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Asymptotic expansion of the Dulac map and time for unfoldings of hyperbolic saddles: general setting

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Abstract. Given a $\mathscr{C}^\infty$ family of planar vector fields $\{X_{\hat\mu}\}_{\hat\mu\in\hat W}$ having a hyperbolic saddle, we study the Dulac map $D(s;\hat\mu)$ and the Dulac time $T(s;\hat\mu)$ between two transverse sections located in the separatrices at arbitrary distance from the saddle. We show (Theorems A and B, respectively) that, for any $\hat\mu_0\in\hat W$ and $L>0$, the functions $T(s;\hat\mu)$ and $D(s;\hat\mu)$ have an asymptotic expansion at $s=0$ for $\hat\mu\approx\hat\mu_0$ with the remainder being uniformly L-flat with respect to the parameters. The principal part of both asymptotic expansions is given in a monomial scale containing a deformation of the logarithm, the so-called Roussarie-Ecalle compensator. The coefficients of these monomials are $\mathscr{C}^\infty$ functions "universally" defined, meaning that their existence is established before fixing the flatness L of the remainder and the unfolded parameter $\hat\mu_0$. Moreover the flatness L of the remainder is preserved after any derivation with respect to the parameters. We also provide (Theorem C) an explicit upper bound for the number of zeros of $T'(s;\hat\mu)$ bifurcating from $s=0$ as $\hat\mu\approx\hat\mu_0$. This result enables to tackle finiteness problems for the number of critical periodic orbits along the lines of those theorems on finite cyclicity around Hilbert's 16th problem. As an application we prove two finiteness results (Corollaries D and E) about the number of critical periodic orbits of polynomial vector fields.

Contents

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1 Introduction and statement of main results

In this paper we study $\mathscr{C}^\infty$ unfoldings of planar vector fields having a hyperbolic saddle. The study of the so-called Dulac map of the saddle has attracted the attention of many authors (see for instance [5, 6, 7, 9, 22, 26] and references there in), mainly due to its close connection with Hilbert's 16th problem (see [10, 28] for details). The *Dulac map* $D(\,\cdot\,;\hat{\mu})$ of a saddle is the transition map from a transverse section Σ_σ in its stable separatrix to a transverse section Σ_τ in its unstable separatrix, whereas the *Dulac time* $T(\,\cdot\,;\hat{\mu})$ is the time that spends the flow to do this transition, see Figure 1. In a previous paper, see [19], we proved a local version of some results presented here. By local we mean that Σ_σ and Σ_τ cannot be at arbitrary distance from the saddle but close enough in order that a suitable (local) normal form for the saddle unfolding can be used. As we will see, those local results constitute a basic building block for the more general ones that we will prove here.

A *polycycle* is a graphic Γ with a well-defined return map on one of its sides. The *cyclicity* of Γ in an unfolding of the vector field is the maximum number of limit cycles that bifurcate from it. If the polycycle is hyperbolic then the return map can be written as the composition of the Dulac maps associated to the passage through each one of its vertices. The limit cycles are fixed points of the return map and to study its number (or even to prove that there are finitely many) a key tool is the asymptotic expansion of the Dulac maps. To this end it is essential that the remainder of the expansion is uniformly flat with respect to the unfolding parameters (see [28, Chapter 5] and the references in the previous paragraph). In case that the return map of Γ is the identity then there is an annulus foliated by periodic orbits where the *period function* (i.e., the time of the return map) is defined. In this context the object of study are the so-called *critical periodic orbits*, which are the critical points of the period function. Similarly as with Hilbert's 16th problem, it arises the notion of *criticality* of a polycycle Γ, i.e., the maximum number of critical periodic orbits that bifurcate from Γ, see [13, 24]. In the same way as for the cyclicity, an asymptotic expansion of the Dulac time with remainder uniformly flat constitutes a key tool to investigate the criticality of a polycycle.

We will consider a $\mathscr{C}^\infty$ unfolding of a hyperbolic saddle with poles along its separatrices. The reason why we permit this "polar" factor is because, when dealing with polynomial vector fields, a special attention must be paid to the study of those polycycles with vertices at infinity in the Poincaré disc. The factor can come from the line at infinity in a saddle at infinity or, more generally, appear in a divisor after desingularizing more general singular points at infinity of a polycycle. It is important to remark that (by means of a reparametrization of time) this factor can be neglected to study the Dulac map but, on the contrary, this cannot be done when dealing with the Dulac time. More precisely, setting $\hat{\mu} := (\lambda, \mu) \in \hat{W} := (0, +\infty) \times W$ with W an open set of $\mathbb{R}^N$, let us take the family of vector fields $\{X_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$ with

$$X_{\hat{\mu}}(x, y) := \frac{1}{x^{n_1} y^{n_2}} \Big(xP(x, y; \hat{\mu})\partial_x + yQ(x, y; \hat{\mu})\partial_y \Big), \tag{1}$$

where

- $n := (n_1, n_2) \in \mathbb{Z}^2_{\geq 0}$,
- P and Q belong to $\mathscr{C}^\infty(V \times \hat{W})$ for some open set V of $\mathbb{R}^2$ containing the origin,
- $P(x, 0; \hat{\mu}) > 0$ and $Q(0, y; \hat{\mu}) < 0$ for all $(x, 0), (0, y) \in V$ and $\hat{\mu} \in \hat{W}$,
- $\lambda = -\frac{Q(0,0;\hat{\mu})}{P(0,0;\hat{\mu})}$.

Note that the hyperbolicity ratio of the saddle is an independent parameter although in the applications we will have $\lambda = \lambda(\mu)$. The hyperbolicity ratio turns out to be the ruling parameter in our study and, besides, having it uncoupled from the rest of parameters simplifies the notation in the computations we shall deal with. Furthermore, see Remark 1.5, we do not lose generality assuming that it is uncoupled.

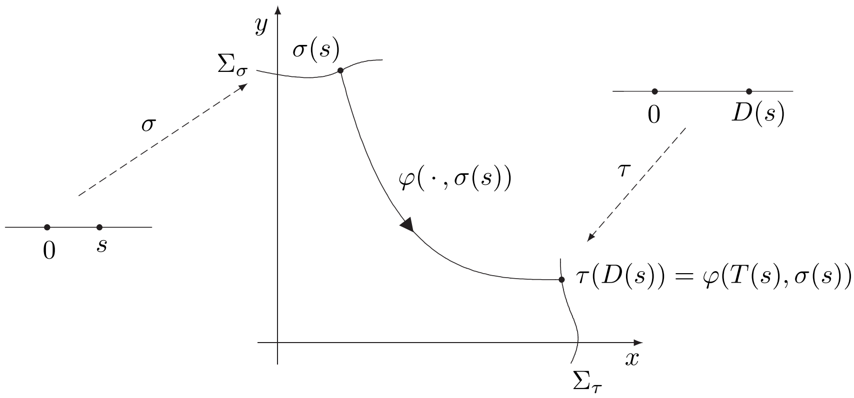

Figure 1: Definition of $T(\,\cdot\,;\hat{\mu})$ and $D(\,\cdot\,;\hat{\mu})$ in Theorems A and B, respectively.

Let $\sigma\colon(-\varepsilon,\varepsilon)\times\hat{W}\to\Sigma_\sigma$ and $\tau\colon(-\varepsilon,\varepsilon)\times\hat{W}\to\Sigma_\tau$ be two $\mathscr{C}^\infty$ transverse sections to $X_{\hat{\mu}}$ defined by

$$\sigma(s;\hat{\mu})=\big(\sigma_1(s;\hat{\mu}),\sigma_2(s;\hat{\mu})\big)\text{ and }\tau(s;\hat{\mu})=\big(\tau_1(s;\hat{\mu}),\tau_2(s;\hat{\mu})\big)$$

such that $\sigma_1(0;\hat{\mu})=0$ and $\tau_2(0;\hat{\mu})=0$ for all $\hat{\mu}\in\hat{W}$. We denote the Dulac map and Dulac time of $X_{\hat{\mu}}$ between the transverse sections Σ_σ and Σ_τ by $R(\,\cdot\,;\hat{\mu})$ and $T(\,\cdot\,;\hat{\mu})$, respectively. More precisely, see Figure 1, if $\varphi(t,p_0;\hat{\mu})$ is the solution of $X_{\hat{\mu}}$ passing through $p_0\in V$ at $t=0$, for each $s\in(0,\varepsilon)$ we define $R(s;\hat{\mu})$ and $T(s;\hat{\mu})$ by means of the relation

$$\varphi\big(T(s;\hat{\mu}),\sigma(s);\hat{\mu}\big)=\tau(R(s;\hat{\mu});\hat{\mu})$$

Definition 1.1. Consider $K\in\mathbb{Z}_{\geq0}\cup\{+\infty\}$ and an open subset $U\subset\hat{W}\subset\mathbb{R}^{N+1}$. We say that a function $\psi(s;\hat{\mu})$ belongs to the class $\mathscr{C}^K_{s>0}(U)$, respectively $\mathcal{E}^K(U)$, if there exist an open neighbourhood V of

$$\{(s,\hat{\mu})\in\mathbb{R}^{N+2};s=0,\hat{\mu}\in U\}=\{0\}\times U$$

in $\mathbb{R}^{N+2}$ such that $(s,\hat{\mu})\mapsto\psi(s;\hat{\mu})$ is $\mathscr{C}^K$ on $V\cap\big((0,+\infty)\times U\big)$, respectively V. Finally we denote

$$\mathcal{E}^K_+(U):=\{\psi(s;\hat{\mu})\in\mathcal{E}^K(U);\ \psi(0;\hat{\mu})>0\text{ for all }\hat{\mu}\in U\}.$$

Here the letter $\mathcal{E}$ stands for functions in $\mathscr{C}^K_{s>0}(U)$ having *extension* to $s=0$. □

More formally, the definition of $\mathscr{C}^K_{s>0}(U)$ and $\mathcal{E}^K(U)$ must be thought in terms of germs with respect to relative neighborhoods of $\{0\}\times U$ in $(0,+\infty)\times U$. In doing so these sets become rings and we have the inclusions $\mathscr{C}^K(U)\subset\mathcal{E}^K(U)\subset\mathscr{C}^K_{s>0}(U)$.

We can now introduce the notion of (finitely) flatness that we shall use in the sequel.

Definition 1.2. Consider $K\in\mathbb{Z}_{\geq0}\cup\{+\infty\}$ and an open subset $U\subset\hat{W}\subset\mathbb{R}^{N+1}$. Given $L\in\mathbb{R}$ and $\hat{\mu}_0\in U$, we say that $\psi(s;\hat{\mu})\in\mathscr{C}^K_{s>0}(U)$ is *(L,K)-flat with respect to s at* $\hat{\mu}_0$, and we write $\psi\in\mathcal{F}^K_L(\hat{\mu}_0)$, if for each $\nu=(\nu_0,\ldots,\nu_{N+1})\in\mathbb{Z}^{N+2}_{\geq0}$ with $|\nu|=\nu_0+\cdots+\nu_{N+1}\leqslant K$ there exist a neighbourhood V of $\hat{\mu}_0$ and $C,s_0>0$ such that

$$\left|\frac{\partial^{|\nu|}\psi(s;\hat{\mu})}{\partial s^{\nu_0}\partial\hat{\mu}_1^{\nu_1}\cdots\partial\hat{\mu}_{N+1}^{\nu_{N+1}}}\right|\leqslant Cs^{L-\nu_0}\text{ for all }s\in(0,s_0)\text{ and }\hat{\mu}\in V.$$

If W is a (not necessarily open) subset of U then define $\mathcal{F}^K_L(W):=\bigcap_{\hat{\mu}_0\in W}\mathcal{F}^K_L(\hat{\mu}_0)$. □

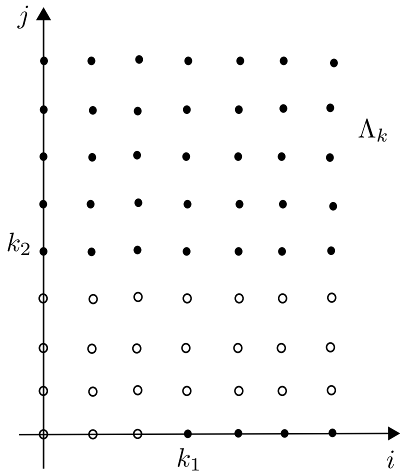

Figure 2: The filled dots are points $(i, j) \in \mathbb{Z}^2_{\geq 0}$ in the set Λ_k for $k = (k_1, k_2)$.

The principal part of the Dulac map and Dulac time will be expressed in terms of the following deformation of the logarithm.

Definition 1.3. The function defined for $s > 0$ and $\alpha \in \mathbb{R}$ by means of

$$\omega(s;\alpha) = \begin{cases} \frac{s^{-\alpha}-1}{\alpha} & \text{if } \alpha \neq 0, \\ -\log s & \text{if } \alpha = 0, \end{cases}$$

is called the *Ecalle-Roussarie compensator*. □

Definition 1.4. Given any $k = (k_1, k_2) \in \mathbb{Z}^2_{\geq 0}$, throughout the paper we shall use the following notation:

- $\Lambda_k := (\mathbb{Z}_{\geq k_1} \times \{0\}) \cup (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq k_2})$, see Figure 2.
- $D^k_{ij} := \big\{\lambda > 0 : \text{there exits } (i', j') \in \Lambda_k \setminus \{(i, j)\} \text{ such that } i + \lambda j = i' + \lambda j'\big\}$ for each $(i, j) \in \Lambda_k$.
- $\mathscr{B}^k_{\lambda,L} := \big\{(i, j) \in \Lambda_k : i + \lambda j \leqslant L\big\}$ for each $L \in \mathbb{R}$ and $\lambda > 0$, see Figure 4.
- $D^k_L := \big\{\lambda > 0 : \text{there exits } (i, j) \in \mathscr{B}^k_{\lambda,L} \text{ such that } \lambda \in D^k_{ij}\big\}$.
- For $\lambda = p/q \in \mathbb{Q}_{>0}$ with $\gcd(p, q) = 1$ and $(i, j) \in \Lambda_k$,

$$\mathscr{A}^k_{ij\lambda} := \begin{cases} \emptyset & \text{if } (i + rp, j - rq) \in \Lambda_k \text{ for some } r \in \mathbb{N}, \\ \{r \in \mathbb{Z}_{\geq 0} : (i - rp, j + rq) \in \Lambda_k\} & \text{otherwise.} \end{cases}$$

Note that if $k_2 = 0$ then $\Lambda_k = \mathbb{Z}^2_{\geq 0} = \Lambda_0$ regardless of the value of k_1. One can prove on the other hand that D^k_{ij} and D^k_L are discrete subsets of $\mathbb{Q}_{>0}$, see Remark 3.3. □

Let us point out that in the previous definition k stands always for a two-dimensional vector with components in $\mathbb{Z}_{\geq 0}$. That being said, if $k = (0, 0)$ then we write Λ_0, D^0_{ij}, $\mathscr{B}^0_{\lambda,L}$, D^0_L and $\mathscr{A}^0_{ij\lambda}$ for shortness.

We are now in position to state our main results. Let us begin with the one regarding the asymptotic development of the Dulac time. In its statement we use the notation introduced so far and denote

$$T_0(\hat{\mu}) = \begin{cases} 0 & \text{if } n \neq (0,0), \\ \frac{-1}{P(0,0;\hat{\mu})} & \text{if } n = (0,0), \end{cases} \tag{2}$$

where recall that the components of the vector $n = (n_1, n_2) \in \mathbb{Z}^2_{\geq 0}$ are the orders of the poles of $X_{\hat{\mu}}$.

Theorem A. *Let $T(s;\hat{\mu})$ be the Dulac time of the hyperbolic saddle (1) between the transverse sections Σ_σ and Σ_τ. For each $(i,j) \in \Lambda_n$ there exists $T_{ij} \in \mathscr{C}^\infty\big(((0,+\infty) \setminus D^n_{ij}) \times W\big)$ such that, for every $L > 0$ and $\lambda_0 > 0$, the following properties hold:*

(a) If $\lambda_0 \notin D^n_L$ then

$$T(s;\hat{\mu}) = T_0(\hat{\mu}) \log s + \sum_{(i,j)\in\mathscr{B}^n_{\lambda_0,L}} T_{ij}(\hat{\mu}) s^{i+\lambda j} + \mathcal{F}^\infty_L(\{\lambda_0\} \times W).$$

(b) If $\lambda_0 \in D^n_L$ then

$$T(s;\hat{\mu}) = T_0(\hat{\mu}) \log s + \sum_{(i,j)\in\mathscr{B}^n_{\lambda_0,L}} \boldsymbol{T}^{\lambda_0}_{ij}\big(\omega_\alpha(s);\hat{\mu}\big) s^{i+\lambda j} + \mathcal{F}^\infty_L(\{\lambda_0\} \times W),$$

where $\lambda_0 = p/q$ with $\gcd(p,q) = 1$, $\alpha(\hat{\mu}) = p - \lambda q$ and

$$\boldsymbol{T}^{\lambda_0}_{ij}(w;\hat{\mu}) := \sum_{r\in\mathscr{A}^n_{ij\lambda_0}} T_{i-rp,j+rq}(\hat{\mu})(1+\alpha w)^r.$$

Moreover the coefficient of these polynomials in w extend $\mathscr{C}^\infty$ to $\{\lambda_0\} \times W$.

This result is a very significant improvement of the ones that we obtained previously in [15, 18]. Indeed, the main result in [15] can be viewed as an embryonic version of Theorem A that (with the notation introduced here) is addressed to the case $n = (0, n_2)$ and $L = \min(1, \lambda_0 n_2) + \varepsilon$, so that its principal part contains just two monomials. More important, it is proved in the analytic setting and assuming additionally that family $\{X_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$ is locally equivalent to its linear part. We referred to this assumption as the family linearization property (FLP, in short). Afterwards we extended in [18] that previous result to $n = (n_1, n_2)$ but still with the same L and the FLP assumption.

Next we state our main result about the Dulac map. Before that let us observe that this map is independent of $n = (n_1, n_2)$ because it does not change after a reparametrization of time in the differential equation. Related with this, and also with regard to the applicability of the theorem, it is important to mention that, thanks to Lemma 4.3, it is not necessary that the separatrices of the saddle are straight lines.

Theorem B. *Let $D(s;\hat{\mu})$ be the Dulac map of the hyperbolic saddle (1) between the transverse sections Σ_σ and Σ_τ. For each $(i,j) \in \Lambda_0$ there exists $\Delta_{ij} \in \mathscr{C}^\infty\big(((0,+\infty) \setminus D^0_{ij}) \times W\big)$ such that, for every $L > 0$ and $\lambda_0 > 0$, the following properties hold:*

(a) If $\lambda_0 \notin D^0_L$ then

$$D(s;\hat{\mu}) = s^\lambda \sum_{(i,j)\in\mathscr{B}^0_{\lambda_0,L-\lambda_0}} \Delta_{ij}(\hat{\mu}) s^{i+\lambda j} + \mathcal{F}^\infty_L(\{\lambda_0\} \times W).$$

(b) If $\lambda_0 \in D^0_L$ then

$$D(s;\hat{\mu}) = s^\lambda \sum_{(i,j)\in\mathscr{B}^0_{\lambda_0,L-\lambda_0}} \boldsymbol{\Delta}^{\lambda_0}_{ij}\big(\omega_\alpha(s);\hat{\mu}\big) s^{i+\lambda j} + \mathcal{F}^\infty_L(\{\lambda_0\} \times W),$$

where $\lambda_0 = p/q$ *with* $\gcd(p,q) = 1$, $\alpha(\hat\mu) = p - \lambda q$ *and*

$$\boldsymbol{\Delta}_{ij}^{\lambda_0}(w;\hat\mu) := \sum_{r\in\mathscr{A}^0_{ij\lambda_0}} \Delta_{i-rp,j+rq}(\hat\mu)(1+\alpha w)^r.$$

Moreover the coefficient of these polynomials in w *extend* $\mathscr{C}^\infty$ *to* $\{\lambda_0\}\times W$.

Finally, $\Delta_{00}(\hat\mu) > 0$ *for all* $\hat\mu \in \hat W$.

Theorem B is closely related with the seminal results by R. Roussarie and his collaborators on the structure of the Dulac map (see [20, 22, 25] and [28, §5.1.3]). Our contribution improves the previous ones in following three aspects (that are also valid for Theorem A with regard to [15, 18]):

1. The monomials appearing in the principal part are completely described, even in the resonant case (i.e., $\lambda_0 \in \mathbb{Q}$). Note moreover that $\max\mathscr{A}^k_{ij\lambda} \leqslant i/p$ for any $k = (k_1,k_2) \in \mathbb{Z}^2_{\geq 0}$ and that if $i < k_2\lambda$ then $\mathscr{A}^k_{i0\lambda} = \{0\}$. This gives a bound on the degrees of $\boldsymbol{T}^{\lambda_0}_{ij}(w;\hat\mu)$ and $\boldsymbol{\Delta}^{\lambda_0}_{ij}(w;\hat\mu)$. (We remark in this respect that all over the paper we follow the usual convention that a summation with the index varying in an empty set is equal to zero.)

