32} t_3^7 t_1^{10} + d_{33} t_3^6 t_1^{10} + d_{34} t_3^5 t_1^{10} + d_{35} t_3^4 t_1^{10} + d_{36} t_3^3 t_1^{10} + d_{37} t_3^2 t_1^{10} \\
& + d_{38} t_3 t_1^{10} + d_{39} t_1^{10} + d_{40} t_3^{12} t_1^9 + d_{41} t_3^{11} t_1^9 + d_{42} t_3^{10} t_1^9 + d_{43} t_3^9 t_1^9 + d_{44} t_3^8 t_1^9 + d_{45} t_3^7 t_1^9 + d_{46} t_3^6 t_1^9 + d_{47} t_3^5 t_1^9 \\
& + d_{48} t_3^4 t_1^9 + d_{49} t_3^3 t_1^9 + d_{50} t_3^2 t_1^9 + d_{51} t_3 t_1^9 + d_{52} t_1^9 + d_{53} t_3^{12} t_1^8 + d_{54} t_3^{11} t_1^8 + d_{55} t_3^{10} t_1^8 + d_{56} t_3^9 t_1^8 + d_{57} t_3^8 t_1^8 \\
& + d_{58} t_3^7 t_1^8 + d_{59} t_3^6 t_1^8 + d_{60} t_3^5 t_1^8 + d_{61} t_3^4 t_1^8 + d_{62} t_3^3 t_1^8 + d_{63} t_3^2 t_1^8 + d_{64} t_3 t_1^8 + d_{65} t_1^8 + d_{66} t_3^{12} t_1^7 + d_{67} t_3^{11} t_1^7 \\
& + d_{68} t_3^{10} t_1^7 + d_{69} t_3^9 t_1^7 + d_{70} t_3^8 t_1^7 + d_{71} t_3^7 t_1^7 + d_{72} t_3^6 t_1^7 + d_{73} t_3^5 t_1^7 + d_{74} t_3^4 t_1^7 + d_{75} t_3^3 t_1^7 + d_{76} t_3^2 t_1^7 + d_{77} t_3 t_1^7 \\
& + d_{78} t_1^7 + d_{79} t_3^{12} t_1^6 + d_{80} t_3^{11} t_1^6 + d_{81} t_3^{10} t_1^6 + d_{82} t_3^9 t_1^6 + d_{83} t_3^8 t_1^6 + d_{84} t_3^7 t_1^6 + d_{85} t_3^6 t_1^6 + d_{86} t_3^5 t_1^6 + d_{87} t_3^4 t_1^6 \\
& + d_{88} t_3^3 t_1^6 + d_{89} t_3^2 t_1^6 + d_{90} t_3 t_1^6 + d_{91} t_1^6 + d_{92} t_3^{12} t_1^5 + d_{93} t_3^{11} t_1^5 + d_{94} t_3^{10} t_1^5 + d_{95} t_3^9 t_1^5 + d_{96} t_3^8 t_1^5 + d_{97} t_3^7 t_1^5
\end{aligned}$$

$$\begin{aligned}
& + d_{98}t_3^6t_1^5 + d_{99}t_3^5t_1^5 + d_{100}t_3^4t_1^5 + d_{101}t_3^3t_1^5 + d_{102}t_3^2t_1^5 + d_{103}t_3t_1^5 + d_{104}t_1^5 + d_{105}t_3^{12}t_1^4 \\
& + d_{106}t_3^{11}t_1^4 + d_{107}t_3^{10}t_1^4 + d_{108}t_3^9t_1^4 + d_{109}t_3^8t_1^4 + d_{110}t_3^7t_1^4 + d_{111}t_3^6t_1^4 + d_{112}t_3^5t_1^4 + d_{113}t_3^4t_1^4 \\
& + d_{114}t_3^3t_1^4 + d_{115}t_3^2t_1^4 + d_{116}t_3t_1^4 + d_{117}t_1^4 + d_{118}t_3^{12}t_1^3 + d_{119}t_3^{11}t_1^3 + d_{120}t_3^{10}t_1^3 + d_{121}t_3^9t_1^3 \\
& + d_{122}t_3^8t_1^3 + d_{123}t_3^7t_1^3 + d_{124}t_3^6t_1^3 + d_{125}t_3^5t_1^3 + d_{126}t_3^4t_1^3 + d_{127}t_3^3t_1^3 + d_{128}t_3^2t_1^3 + d_{129}t_3t_1^3 \\
& + d_{130}t_1^3 + d_{131}t_3^{12}t_1^2 + d_{132}t_3^{11}t_1^2 + d_{133}t_3^{10}t_1^2 + d_{134}t_3^9t_1^2 + d_{135}t_3^8t_1^2 + d_{136}t_3^7t_1^2 + d_{137}t_3^6t_1^2 \\
& + d_{138}t_3^5t_1^2 + d_{139}t_3^4t_1^2 + d_{140}t_3^3t_1^2 + d_{141}t_3^2t_1^2 + d_{142}t_3t_1^2 + d_{143}t_1^2 + d_{144}t_3^{12}t_1 + d_{145}t_3^{11}t_1 \\
& + d_{146}t_3^{10}t_1 + d_{147}t_3^9t_1 + d_{148}t_3^8t_1 + d_{149}t_3^7t_1 + d_{150}t_3^6t_1 + d_{151}t_3^5t_1 + d_{152}t_3^4t_1 + d_{153}t_3^3t_1 \\
& + d_{154}t_3^2t_1 + d_{155}t_3t_1 + d_{156}t_1 + d_{157}t_3^{12} + d_{158}t_3^{11} + d_{159}t_3^{10} + d_{160}t_3^9 + d_{161}t_3^8 + d_{162}t_3^7 \\
& + d_{163}t_3^6 + d_{164}t_3^5 + d_{165}t_3^4 + d_{166}t_3^3 + d_{167}t_3^2 + d_{168}t_3 + d_{169},
\end{aligned}$$