2. The coefficients T_{ij} and Δ_{ij} are $\mathscr{C}^\infty$ functions "universally" defined. More precisely, their existence is established before fixing the flatness L of the remainder and, more important, the parameter λ_0 (see the order of quantifiers in the statements).

3. The remainder is given by a function $\mathscr{R}(s;\hat\mu)$ in $\mathcal{F}^\infty_L(\{\lambda_0\}\times W)$. The application of Lemma A.1 shows that if we take any $K \in \mathbb{Z}_{\geq 0}$ with $K < L$ then $\mathscr{R}$ extends to a $\mathscr{C}^K$-function $\hat{\mathscr{R}}$ defined in some open neighbourhood of $\{0\}\times\{\lambda_0\}\times W$ inside $\mathbb{R}\times\mathbb{R}\times\mathbb{R}^N$ and satisfying $\partial^\nu\hat{\mathscr{R}}(0;\hat\mu) \equiv 0$ for all $\nu \in \mathbb{Z}^{N+2}_{\geq 0}$ with $|\nu| \leqslant K$. To the best of our knowledge this constitutes a new result. The flatness of the remainder in the previous results (see [9, §3] and [25] for the Dulac map) holds for derivation with respect to s, which suffices to bound the number of critical periodic orbits or limit cycles. However it does not enable to conclude that $\mathscr{R}$ extends smoothly with respect to the parameters (see [19, Appendix A] for a counterexample), which is crucial to study the smoothness properties of the corresponding bifurcation diagram (see [27] for limit cycles). We refer the interested reader to [19, Remark 1.4] for more details on this issue.

With regard to second point above let us comment that in a forthcoming paper we will give the explicit expression of T_{ij} and Δ_{ij} for particular (i,j) in terms only the parametrization of the transversal sections and the functions P and Q in (1). For instance, in the context of Hilbert's 16th problem, to have these expressions when dealing with a particular polycycle is essential in order to compute its cyclicity and to fully understand the bifurcation diagram of emergence/disappearance of limit cycles.

Remark 1.5. As we already mentioned we consider the hyperbolicity ratio $\lambda > 0$ of the saddle as an independent parameter in $\hat\mu = (\lambda,\mu)$. This is by no means a restriction because if we deal with a family

$$X_\nu(x,y) = \frac{1}{x^{n_1}y^{n_2}}\Big(xP(x,y;\nu)\partial_x + yQ(x,y;\nu)\partial_y\Big)$$

with hyperbolicity ratio $\lambda(\nu) = -\frac{Q(0,0;\nu)}{P(0,0;\nu)}$ then instead we can consider

$$\bar X_{(\lambda,\nu)}(x,y) = \frac{1}{x^{n_1}y^{n_2}}\Big(x\bar P(x,y;\lambda,\nu)\partial_x + y\bar Q(x,y;\lambda,\nu)\partial_y\Big)$$

with $\bar P(x,y;\lambda,\nu) := P(x,y;\nu)$ and $\bar Q(x,y;\lambda,\nu) := P(0,0;\nu)(-\lambda+\lambda(\nu)) + Q(x,y;\nu)$. Clearly $\{\bar X_{(\lambda,\nu)}\}$ is a family with a saddle with hyperbolicity ratio λ that restricted to $\lambda = \lambda(\nu)$ coincides with $\{X_\nu\}$. □

Next we make some further considerations about the consequences of Theorem A. Let us point out that although we focus on the Dulac time for simplicity in the exposition, they are also valid for Theorem B on the Dulac with the obvious modifications. On account of the second point above the coefficients T_{ij} (and so the polynomials $\boldsymbol{T}_{ij}^{\lambda_0}$) are independent of the remainder's flatness L. This endows the asymptotic development of the Dulac time given in Theorem A with a property similar to the unicity of the Taylor series. More precisely, we have a well defined (formal) series

$$T_0(\hat{\mu})\log s + \begin{cases} \displaystyle\sum_{(i,j)\in\Lambda_n} T_{ij}(\hat{\mu})s^{i+\lambda j} & \text{if } \lambda_0 \notin \mathbb{Q}, \\ \displaystyle\sum_{(i,j)\in\Lambda_n} \boldsymbol{T}_{ij}^{\lambda_0}\big(\omega_\alpha(s);\hat{\mu}\big)s^{i+\lambda j} & \text{if } \lambda_0 \in \mathbb{Q}, \end{cases} \tag{3}$$

which is asymptotic to $T(s;\hat{\mu})$ as $(s,\hat{\mu}) \to (0,\hat{\mu}_0)$ in the sense established by Theorem A. In this respect it is to be referred the works of Saavedra on the Dulac time of (single) analytic vector fields (see [29, 30]). She proves that the Dulac time T (and its derivative T') of a hyperbolic saddle is asymptotic to a formal series

$$\hat{T}(s) = \sum_{k\in\mathbb{N}} s^{\nu_k} P_k(\log s), \tag{4}$$

where $\{\nu_k\}_{k\in\mathbb{N}}$ is a strictly increasing unbounded sequence of real numbers and P_k is a real polynomial. In other words, $|T(s;\hat{\mu}_0) - \sum_{i=1}^k s^{\nu_i}P_i(\log s)| < \mathrm{o}(s^{\nu_k})$ for all $k \in \mathbb{N}$. We note that in her result a finite number of exponents ν_k may be negative because she contemplates the case $n_1, n_2 \in \mathbb{Z}_{<0}$ as well.

Definition 1.6. For $\lambda_0 \notin \mathbb{Q}$ we write $s^{i+\lambda j} \prec_{\lambda_0} s^{i'+\lambda j'}$ if, and only if, $i + \lambda_0 j < i' + \lambda_0 j'$. If $\lambda_0 \in \mathbb{Q}$ then we write

$$s^{i+\lambda j}\omega^k \prec_{\lambda_0} s^{i'+\lambda j'}\omega^{k'} \;\Leftrightarrow\; \begin{cases} i + \lambda_0 j < i' + \lambda_0 j' \\ \text{or} \\ (i,j) = (i',j') \text{ and } k > k' \end{cases}$$

where as usual, setting $\lambda_0 = p/q$ with $\gcd(p,q) = 1$, ω stands for $\omega_\alpha(s) = \omega(s; p - \lambda q)$. □

It is clear that $\prec_{\lambda_0}$ is a strict *partial* order among the monomials $s^{i+\lambda j}\omega^k$ with $i,j,k \in \mathbb{Z}_{\geq 0}$. Moreover, if f and g are two monomials such that $f \prec_{\lambda_0} g$ then, by applying (a) in Lemma A.3,

$$\lim_{(s,\lambda)\to(0,\lambda_0)} \frac{g(s;\lambda)}{f(s;\lambda)} = 0,$$

where the limit is taken with $(s,\lambda) \in (0,\varepsilon) \times (0,+\infty)$.

For a fixed $\hat{\mu}_0 = (\lambda_0,\mu_0) \in \hat{W}$, our main concern is to study the number of zeros of $T'(\,\cdot\,;\hat{\mu})$ that are close to $s = 0$ when $\hat{\mu} \approx \hat{\mu}_0$. We shall define precisely this notion below but let us advance that, in this respect, the case $n = (0,0)$ is trivial. This is so because, by applying Theorem A and (d) in Lemma A.3,

$$\lim_{(s,\hat{\mu})\to(0,\hat{\mu}_0)} sT'(s;\hat{\mu}) = T_0(\hat{\mu}_0),$$

which is different from zero due to (2). Clearly this fact prevents the existence of zeros of $T'(s;\hat{\mu})$ with $(s,\hat{\mu}) \approx (0,\hat{\mu}_0)$. This is the reason why we will suppose $n \neq (0,0)$ in the statement of Theorem C, which constitutes our third main result. Before that let us make the following key observation.

Remark 1.7. We note that $\prec_{\lambda_0}$ is a strict *total* order among the monomials that can appear in the formal series (3) of the Dulac time, namely $s^{i+\lambda j}$ for $\lambda_0 \notin \mathbb{Q}$ and $s^{i+\lambda j}\omega^k$ for $\lambda_0 \in \mathbb{Q}$. Indeed, this is obvious for $\lambda_0 \notin \mathbb{Q}$ because then $i + \lambda_0 j \neq i' + \lambda_0 j'$ for $(i,j) \neq (i',j')$. In case that $\lambda_0 \in \mathbb{Q}$, say $\lambda_0 = p/q$ with $\gcd(p,q) = 1$, this is due to the fact that if $i+\lambda_0 j = i'+\lambda_0 j'$ with $(i,j) \neq (i',j')$ then there exists $r \in \mathbb{Z}\setminus\{0\}$ such that $(i',j') = (i - rp, j + rq)$ and consequently (see Definition 1.4) either $\mathscr{A}_{ij\lambda_0}^n = \emptyset$ or $\mathscr{A}_{i'j'\lambda_0}^n = \emptyset$, which implies $\boldsymbol{T}_{ij}^{\lambda_0} \equiv 0$ and $\boldsymbol{T}_{i'j'}^{\lambda_0} \equiv 0$, respectively. □

On account of the previous remark, for a given $\lambda_0>0$ we denote by $t_i(\hat{\mu})$ the coefficient of the monomial in the i-th position (with respect to $\prec_{\lambda_0}$) among all the monomials appearing in the formal series (3).

Definition 1.8. Assume $n\neq(0,0)$ and take $\hat{\mu}_0=(\lambda_0,\mu_0)\in\hat{W}$. Let $\{t_i\}_{i\in\mathbb{N}}$ be the sequence of coefficients with respect to $\prec_{\lambda_0}$ in the formal series (3). We define $\ell_{\hat{\mu}_0}:=\inf\{i\in\mathbb{N}:t_i(\hat{\mu}_0)\neq 0\}-\eta$, where $\eta=1$ if $n_1n_2=0$ and $\eta=0$ otherwise. □

Notice that if $n\neq(0,0)$ but $n_1n_2=0$ then the first monomial (with respect to the order $\prec_{\lambda_0}$) in the formal series (3) is the constant one, which follows taking $(i,j)=(0,0)\in\Lambda_n$. This monomial does not appear in the asymptotic development of $T'(\,\cdot\,;\hat{\mu})$ and this is the reason why we subtract η in the previous definition.

Definition 1.9. Let $h(s;\hat{\mu})$ be a function in $\mathscr{C}^{\infty}_{s>0}(U)$ for some open set $U\subset\hat{W}$. Given any $\hat{\mu}_0\in U$ we define $\mathscr{Z}_0(h(\,\cdot\,;\hat{\mu}),\hat{\mu}_0)$ to be the smallest integer N having the property that there exist $\delta>0$ and a neighbourhood V of $\hat{\mu}_0$ such that for every $\hat{\mu}\in V$ the function $h(s;\hat{\mu})$ has no more than N zeros on $(0,\delta)$ counted with multiplicities. □

We can now state our third main result, that gives a uniform bound for the number of zeros of $T'(\,\cdot\,;\hat{\mu})$ bifurcating from $s=0$ when $\hat{\mu}\approx\hat{\mu}_0$.

Theorem C. *Consider the family of vector fields $\{X_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$ in (1) and assume that $n\neq(0,0)$. Let $T(\,\cdot\,;\hat{\mu})$ be the Dulac time between the transverse sections Σ_σ and Σ_τ and fix $\hat{\mu}_0\in\hat{W}$. Then $\mathscr{Z}_0(T'(\,\cdot\,;\hat{\mu}),\hat{\mu}_0)\leqslant\ell_{\hat{\mu}_0}$.*

If $\ell_{\hat{\mu}_0}<+\infty$ then Theorem C gives an upper bound for the number of zeros of $T'(\,\cdot\,;\hat{\mu})$ that can emerge/disappear from $s=0$ when we perturb $\hat{\mu}\approx\hat{\mu}_0$. It is important to remark, and this is a key feature, that $\ell_{\hat{\mu}_0}$ depends only on the vector field $X_{\hat{\mu}_0}$ and not in the family $\{X_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$. Note in this respect, cf. (4), that if $T'(s;\hat{\mu}_0)=\beta s^{\nu}\log^m s+\mathrm{o}(s^{\nu}\log^m s)$ with $\beta\neq 0$ then $\ell_{\hat{\mu}_0}<+\infty$. Indeed, Theorem A shows that

$$\ell_{\hat{\mu}_0}=\begin{cases}\#\mathscr{B}^n_{\lambda_0,\nu}-\eta & \text{if } \lambda_0\notin\mathbb{Q},\\ \displaystyle\sum_{(i,j)\in\mathscr{B}^n_{\lambda_0,\nu}}\Big(\max\{r\in\mathscr{A}^n_{ij\lambda_0}\}+1\Big)-m-\eta & \text{if } \lambda_0\in\mathbb{Q},\end{cases}$$

where $\eta=1$ if $n_1n_2=0$ and $\eta=0$ otherwise. It is to be referred here the paper by Saavedra and Mardešić [17] because (in the analytic context) they prove that if $T'(\,\cdot\,;\hat{\mu}_0)\not\equiv 0$ then there exist some $\nu\in\mathbb{R}$ and $m\in\mathbb{Z}_{\geq 0}$ such that $T'(s;\hat{\mu}_0)=\beta s^{\nu}\log^m s+\mathrm{o}(s^{\nu}\log^m s)$ with $\beta\neq 0$. By analytic context we mean assuming that the functions $P(\,\cdot\,;\hat{\mu}_0)$ and $Q(\,\cdot\,;\hat{\mu}_0)$ in (1), together with the parametrizations $\sigma(\,\cdot\,;\hat{\mu}_0)$ and $\tau(\,\cdot\,;\hat{\mu}_0)$ of the transverse sections, are analytic. On a rather different tack, but also in this setting, it is to be mentioned that the application of Proposition 4.2 provides a lower bound for $\mathscr{Z}_0(T'(\,\cdot\,;\hat{\mu}),\hat{\mu}_0)$.

We conclude this section by explaining two applications of the tools and results introduced so far in the context of the study of the period function, which was our initial motivation for considering this kind of problems. To this end some additional definitions are needed. A singular point p of a planar differential system is a *center* if it has a punctured neighbourhood that consists entirely of periodic orbits surrounding p. The *period annulus* of the center is the largest punctured neighbourhood with this property and we will denote it by $\mathscr{P}$. We embed $\mathscr{P}$ in $\mathbb{RP}^2$ and denote its boundary by $\partial\mathscr{P}$. Clearly the center p belongs to $\partial\mathscr{P}$, and in what follows we will call it the *inner boundary* of the period annulus. We also define the *outer boundary* of the period annulus to be $\Pi:=\partial\mathscr{P}\setminus\{p\}$. We point out that Π is a nonempty compact subset of $\mathbb{RP}^2$. The *period function* of the center assigns to each periodic orbits in $\mathscr{P}$ its period. Since the period function is defined on the set of periodic orbits in $\mathscr{P}$, in order to study its qualitative properties usually the first step is to parametrize this set. This can be done by taking a transverse section to the vector field on $\mathscr{P}$, for instance an orbit of the orthogonal vector field. If $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization, then $s\mapsto P(s):=\{\text{period of }\gamma_s\}$ is a function that provides the qualitative properties of the period function that we are interested in. (Note that the function P is as smooth as the vector field and the parametrization used.)

The *critical periods* are the isolated critical points of P, i.e. $\hat{s} \in (0,1)$ such that $P'(s) = \alpha(s-\hat{s})^k + \mathrm{o}\big((s-\hat{s})^k\big)$ with $\alpha \neq 0$ and $k \geqslant 1$. In this case, more geometrically, we shall say that $\gamma_{\hat{s}}$ is a *critical periodic orbit* of multiplicity k of the center. One can easily see that this definition does not depend on the particular parametrization of the set of periodic orbits used.

In the next definition d_H stands for the Hausdorff distance between compact sets of $\mathbb{RP}^2$.

Definition 1.10. Consider a $\mathscr{C}^\infty$ family $\{X_\nu\}_{\nu\in U}$ of planar vector fields with a center and fix some $\nu_0 \in U$. Suppose that the outer boundary of the period annulus varies continuously at $\nu_0 \in U$, meaning that for any $\varepsilon > 0$ there exists $\delta > 0$ such that $d_H(\Pi_\nu, \Pi_{\nu_0}) \leqslant \varepsilon$ for all $\nu \in U$ with $\|\nu - \nu_0\| \leqslant \delta$. Then, setting

$$N(\delta,\varepsilon) = \sup\{\#\text{ critical periodic orbits } \gamma \text{ of } X_\nu \text{ in } \mathscr{P}_\nu \text{ with } d_H(\gamma,\Pi_{\nu_0}) \leqslant \varepsilon \text{ and } \|\nu-\nu_0\| \leqslant \delta\},$$

the *criticality* of (Π_{ν_0}, X_{ν_0}) with respect to the deformation X_ν is $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) := \inf_{\delta,\varepsilon} N(\delta,\varepsilon)$. □

Let us stress that the number of critical periodic orbits γ of X_ν in $\mathscr{P}_\nu$ is counted with multiplicities. Note furthermore that in this definition each vector field X_ν is assumed to be $\mathscr{C}^\infty$. To define the outer boundary Π_ν of the period annulus $\mathscr{P}_\nu$ of X_ν we do not compactify the vector field but only the set $\mathscr{P}_\nu$ and to this end there is no need that X_ν is polynomial. Certainly $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big)$ may be infinite but, if it is not, then it gives the maximal number of critical periodic orbits of X_ν that tend to Π_{ν_0} in the Hausdorff sense as $\nu \to \nu_0$. It is clear on the other hand that, for a given $\nu_0 \in U$, the contour of the period annulus $\mathscr{P}_{\nu_0}$ changes as we move $\nu \approx \nu_0$. The assumption that the period annulus varies continuously ensures that these changes do not occur abruptly. In this regard note that $X_\nu = -y\partial_x + (x + \nu x^3 + x^5)\partial_y$, with $\nu \in \mathbb{R}$, is a polynomial family of vector fields with a center at the origin for which the outer boundary does not vary continuously at $\nu = 2$. Indeed, the period annulus $\mathscr{P}_\nu$ is the whole plane for $\nu < 2$, whereas is bounded for $\nu = 2$ (see [14] for details). Clearly in this situation the criticality of Π_ν at $\nu = \nu_0$ has no point.

Definition 1.11. Let X be a vector field on $\mathbb{R}^2$ (or $\mathbb{S}^2$). A *graphic* Γ for X is a compact, non-empty invariant subset which is a continuous image of $\mathbb{S}^1$ and consists of a finite number of isolated singular points $\{p_1,\ldots,p_m,p_{m+1}=p_1\}$ (not necessarily distinct) and compatibly oriented separatrices $\{s_1,\ldots,s_m\}$ connecting them (i.e., such that the α-limit set of s_j is p_j and the ω-limit set of s_j is p_{j+1}). A graphic is said to be *hyperbolic* if all its singular points are hyperbolic saddles. A *polycycle* is a graphic with a return map defined on one of its sides. Consider now a $\mathscr{C}^\infty$ family of vector fields $\{X_\nu\}$ such that, for $\nu = \nu_0$, Γ is a hyperbolic polycycle for X_{ν_0}. Then, in the context of Hilbert's 16th problem, in order to study the cyclicity of (X_ν, Γ) it is sometimes assumed that some connections between the hyperbolic saddles in Γ remain unbroken inside the family $\{X_\nu\}$, see for instance [5, 12, 21]. Here, in case that all the separatrix connections remain unbroken we shall say that the hyperbolic polycycle Γ is *persistent* inside the family $\{X_\nu\}$. When studying the criticality this will be a non-degeneracy condition for the polycycle at the outer boundary of the period annulus. □

In order to study the behaviour of the trajectories of a polynomial vector field Y near infinity we can consider its Poincaré compactification $p(Y)$, see [1, §5] for details, which is an analytically equivalent vector field defined on the sphere $\mathbb{S}^2$. The points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator of $\mathbb{S}^2$, that we denote by ℓ_∞. Furthermore the trajectories of $p(Y)$ in $\mathbb{S}^2$ are symmetric with respect to the origin and so it suffices to draw its flow in the closed northern hemisphere only, the so called Poincaré disk. Taking this notation into account we can now state two finiteness results for polynomial vector fields that follow from our main results.

Corollary D. *Let $\{X_\nu\}_{\nu\in U}$ be a $\mathscr{C}^\infty$ family of planar polynomial vector fields. Assume that, for all $\nu \in U$,*

(*a*) *X_ν has a center such that the outer boundary Π_ν of its period annulus is a hyperbolic polycycle that is persistent and varies continuously inside the family,* (*b*) *the infinite line* ℓ_∞ *is invariant for* $p(X_\nu)$ *in case that* $\Pi_\nu \cap \ell_\infty \neq \emptyset$, *and*

(*c*) *not all the singularities of* X_ν *in* Π_ν *are in* ℓ_∞.

Then $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) = 0$ *for any* $\nu_0 \in U$.

Recall that, for a given polynomial vector field Y, it is well known that the infinite line ℓ_∞ is invariant for $p(Y)$ if, and only if, the homogeneous part of highest degree of Y is not a multiple of the radial vector field $x\partial_x + y\partial_y$. On the other hand, if the family $\{X_\nu\}_{\nu\in U}$ is analytic (instead of $\mathscr{C}^\infty$) and each center is non degenerated then Corollary D implies the existence of a uniform *finite* upper bound for the number of critical periodic orbits in the whole period annulus. With regard to this aftermath it is to be referred a paper by Chicone and Dumortier [3] because from their main theorem it follows that a center of a (single) analytic vector field with a bounded period annulus cannot have an infinite number of critical periodic orbits.

Turning to Hilbert's 16th problem, the saddle loop constitutes the first type of polycycle that was proved to have finite cyclicity (see [25, Theorem A] and also [9, Theorem 9]). Corollary D shows that the saddle loop has zero criticality, so the next polycycle to consider is the one having two hyperbolic saddles as vertices. Our last result is addressed to this situation and let us advance that the non-trivial case is when both saddles are at infinity, see Figure 7.

Corollary E. *Let* $\{X_\nu\}_{\nu\in U}$ *be a* $\mathscr{C}^\infty$ *family of planar polynomial vector fields. Assume that, for all* $\nu \in U$,

(*a*) X_ν *has a center such that the outer boundary* Π_ν *of its period annulus is a hyperbolic polycycle with exactly two singularities that is persistent and varies continuously inside the family, and*

(*b*) *the infinite line* ℓ_∞ *is invariant for* $p(X_\nu)$ *in case that* $\Pi_\nu \cap \ell_\infty \neq \emptyset$

Then $\mathrm{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) < +\infty$ *for any* $\nu_0 \in U$ *such that the center* q_{ν_0} *is non-isochronous.*

The paper is organized as follows. In Section 2 we prove Theorem 2.5, which constitutes a fundamental preliminary result addressed to both, the Dulac map and the Dulac time. Briefly, for every $\hat\mu_0 \in \hat W$, $K \in \mathbb{N}$ and $L > 0$, Theorem 2.5 provides an asymptotic expansion of $D(\,\cdot\,;\hat\mu)$ and $T(\,\cdot\,;\hat\mu)$ with the coefficients $\mathscr{C}^K$ functions in a neighbourhood of $\hat\mu_0$ and the remainder in $\mathcal{F}_L^K(\hat\mu_0)$. Next, relying on this result, Theorems A and B follow by showing that in fact these coefficients do not depend on $\hat\mu_0$, K and L. More concretely, that we can change the order of the quantifiers $\forall \ldots \exists$ in the statement of Theorem 2.5 to $\exists \ldots \forall$. The formalization of this simple idea is rather long and technical but let us advance that the proof of Theorem A from Theorem 2.5 gives as particular case the proof of Theorem B. That being said, Section 3 is devoted to the proofs of Theorems A, B and C. Next, in Section 4, we prove Corollaries D and E, together with a result, namely Lemma 4.3, that straightens globally the separatrices of a saddle depending on parameters. This result, which is well-known to be true locally, is relevant with regard to the applicability of Theorems A and B because it enables to set aside the condition that the separatrices lay on the coordinate axis. Finally, for reader's convenience, in Appendix A we state some results from [19] about the class of functions $\mathcal{F}_L^K(W)$ that we shall use in the present paper.

2 A preliminary result

This section is completely devoted to the proof of Theorem 2.5. With this aim in view we begin with two results about the class of functions $\mathcal{E}^K(U)$, recall Definition 1.1.

Lemma 2.1. *If* $f \in \mathcal{E}^K(U)$ *with* $K \in \mathbb{N}$ *verifies* $f(0;\hat\mu) = 0$ *for all* $\hat\mu \in U$ *then* $f = sg$ *some* $g \in \mathcal{E}^{K-1}(U)$.

Proof. By definition $f(s;\hat\mu) \in \mathscr{C}^K(V)$ for some open neighborhood V of $\{0\} \times U$. Fix any $(s_0, \hat\mu_0) \in V$ and observe that the existence of g is clear in a neighbourhood of any point with $s_0 \neq 0$. So assume that $s_0 = 0$

and take $\varepsilon > 0$ small enough and a closed disc D centered at $\hat{\mu}_0$ with $D \subset U$ such that $[-\varepsilon, \varepsilon] \times D \subset V$. Then, by Proposition 2 and Remark 3 in [31, §17.1.3], $g(s;\hat{\mu}) := \int_0^1 \partial_s f(st;\hat{\mu})dt$ is $\mathscr{C}^{K-1}$ on $[-\varepsilon, \varepsilon] \times D$. Furthermore

$$sg(s;\hat{\mu}) = \int_0^1 s\partial_s f(st;\hat{\mu})dt = \int_0^s \partial_s f(u;\hat{\mu})du = f(s;\hat{\mu}) - f(0;\hat{\mu}) = f(s;\hat{\mu}),$$

for all $(s,\hat{\mu}) \in [-\varepsilon, \varepsilon] \times D$. This proves the existence of $g \in \mathscr{C}^K(V')$ with $V' := \{(s,\hat{\mu}) \in V : \hat{\mu} \in U\}$, which is an open neighborhood of $\{0\} \times U$, verifying $f = sg$ as desired. ■

Corollary 2.2. *If $f \in \mathcal{E}^K(U)$ and $m \in \mathbb{N}$ with $m \leqslant K$ then there exist $f_i \in \mathscr{C}^{K-i}(U)$, $i = 0, 1, \ldots, m-1$, and $g \in \mathcal{E}^{K-m}(U)$ such that $f(s;\hat{\mu}) = \sum_{i=0}^{m-1} f_i(\hat{\mu})s^i + s^m g(s;\hat{\mu})$. In particular, for any $L \geqslant 0$,*

$$\mathcal{E}^{K'}(U) \subset \mathscr{C}^K(U)[s] + \mathcal{F}_L^K(U),$$

provided that $K' \geqslant K + L$.

Proof. The first assertion follows by applying Lemma 2.1 recursively. The second assertion for $L = 0$ follows by (c) and (d) in Lemma A.2, which show that $\mathcal{E}^{K'}(U) \subset \mathcal{F}_0^{K'}(U) \subset \mathcal{F}_0^K(U)$. The second assertion for $L > 0$ follows by applying the first one with $m = \lceil L \rceil \geqslant 1$ and using then Lemma A.2 to show that

$$s^{\lceil L \rceil} g(s;\hat{\mu}) \in \mathcal{F}_{\lceil L \rceil}^{\infty}(U)\mathcal{E}^{K'-\lceil L \rceil}(U) \subset \mathcal{F}_{\lceil L \rceil}^{\infty}(U)\mathcal{F}_0^K(U) \subset \mathcal{F}_{\lceil L \rceil}^K(U) \subset \mathcal{F}_L^K(U),$$

thanks to $s^{\lceil L \rceil} \in \mathcal{F}_{\lceil L \rceil}^{\infty}(U)$ and $K' - \lceil L \rceil \geqslant K$. ■

In the previous statement $\mathscr{C}^K(U)[s]$ stands for the set of functions $h(s;\hat{\mu})$ that are polynomial in s with coefficients in $\mathscr{C}^K(U)$, i.e., that can be written as $h(s;\hat{\mu}) = h_0(\hat{\mu}) + h_1(\hat{\mu})s + \ldots + h_m(\hat{\mu})s^m$ for some $n \in \mathbb{Z}_{\geq 0}$ and $h_i \in \mathscr{C}^K(U)$ for $i = 0, 1, \ldots, m$. In the next result we extend this compact notation in the obvious way, for instance to refer to functions $h(s;\hat{\mu}) \in \mathscr{C}^K(U)[s] + \mathcal{F}_L^K(U)$. The following is a key result in order to study the composition of functions inside these type of sets. For the sake of clarity let us remark that, given $h(s;\hat{\mu})$ and $g(s;\hat{\mu})$, by $h \circ g$ we mean $h\big(g(s;\hat{\mu});\hat{\mu}\big)$.

Lemma 2.3. *Let us consider $\alpha, \beta \in \mathscr{C}^K(U)$ with $K \in \mathbb{N}$ and suppose that $\beta(\mu) > 0$ for all $\mu \in U$. Then, for $L \in \mathbb{R}_{\geq 0}$ and $p \in \mathbb{N}$, the following holds:*

(a) $s^\beta \circ s\mathcal{E}_+^{K'}(U) \subset s^\beta\big(\mathscr{C}^K(U)[s] + \mathcal{F}_L^K(U)\big) \subset s^\beta \mathscr{C}^K(U)[s] + \mathcal{F}_L^K(U)$ *if* $K' \geqslant K + L$.

(b) $s^p\omega_\alpha \circ s\mathcal{E}_+^{K'}(U) \subset \mathscr{C}^K(U)[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})$ *if* $K' \geqslant K + L$.

(c) $\big(s^\beta \mathscr{C}^{K'}(U)[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})\big) \circ s\mathcal{E}_+^{K'}(U) \subset s^\beta \mathscr{C}^K(U)[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})$ *if* $K' \geqslant K + L$.

(d) $s^\beta \mathscr{C}^K(U)[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\}) \subset \mathcal{F}_\ell(\{\alpha = 0, \beta > \ell\})$ *for every* $\ell \leqslant L$.

(e) If $L > 0$ and $g(s;\hat{\mu}) \in s^\beta \mathscr{C}^K(U)[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})$ with $g(s;\hat{\mu}) > 0$ for all $s > 0$ small enough, then

$$\big(s^m \mathscr{C}^K(U)[s] + \mathcal{F}_{L'}^K(\{\alpha = 0\})\big) \circ g \subset s^{\beta m} \mathscr{C}^K(U)[s, s^\beta, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0, \beta > L/L'\})$$

for every $L' \geqslant 1$ and $m \in \mathbb{Z}_{\geq 0}$.

Proof. For shortness we will write $\mathscr{C}^K$, $\mathcal{E}^K$ and $\mathcal{E}_+^K$ instead of $\mathscr{C}^K(U)$, $\mathcal{E}^K(U)$ and $\mathcal{E}_+^K(U)$, respectively. The assertion in (a) follows noting that

$$s^\beta \circ s\mathcal{E}_+^{K'} \subset s^\beta \mathcal{E}_+^{K'} \subset s^\beta(\mathscr{C}^K[s] + \mathcal{F}_L^K(U)) \subset s^\beta \mathscr{C}^K[s] + \mathcal{F}_L^K(U).$$

Here we apply Corollary 2.2 in the second inclusion whereas in the third one we use that $s^\beta \in \mathcal{F}_0(U)$ by (c) in Lemma A.3 (thanks to the asumption $\beta > 0$) and that $\mathcal{F}_0(U)\cdot\mathcal{F}_L(U) \subset \mathcal{F}_L(U)$ by (g) in Lemma A.2. This proves (a). Due to $\{\alpha = 0\} \subset U$, by applying (a) in Lemma A.2, the second inclusion above yields

$$s^\beta \circ s\mathcal{E}_+^{K'} \subset s^\beta(\mathscr{C}^K[s] + \mathcal{F}_L^K(U)) \subset s^\beta(\mathscr{C}^K[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})). \tag{5}$$

We claim at this point that $\mathscr{C}^K[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})$ is closed under sum and product. That it is closed under the sum follows directly from (e) in Lemma A.2. In order to show that it is also closed with respect to the product we note that $s^p\omega_\alpha \in \mathcal{F}_0^\infty(\{\alpha = 0\})$ by (d) in Lemma A.3 and that, on the other hand, $s \in \mathcal{F}_0^\infty(U) \subset \mathcal{F}_0^\infty(\{\alpha = 0\})$. Consequently, by applying (g), (c) and (e) in Lemma A.2, we can assert that $\mathscr{C}^K[s, s^p\omega_\alpha] \subset \mathcal{F}_0^K(\{\alpha = 0\})$. Hence, by (g) in Lemma A.2 again,

$$\mathscr{C}^K[s, s^p\omega_\alpha]\cdot\mathcal{F}_L^K(\{\alpha = 0\}) \subset \mathcal{F}_0^K(\{\alpha = 0\})\cdot\mathcal{F}_L^K(\{\alpha = 0\}) \subset \mathcal{F}_L^K(\{\alpha = 0\}).$$

Thus, since $\mathcal{F}_L^K(\{\alpha = 0\})\cdot\mathcal{F}_L^K(\{\alpha = 0\}) \subset \mathcal{F}_{2L}^K(\{\alpha = 0\}) \subset \mathcal{F}_L^K(\{\alpha = 0\})$ by (g) and (d) in Lemma A.2, and $\mathscr{C}^K[s, s^p\omega_\alpha]\cdot\mathscr{C}^K[s, s^p\omega_\alpha] \subset \mathscr{C}^K[s, s^p\omega_\alpha]$, the claim is true with respect to the product as well.

To prove (b) we observe that

$$\begin{aligned} s^p\omega_\alpha \circ s\mathcal{E}_+^{K'} &\subset s^p\mathcal{E}_+^{K'}(\mathcal{E}_+^{K'}\omega_\alpha + \omega_\alpha \circ \mathcal{E}_+^{K'}) \subset s^p\mathcal{E}_+^{K'}(\omega_\alpha + \mathcal{E}^{K'}) \subset \mathcal{E}_+^{K'}(s^p\omega_\alpha + \mathcal{E}^{K'}) \\ &\subset (\mathscr{C}^K[s] + \mathcal{F}_L^K(U))(s^p\omega_\alpha + \mathscr{C}^K[s] + \mathcal{F}_L^K(U)) \subset \mathscr{C}^K[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\}), \end{aligned}$$

where we use $\omega_\alpha(su) = u^{-\alpha}\omega_\alpha(s) + \omega_\alpha(u)$ in the first inclusion and in the second one that $\omega_\alpha \circ \mathcal{E}_+^{K'} \subset \mathcal{E}^{K'}$ due to $\omega_\alpha(s) = F(\alpha\log s)\log s$ with $F(z) = \frac{e^{-z}-1}{z}$, which is an entire function. Finally, in the fourth inclusion we apply Corollary 2.2, and the last one follows by taking the claim into account.

Let us turn now to the proof of (c). With this aim in view note that, taking assertions (a) and (b) into account together with the claim, we get

$$\mathscr{C}^{K'}[s, s^p\omega_\alpha] \circ s\mathcal{E}_+^{K'} \subset \mathscr{C}^K[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\}).$$

Accordingly, using also (5) and applying the claim once again,

$$s^\beta\mathscr{C}^{K'}[s, s^p\omega_\alpha] \circ s\mathcal{E}_+^{K'} \subset s^\beta(\mathscr{C}^K[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})) \subset s^\beta\mathscr{C}^K[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\}).$$

In the second inclusion above we use first that $s^\beta \in \mathcal{F}_0^\infty(U)$ by (c) in Lemma A.3 and $\beta > 0$, and then Lemma A.2 to conclude that $\mathcal{F}_0^\infty(U)\cdot\mathcal{F}_L^K(\{\alpha = 0\}) \subset \mathcal{F}_L^K(\{\alpha = 0\})$. On account of the above inclusion and (e) in Lemma A.2, to prove (c) it suffices to verify that $\mathcal{F}_L^K(\{\alpha = 0\}) \circ s\mathcal{E}_+^{K'} \subset \mathcal{F}_L^K(\{\alpha = 0\})$. To show this we firstly note that, by (c), (d), (g) and (a) in Lemma A.2,

$$s\mathcal{E}_+^{K'} \subset \mathcal{F}_1^\infty(U)\cdot\mathcal{F}_0^{K'}(U) \subset \mathcal{F}_1^{K'}(U) \subset \mathcal{F}_1^K(U) \subset \mathcal{F}_1^K(\{\alpha = 0\}),$$

and secondly we apply (h) in Lemma A.2. This proves the validity of (c).

In order to prove (d) note first that $s^\beta\mathscr{C}^K[s, s^p\omega_\alpha] \subset \mathcal{F}_\ell^K(\{\alpha = 0, \beta > \ell\})$ by (c) in Lemma A.3. On the other hand, since $\ell \leqslant L$ by assumption, $\mathcal{F}_L^K(\{\alpha = 0\}) \subset \mathcal{F}_\ell^K(\{\alpha = 0\}) \subset \mathcal{F}_\ell^K(\{\alpha = 0, \beta > \ell\})$ by (d) and (a) in Lemma A.2. Hence $s^\beta\mathscr{C}^K[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\}) \subset \mathcal{F}_\ell^K(\{\alpha = 0, \beta > \ell\})$ by (e) in Lemma A.2, and this proves the assertion in (d).

Finally let us show (e). To this end observe firstly that, since $g(s;\hat{\mu}) \in s^\beta\mathscr{C}^K(U)[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})$,

$$(s^m\mathscr{C}^K[s] + \mathcal{F}_{L'}^K(\{\alpha = 0\})) \circ g \subset s^m\mathscr{C}^K[s] \circ (s^\beta\mathscr{C}^K[s, s^p\omega_\alpha] + \mathcal{F}_L^K(\{\alpha = 0\})) + \mathcal{F}_{L'}^K(\{\alpha = 0\}) \circ g. \tag{6}$$

Note next that, thanks to assertion (d) in the present result, $g \in \mathcal{F}_\ell^K(\{\alpha = 0, \beta > \ell\})$ with $\ell = L/L'$ because $L' \geqslant 1$ by assumption. Hence, since $g > 0$ and $\ell > 0$, the application of (a) and (h) in Lemma A.2 yields

$$\mathcal{F}_{L'}^K(\{\alpha = 0\}) \circ g \subset \mathcal{F}_L^K(\{\alpha = 0, \beta > L/L'\}).$$

It is clear on the other hand that the first summand in (6) is a finite linear combination of terms of the form $(s^{\beta}\mathscr{C}^K[s,s^p\omega_\alpha]+\mathcal{F}_L^K(\{\alpha=0\}))^{m+i}$ with coefficients in $\mathscr{C}^K$ and $i\in\mathbb{Z}_{\geq 0}$. In this respect note that

$$
\begin{aligned}
(s^{\beta}\mathscr{C}^K[s,s^p\omega_\alpha]+\mathcal{F}_L^K(\{\alpha=0\}))^{m+i} &\subset s^{\beta(m+i)}\mathscr{C}^K[s,s^p\omega_\alpha]+\mathcal{F}_L^K(\{\alpha=0\})\\
&\subset s^{\beta m}\mathscr{C}^K[s,s^{\beta},s^p\omega_\alpha]+\mathcal{F}_L^K(\{\alpha=0\}),
\end{aligned}
$$

where the first inclusion follows from the fact that $s^{\beta}\mathscr{C}^K[s,s^p\omega_\alpha]\subset\mathcal{F}_0^K(\{\alpha=0\})$, due to $\beta>0$ and (d) in Lemma A.3, and that $\mathcal{F}_0^K(\{\alpha=0\})\cdot\mathcal{F}_L^K(\{\alpha=0\})\subset\mathcal{F}_L^K(\{\alpha=0\})$, by (g) in Lemma A.2. Finally, from (6) and due to $\mathcal{F}_L^K(\{\alpha=0\})+\mathcal{F}_L^K(\{\alpha=0,\beta>L/L'\})\subset\mathcal{F}_L^K(\{\alpha=0,\beta>L/L'\})$, the inclusion in (e) follows. This concludes the proof of the result. ■

The next result is addressed to regularity properties of the Poincaré transition map (and its associated time) between two transverse sections.

Lemma 2.4. *Let U be an open set of $\mathbb{R}^N$ and consider a family of vector fields $\{Y_\nu\}_{\nu\in U}$ of the form*

$$
Y_\nu=\frac{1}{y^{\ell}f(x,y;\nu)}(\partial_x+h(x,y;\nu)y\partial_y),
$$

where $\ell\in\mathbb{Z}$ and $f,h\in\mathscr{C}^K(V\times U)$ with $V=(a,b)\times(-c,c)\subset\mathbb{R}^2$ for some $a<b$ and $c>0$. Suppose also that $f(x,0;\nu)>0$ for all $x\in(a,b)$ and $\nu\in U$. Consider two $\mathscr{C}^K$ families of transverse sections

$$
\xi(\,\cdot\,;\nu):(-\varepsilon,\varepsilon)\to\Pi_1 \text{ and } \zeta(\,\cdot\,;\nu):(-\varepsilon,\varepsilon)\to\Pi_2
$$

to the straight line $\{y=0\}$ with $\xi_2(0)=\zeta_2(0)=0$ and $\xi_2'(0)\zeta_2'(0)>0$. If $P(s;\nu)$ and $T(s;\nu)$ are, respectively, the Poincaré and time maps from Π_1 to Π_2 of Y_ν then the following holds:

(a) $P\in\mathscr{C}^K((-\varepsilon,\varepsilon)\times U)$, $P(0;\nu)=0$ *and* $P'(0;\nu)>0$.

(b) $T(s;\nu)=s^{\ell}\bar{T}(s;\nu)$ *with* $\bar{T}\in\mathscr{C}^{K-1}((-\varepsilon,\varepsilon)\times U)$.