where the coefficients d_i are

$d_1 = 9658413369501255163276163798755600773$	$d_2 = 35317295585910647321645419281661863456$
$d_3 = 15882166700089440555049128988946135766$	$d_4 = -105388392832058172214709783809655818464$
$d_5 = -350325349443127055311977332321980548393$	$d_6 = -428381522780610943179781647065840260416$
$d_7 = -15809160534680777659466556344143159372$	$d_8 = 147223860351680840789833153696442108096$
$d_9 = -6615219861428775459196868998657919693$	$d_{10} = -130457607152598424884258075670839593696$
$d_{11} = -66028157665812914294761894112469629034$	$d_{12} = -18827813456539996093060500369404280288$
$d_{13} = -4438557146599625422697512301083785327$	$d_{14} = -18862342571408613262814141417644415424$
$d_{15} = -27098581623582545267174529043905218880$	$d_{16} = -164348132527117286631607204863806542344$
$d_{17} = -108963336738218450613804747575390174400$	$d_{18} = -302829116835585602363956236127401975960$
$d_{19} = -206842844152946337723563587261549468800$	$d_{20} = -273865734699193772482856811909138308880$
$d_{21} = -231398698055413101271840354030223548800$	$d_{22} = -214120289749485949651766944724108262960$
$d_{23} = -142629302486948265492336291357833294400$	$d_{24} = -109393235536598237729637222417243283944$
$d_{25} = -36208693469845596597429306057674258880$	$d_{26} = -11795353606429131565563916921033480824$
$d_{27} = 57729284157066353866155204954743858226$	$d_{28} = 234656987160644447922639514794924595296$
$d_{29} = 121820635321241028646077092132716546224$	$d_{30} = -559506549687102693511196488061990037984$
$d_{31} = -2118567476935918029212416222051124573838$	$d_{32} = -2477495357417913260398615281171412696896$
$d_{33} = -203625537106801792577927445360122094672$	$d_{34} = 947931986191336347426324901057572154176$
$d_{35} = -102183554730617865953018442800748720138$	$d_{36} = -754420171555262600826706436279072729376$
$d_{37} = -395268568494608310922992670905533033376$	$d_{38} = -106507047854476707871067577816742947168$
$d_{39} = -28645092258199231344060934367609146074$	$d_{40} = -73970283591967734228551806698844431876$
$d_{41} = -134189170000435600917229454708240589120$	$d_{42} = -676324943209845484012609069384477401456$
$d_{43} = -513753920087134271811361522644923025600$	$d_{44} = -1166568377609732222391110648329955513540$
$d_{45} = -912386100447926698984152962989285251200$	$d_{46} = -943207740013608747433032966943709157120$
$d_{47} = -966450443162048947134195117857859491200$	$d_{48} = -766862279563437264907714826444795926540$
$d_{49} = -579812415326676879975230163786352385600$	$d_{50} = -432571835027556162389038838648966019856$
$d_{51} = -146183322525855960931055940981095709120$	$d_{52} = -44703577485873172307303593503556706476$
$d_{53} = 191315042643624893059513334646493610352$	$d_{54} = 679843111749526518876465654522124399680$
$d_{55} = 740250951786465306163790653404374475432$	$d_{56} = -1051495204182701241337520378923832357760$
$d_{57} = -4567298913320431082807614362880426101740$	$d_{58} = -5601443897393791581320665217099419427840$
$d_{59} = 49378570712444040552799773994988051760$	$d_{60} = 2974683728903811939898123617369472019840$
$d_{61} = 210689382944802130009482394572683194760$	$d_{62} = -1530144929589544094017213291483429106240$
$d_{63} = -711268829965776415000812749116172887768$	$d_{64} = -176991913799198956655158497920363796480$
$d_{65} = -42828243453612253016847113385061063948$	$d_{66} = -71403228819803075647235037594292987164$
$d_{67} = -276340325428861805181413653270972231680$	$d_{68} = -826830748066137855145509801288955145784$
$d_{69} = -958698623616697982429997693878013158400$	$d_{70} = -1028876778950424268929788905061289143060$
$d_{71} = -1448686586234176482148961046779916876800$	$d_{72} = -230538758872980450035127216361479577680$
$d_{73} = -1300323422737240100069148907901297996800$	$d_{74} = -357915239475244484172345572739976525060$
$d_{75} = -707679954094433657504337184620036838400$	$d_{76} = -433986950721373872114326335761653083384$
$d_{77} = -173684819403533862335565282891614791680$	$d_{78} = -33161210415020348545593838016529151564$
$d_{79} = 350871651626256743176411109116152088858$	$d_{80} = 1089714202226686546548282214306454394880$
$d_{81} = 1859003338875912157376044120131439040308$	$d_{82} = -725364496690152082851904149676874420480$
$d_{83} = -4615250140408917891828870452214289122690$	$d_{84} = -6330834902933215056529571936973326750720$
$d_{85} = 1225792967970035594750060525069677451320$	$d_{86} = 5112490037263997374490230331701725099520$
$d_{87} = 1247826542758169314920002701108756600310$	$d_{88} = -1370741153366541356303913048202579205120$
$d_{89} = -381782134472456115772808601958238312492$	$d_{90} = -68397626439287293434385493446302240000$
$d_{91} = 4254886772984477457428118126258612658$	$d_{92} = 12133046104963011972623401053615565724$
$d_{93} = -291844218340594690554353431215560197120$	$d_{94} = -244846173890652927950500933729293270856$
$d_{95} = -885120217451118186476551063313792665600$	$d_{96} = 431235518431186969261268616512628765460$
$d_{97} = -978105138303039807026880710070743731200$	$d_{98} = 1627071619724896597197103977159274604880$
$d_{99} = -488292671827481256785879223370613171200$	$d_{100} = 944651435641691915646022513671241027460$