Proof. Let us consider the vector field $\partial_x+h(x,y;\nu)\partial_y$, which is equivalent to Y_ν, and denote by $\phi(t,p_0;\nu)$ its solution passing through $p_0\in\mathbb{R}^2$ at time $t=0$. Clearly if $p_0=(x_0,y_0)$ then $\phi(t,p_0;\nu)=\big(t+x_0,\phi_2(t,p_0;\nu)\big)$. Therefore, by definition,

$$
\zeta_2(P(s))=\phi_2\big(\zeta_1(P(s))-\xi_1(s),\xi(s)\big) \text{ for all } s\in(-\varepsilon,\varepsilon).
$$

(Here, and in what follows when there is no risk of ambiguity, we omit the parameter dependence for the sake of shortness.) Then the assertion in (a) follows easily by the implicit function theorem, the smooth dependence of solutions with respect to initial conditions and parameters (see [4, Theorem 1.1]) and the assumption $\xi_2'(0)\zeta_2'(0)>0$. In order to prove (b) note first that if we define $\Theta(x,s;\nu):=\phi_2(x-\xi_1(s),\xi(s))$, which is a $\mathscr{C}^K$ function, then

$$
T(s;\nu)=\int_{\xi_1(s)}^{\zeta_1(P(s))}\Theta(x,s)^{\ell}f(x,\Theta(x,s))dx.
$$

Since $\Theta(x,0)=0$ due to the invariance of $\{y=0\}$, Lemma 2.1 shows that there exists a $\mathscr{C}^{K-1}$ function $\bar{\Theta}(x,s;\nu)$ such that $\Theta(x,s)=s\bar{\Theta}(x;s)$ and then we can write $T(s;\nu)=s^{\ell}\bar{T}(s;\nu)$ with

$$
\bar{T}(s;\nu):=\int_{\xi_1(s)}^{\zeta_1(P(s))}\bar{\Theta}(x,s)^{\ell}f(x,\Theta(x,s))dx,
$$

which is also $\mathscr{C}^{K-1}$ in $(-\varepsilon,\varepsilon)\times U$ for $\varepsilon>0$ small enough. This shows (b) and completes the proof. ■

We can now state and prove the following fundamental result concerning the Dulac map $D(\,\cdot\,;\hat{\mu})$ and the Dulac time $T(\,\cdot\,;\hat{\mu})$ of the hyperbolic saddle (1) between the transverse sections Σ_σ and Σ_τ.

Theorem 2.5. *For every $\hat{\mu}_0=(\lambda_0,\mu_0)\in\hat{W}$, $K\in\mathbb{N}$ and $L>0$ there exists a neighborhood U of $\hat{\mu}_0$ so that*

$$D(s;\hat{\mu})=s^{\lambda}\sum_{(i,j)\in\mathscr{B}^0_{\lambda_0,L-\lambda_0}}\boldsymbol{\Delta}_{ij}(\omega_\alpha(s);\hat{\mu})s^{i+\lambda j}+\mathcal{F}^K_L(\hat{\mu}_0),$$

and

$$T(s;\hat{\mu})=T_0(\hat{\mu})\log s+\sum_{(i,j)\in\mathscr{B}^n_{\lambda_0,L}}\boldsymbol{T}_{ij}(\omega_\alpha(s);\hat{\mu})s^{i+\lambda j}+\mathcal{F}^K_L(\hat{\mu}_0),$$

where T_0 is given in (2) and $\boldsymbol{\Delta}_{ij}(w;\hat{\mu}),\boldsymbol{T}_{ij}(w;\hat{\mu})\in\mathscr{C}^K(U)[w]$. Furthermore $\deg_w\boldsymbol{\Delta}_{ij}=\deg_w\boldsymbol{T}_{ij}=0$ if $\lambda_0\notin\mathbb{Q}$ and, otherwise, $\alpha=p-\lambda q$ if $\lambda_0=p/q$ with $\gcd(p,q)=1$. In the latter case the following additional properties hold:

(a) $\boldsymbol{\Delta}_{ij}\equiv 0$ *(respectively, $\boldsymbol{T}_{ij}\equiv 0$) if $(i+rp,j-rq)$ belongs to Λ_0 (respectively, Λ_n) for some $r\in\mathbb{N}$,*

(b) $\deg_w\boldsymbol{\Delta}_{ij}\leqslant i/p$ *and* $\deg_w\boldsymbol{T}_{ij}\leqslant i/p$,

(c) if $p/q\notin D^0_{ij}$ (respectively, $p/q\notin D^n_{ij}$) then $\deg_w\boldsymbol{\Delta}_{ij}=0$ (respectively, $\deg_w\boldsymbol{T}_{ij}=0$),

In particular $\deg_w\boldsymbol{\Delta}_{00}=0$ and, in case that $L\geqslant\lambda_0$, $\boldsymbol{\Delta}_{00}(0;\hat{\mu})>0$ for all $\hat{\mu}\in U$.

Proof. Consider the parameter $\hat{\mu}_0=(\lambda_0,\mu_0)\in(0,+\infty)\times W$, the integer K and the real number $L>0$ given in the statement and let us define

$$K'=K+1+\lceil\max(L,2L/\lambda_0)\rceil. \tag{7}$$

By applying [16, Theorem A], the family $\{X_{\hat{\mu}}\}_{\hat{\mu}\in\hat{W}}$ is $\mathscr{C}^{K'}$ conjugated by a diffeomorphism of the form $\Phi(x,y,\hat{\mu})=(\Phi_{\hat{\mu}}(x,y),\hat{\mu})$ defined in a neighbourhood of $(0,0,\hat{\mu}_0)\in\mathbb{R}^2\times\hat{W}$ to

$$Y^{NF}_{\hat{\mu}}=\frac{1}{\eta x^{n_1}y^{n_2}+u^{\ell}A(u;\hat{\mu})}\Big(x\partial_x+\big(-\lambda+B(u;\hat{\mu})\big)y\partial_y\Big), \tag{8}$$

where $\eta=\frac{1}{P(0,0;\hat{\mu})}$, $\ell\in\mathbb{N}$ and

(*i*) if $\lambda_0\notin\mathbb{Q}$ then $A=B=0$, and

(*ii*) if $\lambda_0=p/q$ with $\gcd(p,q)=1$, then $A(u;\hat{\mu})$ and $B(u;\hat{\mu})$ are polynomials in the resonant monomial $u:=x^py^q$ with coefficients in $\mathscr{C}^{K'}(\hat{W})$, i.e., $A,B\in\mathscr{C}^{K'}(\hat{W})[u]$.

Without lost of generality, we can assume that the normalizing domain of Φ is $V\times U$, where V and U are, respectively, neighbourhoods of $(0,0)\in\mathbb{R}^2$ and $\hat{\mu}_0\in\hat{W}$.

During the computation of the development of the Dulac map and Dulac time we shall lose order of differentiability of the involved functions. As it is usual in this kind of study, the idea is that we can take K' arbitrarily large and that the lost of differentiability is well controlled. In any case, in order to avoid any ambiguity, we give a specific value of K' in (7) which is large enough for our purposes.

Fix $\delta>0$ and $\varepsilon>0$ small enough so that the points $(0,\delta)$ and $(\varepsilon,0)$ are inside V. We define two auxiliary transverse sections Σ_δ and Σ_ε to $X_{\hat{\mu}}$, see Figure 3, parametrized by $s\longmapsto\Phi_{\hat{\mu}}(s,\delta)$ and $s\longmapsto\Phi_{\hat{\mu}}(\varepsilon,s)$, respectively. Let $P_1(\cdot\,;\hat{\mu}),D_2(\cdot\,;\hat{\mu})$ and $P_3(\cdot\,;\hat{\mu})$ be respectively the transition maps from Σ_σ to Σ_δ, Σ_δ to Σ_ε and

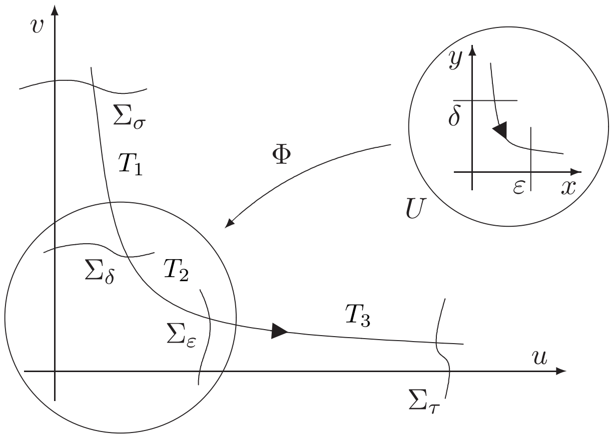

Figure 3: Auxiliary transverse sections in the decomposition of T.

Σ_ε to Σ_τ. (Here P stands for Poincaré and D stands for Dulac.) Consider also $T_1(\cdot\,;\hat\mu), T_2(\cdot\,;\hat\mu)$ and $T_3(\cdot\,;\hat\mu)$ the corresponding time functions. Omitting the dependence on $\hat\mu$ we have that $D(s) = P_3(D_2(P_1(s)))$ and

$$T(s) = T_1(s) + T_2\big(P_1(s)\big) + T_3\big(D_2(P_1(s))\big).$$

In what follows, for the sake of shortness we shall use the compact notation $\mathscr{C}^K$, $\mathcal{E}^K$ and $\mathcal{E}^K_+$ instead of $\mathscr{C}^K(U)$, $\mathcal{E}^K(U)$ and $\mathcal{E}^K_+(U)$, respectively. That being said, by Lemma 2.4 and Corollary 2.2 we have that

$$\begin{array}{ll} P_1(s;\hat\mu) \in s\mathcal{E}_+^{K'-1} & T_1(s;\hat\mu) \in s^{n_1}\mathcal{E}^{K'-1} \subset s^{n_1}\mathscr{C}^K[s] + \mathcal{F}^K_L(\hat\mu_0), \\ P_3(s;\hat\mu) \in s\mathcal{E}_+^{K'-1} \subset s\mathscr{C}^K[s] + \mathcal{F}^K_{L'}(\hat\mu_0) & T_3(s;\hat\mu) \in s^{n_2}\mathcal{E}^{K'-1} \subset s^{n_2}\mathscr{C}^K[s] + \mathcal{F}^K_{L'}(\hat\mu_0), \end{array} \tag{9}$$

where we set

$$L' := \max(1, 2L/\lambda_0)$$

and we take (7) into account. Furthermore, setting $n = (n_1, n_2)$, if $\lambda_0 \notin \mathbb{Q}$ then the local Dulac map and the local Dulac time are given by

$$D_2(s;\hat\mu) = \delta\varepsilon^{-\lambda}s^\lambda \text{ and } T_2(s;\hat\mu) = \begin{cases} \frac{\delta^{n_2}}{P(0,0)}\frac{s^{\lambda n_2}\varepsilon^{n_1-\lambda n_2} - s^{n_2}}{n_1 - \lambda n_2} & \text{if } n \neq (0,0), \\ \frac{-1}{P(0,0)}\log(\frac{s}{\varepsilon}) & \text{if } n = (0,0), \end{cases} \tag{10}$$

respectively. (Here *local* means that we work close enough to the saddle so that the normalizing coordinates provided by the normal form $Y^{NF}_{\hat\mu}$ in (8) can be used.) Of course these maps cannot be computed explicitly when $\lambda_0 \in \mathbb{Q}$, and in this case we apply Theorems A and B in [19] which show, respectively, that if $\lambda_0 = p/q$ with $\gcd(p,q) = 1$ then

$$D_2(s;\hat\mu) \in s^\lambda\mathscr{C}^{K'}[s^p, s^p\omega_\alpha] + \mathcal{F}^{K'}_L(\hat\mu_0) \tag{11}$$

and, setting $\kappa := \left\lceil \max\left(\frac{n_1}{p}, \frac{n_2}{q}\right)\right\rceil$,

$$T_2(s;\hat\mu) \in T_0(\hat\mu)(\log s - \log\varepsilon) + s^{\lambda n_2}\mathscr{C}^{K'}[s^p, s^p\omega_\alpha] + s^{n_1}\mathscr{C}^{K'}[s^p] + s^{\kappa p}\mathscr{C}^{K'}[s^p, s^p\omega_\alpha] + \mathcal{F}^{K'}_L(\hat\mu_0). \tag{12}$$

Here T_0 is the one given in (2), $\alpha = \alpha(\hat\mu) = p - \lambda q$, so that $\alpha(\hat\mu_0) = 0$, and we use the compact notation $\omega_\alpha = \omega\big(s;\alpha(\hat\mu)\big)$. There are some remarks to be made about the application of Theorems A and B in [19]:

- They provide the local Dulac map and local Dulac time with transversal sections *normalized* to $\varepsilon = \delta = 1$, say D_N and T_N, respectively. To bypass this technical inconvenience we consider the new local change of coordinates $\tilde{\Phi} = \Phi \circ h$ where $h(x,y) := (\varepsilon x, \delta y)$, so that $D_2(s;\hat{\mu}) = \delta D_N(s/\varepsilon;\hat{\mu})$ and $T_2(s;\hat{\mu}) = T_N(s/\varepsilon;\hat{\mu})$. Then the principal parts in (11) and (12) follow noting, firstly, that the pull-back of (8) by h preserves the normal form (it only changes η and the coefficients of A and B) and, secondly, that $\omega_\alpha(s/\varepsilon) = \varepsilon^\alpha \omega_\alpha(s) + \omega_\alpha(\varepsilon)$.
- In both results the coefficients of A and B are treated as independent parameters, i.e.,

$$Y_{\alpha,\beta} = \frac{1}{\beta_0 x^{n_1} y^{n_2} + u^\ell \sum_{i=1}^M \beta_i u^{i-1}} \Big(x\partial_x + \frac{1}{q}\Big(-p + \sum_{i=0}^{N-1} \alpha_{i+1} u^i \Big) y \partial_y \Big),$$

  where $\alpha_1 := p - \lambda q$. In particular, they show that the remainder of $D_N(s;\alpha)$ and $T_N(s;\alpha,\beta)$ belong to $\mathcal{F}_L^\infty(U_0)$ and $\mathcal{F}_L^\infty(U_0 \times \mathbb{R}^{M+1})$, respectively, where U_0 is a neighbourhood of $\{\alpha_1 = 0\}$ in $\mathbb{R}^N$. In our application we have $\alpha_i = \alpha_i(\hat{\mu})$ and $\beta_i = \beta_i(\hat{\mu})$, which are $\mathscr{C}^{K'}$ functions, and consequently to obtain the remainder in (11) and (12) we must also use (h) in Lemma A.2.

Let us consider the case $\lambda_0 = p/q \in \mathbb{Q}$ first. Then, from (9) and (11), by applying (c) in Lemma 2.3 and taking (7) also into account, we get

$$(D_2 \circ P_1)(s;\hat{\mu}) \in s^\lambda \mathscr{C}^K[s, s^p \omega_\alpha] + \mathcal{F}_L^K(\hat{\mu}_0). \tag{13}$$

(Here, and in what follows, we take $\alpha(\hat{\mu}_0) = 0$ also into account.) Moreover, from (9) and (12), the application of (a) and (c) in Lemma 2.3 shows that

$$(T_2 \circ P_1)(s;\hat{\mu}) \in T_0\left(\log s + \mathscr{C}^K[s]\right) + s^{n_1}\mathscr{C}^K[s] + s^{\lambda n_2}\mathscr{C}^K[s, s^p\omega_\alpha] + s^{\kappa p}\mathscr{C}^K[s, s^p \omega_\alpha] + \mathcal{F}_L^K(\hat{\mu}_0), \tag{14}$$

where we also use that $\log(s\mathcal{E}_+^{K'}) = \log s + \mathcal{E}^{K'} \subset \log s + \mathscr{C}^K[s] + \mathcal{F}_L^K(\hat{\mu}_0)$ thanks to Corollary 2.2. Finally, if $f(s;\hat{\mu}) \in s^m \mathscr{C}^K[s] + \mathcal{F}_{L'}^K(\hat{\mu}_0)$ then, from (13) and applying (e) in Lemma 2.3,

$$(f \circ D_2 \circ P_1)(s;\hat{\mu}) \in s^{\lambda m}\mathscr{C}^K[s, s^\lambda, s^p \omega_\alpha] + \mathcal{F}_L^K(\hat{\mu}_0)$$

due to $\lambda(\hat{\mu}_0) = \lambda_0 > L/L' = \min(L, \lambda_0/2)$. Using the above inclusion with $\{f = T_3, m = n_2\}$ and $\{f = P_3, m = 1\}$, from (9) and (14) we get, respectively,

$$\begin{aligned} T &= T_1 + T_2 \circ P_1 + T_3 \circ D_2 \circ P_1 \\ &\in T_0 \log s + T_0 \mathscr{C}^K[s] + s^{n_1}\mathscr{C}^K[s] + s^{\kappa p}\mathscr{C}^K[s, s^p\omega_\alpha] + s^{\lambda n_2}\mathscr{C}^K[s, s^\lambda, s^p \omega_\alpha] + \mathcal{F}_L^K(\hat{\mu}_0) \end{aligned} \tag{15}$$

and $D = P_3 \circ D_2 \circ P_1 \in s^\lambda \mathscr{C}^K[s, s^\lambda, s^p\omega_\alpha] + \mathcal{F}_L^K(\hat{\mu}_0)$. Recall that $\mathscr{B}_{\lambda,L}^k = \{(i,j) \in \Lambda_k : i + \lambda j \leqslant L\}$ with $\Lambda_k = (\mathbb{Z}_{\geq k_1} \times \{0\}) \cup (\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq k_2})$ by definition (see also Figure 4). On account of this the above two inclusions imply, respectively, that we can write

$$T(s;\hat{\mu}) = T_0(\hat{\mu}) \log s + \sum_{(i,j) \in \mathscr{B}_{\lambda_0,L}^n} \hat{T}_{ij}(\omega_\alpha(s);\hat{\mu}) s^{i+\lambda j} + \mathcal{F}_L^K(\hat{\mu}_0) \tag{16}$$

and

$$D(s;\hat{\mu}) = s^\lambda \sum_{(i,j) \in \mathscr{B}_{\lambda_0, L-\lambda_0}^0} \hat{\Delta}_{ij}(\omega_\alpha(s);\hat{\mu}) s^{i+\lambda j} + \mathcal{F}_L^K(\hat{\mu}_0), \tag{17}$$

where $\hat{\Delta}_{ij}(w;\hat{\mu})$ and $\hat{T}_{ij}(w;\hat{\mu})$ are polynomials in w with coefficients $\mathscr{C}^K$ functions in a neighbourhood of $\hat{\mu}_0$. Here we use that, thanks to (c) in Lemma A.3, if $i + \lambda_0 j > L$ then $s^{i+\lambda j}\omega_\alpha^\ell \in \mathcal{F}_L^\infty(\hat{\mu}_0)$ for any $\ell \in \mathbb{Z}_{\geq 0}$. Observe moreover that these polynomials can be taken with $\deg_w \hat{\Delta}_{ij} \leqslant i/p$ and $\deg_w \hat{T}_{ij} \leqslant i/p$.

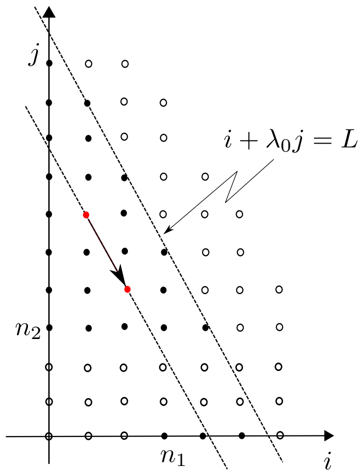

Figure 4: The set $\mathscr{B}^n_{\lambda_0,L}$ (filled dots) with two pairs (in red) yielding to a replacement.

Let us focus now on the expression in (16) and consider the summation's grid where (i,j) varies. For each fixed $(i,j) \in \mathscr{B}^n_{\lambda_0,L}$ we define, see Figure 4,

$$r_\star := \max\{r \in \mathbb{Z}_{\geq 0} : (i,j) + r(p,-q) \in \Lambda_n\} \text{ and } (i_\star, j_\star) := (i,j) + r_\star(p,-q).$$

Accordingly $(i_\star, j_\star) \in \mathscr{B}^n_{\lambda_0,L}$ and we can write

$$s^{i+\lambda j} = s^{i_\star - r_\star p + \lambda(j_\star + r_\star q)} = s^{i_\star + \lambda j_\star} s^{-r_\star(p-\lambda q)} = s^{i_\star + \lambda j_\star}(1 + \alpha\omega_\alpha)^{r_\star}.$$

(Certainly $r_\star$ depends on the particular pair (i,j) considered as well as on n and $\lambda_0 = p/q$. We omit this dependence for the sake of shortness.) Then, for each pair $(i,j) \in \mathscr{B}^n_{\lambda_0,L}$ and $\ell = 0, 1, \ldots, \deg_w \hat{\Delta}_{ij}$, we replace the monomial $s^{i+\lambda j}\omega_\alpha^\ell$ in (16) by $s^{i_\star + \lambda j_\star}\omega_\alpha^\ell(1+\alpha\omega_\alpha)^{r_\star}$ to obtain

$$T(s;\hat{\mu}) = T_0(\hat{\mu}) \log s + \sum_{(i,j) \in \mathscr{B}^n_{\lambda_0,L}} \boldsymbol{T}_{ij}(\omega_\alpha(s);\hat{\mu}) s^{i+\lambda j} + \mathcal{F}^K_L(\hat{\mu}_0),$$

with new polynomials $\boldsymbol{T}_{ij}(w;\hat{\mu})$ such that $\boldsymbol{T}_{ij} \equiv 0$ if $(i + rp, j - rq) \in \Lambda_n$ for some $r \in \mathbb{N}$, i.e., assertion (a) in the statement.

Next we replace similarly the monomials in the grid $\mathscr{B}^0_{\lambda_0, L-\lambda_0}$ for the Dulac map in (17). (Note that in this case if $\lambda_0 > L$ then the grid is empty.) We proceed exactly the same way as before but taking Λ_0 instead of Λ_n in the definition of $r_\star$ given above. In doing so we get

$$D(s;\hat{\mu}) = s^\lambda \sum_{(i,j) \in \mathscr{B}^0_{\lambda_0, L-\lambda_0}} \boldsymbol{\Delta}_{ij}(\omega_\alpha(s);\hat{\mu}) s^{i+\lambda j} + \mathcal{F}^K_L(\hat{\mu}_0)$$

with new polynomials $\boldsymbol{\Delta}_{ij}(w;\hat{\mu})$ satisfying the assertion in (a).

Let us turn to the proof of the other assertions. That $\deg_w \boldsymbol{\Delta}_{ij} \leqslant i/p$ and $\deg_w \boldsymbol{T}_{ij} \leqslant i/p$ follows noting that this is true for the polynomials $\hat{\Delta}_{ij}$ and $\hat{T}_{ij}$ and that, on account of this, after the replacement of $s^{i+\lambda j}\omega_\alpha^\ell$ by $s^{i_\star + \lambda j_\star}\omega_\alpha^\ell(1+\alpha\omega_\alpha)^{r_\star}$ we have $r_\star + \ell \leqslant r_\star + i/p = (i + r_\star p)/p = i_\star/p$. This proves (b).

Let us prove the validity of (c). We prove first the assertion with regard to the polynomials $\boldsymbol{T}_{ij}$ by contradiction. Suppose $\lambda_0 = p/q \notin D^n_{ij}$, which implies $\boldsymbol{T}_{ij} = \hat{T}_{ij}$, and $\ell := \deg_w \boldsymbol{T}_{ij} > 0$. We distinguish the cases $j = 0$ and $j > 0$:

- If $j = 0$ and $n_2 > 0$ then, due to $\boldsymbol{T}_{ij} = \hat{T}_{ij}$, the monomial $s^i \omega_\alpha^\ell$ in $T(s)$ comes necessarily from the summand $s^{\kappa p} \mathscr{C}^K[s, s^p \omega_\alpha]$ in (15), where recall that $\kappa = \lceil \max\left(\frac{n_1}{p}, \frac{n_2}{q}\right) \rceil$. Hence $i = (\kappa + \ell)p + i'$ for some $i' \in \mathbb{Z}_{\geq 0}$. Then, setting $j' := (\kappa + \ell)q$, on account of $\ell > 0$ and $\kappa \geqslant n_2/q$, it follows that $(i', j') \in \Lambda_n \setminus \{(i, 0)\}$ satisfies $i + \lambda_0 j = i' + \lambda_0 j'$, which implies $\lambda_0 \in D_{i0}^n$, a contradiction.

  If $j = n_2 = 0$ then $s^i \omega_\alpha^\ell$ can also come from the summand $s^{\lambda n_2} \mathscr{C}^K[s, s^\lambda, s^p \omega_\alpha] = \mathscr{C}^K[s, s^\lambda, s^p \omega_\alpha]$ in (15). Then $i = i' + \ell p$ for some $i' \in \mathbb{Z}_{\geq 0}$ and setting $j' = \ell q$ it follows that $(i', j') \in \mathbb{Z}_{\geq 0}^2$ (which is equal to Λ_n due to $n_2 = 0$) with $(i', j') \neq (i, 0)$ and, on the other hand,

  $$i + \lambda_0 0 = i' + \ell p = i' + \ell q \lambda_0 = i' + \lambda_0 j'.$$

  Consequently $\lambda_0 \in D_{i0}^n$, which contradicts the assumption on λ_0 again.

- Consider finally the case $j > 0$, which implies $j \geqslant n_2$ (see Figure 4). Then, due to $\boldsymbol{T}_{ij} = \hat{T}_{ij}$, the monomial $s^{i+\lambda j} \omega_\alpha^\ell$ in $T(s)$ comes unavoidably from the summand $s^{\lambda n_2} \mathscr{C}^K[s, s^\lambda, s^p \omega_\alpha]$ in (15). Thus $i = i' + \ell p$ for some $i' \in \mathbb{Z}_{\geq 0}$. In this case, setting $j' := j + \ell q \in \mathbb{Z}_{\geq n_2}$, we have $(i', j') \in \Lambda_n \setminus \{(i, j)\}$, due to $\ell > 0$, and $i + \lambda_0 j = i' + \lambda_0 j'$, which implies $\lambda_0 \in D_{ij}^n$, a contradiction.

We show next that if $p/q \notin D_{ij}^0$ then $\ell := \deg_w \boldsymbol{\Delta}_{ij} = 0$. To this end note first that $\boldsymbol{\Delta}_{ij} = \hat{\Delta}_{ij}$ by construction. Moreover, due to $D(s; \hat{\mu}) \in s^\lambda \mathscr{C}^K[s, s^\lambda, s^p \omega_\alpha] + \mathcal{F}_L^K(\hat{\mu}_0)$, the monomial $s^{i+\lambda j} \omega_\alpha^\ell$ in (17) verifies that $i = i' + p\ell$ for some $i' \in \mathbb{Z}_{\geq 0}$. Thus, setting $j' := j + \ell q$, we have $(i', j') \in \Lambda_0$ and $i + \lambda_0 j = i' + \lambda_0 j'$. Since $(i, j) \neq (i', j')$ if and only if $\ell > 0$, this implies that $\ell = 0$, otherwise $\lambda_0 \in D_{ij}^0$, a contradiction.

This concludes the proof for the case $\lambda_0 \in \mathbb{Q}$. Let us consider next the case $\lambda_0 \notin \mathbb{Q}$. Note first that, on account of (9) and (10), the preceding expressions are valid without using the compensator, i.e.,

$$D_2 \circ P_1 \in s^\lambda \mathscr{C}^K[s] + \mathcal{F}_L^K(\hat{\mu}_0) \text{ and } T_2 \circ P_1 \in \tau_0(\log s + \mathscr{C}^K[s]) + s^{n_1} \mathscr{C}^K[s] + s^{\lambda n_2} \mathscr{C}^K[s] + \mathcal{F}_L^K(\hat{\mu}_0).$$

Then, exactly as before, using that if $f \in s^m \mathscr{C}^K[s] + \mathcal{F}_{L'}^K(\hat{\mu}_0)$ then $f \circ D_2 \circ P_1 \in s^{\lambda m} \mathscr{C}^K[s, s^\lambda] + \mathcal{F}_L^K(\hat{\mu}_0)$, we can conclude that $D = P_3 \circ D_2 \circ P_1 \in s^\lambda \mathscr{C}^K[s, s^\lambda] + \mathcal{F}_L^K(\hat{\mu}_0)$ and

$$T = T_1 + T_2 \circ P_1 + T_3 \circ D_2 \circ P_1 \in \tau_0 \log s + \tau_0 \mathscr{C}^K[s] + s^{n_1} \mathscr{C}^K[s] + s^{\lambda n_2} \mathscr{C}^K[s, s^\lambda] + \mathcal{F}_L^K(\hat{\mu}_0).$$

write as (17) and (16), respectively, but with $\deg_w \hat{\Delta}_{ij} = \deg_w \hat{T}_{ij} = 0$. This shows the validity of the result for the case $\lambda_0 \notin \mathbb{Q}$.

With regard to the last assertion in the statement we can already assert that $\deg_w \Delta_{00} = 0$ in both cases, $\lambda_0 \in \mathbb{Q}$ and $\lambda_0 \notin \mathbb{Q}$. Recall on the other hand that $D(s) = P_3(D_2(P_1(s)))$ with P_1 and P_3 regular transitions maps, verifying $P_i(0; \hat{\mu}) = 0$ and $P_i'(0; \hat{\mu}) > 0$ by Lemma 2.4, and $D_2(s; \hat{\mu}) = \delta D_N(s/\varepsilon; \hat{\mu})$, where D_N is the local Dulac map with transversal sections normalized to $\varepsilon = \delta = 1$. Since $\lim_{s \to 0^+} \frac{D_N(s; \hat{\mu})}{s^\lambda} = 1$ thanks to [19, Theorem A], we conclude that

$$\boldsymbol{\Delta}_{00}(0; \hat{\mu}) = \Delta_{00}(\hat{\mu}) = P_3'(0; \hat{\mu}) \delta \left(P_1'(0; \hat{\mu})/\varepsilon\right)^\lambda > 0$$

and so the last assertion is true. This completes the proof of the result. ∎

3 Proof of the main results

This section is devoted to the proofs of Theorems A, B and C. In order to show the first one we shall apply the following two lemmas.

Lemma 3.1. *Let U be an open subset of $\mathbb{R}^N$ and C a closed subset inside U. Assume that $f \in \mathscr{C}^\infty(U \setminus C)$ and that, for each $K \in \mathbb{Z}_{\geq 0}$, there exist a collection of functions $\{g_K^m\}_{m \in \mathbb{N}}$ and a collection of open subsets $\{V_K^m\}_{m \in \mathbb{N}}$ with $g_K^m \in \mathscr{C}^K(V_K^m)$ and $C \subset \bigcup_{m \in \mathbb{N}} V_K^m$ verifying that*

(a) $g_K^m = g_{K'}^{m'}$ in $V_K^m \cap V_{K'}^{m'} \cap C$,

(b) $f = g_K^m$ in $(U \setminus C) \cap V_K^m$.

Then the function

$$\hat{f}(p) := \begin{cases} f(p) & \text{if } p \in U \setminus C, \\ g_K^m(p) & \text{if } p \in C \cap V_K^m, \end{cases}$$

is well defined and belongs to $\mathscr{C}^\infty(U)$.

Proof. We will prove that $\hat{f} \in \mathscr{C}^K(U)$ for all $K \in \mathbb{Z}_{\geq 0}$ by induction on K. The base case $K = 0$ is clear. For the induction step assume that $\hat{f} \in \mathscr{C}^K(U)$. Let us fix any $p_0 \in C$ and take m such that $p_0 \in C \cap V_{K+1}^m$. Then the partial derivatives of $\hat{f}$ of order $K+1$ at p_0 exist and are continuous because

$$\partial_w \left(\partial_v^K \hat{f} \right)(p_0) = \lim_{h \to 0} \frac{\partial_v^K g_{K+1}^m(p_0 + hw) - \partial_v^K g_{K+1}^m(p_0)}{h} = \partial_w \partial_v^K g_{K+1}^m(p_0) = \partial_{v+w}^{K+1} g_{K+1}^m(p_0)$$

for any $v, w \in \mathbb{Z}_{\geq 0}^N$ with $|v| = K$ and $|w| = 1$. This proves the validity of the result. ■

Lemma 3.2. *For any* $k \in \mathbb{Z}_{\geq 0}^2$ *and* $(i_\star, j_\star) \in \Lambda_k$, *the set*

$$\hat{D}_{i_\star j_\star}^k := \big\{ \lambda > 0 : \text{there exist } (i,j), (i',j') \in \Lambda_k \text{ with } (i,j) \neq (i',j') \text{ and } i + \lambda j = i' + \lambda j' \leqslant i_\star + \lambda j_\star \big\}$$

is discrete in $(0, +\infty)$.

Proof. Suppose that $\lim_{m \to \infty} \lambda_m = \lambda_\star \in (0, +\infty)$ with $\lambda_m \in \hat{D}_{i_\star j_\star}^k$ for all $m \in \mathbb{N}$. Then, for each $m \in \mathbb{N}$ there exist two different pairs (i_m, j_m) and (i'_m, j'_m) in Λ_n such that

$$i_m + \lambda_m j_m = i'_m + \lambda_m j'_m \leqslant i_\star + \lambda_m j_\star.$$

Therefore, since $j_m - j'_m \neq 0$, $\lambda_m = \frac{i'_m - i_m}{j_m - j'_m}$. In addition, due to $\lambda_\star \in (0, +\infty)$, it follows that the sequences $(j_m)_m$ and $(j'_m)_m$ are bounded. Thus $(j_m - j'_m)_m$ is a bounded sequence of integers and, consequently, λ_m belongs to the set of rational numbers with bounded denominator, which is discrete. Hence $\lambda_m = \lambda_\star$ for all $m \in \mathbb{N}$ large enough and the result follows. ■

Remark 3.3. By Lemma 3.2 we can assert that D_{ij}^k is a discrete subset of $(0, +\infty)$ because $D_{i_\star j_\star}^k \subset \hat{D}_{i_\star j_\star}^k$, see Definition 1.4. Exactly the same proof shows that D_L^k is a discrete set in $(0, +\infty)$ as well. □

We are now ready to tackle the proof Theorem A, that follows fundamentally by applying Theorem 2.5. Notice that basic difference between Theorem 2.5 and Theorem A is the order of quantifiers in the statements, $\forall \ldots \exists$ in the first one and $\exists \ldots \forall$ in the second one. The key idea to prove that the coefficients T_{ij} in the asymptotic expansion of $T(\,\cdot\,; \hat{\mu})$ are $\mathscr{C}^\infty$ is to reverse the order of these quantifiers. Before starting the proof let us make the following easy observation that we shall use several times.

Remark 3.4. If $\sum_{i=1}^m a_i s^{\lambda_i} + f(s) = 0$ for all $s \in (0, \varepsilon)$, where $\lambda_i \in \mathbb{R}$ with $\lambda_1 < \lambda_2 < \cdots < \lambda_m$, $a_1, a_2, \ldots, a_m \in \mathbb{R}$ and $f(s) = \mathrm{o}(s^{\lambda_m})$ then $a_1 = a_2 = \cdots = a_m = 0$. □

Proof of Theorem A. The application of Theorem 2.5 shows that, for every $\hat{\mu}_0 = (\lambda_0, \mu_0) \in \hat{W}$, $K \in \mathbb{N}$ and $L > 0$, there exists a neighborhood $U = U_{\hat{\mu}_0 K L}$ of $\hat{\mu}_0$ such that the Dulac time can be written as

$$T(s; \hat{\mu}) = T_0(\hat{\mu}) \log s + \sum_{(i,j) \in \mathscr{B}_{\lambda_0, L}^n} \boldsymbol{T}_{ij}^{\hat{\mu}_0 K L}\big(\omega_\alpha(s); \hat{\mu}\big) s^{i + \lambda j} + \mathscr{R}(s; \hat{\mu}), \tag{18}$$

where T_0 is given in (2), $\boldsymbol{T}_{ij}^{\hat{\mu}_0 KL}(w;\hat{\mu}) \in \mathscr{C}^K(U)[w]$ and $\mathscr{R} \in \mathcal{F}_L^K(\hat{\mu}_0)$. Furthermore we know that if $\lambda_0 \notin D_{ij}^n$ then $\deg_w \boldsymbol{T}_{ij}^{\hat{\mu}_0 KL} = 0$. Taking this property (and notation) into account, we shall prove the following claim, where recall that $D_{ij}^n \subset \mathbb{Q}_{>0}$ is a discrete set in $(0,+\infty)$, see Definition 1.4.

KEY ASSERTION: For each $(i,j) \in \Lambda_n$ there exists $T_{ij} \in \mathscr{C}^\infty\big(((0,+\infty)\setminus D_{ij}^n)\times W\big)$ such that, for every $\hat{\mu}_0 \in ((0,+\infty)\setminus D_{ij}^n)\times W$, $K \in \mathbb{N}$ and $L>0$ large enough, we have

$$T_{ij}(\hat{\mu}) = \boldsymbol{T}_{ij}^{\hat{\mu}_0 KL}(0;\hat{\mu}) \text{ for all } \hat{\mu}=(\lambda,\mu) \in U_{\hat{\mu}_0 KL} \text{ with } \lambda \notin D_{ij}^n. \tag{19}$$

The proof of this is rather long and technical but the theorem will follow almost right away once we prove it. With this end in mind let us fix any $(i_\star,j_\star)\in\Lambda_n$ and recall that, by Lemma 3.2, $\hat{D}_{i_\star j_\star}^n$ is a discrete set in $(0,+\infty)$ that contains $D_{i_\star j_\star}^n$. We write $(0,+\infty)\setminus\hat{D}_{i_\star j_\star}^n$ as a disjoint union

$$(0,+\infty)\setminus\hat{D}_{i_\star j_\star}^n = \bigsqcup_{r\in\mathbb{N}} J_{i_\star j_\star}^r, \tag{20}$$

where each $J_{i_\star j_\star}^r$ is a open interval in $(0,+\infty)$. Since $\sup \hat{D}_{i_\star j_\star}^n < \infty$ if and only if $j_\star = 0$, it turns out that there exists an unbounded interval $J_{i_\star j_\star}^r$ if and only if $j_\star=0$. Therefore

$$M_{i_\star j_\star}^r := \sup\{i_\star + \lambda j_\star : \lambda \in J_{i_\star j_\star}^r\}$$

is well defined and finite for all $r\in\mathbb{N}$. Furthermore, for each $r\in\mathbb{N}$ such that $J_{i_\star j_\star}^r \neq \emptyset$ we also define

$$\mathscr{Z}_{i_\star j_\star}^r := \big\{(i,j)\in\Lambda_n : i+\lambda j \leqslant i_\star + \lambda j_\star \text{ for all } \lambda \in J_{i_\star j_\star}^r\big\}. \tag{21}$$

Observe that this is a finite subset that contains the pair $(i_\star,j_\star)$ by definition. Moreover if we say that $\mathscr{Z}_{i_\star j_\star}^r = \{(i_1,j_1),(i_2,j_2),\ldots,(i_m,j_m),(i_\star,j_\star)\}$ then, on account of (20) and the definition of $\hat{D}_{i_\star j_\star}^n$, it can be indexed so that

$$i_1+\lambda j_1 < i_2 + \lambda j_2 < \cdots < i_m + \lambda j_m < i_\star + \lambda j_\star \text{ for all } \lambda \in J_{i_\star j_\star}^r. \tag{22}$$

We claim at this point that

$$i+\lambda j > i_\star + \lambda j_\star \text{ for all } (i,j)\in\Lambda_n\setminus\mathscr{Z}_{i_\star j_\star}^r \text{ and } \lambda\in J_{i_\star j_\star}^r. \tag{23}$$

Let us show this by contradiction. If there exists $\lambda_1 \in J_{i_\star j_\star}^r$ such that $i+\lambda_1 j \leqslant i_\star + \lambda_1 j_\star$ then the fact that $(i,j)\notin\mathscr{Z}_{i_\star j_\star}^r$ implies the existence of some $\lambda_2\in J_{i_\star j_\star}^r$ such that $i+\lambda_2 j > i_\star + \lambda_2 j_\star$. Accordingly, by continuity, there exists $\lambda_3 \in J_{i_\star j_\star}^r$ verifying $i+\lambda_3 j = i_\star + \lambda_3 j_\star$ and then $\lambda_3 \in \hat{D}_{i_\star j_\star}^n$ by definition, which contradicts (20). This proves the validity of the claim.

Let us fix now $r\in\mathbb{N}$ and take any $\lambda_0 \in J_{i_\star j_\star}^r$. Then, by applying Theorem 2.5 with any $\hat{\mu}_0 = (\lambda_0,\mu_0)$, $K\in\mathbb{N}$ and $L > M_{i_\star j_\star}^r$ (which in particular implies $\mathscr{Z}_{i_\star j_\star}^r \subset \mathscr{B}_{\lambda_0,L}^n$), we can assert the existence of an open neighbourhood U of $\hat{\mu}_0$ such that $T(s;\hat{\mu})$ can be written as in (18) with $\boldsymbol{T}_{ij}^{\hat{\mu}_0 KL}(w;\hat{\mu})\in\mathscr{C}^K(U)[w]$ for all $(i,j)\in\mathscr{B}_{\lambda_0,L}^n$ and $\mathscr{R}\in\mathcal{F}_L^K(\hat{\mu}_0)$. On account of this there exist a neighborhood V of $\hat{\mu}_0$ and constants $C, s_0>0$ such that $|\mathscr{R}(s;\hat{\mu})|\leqslant Cs^L$ for all $\hat{\mu}\in V$ and $s\in(0,s_0)$. At this point we replace U by a smaller open neighbourhood of $\hat{\mu}_0$ given by the finite intersection

$$\bigcap_{(i,j)\in\mathscr{B}_{\lambda_0,L}^n\setminus\mathscr{Z}_{i_\star j_\star}^r} \{(\lambda,\mu)\in U\cap V : i+\lambda j - i_\star - \lambda j_\star - \alpha^+(\lambda) > 0\}, \tag{24}$$

where we use the notation $x^+ = \max(x,0)$ for shortness. Note that $\hat{\mu}_0$ belongs to the above set due to (23) and the fact that $\alpha(\hat{\mu}_0)=0$.