$d_{101} = -123529706847342385625553413706364825600$	$d_{102} = 3005923191648462833405263688333050744$
$d_{103} = -20066174212377439944357268308262917120$	$d_{104} = 24264677572886400700741149075829026124$
$d_{105} = 298947609281456947471060011296941304352$	$d_{106} = 1015571184479274887945888415901154623680$
$d_{107} = 1870011388733431021645249868715687489432$	$d_{108} = 113151275426306775770268615262777482240$
$d_{109} = -3004180338976207791453575875424774221740$	$d_{110} = -3841787582529920346766663371050306307840$
$d_{111} = 747067120340221896655180543948731311760$	$d_{112} = 4553872354179739072488325653565854579840$
$d_{113} = 958780374827108496447812202232516424760$	$d_{114} = -670791068842013309414772587386715106240$
$d_{115} = -58684982615440335953157510489007593768$	$d_{116} = 33911229657016281872715783221501147520$
$d_{117} = 18362457607105685410733138741227700052$	$d_{118} = 23269440792038131824508691360632637556$
$d_{119} = -155931232278433642897511076231639961280$	$d_{120} = 25419288014496715869287060255471171536$
$d_{121} = -404343560709587259839531397696287086400$	$d_{122} = 592419701955607140632641978191485710740$
$d_{123} = -225480698930525274925672195451130492800$	$d_{124} = 1269518902180883447935607718727406758720$
$d_{125} = 215576642185627504458109575682347427200$	$d_{126} = 761647174705759245502456606068942163740$
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$d_{167} = -9990020729345629479621282675926173754$	$d_{168} = -28380134726799150205020914227035374688$
$d_{169} = 1756153198289277660124414662768185553$	

4. ACKNOWLEDGEMENTS

The second author is supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The third author is partially supported by FCT/Portugal through UID/MAT/04459/2013. The present paper is part of the thesis of the first author.

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