Next we show that $\deg_w \boldsymbol{T}_{ij}^{\hat\mu_0 KL}=0$ for all $(i,j)\in\mathscr{Z}^r_{i_\star j_\star}$. By contradiction again, if $\deg_w \boldsymbol{T}_{ij}^{\hat\mu_0 KL}>0$ then $\lambda_0\in D^n_{ij}$ by assertion (c) in Theorem 2.5, which implies the existence of some pair $(i',j')\in\Lambda_n\setminus\{(i,j)\}$ such that $i'+\lambda_0 j'=i+\lambda_0 j\leqslant i_\star+\lambda_0 j_\star$, where the inequality holds thanks to (22). Accordingly $\lambda_0\in\hat D^n_{i_\star j_\star}$, that contradicts (20). Thus the assertion is true and, from (18), we can write

$$T(s;\hat\mu)=T_0(\hat\mu)\log s+\sum_{(i,j)\in\mathscr{Z}^r_{i_\star j_\star}} T_{ij}^{\hat\mu_0 KL}(\hat\mu)s^{i+\lambda j}+\sum_{(i,j)\in\mathscr{B}^n_{\lambda_0,L}\setminus\mathscr{Z}^r_{i_\star j_\star}}\boldsymbol{T}_{ij}^{\hat\mu_0 KL}(\omega_\alpha(s);\hat\mu)s^{i+\lambda j}+\mathscr{R}(s;\hat\mu),$$

where $T_{ij}^{\hat\mu_0 KL}(\hat\mu):=\boldsymbol{T}_{ij}^{\hat\mu_0 KL}(0;\hat\mu)$ is a $\mathscr{C}^K$ function in U. Note that the two last summands above belong to $\mathrm{o}(s^{i_\star+\lambda j_\star})$ for each fixed $(\lambda,\mu)\in U$. Indeed, the assertion with regard to the third summand is a consequence of (24) using that, by applying (a) in Lemma A.3 with $\nu=(0,0)$, for each fixed α there exists $C>0$ such that $|\omega(s;\alpha)|\leqslant Cs^{-\alpha^+}|\log s|$ for all $s\in(0,1/e)$. On the other hand, the fact that $\mathscr{R}\in\mathrm{o}(s^{i_\star+\lambda j_\star})$ is clear because $\mathscr{R}\in\mathrm{O}(s^L)$ and, by construction, $L>M^r_{i_\star j_\star}\geqslant i_\star+\lambda j_\star$ for all $\lambda\in J^r_{i_\star j_\star}$. Therefore

$$T(s;\hat\mu)=T_0(\hat\mu)\log s+\sum_{(i,j)\in\mathscr{Z}^r_{i_\star j_\star}} T_{ij}^{\hat\mu_0 KL}(\hat\mu)s^{i+\lambda j}+\mathrm{o}(s^{i_\star+\lambda j_\star}). \tag{25}$$

Now we take any other $\lambda_0'\in J^r_{i_\star j_\star}$ and apply Theorem 2.5 with $\hat\mu_0'=(\lambda_0',\mu_0')$, $K'\geqslant K$ and $L'>M^r_{i_\star j_\star}$. In doing so, exactly as before, we obtain an open neighborhood U' of $\hat\mu_0'$ and $T_{ij}^{\hat\mu_0' K'L'}\in\mathscr{C}^{K'}(U')$ so that

$$T(s;\hat\mu)=T_0(\hat\mu)\log s+\sum_{(i,j)\in\mathscr{Z}^r_{i_\star j_\star}} T_{ij}^{\hat\mu_0' K'L'}(\hat\mu)s^{i+\lambda j}+\mathrm{o}(s^{i_\star+\lambda j_\star}).$$

Consequently, for each fixed $\hat\mu=(\lambda,\mu)\in U\cap U'$ with $\lambda\in J^r_{i_\star j_\star}$, we get that

$$0=T(s;\hat\mu)-T(s;\hat\mu)=\sum_{(i,j)\in\mathscr{Z}^r_{i_\star j_\star}}\left(T_{ij}^{\hat\mu_0 KL}(\hat\mu)-T_{ij}^{\hat\mu_0' K'L'}(\hat\mu)\right)s^{i+\lambda j}+\mathrm{o}(s^{i_\star+\lambda j_\star})$$

for all $s>0$ small enough. This equality, on account of (22) and Remark 3.4, shows that $T_{ij}^{\hat\mu_0 KL}=T_{ij}^{\hat\mu_0' K'L'}$ on $U\cap U'$ for each $(i,j)\in\mathscr{Z}^r_{i_\star j_\star}$. Hence, since $(i_\star,j_\star)\in\mathscr{Z}^r_{i_\star j_\star}$, all the local functions $T_{i_\star j_\star}^{\hat\mu_0 KL}$ and $T_{i_\star j_\star}^{\hat\mu_0' K'L'}$ glue together and provide a well defined function $T_{i_\star j_\star}$ on $J^r_{i_\star j_\star}\times W$, which is of class $\mathscr{C}^K$ for arbitrarily $K\in\mathbb{N}$, i.e., $T_{i_\star j_\star}\in\mathscr{C}^\infty(J^r_{i_\star j_\star}\times W)$. In addition, due to (20) and the fact that r is arbitrary, we get that $T_{i_\star j_\star}\in\mathscr{C}^\infty\big(((0,+\infty)\setminus\hat D^n_{i_\star j_\star})\times W\big)$. It is clear then that $T_{i_\star j_\star}$ does not depend on K, and we remark that it does not depend on $\hat\mu_0$ or L neither.

So far we have proved the existence of $T_{i_\star j_\star}\in\mathscr{C}^\infty\big(((0,+\infty)\setminus\hat D^n_{i_\star j_\star})\times W\big)$ verifying the equality in (19) for every $\hat\mu_0\in((0,+\infty)\setminus\hat D^n_{i_\star j_\star})\times W$, $K\in\mathbb{N}$ and $L>0$. In other words, that the key assertion is true replacing $D^n_{i_\star j_\star}$ by $\hat D^n_{i_\star j_\star}$.

Our next goal (recall that $D^n_{i_\star j_\star}\subset\hat D^n_{i_\star j_\star}$) is to prove that $T_{i_\star j_\star}$ extends smoothly to $(\hat D^n_{i_\star j_\star}\setminus D^n_{i_\star j_\star})\times W$. With this aim in view we fix $\lambda_0\in\hat D^n_{i_\star j_\star}\setminus D^n_{i_\star j_\star}$ and apply Theorem 2.5 with any $\hat\mu_0=(\lambda_0,\mu_0)$, $K\in\mathbb{N}$ and $L>0$ large enough to obtain a neighbourhood U of $\hat\mu_0$ such that the Dulac time $T(s;\hat\mu)$ can be written as in (18), where T_0 is given in (2), $\boldsymbol{T}_{ij}^{\hat\mu_0 KL}(w;\hat\mu)\in\mathscr{C}^K(U)[w]$ and $\mathscr{R}\in\mathcal{F}_L^K(\hat\mu_0)$. Let us note that $\deg_w\boldsymbol{T}_{i_\star j_\star}^{\hat\mu_0 KL}=0$ due to $\lambda_0\notin D^n_{i_\star j_\star}$, by (c) in Theorem 2.5, and define $T_{i_\star j_\star}^{\hat\mu_0 KL}(\hat\mu):=\boldsymbol{T}_{i_\star j_\star}^{\hat\mu_0 KL}(0;\hat\mu)$, which is a $\mathscr{C}^K$ function in U. Take an open interval $I\subset(0,+\infty)$ verifying $I\cap\hat D^0_{i_\star j_\star}=\{\lambda_0\}$, which exists because $\hat D^0_{i_\star j_\star}$ is a discrete set in $(0,+\infty)$, and define $V:=U\cap(I\times W)$. We claim that

$$T_{i_\star j_\star}(\hat\mu)=T_{i_\star j_\star}^{\hat\mu_0 KL}(\hat\mu)\text{ for all }\hat\mu\in V\setminus\{\lambda=\lambda_0\}. \tag{26}$$

(Note that $T_{i_\star j_\star}\in\mathscr{C}^\infty(V\setminus\{\lambda=\lambda_0\})$, whereas $T_{i_\star j_\star}^{\hat\mu_0 KL}\in\mathscr{C}^K(V)$. This will be the key point later on.) With regard to the proof of this claim we stress that although we have already proved the key assertion replacing

$D^n_{i_\star j_\star}$ by $\hat{D}^n_{i_\star j_\star}$, we can neither apply this weaker version because $\hat{\mu}_0 \notin ((0,+\infty)\setminus \hat{D}^n_{i_\star j_\star})\times W$. That being said, in order to prove (26) we observe first that, taking $s^{-\alpha}=1+\alpha\omega_\alpha(s)$ into account and working on $\alpha\neq 0$, we can "deconstruct" any polynomial in $\omega_\alpha(s)$ and write it as a polynomial in $s^{-\alpha}$. More generally, one can readily show using the Newton's binomial formula that the identity $\sum_{k=0}^{\ell} A_k\omega^k(s;\alpha)=\sum_{r=0}^{\ell} B_r s^{-\alpha r}$ holds on $\alpha\neq 0$ if, and only if,

$$A_k=\alpha^k\sum_{r=k}^{\ell}\binom{r}{k}B_r \text{ and } B_r=\sum_{k=r}^{\ell}\alpha^{-k}\binom{k}{r}(-1)^{k-r}A_k. \tag{27}$$

In this way, since $\alpha=0$ if and only if $\lambda=\lambda_0$ (recall that $\alpha := p-\lambda q$ and $\lambda_0=p/q$) and, on the other hand, $\deg_w T^{\hat{\mu}_0 KL}_{ij}\leqslant \lfloor i/p\rfloor$ by (b) in Theorem 2.5, we can write

$$\boldsymbol{T}^{\hat{\mu}_0 KL}_{ij}\big(\omega_\alpha(s);\hat{\mu}\big)s^{i+\lambda j}=\sum_{k=0}^{\lfloor i/p\rfloor}\widetilde{T}^{\hat{\mu}_0 KL}_{i-kp,j+kq}(\hat{\mu})s^{i-kp+\lambda(j+kq)} \text{ for all } \hat{\mu}\in V\setminus\{\lambda=\lambda_0\}, \tag{28}$$

where the functions $\widetilde{T}^{\hat{\mu}_0 KL}_{i-kp,j+kq}$ for $k=0,1,\ldots,\lfloor i/p\rfloor$ are defined univocally in terms of the coefficients of the polynomial $\boldsymbol{T}^{\hat{\mu}_0 KL}_{ij}\in\mathscr{C}^K(V)[w]$ by means of the second equality in (27) with $\ell=\lfloor i/p\rfloor$. Thus, in particular, $\widetilde{T}^{\hat{\mu}_0 KL}_{i-kp,j+kq}\in\mathscr{C}^K(V\setminus\{\lambda=\lambda_0\})$. In doing so for all the polynomials in (18) we obtain that

$$T(s;\hat{\mu})=T_0(\hat{\mu})\log s+\sum_{(i,j)\in\mathscr{B}^0_{\lambda_0,L}}\widetilde{T}^{\hat{\mu}_0 KL}_{ij}(\hat{\mu})s^{i+\lambda j}+\mathcal{F}^K_L(\hat{\mu}_0) \text{ for all } \hat{\mu}\in V\setminus\{\lambda=\lambda_0\}. \tag{29}$$

(We remark that the summation grid is $\mathscr{B}^0_{\lambda_0,L}$ instead of $\mathscr{B}^n_{\lambda_0,L}$.) In this respect we observe that

$$\widetilde{T}^{\hat{\mu}_0 KL}_{i_\star j_\star}=T^{\hat{\mu}_0 KL}_{i_\star j_\star} \text{ on } V\setminus\{\lambda=\lambda_0\},$$

i.e., the coefficient of $s^{i_\star+\lambda j_\star}$ remains unchanged after the "deconstruction process" that brings (18) to (29). This equality is a consequence of the following facts:

(a) $\deg_w \boldsymbol{T}^{\hat{\mu}_0 KL}_{i_\star j_\star}=0$ due to $\lambda_0\notin D^n_{i_\star j_\star}$, by (c) in Theorem 2.5, and

(b) the values of the exponents in $s^{i+\lambda j}$ and $s^{i-kp+\lambda(j+kq)}$ in (28) are the same at $\lambda=\lambda_0=p/q$.

Thus, on account of the above equality and setting $I\setminus\{\lambda_0\}=I_-\sqcup I_+$, the claim in (26) will follow once we show that

$$T_{i_\star j_\star}=\widetilde{T}^{\hat{\mu}_0 KL}_{i_\star j_\star} \text{ on } V\cap(I_+\times W) \text{ and } V\cap(I_-\times W). \tag{30}$$

In order to prove this let $J^{r-}_{i_\star j_\star}$ and $J^{r+}_{i_\star j_\star}$ be the intervals of $(0,+\infty)\setminus\hat{D}^n_{i_\star j_\star}$ with $I_\pm\subset J^{r\pm}_{i_\star j_\star}$ and note that if $\lambda\in J^{r\pm}_{i_\star j_\star}$ then $\lambda\notin\hat{D}^n_{ij}$ for all $(i,j)\in\mathscr{Z}^{r\pm}_{i_\star j_\star}$. (This is so because otherwise there would exist $(i,j)\in\mathscr{Z}^{r\pm}_{i_\star j_\star}$ and two different pairs $(i_1,j_1),(i_2,j_2)\in\Lambda_n$ such that $i_1+\lambda j_1=i_2+\lambda j_2\leqslant i+\lambda j\leqslant i_\star+\lambda j_\star$, which would imply $\lambda\in\hat{D}^n_{i_\star j_\star}$, contradicting that $\lambda\in J^{r\pm}_{i_\star j_\star}$.) Taking $L>0$ large enough, specifically $L>\max\{M^{r-}_{i_\star j_\star},M^{r+}_{i_\star j_\star}\}$, this implies that

$$T(s;\hat{\mu})=T_0(\hat{\mu})\log s+\sum_{(i,j)\in\mathscr{Z}^{r\pm}_{i_\star j_\star}}T_{ij}(\hat{\mu})s^{i+\lambda j}+\mathrm{o}(s^{i_\star+\lambda j_\star}) \text{ for all } \hat{\mu}\in V\cap(I_\pm\times W). \tag{31}$$

Indeed, to verify this at some $\hat{\mu}_1=(\lambda_1,\mu_1)\in V$ with $\lambda_1\in I_\pm\subset J^{r\pm}_{i_\star j_\star}$ we use first the identity in (25) taking $\hat{\mu}_1$ instead of $\hat{\mu}_0$ (we can do this due to the fact that $\lambda_1\notin\hat{D}^n_{i_\star j_\star}$) and then the equality follows by applying the key assertion in (19) to each $(i,j)\in\mathscr{Z}^{r\pm}_{i_\star j_\star}$. (For this effect recall that the key assertion has already been proved on $((0,+\infty)\setminus\hat{D}^n_{ij})\times W$ and that $\lambda_1\in J^{r\pm}_{i_\star j_\star}$ implies $\lambda_1\notin\hat{D}^n_{ij}$ for all $(i,j)\in\mathscr{Z}^{r\pm}_{i_\star j_\star}$.) Let us define in addition the sets $\tilde{\mathscr{Z}}_\pm:=\{(i,j)\in\Lambda_0: i+\lambda j\leqslant i_\star+\lambda j_\star \text{ for all } \lambda\in I_\pm\}$. Note that if $(i,j)\in\mathscr{B}^0_{\lambda_0,L}\setminus\tilde{\mathscr{Z}}_\pm$ then $i+\lambda j>i_\star+\lambda j_\star$ for all $\lambda\in I_\pm$. This is so because if there exists $\lambda_1\in I_\pm$ such that $i+\lambda_1 j\leqslant i_\star+\lambda_1 j_\star$ then the fact that $(i,j)\notin\tilde{\mathscr{Z}}_\pm$ implies the existence of some $\lambda_2\in I_\pm$ such that $i+\lambda_2 j>i_\star+\lambda_2 j_\star$. Then, by continuity, there would exist $\lambda_3\in I_\pm$ such that $i+\lambda_3 j=i_\star+\lambda_3 j_\star$ and so, by definition, $\lambda_3\in\hat{D}^0_{i_\star j_\star}$ which contradicts that $I\cap\hat{D}^0_{i_\star j_\star}=\{\lambda_0\}$. Accordingly, from (29),

$$T(s;\hat{\mu})=T_0(\hat{\mu})\log s+\sum_{(i,j)\in\tilde{\mathscr{Z}}_\pm}\tilde{T}^{\hat{\mu}_0KL}_{ij}(\hat{\mu})s^{i+\lambda j}+\mathrm{o}(s^{i_\star+\lambda j_\star})\text{ for all }\hat{\mu}\in V\cap(I_\pm\times W).\tag{32}$$

Notice that $\mathscr{Z}^{r\pm}_{i_\star j_\star}\subset\tilde{\mathscr{Z}}_\pm$ due to, recall (21), $I_\pm\subset J^{r\pm}_{i_\star j_\star}$ and $\Lambda_n\subset\Lambda_0$. Consequently, from (31) and (32),

$$\sum_{(i,j)\in\mathscr{Z}^{r\pm}_{i_\star j_\star}}\left(\tilde{T}^{\hat{\mu}_0KL}_{ij}(\hat{\mu})-T_{ij}(\hat{\mu})\right)s^{i+\lambda j}+\sum_{(i,j)\in\tilde{\mathscr{Z}}_\pm\setminus\mathscr{Z}^{r\pm}_{i_\star j_\star}}\tilde{T}^{\hat{\mu}_0KL}_{ij}(\hat{\mu})s^{i+\lambda j}+\mathrm{o}(s^{i_\star+\lambda j_\star})=0\tag{33}$$

for all $\hat{\mu}\in V\cap(I_\pm\times W)$. Therefore, since $i+\lambda j\neq i'+\lambda j'$ for all $\lambda\in I_\pm$ and any two different pairs $(i,j),(i',j')\in\tilde{\mathscr{Z}}_\pm$, by Remark 3.4 we deduce that $\tilde{T}^{\hat{\mu}_0KL}_{ij}=T_{ij}$ on $V\cap(I_\pm\times W)$ for any $(i,j)\in\mathscr{Z}^{r\pm}_{i_\star j_\star}$. This, on account of $(i_\star,j_\star)\in\mathscr{Z}^{r-}_{i_\star j_\star}\cap\mathscr{Z}^{r+}_{i_\star j_\star}$, shows (30) and proves that the claim in (26) is true.

At this point we are in position to apply Lemma 3.1. To this end, for reader's convenience, let us denote $C=\{\lambda_0\}\times W$ and observe that $T_{i_\star j_\star}\in\mathscr{C}^\infty((I\times W)\setminus C)$. Furthermore the claim in (26) shows that for each $\hat{\mu}_0\in C$ and $K\in\mathbb{N}$ there exist a neighbourhood $V^{\hat{\mu}_0}_K$ of $\hat{\mu}_0$ and a function $T^{\hat{\mu}_0KL}_{i_\star j_\star}\in\mathscr{C}^K(V^{\hat{\mu}_0}_K)$ such that $T_{i_\star j_\star}=T^{\hat{\mu}_0KL}_{i_\star j_\star}$ on $V^{\hat{\mu}_0}_K\setminus C$. We take a sequence $\{\hat{\mu}_r\}_{r\in\mathbb{N}}$ with $\hat{\mu}_r\in C=\{\lambda_0\}\times W$ for all $r\in\mathbb{N}$ such that $C\subset\bigcup_{r\in\mathbb{N}}V^{\hat{\mu}_r}_K$ and we apply Lemma 3.1 with $f=T_{i_\star j_\star}$ and $g^r_K=T^{\hat{\mu}_rKL}_{i_\star j_\star}$ in order to conclude that $T_{i_\star j_\star}$ extends smoothly to $C=\{\lambda_0\}\times W$. Since $\lambda_0\in\hat{D}^n_{i_\star j_\star}\setminus D^n_{i_\star j_\star}$ is arbitrary and we have previously proved that $T_{i_\star j_\star}\in\mathscr{C}^\infty\big(((0,+\infty)\setminus\hat{D}^n_{i_\star j_\star})\times W\big)$, we can assert that $T_{i_\star j_\star}\in\mathscr{C}^\infty(((0,+\infty)\setminus D^n_{i_\star j_\star})\times W)$.

So far we have proved the key assertion in (19) with regard to the functions T_{ij} with $(i,j)\in\Lambda_n$ is true. Let us fix now $\lambda_0>0$ and $L>0$ and, in order to prove (a), suppose that $\lambda_0\notin D^n_L$, i.e., $\lambda_0\notin D^n_{ij}$ for all $(i,j)\in\mathscr{B}^n_{\lambda_0,L}$. Then, on account of (19), the application of Theorem 2.5 to any $K\in\mathbb{N}$ and $\mu_0\in W$ yields

$$T(s;\hat{\mu})-T_0(\hat{\mu})\log s-\sum_{(i,j)\in\mathscr{B}^n_{\lambda_0,L}}T_{ij}(\hat{\mu})s^{i+\lambda j}\in\bigcap_{K\in\mathbb{N}}\bigcap_{\mu_0\in W}\mathcal{F}^K_L((\lambda_0,\mu_0)).$$

This proves the assertion in (a) because from Definition 1.2 one can readily verify that

$$\bigcap_{K\in\mathbb{N}}\bigcap_{\mu_0\in W}\mathcal{F}^K_L((\lambda_0,\mu_0))=\bigcap_{K\in\mathbb{N}}\mathcal{F}^K_L(\{\lambda_0\}\times W)=\mathcal{F}^\infty_L(\{\lambda_0\}\times W).$$

Let us proceed with the proof of (b). So assume $\lambda_0\in D^n_L$ and take any $\mu_0\in W$, $K\in\mathbb{N}$ and $L>0$ large enough. We claim that if $(i_0,j_0)\in\Lambda_0\setminus\Lambda_n$ then there exists a neighborhood V of $\hat{\mu}_0=(\lambda_0,\mu_0)$ such that

$$\tilde{T}^{\hat{\mu}_0KL}_{i_0j_0}(\hat{\mu})=0\text{ for all }\hat{\mu}\in V\setminus\{\lambda=\lambda_0\}.$$

Indeed, this follows from (33) particularized with $(i_\star,j_\star):=(i_0,j_0+n_2)\in\Lambda_n$ because in this case one can easily verify that $(i_0,j_0)\in\tilde{\mathscr{Z}}_\pm\setminus\mathscr{Z}^{r\pm}_{i_\star j_\star}$ and, exactly as before, by applying Remark 3.4 we deduce that $\tilde{T}^{\hat{\mu}_0KL}_{i_0j_0}=0$ on $V\cap(I_\pm\times W)$. (Let us remark in this respect that, regardless of $\lambda_0\in D^n_{i_\star j_\star}$, to obtain (33) we only need to work in an interval I such that $I\cap\hat{D}^0_{i_\star j_\star}=\{\lambda_0\}$.) Consequently, together with (30), this shows that for all $\mu_0\in W$, $K\in\mathbb{N}$ and $L>0$ large enough, there exists a neighbourhood V of $\hat{\mu}_0$ such that

$$\tilde{T}^{\hat{\mu}_0KL}_{ij}(\hat{\mu})=\begin{cases}T_{ij}(\hat{\mu}) & \text{if }(i,j)\in\Lambda_n,\\ 0 & \text{if }(i,j)\in\Lambda_0\setminus\Lambda_n,\end{cases}$$

for all $\hat{\mu} \in V \setminus \{\lambda = \lambda_0\}$. Accordingly, if $\hat{\mu} \in V \setminus \{\lambda = \lambda_0\}$ then we can assert that

$$\boldsymbol{T}_{ij}^{\lambda_0}(w;\hat{\mu}) := \sum_{r\in\mathscr{A}_{ij\lambda_0}^n} T_{i-rp,j+rq}(\hat{\mu})(1+\alpha w)^r = \sum_{r=0}^{\lfloor i/p\rfloor} \tilde{T}_{i-rp,j+rq}^{\hat{\mu}_0KL}(\hat{\mu})(1+\alpha w)^r = \boldsymbol{T}_{ij}^{\hat{\mu}_0KL}(w;\hat{\mu}) \in \mathscr{C}^K(U)[w],$$

where the third equality follows from (28) and $s^{i-kp+\lambda(j+kq)} = s^{i+\lambda j}s^{-\alpha k} = s^{i+\lambda j}(1+\alpha\omega_\alpha(s))^k$. Thus, the coefficients of the w-polynomial $\boldsymbol{T}_{ij}^{\lambda_0}(w;\hat{\mu})$, which are $\mathscr{C}^\infty$ on the open set

$$\left\{\hat{\mu} = (\lambda,\mu) \in \hat{W} : \lambda \notin \cup_{r\in\mathscr{A}_{ij\lambda_0}^n} D_{i-rp,j+rq}^n\right\},$$

are $\mathscr{C}^K$ on a neighborhood of $C = \{\lambda_0\}\times W$. Exactly as before, the application of Lemma 3.1 shows that these coefficients extend smoothly to C. The above equality also shows the application of Theorem 2.5 to any $K\in\mathbb{N}$ and $\mu_0 \in W$ gives

$$T(s;\hat{\mu}) - T_0(\hat{\mu})\log s - \sum_{(i,j)\in\mathscr{B}_{\lambda_0,L}^n} \boldsymbol{T}_{ij}^{\lambda_0}(\omega_\alpha(s);\hat{\mu})s^{i+\lambda j} \in \bigcap_{K\in\mathbb{N}}\bigcap_{\mu_0\in W} \mathcal{F}_L^K((\lambda_0,\mu_0)) = \mathcal{F}_L^\infty(\{\lambda_0\}\times W).$$

This shows the validity of the assertion in (b) and completes the proof of the result. ■

Proof of Theorem B. The application of Theorem 2.5 shows that, for every $\hat{\mu}_0 = (\lambda_0,\mu_0) \in \hat{W}$, $K\in\mathbb{N}$ and $L>0$, there exists a neighborhood $U = U_{\hat{\mu}_0KL}$ of $\hat{\mu}_0$ such that the Dulac map can be written as

$$D(s;\hat{\mu}) = s^\lambda \sum_{(i,j)\in\mathscr{B}_{\lambda_0,L-\lambda_0}^0} \boldsymbol{\Delta}_{ij}^{\hat{\mu}_0KL}(\omega_\alpha(s);\hat{\mu})s^{i+\lambda j} + \mathcal{F}_L^K(\hat{\mu}_0),$$

where $\boldsymbol{\Delta}_{ij}^{\hat{\mu}_0KL}(w;\hat{\mu}) \in \mathscr{C}^K(U)[w]$ verify assertions (a), (b) and (c). Moreover $\deg_w \boldsymbol{\Delta}_{00}^{\hat{\mu}_0KL} = 0$ and if $L \geqslant \lambda_0$ then $\boldsymbol{\Delta}_{00}^{\hat{\mu}_0KL}(0;\hat{\mu}) > 0$ for all $\hat{\mu}\in U$.

The idea now is to put the factor s^λ inside the summation in order to get a more convenient expression. With this aim let us fix $\varepsilon>0$ small enough such that $\mathscr{B}_{\lambda_0,L+\varepsilon}^0 = \mathscr{B}_{\lambda_0,L}^0$ and shrink U so that $\lambda-\lambda_0<\varepsilon$ for all $\hat{\mu} = (\lambda,\mu)\in U$. On account of this, if $\hat{\mu}\in U$ and $(i,j)\in\mathscr{B}_{\lambda_0,L-\lambda_0}^0$ then $i+\lambda(j+1) \leqslant L-\lambda_0+\lambda < L+\varepsilon$ and, setting

$$\boldsymbol{Z}_{ij}^{\hat{\mu}_0KL}(w;\hat{\mu}) := \begin{cases} \boldsymbol{\Delta}_{i,j-1}^{\hat{\mu}_0KL}(w;\hat{\mu}) & \text{if } j\geqslant 1,\\ 0 & \text{if } j=0,\end{cases}$$

we can write

$$D(s;\hat{\mu}) = \sum_{(i,j)\in\mathscr{B}_{\lambda_0,L}^0} \boldsymbol{Z}_{ij}^{\hat{\mu}_0KL}(\omega_\alpha(s);\hat{\mu})s^{i+\lambda j} + \mathcal{F}_L^K(\hat{\mu}_0).$$

Certainly $\boldsymbol{Z}_{ij}^{\hat{\mu}_0KL}(w;\hat{\mu}) \in \mathscr{C}^K(U)[w]$ and, more important, one can easily show that these polynomials also verify assertions (a), (b) and (c) in the statement of Theorem 2.5. Taking this into account it is clear from the above expression that the proof of Theorem A from Theorem 2.5 gives as particular case, taking $T_0\equiv 0$ and $n=(0,0)$, the proof of the present result. That being said, since it is only a matter of adapting the notation in the obvious way, we do not include it for the sake of shortness. ■

Proof of Theorem C. Suppose that $\ell_{\hat{\mu}_0} < +\infty$, otherwise there is nothing to be proved. To show the result we adapt the derivation-division algorithm used by R. Roussarie to bound the cyclicity of an unfolding of a saddle loop (see Theorem 19 in page 113 of [28]). Here for convenience we will use the derivation $\mathscr{D} := s\partial_s$ instead of the usual one.

Let us take the given $\hat{\mu}_0 = (\lambda_0,\mu_0)$ and consider the case $\lambda_0\in\mathbb{Q}$ first. Suppose that $\lambda_0 = p/q$ with $\gcd(p,q) = 1$ and denote $\alpha(\lambda) = p-\lambda q$ for the sake of shortness. Note in particular that $\alpha(\lambda_0) = 0$.

In addition, when there is not risk of confusion, we simply write ω or $\omega(s;\alpha)$ instead of $\omega(s;p-\lambda q)$. In what follows the notation $\omega^k+\cdots$ will mean that after the sign $+$ there is an unwritten polynomial $a_1(\lambda)\omega^{k-1}+\ldots+a_k(\lambda)$ with the coefficients $a_i(\lambda)$ continuous in a neighbourhood of λ_0. Taking this notation into account one can readily show the following properties:

(*a*) $\mathscr{D}(s^a\omega^k)=(a-k\alpha)s^a\omega^k-ks^a\omega^{k-1}=s^a\omega^{k-1}\big((a-k\alpha)\omega-k\big)$ for any continuous function $a(\lambda)$ in a neighbourhood of λ_0. In particular, $\mathscr{D}(\omega)=-(1+\alpha\omega)=-s^{-\alpha}$ and

$$\mathscr{D}(s^a\omega^k)=s^a(*\omega^k+\cdots)\text{ in case that }a(\lambda_0)\neq 0,$$

where here (and in what follows) we use the symbol $*$ to replace any continuous function of λ which is non-zero at $\lambda=\lambda_0$.

(*b*) On account of the chain rule, $\mathscr{D}(R(\omega))=R'(\omega)\mathscr{D}(\omega)=-R'(\omega)s^{-\alpha}$ for any function R. Hence

$$\mathscr{D}\left(\frac{\omega^k+\cdots}{\omega^r+\cdots}\right)=s^{-\alpha}\frac{(k-r)\omega^{k+r-1}+\cdots}{(\omega^r+\cdots)^2}.$$

(*c*) If $a(\lambda)$ is a continuous function with $a(\lambda_0)\neq 0$ then, from (*b*),

$$\begin{aligned}\mathscr{D}\left(s^a\frac{\omega^k+\cdots}{\omega^r+\cdots}\right)&=s^a\left(a\frac{\omega^k+\cdots}{\omega^r+\cdots}+s^{-\alpha}\frac{(k-r)\omega^{k+r-1}+\cdots}{(\omega^r+\cdots)^2}\right)\\&=s^a\left(a\frac{\omega^k+\cdots}{\omega^r+\cdots}+(1+\alpha\omega)\frac{(k-r)\omega^{k+r-1}+\cdots}{(\omega^r+\cdots)^2}\right)\\&=s^a\frac{(a+\alpha(k-r))\omega^{k+r}+\cdots}{(\omega^r+\cdots)^2}=s^a\frac{*\omega^{k+r}+\cdots}{(\omega^r+\cdots)^2}.\end{aligned}$$

That being said, for any $L>0$, by applying Theorem A we can write

$$\begin{aligned}T(s;\hat{\mu})=\beta_0^0+s^{i_1+\lambda j_1}(\beta_0^1\omega^{k_1}+&\beta_1^1\omega^{k_1-1}+\ldots+\beta_{k_1}^1)+\ldots\\&+s^{i_m+\lambda j_m}(\beta_0^m\omega^{k_m}+\beta_1^m\omega^{k_m-1}+\ldots+\beta_{k_m}^m)+R(s;\hat{\mu}),\end{aligned}$$

where $k_i\in\mathbb{Z}_{\geq 0}$ for $i=1,2,\ldots,m$ and $R\in\mathcal{F}_L^\infty(\hat{\mu}_0)$. Here we do not have logarithmic term since the assumption $n=(n_1,n_2)\neq(0,0)$ implies $T_0=0$, see (2). Note on the other hand that the constant term β_0^0 only appears if $n_1n_2=0$ because otherwise $(0,0)\notin\Lambda_n$. Furthermore (recall Remark 1.7)

- $0<i_1+\lambda_0 j_1<i_2+\lambda_0 j_2<\cdots<i_m+\lambda_0 j_m\leqslant L$.
- The coefficients $\beta_r^d=\beta_r^d(\hat{\mu})$ are labelled according to the position of its corresponding monomial with respect to the order $\prec_{\lambda_0}$, see Definition 1.6. More precisely, β_r^d is the coefficient of $s^{i_d+\lambda j_d}\omega^{k_d-r}$ and

  $$s^{i_d+\lambda j_d}\omega^{k_d-r}\prec_{\lambda_0}s^{i_{d'}+\lambda j_{d'}}\omega^{k_{d'}-r'}\ \Leftrightarrow\ \begin{cases}d<d'\\ \text{or}\\ d=d'\text{ and }r<r'.\end{cases}$$

  In particular, if $\{t_i\}_{i\in\mathbb{N}}$ is the sequence as introduced in Definition 1.8 then it turns out that $t_i=\beta_r^d$ with $i=r+d+1-\eta+\sum_{m=1}^{d-1}k_m$, where recall that $\eta=1$ if $n_1n_2=0$ and $\eta=0$ if $n_1n_2\neq 0$.
- We fix $a,b\in\mathbb{Z}_{\geq 0}$ with $a\leqslant k_b$ to be the ones that $\ell_{\hat{\mu}_0}=a+b+1-\eta+\sum_{m=1}^{b-1}k_m$, see Definition 1.8. Then, and this is crucial,

  $$\beta_a^b(\hat{\mu}_0)=t_{\ell_{\hat{\mu}_0}}(\hat{\mu}_0)\neq 0.$$

Define $\xi_0(s;\hat\mu) := s^{-i_1-\lambda j_1}\mathscr{D}T_{\hat\mu}(s)$ and note that $\mathcal{Z}_0(T'(\,\cdot\,;\hat\mu),\hat\mu_0)=\mathcal{Z}_0(\xi_0,\hat\mu_0)$. Due to $i_d+\lambda_0 j_d\neq 0$ for all $d=1,2,\ldots,m$, by applying (a) we get

$$\begin{aligned}\xi_0(s;\hat\mu)=&\beta_0^1(*\omega^{k_1}+\ldots)+\beta_1^1(*\omega^{k_1-1}+\cdots)+\ldots+*\beta_{k_1}^1\\&+s^{-i_1-\lambda j_1}\sum_{d=2}^m s^{i_d+\lambda j_d}\Big[\beta_0^d(*\omega^{k_d}+\cdots)+\beta_1^d(*\omega^{k_d-1}+\cdots)+\ldots+*\beta_{k_d}^d\Big]+R_0(s;\hat\mu),\end{aligned}$$

where the remainder is given by $R_0 := s^{-i_1-\lambda j_1}\mathscr{D}(R)$.With regard to the flatness properties of the remainder note first that $\mathscr{D}(R)=sR'\in\mathcal{F}_L^\infty(\hat\mu_0)$ by applying (f) and (g) in Lemma A.2. On the other hand, setting $L_1 := i_1+\lambda_0 j_1$, by applying (c) in Lemma A.3 we get $s^{-i_1-\lambda j_1}\in\mathcal{F}_{-L_1-\varepsilon}^\infty(\hat\mu_0)$ for some $\varepsilon>0$ (that we fix from now on). Accordingly (g) in Lemma A.2 shows that $R_0=s^{-i_1-\lambda j_1}\mathscr{D}(R)\in\mathcal{F}_{L-L_1-\varepsilon}^\infty(\hat\mu_0)$.

From this point on we will make $\ell=\ell_{\hat\mu_0}$ steps of a derivation-division algorithm to construct a sequence of functions $\xi_1,\xi_2,\ldots,\xi_\ell$ such that the last one is locally non-zero and the bound for $\mathcal{Z}_0(\xi_0,\hat\mu_0)$ will follow from a recurrent application of Rolle's Theorem. We gather these steps in several stages. In the first stage we eliminate, one by one and in this order, the coefficients $\beta_0^1,\beta_1^1,\ldots,\beta_{k_1}^1$, in the second stage we get rid of the coefficients $\beta_0^2,\beta_1^2,\ldots,\beta_{k_2}^2$, and so on until the last stage, in which we remove $\beta_0^b,\beta_1^b,\ldots,\beta_{a-1}^b$. Since all the steps in each stage are exactly the same, we only explain in detail the first and last stages for the sake of shortness. Certainly the flatness of the remainder will decrease in each step but this will not be a problem because, thanks to Theorem A, the coefficients β_j^i are independent of L and we can take this number arbitrarily large. This will guarantee that the remainder in the last step is flat enough for our purposes. However, but only in the first steps of the algorithm, we will pay attention on the flatness of the remainder for reader's convenience.

First stage:

Let $u_1(s;\lambda)=*\omega^{k_1}+\cdots$ be the function multiplying the coefficient β_0^1 in ξ_0. This function does not vanish in a neighbourhood of $(0,\hat\mu_0)$ because $u_1(s;\lambda)/\omega^{k_1}$ tends to some non-zero value as $(s,\lambda)\to(0,\lambda_0)$ since $\lim_{(s,\lambda)\to(0,\lambda_0)}\frac{1}{\omega(s;\alpha)}=0$ by (a) in Lemma A.3. Consequently if we define $\xi_1 := s^\alpha(u_1)^2\mathscr{D}(\xi_0/u_1)$ then, by Rolle's Theorem, $\mathcal{Z}_0(\xi_0,\hat\mu_0)\leqslant\mathcal{Z}_0(\xi_1,\hat\mu_0)+1$. Due to $i_d+\lambda_0 j_d-(i_1+\lambda_0 j_1)\neq 0$ for $d=2,3,\ldots,m$, taking properties (b) and (c) into account we get

$$\begin{aligned}\xi_1=&s^\alpha(u_1)^2\mathscr{D}(\xi_0/u_1)=\beta_1^1(*\omega^{2(k_1-1)}+\cdots)+\ldots+\beta_{k_1}^1(*\omega^{k_1-1}+\cdots)\\&+s^{-i_1-\lambda j_1+\alpha}\sum_{d=2}^m s^{i_d+\lambda j_d}\Big[\beta_0^d(*\omega^{k_1+k_d}+\cdots)+\beta_1^d(*\omega^{k_1+k_d-1}+\cdots)+\ldots+\beta_{k_d}^d(*\omega^{k_1}+\cdots)\Big]\\&+s^\alpha(\mathscr{D}(R_0)u_1-R_0\mathscr{D}(u_1)).\end{aligned}$$

Moreover, since $\mathscr{D}(\omega)=-s^{-\alpha}$, the remainder can be written as

$$R_1 := s^\alpha(\mathscr{D}(R_0)u_1-R_0\mathscr{D}(u_1))=s^\alpha\mathscr{D}(R_0)(*\omega^{k_1}+\cdots)+R_0(*\omega^{k_1-1}+\cdots).$$

Note that the functions s^α, $*\omega^{k_1}+\cdots$ and $*\omega^{k_1-1}+\cdots$ belong to $\mathcal{F}_{-\varepsilon}^\infty(\hat\mu_0)$ thanks to (c) in Lemma A.3 and, consequently, $R_1\in\mathcal{F}_{L-L_1-3\varepsilon}^\infty(\hat\mu_0)$ by applying Lemma A.2. Exactly for the same reasons as before, if $u_2=*\omega^{2(k_1-1)}+\cdots$ is the function multiplying the coefficient β_1^1 in ξ_1 then $\mathcal{Z}_0(\xi_1,\hat\mu_0)\leqslant\mathcal{Z}_0(\xi_2,\hat\mu_0)+1$ with

$$\begin{aligned}\xi_2 :=&s^\alpha(u_2)^2\mathscr{D}(\xi_1/u_2)=\beta_2^1(*\omega^{4k_1-6}+\cdots)+\ldots+\beta_{k_1}^1(*\omega^{3k_1-4}+\cdots)\\&+s^{-i_1-\lambda j_1+2\alpha}\sum_{d=2}^m s^{i_d+\lambda j_d}\Big[\beta_0^d(*\omega^{3k_1+k_d-2}+\cdots)+\ldots+\beta_{k_d}^d(*\omega^{3k_1-2}+\cdots)\Big]+R_2\end{aligned}$$

and where $R_2 := s^{\alpha}(\mathscr{D}(R_1)u_2 - R_1\mathscr{D}(u_2)) \in \mathcal{F}^{\infty}_{L-L_1-5\varepsilon}(\hat{\mu}_0)$. We get in this way a sequence of functions $\xi_1, \xi_2, \ldots, \xi_{k_1+1}$ such that $\mathcal{Z}_0(\xi_{i-1}, \hat{\mu}_0) \leqslant \mathcal{Z}_0(\xi_i, \hat{\mu}_0) + 1$ and where

$$\xi_{k_1+1} := s^{-i_1-\lambda j_1+(k_1+1)\alpha} \sum_{d=2}^{m} s^{i_d+\lambda j_d}\Big[\beta_0^d(*\omega^{r+k_d}+\cdots)+\ldots+\beta_{k_d}^d(*\omega^r+\cdots)\Big] + R_{k_1+1}$$

with $R_{k_1+1} \in \mathcal{F}^{\infty}_{L-L_1-(2k_1+3)\varepsilon}(\hat{\mu}_0)$. Here r is a natural number depending on k_1 that we do not specify because its expression is not relevant. Thus, at the end of the first stage we removed the first k_1+1 coefficients of the asymptotic development of ξ_0 and we get ξ_{k_1+1} such that $\mathcal{Z}_0(\xi_0, \hat{\mu}_0) \leqslant \mathcal{Z}_0(\xi_{k_1+1}, \hat{\mu}_0) + k_1 + 1$.

Next, in the second stage, we begin with the function

$$\begin{aligned} s^{i_1+\lambda j_1-(i_2+\lambda j_2)-(k_1+1)\alpha}\xi_{k_1+1} =& \beta_0^2(*\omega^{r+k_2}+\cdots)+\ldots+\beta_{k_2}^2(*\omega^r+\cdots) \\ &+ s^{-i_2-\lambda j_2}\sum_{d=3}^{m} s^{i_d+\lambda j_d}\Big[\beta_0^d(*\omega^{r+k_d}+\cdots)+\ldots+\beta_{k_d}^d(*\omega^r+\cdots)\Big] \\ &+ s^{i_1+\lambda j_1-(i_2+\lambda j_2)-(k_1+1)\alpha}R_{k_1+1}. \end{aligned}$$

The application of Lemmas A.2 and A.3 shows that the remainder belongs to $\mathcal{F}^{\infty}_{L-L_2-2(k_1+2)\varepsilon}(\hat{\mu}_0)$ where $L_2 := i_2 + \lambda_0 j_2$ since $s^{i_1+\lambda j_1-(i_2+\lambda j_2)-(k_1+1)\alpha} \in \mathcal{F}_{L_1-L_2-\varepsilon}(\hat{\mu}_0)$. We eliminate, one by one and in this order, the coefficients $\beta_0^2, \beta_1^2, \ldots, \beta_{k_2}^2$ following verbatim the steps that we carried out in the first stage. (It is in this stage that we use the inequalities $i_d + \lambda_0 j_d \neq i_2 + \lambda_0 j_2$ for $d = 3, 4, \ldots, m$.) In doing so we get a sequence of functions $\xi_{k_1+2}, \xi_{k_1+3}, \ldots, \xi_{k_1+k_2+2}$ such that, exactly as before, $\mathcal{Z}_0(\xi_{i-1}, \hat{\mu}_0) \leqslant \mathcal{Z}_0(\xi_i, \hat{\mu}_0) + 1$.

Last stage:

In the final stage, since $\sum_{n=1}^{b-1}(k_n+1) = \ell - a$ by construction, we begin with

$$\begin{aligned} \xi_{\ell-a} :=& \beta_0^b(*\omega^{r+k_b}+\cdots)+\ldots+\beta_a^b(*\omega^{r+k_b-a}+\cdots)+\ldots+\beta_{k_b}^b(*\omega^r+\cdots) \\ &+ s^{-i_b-\lambda j_b}\sum_{d=b+1}^{m} s^{i_d+\lambda j_d}\Big[\beta_0^d(*\omega^{r+k_d}+\cdots)+\ldots+\beta_{k_d}^d(*\omega^r+\cdots)\Big] + R_{\ell-a}, \end{aligned}$$

where r is now a natural number that depends on $k_1, k_2, \ldots, k_{b-1}$. Consequently, with a additional steps of the derivation-division algorithm as we did to ξ_0 in the first stage, by using properties (b) and (c) we get

$$\begin{aligned} \xi_\ell :=& \beta_a^b(*\omega^{\hat{r}+k_b-a}+\cdots)+\beta_{a+1}^b(*\omega^{\hat{r}+k_b-a-1}+\cdots)+\ldots+\beta_{k_b}^b(*\omega^{\hat{r}}+\cdots) \\ &+ s^{-i_b-\lambda j_b+a\alpha}\sum_{d=b+1}^{m} s^{i_d+\lambda j_d}\Big[\beta_0^d(*\omega^{\hat{r}+k_d+a}+\cdots)+\ldots+\beta_{k_d}^d(*\omega^{\hat{r}+a}+\cdots)\Big] + R_\ell, \end{aligned}$$

where $\hat{r}$ is once again a natural number and, taking $L > 0$ large enough, we can guarantee that $R_\ell \in \mathcal{F}_1^{\infty}(\hat{\mu}_0)$. Finally, after dividing by the function multiplying the coefficient β_a^b,

$$\begin{aligned} \frac{\xi_\ell}{*\omega^{\hat{r}+k_b-a}+\cdots} =& \beta_a^b + \beta_{a+1}^b\frac{*\omega^{\hat{r}+k_b-a-1}+\cdots}{*\omega^{\hat{r}+k_b-a}+\cdots}+\ldots+\beta_{k_b}^b\frac{*\omega^{\hat{r}}+\cdots}{*\omega^{\hat{r}+k_b-a}+\cdots} \\ &+ s^{-i_b-\lambda j_b+a\alpha}\sum_{d=b+1}^{m} s^{i_d+\lambda j_d}\left[\beta_0^d\frac{*\omega^{\hat{r}+k_d+a}+\cdots}{*\omega^{\hat{r}+k_b-a}+\cdots}+\ldots+\beta_{k_d}^d\frac{*\omega^{\hat{r}+a}+\cdots}{*\omega^{\hat{r}+k_b-a}+\cdots}\right] + \frac{R_\ell}{*\omega^{\hat{r}+k_b-a}+\cdots}. \end{aligned}$$

Accordingly, on account of $i_d + \lambda j_d - i_b - \lambda j_b + a\alpha\,|_{\lambda=\lambda_0} = i_d + \lambda_0 j_d - (i_b + \lambda_0 j_b) > 0$ for all $d = b+1, \ldots, m$, by applying (a) in Lemma A.3 with $\nu = (0,0)$ and thanks to the flatness of the remainder, from the above expression we can conclude that

$$\lim_{(s,\hat{\mu})\to(0,\hat{\mu}_0)} \frac{\xi_\ell}{*\omega^{\hat{r}+k_b-a}+\cdots} = \beta_a^b(\hat{\mu}_0) \neq 0.$$

Hence $\mathcal{Z}_0(\xi_\ell,\hat\mu_0)=0$. Consequently, due to $\mathcal{Z}_0(\xi_{i-1},\hat\mu_0)\leqslant\mathcal{Z}_0(\xi_i,\hat\mu_0)+1$ for all i, this implies that

$$\mathcal{Z}_0(T'(\,\cdot\,;\hat\mu),\hat\mu_0)=\mathcal{Z}_0(\xi_0,\hat\mu_0)\leqslant\ell,$$

as desired.

Consider finally the case $\lambda_0\notin\mathbb{Q}$. This is an easier situation than the previous one because, by Theorem A, we know that

$$T(s;\hat\mu)=t_1+t_2s^{i_1+\lambda j_1}+t_3s^{i_2+\lambda j_2}+\ldots+t_ms^{i_m+\lambda j_m}+R(s;\hat\mu),$$

where, once again, $R\in\mathcal{F}_L^\infty(\hat\mu_0)$ and, on the other hand, $0<i_1+\lambda_0j_1<i_2+\lambda_0j_2<\cdots<i_m+\lambda_0j_m\leqslant L$. Note then that we can treat this case using the previous approach particularised with $k_1=\ldots=k_m=0$. We obtain in this way the desired bound $\mathcal{Z}_0(T'(\,\cdot\,;\hat\mu),\hat\mu_0)\leqslant\ell$. This concludes the proof of the result. ■

Remark 3.5. It is clear from its proof that Theorem C is valid if $\{T(s;\hat\mu)\}_{\hat\mu\in\hat W}$ is any family of functions in $\mathscr{C}^\infty_{s>0}(\hat W)$ verifying the conclusion of Theorem A. □

4 Applications

Theorem C establishes an upper bound for $\mathcal{Z}_0(T'(\,\cdot\,;\hat\mu),\hat\mu_0)$. It will be also convenient to have some tool in order to ensure a lower bound. We begin this section with a result that is addressed to this issue. It is in fact an adaptation of a well-known technique used to study the bifurcation of zeros (see [2, Theorem 2.1] or [9, Lemma 15] for instance).

Definition 4.1. Consider the functions $g_i\colon\hat W\to\mathbb{R}$ for $i=1,2,\ldots,k$. The *real variety* $V(g_1,g_2,\ldots,g_k)$ is defined to be the set of $\hat\mu\in\hat W$ such that $g_i(\hat\mu)=0$ for $i=1,2,\ldots,k$. We say that $g_1,g_2,\ldots,g_k$ are *independent* at $\hat\mu_\star\in V(g_1,g_2,\ldots,g_k)$ if the following conditions are satisfied:

(1) Every neighbourhood of $\hat\mu_\star$ contains two points $\hat\mu_1,\hat\mu_2\in V(g_1,\ldots,g_{k-1})$ such that $g_k(\hat\mu_1)g_k(\hat\mu_2)<0$ (if $k=1$ then we set $V(g_1,\ldots,g_{k-1})=V(0)=\hat W$ for this to hold).

(2) The varieties $V(g_1,\ldots,g_i)$, $2\leqslant i\leqslant k-1$, are such that if $\hat\mu_0\in V(g_1,\ldots,g_i)$ and $g_{i+1}(\hat\mu_0)\neq0$, then every neighbourhood of $\hat\mu_0$ contains a point $\hat\mu\in V(g_1,\ldots,g_{i-1})$ such that $g_i(\hat\mu)g_{i+1}(\hat\mu_0)<0$.

(3) If $\hat\mu_0\in V(g_1)$ and $g_2(\hat\mu_0)\neq0$, then every open neighbourhood of $\hat\mu_0$ contains a point $\hat\mu$ such that $g_1(\hat\mu)g_2(\hat\mu_0)<0$.

It is clear that if $g_i\in\mathscr{C}^1(\hat W)$ for $i=1,2,\ldots,k$ then a sufficient condition for $g_1,g_2,\ldots,g_k$ to be independent at $\hat\mu_\star$ is that the gradients $\nabla g_1(\hat\mu_\star),\nabla g_2(\hat\mu_\star)\ldots,\nabla g_k(\hat\mu_\star)$ are linearly independent vectors of $\mathbb{R}^{N+1}$. □

Proposition 4.2. *Consider $F(s;\hat\mu)=\sum_{i=1}^n\delta_i(\hat\mu)f_i(s;\hat\mu)+f_{n+1}(s;\hat\mu)$, where $f_i\in\mathscr{C}^\infty((0,\varepsilon)\times\hat W)$ and $\delta_i\in\mathscr{C}^0(\hat W)$. If $\hat\mu_\star\in V(\delta_1,\delta_2,\ldots,\delta_n)$ satisfies*

(a) $F(s;\hat\mu_\star)$ is not identically zero on $(0,\rho)$ for every $\rho\in(0,\varepsilon)$,

(b) $f_i(s;\hat\mu)>0$, $1\leqslant i\leqslant n$, for all $(s,\hat\mu)$ in a neighbourhood of $(0,\hat\mu_\star)$,

(c) $\lim_{s\to0}\frac{f_{i+1}(s;\hat\mu)}{f_i(s;\hat\mu)}=0$, $1\leqslant i\leqslant n$, for every $\hat\mu$ in a neighbourhood of $\hat\mu_\star$, and

(d) $\delta_1,\delta_2,\ldots,\delta_n$ are independent at $\mu_\star$,

then $\mathcal{Z}_0(F(\,\cdot\,;\hat\mu),\hat\mu_\star)\geqslant n$.

Proof. Fix any $\rho > 0$ and any neighbourhood U of $\hat{\mu}_\star$. Then, by the assumption (a), there exists $s_1 \in (0, \rho)$ such that $F(s_1; \hat{\mu}_\star) = f_{n+1}(s_1; \hat{\mu}_\star) \neq 0$. Suppose for instance that $F(s_1; \hat{\mu}_\star) > 0$. Then, on account of (1) in Definition 4.1, we can take $\hat{\mu}_1 \in U \cap V(\delta_1, \delta_2, \ldots, \delta_{n-1})$ such that $\delta_n(\hat{\mu}_1) < 0$ and close enough to $\hat{\mu}_\star$ so that, by continuity, $F(s_1; \hat{\mu}_1) > 0$. Observe that

$$F(s; \hat{\mu}_1) = \delta_n(\hat{\mu}_1) f_n(s; \hat{\mu}_1) + f_{n+1}(s; \hat{\mu}_1).$$

Thus, by (b) and (c), $\lim_{s \to 0} \frac{F(s;\hat{\mu}_1)}{f_n(s;\hat{\mu}_1)} = \delta_n(\hat{\mu}_1) < 0$ and we can take $s_2 \in (0, s_1)$ such that $F(s_2; \hat{\mu}_1) < 0$. Next, thanks to (2) in Definition 4.1, we can choose $\hat{\mu}_2 \in U \cap V(\delta_1, \delta_2, \ldots, \delta_{n-2})$ with $\delta_{n-1}(\hat{\mu}_2) > 0$ and close enough to $\hat{\mu}_1$ so that $F(s_1; \hat{\mu}_2) > 0$ and $F(s_2; \hat{\mu}_2) < 0$. Note that

$$F(s; \hat{\mu}_2) = \delta_{n-1}(\hat{\mu}_2) f_{n-1}(s; \hat{\mu}_2) + \delta_n(\hat{\mu}_2) f_n(s; \hat{\mu}_2) + f_{n+1}(s; \hat{\mu}_2).$$

Consequently, by (b) and (c), $\lim_{s \to 0} \frac{F(s;\hat{\mu}_2)}{f_{n-1}(s;\hat{\mu}_2)} = \delta_{n-1}(\hat{\mu}_2) > 0$ and we can choose $s_3 \in (0, s_2)$ such that $F(s_3; \hat{\mu}_2) > 0$. Next we take $\hat{\mu}_3 \in U \cap V(\delta_1, \delta_2, \ldots, \delta_{n-3})$ with $\delta_{n-2}(\hat{\mu}_3) < 0$ and close enough to $\hat{\mu}_2$ so that $F(s_1; \hat{\mu}_3) > 0$, $F(s_2; \hat{\mu}_3) < 0$ and $F(s_3; \hat{\mu}_3) > 0$. We repeat this process $n-2$ times after which we find a parameter $\hat{\mu}_{n+1} \in U$ and $0 < s_{n+1} < s_n < \ldots < s_2 < s_1 < \rho$, such that $(-1)^{i+1} F(s_i; \hat{\mu}_{n+1}) > 0$ for all $i = 1, 2, \ldots, n+1$. By applying Bolzano's theorem we can assert the existence of at least n zeros of $F(\,\cdot\,; \hat{\mu}_{n+1})$ inside the interval $(0, \delta)$. Accordingly $\mathcal{Z}_0(F(\,\cdot\,; \hat{\mu}), \hat{\mu}_\star) \geqslant n$ and this concludes the proof. ■

We prove next an auxiliary result that enables to straighten globally the separatrices of a saddle depending on parameters. This result, which is well-known to be true locally, is relevant with regard to the applicability of Theorems A and B. It will be essential, for instance, in the proof of Corollaries D and E, in which we do not have any assumption regarding the separatrices of the saddles.

Lemma 4.3. *Consider a $\mathscr{C}^\infty$ family $\{X_\nu\}_{\nu \in \mathbb{R}^N}$ of planar vector fields defined in some open set W of $\mathbb{R}^2$. Let us fix some $\nu_0 \in \mathbb{R}^N$ and assume that, for all $\nu \approx \nu_0$, X_ν has a hyperbolic saddle point at $p_\nu \in W$ with (global) stable and unstable separatrices S_ν^+ and S_ν^-, respectively. Consider two closed connected arcs $\ell^\pm \subset S_{\nu_0}^\pm$, having both an endpoint at p_{ν_0}. In case of a homoclinic connection (i.e., $S_{\nu_0}^+ = S_{\nu_0}^-$) we require additionally that $\ell^+ \cap \ell^- = \{p_{\nu_0}\}$. Then there exists a neighborhood V of $(\ell^+ \cup \ell^-) \times \{\nu_0\}$ in $\mathbb{R}^2 \times \mathbb{R}^N$ and a $\mathscr{C}^\infty$ diffeomorphism $\Phi : V \to \Phi(V) \subset \mathbb{R}^2 \times \mathbb{R}^N$ with $\Phi(x, y, \nu) = (\phi_\nu(x, y), \nu)$ such that*

$$\Phi((S_\nu^+ \times \{\nu\}) \cap V) \subset \{x = 0\} \times \{\nu\} \text{ and } \Phi((S_\nu^- \times \{\nu\}) \cap V) \subset \{y = 0\} \times \{\nu\}.$$

In other words, $(\phi_\nu)_\star(X_\nu) = \hat{X}_\nu$ where $\hat{X}_\nu(x, y) = xP(x, y; \nu)\partial_x + yQ(x, y; \nu)\partial_y$, with $P, Q \in \mathscr{C}^\infty(\Phi(V))$.

Proof. The existence of such a diffeomorphism in a neighbourhood U of $(p_{\nu_0}, \nu_0) \in \mathbb{R}^{N+2}$ is well-known (see for instance [9, p. 11] or [28, p. 92]). The proof of this local result is based on the existence and smoothness of the center-stable and center-unstable manifolds (see [8, 11]) for the system of differential equations in $\mathbb{R}^2 \times \mathbb{R}^N$ obtained by adding the equation $\dot{\nu} = 0$ to $(\dot{x}, \dot{y}) = X_\nu(x, y)$, and the fact that (in this context) these manifolds are unique (see [23, p. 165]). Let $\Phi_L : U \to \Phi_L(U) \subset \mathbb{R}^{N+2}$ be this diffeomorphism that straightens locally the separatrices $S_\nu^\pm$. We also denote by X the vector field associated to the above-mentioned system in $\mathbb{R}^{N+2}$ and by φ its flow. Furthermore let ψ be the flow of the "straightened" vector field $(\Phi_L)_\star(X)$ that leaves the coordinate planes invariant.

The idea is to extend $\Phi_L : U \to R^{N+2}$ taking advantage of the fact that small enough neighbourhoods of $\ell^+ \times \{\nu_0\}$ and $\ell^- \times \{\nu_0\}$ will be mapped by φ (in forward and backward time, respectively) inside U. With this aim in view we take $\delta > 0$ and consider the open set $B_\nu \subset \mathbb{R}^2$ in Figure 5. The boundary of B_ν consists in four (pieces of) trajectories of the flow ψ_ν together with four segments (inside $x = \pm\delta$ and $y = \pm\delta$) where the straightened vector field is transversal. We define $B := \{(p, \nu) \in \mathbb{R}^{N+2} : p \in B_\nu, |\nu - \nu_0| < \delta\} \subset \Phi_L(U)$, which is an open neighbourhood of $(0, 0, \nu_0)$ by the continuous dependence with respect to initial conditions.

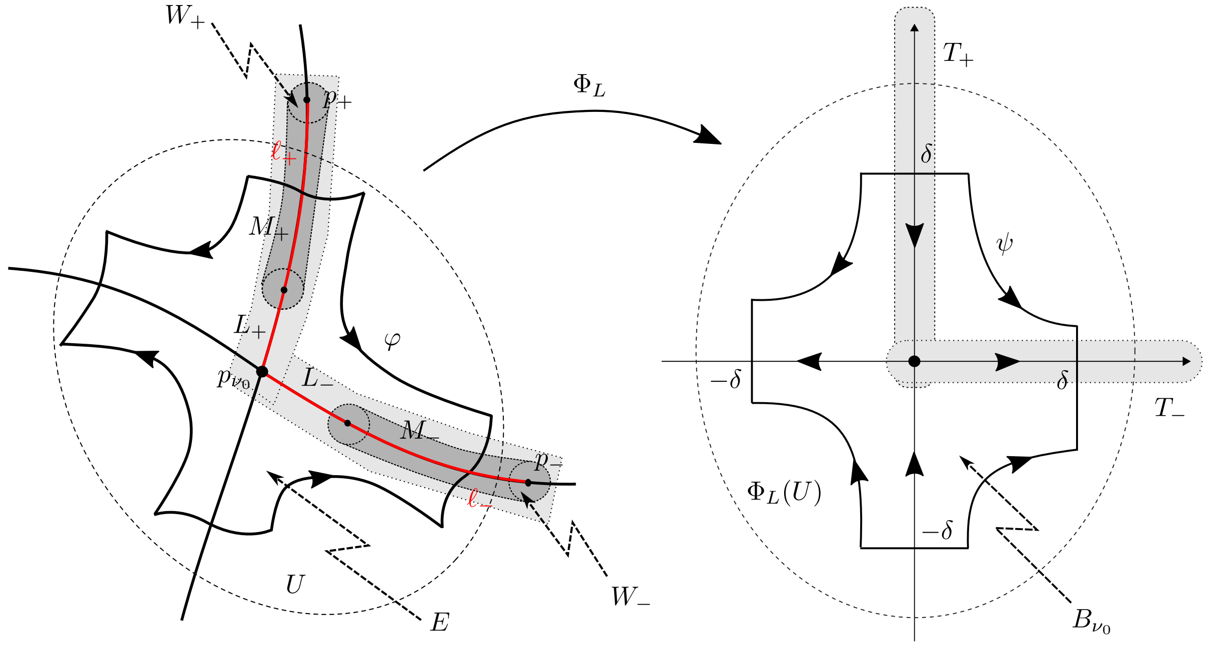

Figure 5: Slice $\nu = \nu_0$ of the sets involved in the construction of $V = M_+ \cup E \cup M_-$.

Next we take a $\mathscr{C}^\infty$ bump function $\rho\colon \mathbb{R}^{N+2} \to \mathbb{R}$ such that $\rho|_{\bar{B}} \equiv 1$ and $\rho|_{A^c} \equiv 0$, where A is an open set with $\bar{B} \subset A \subset \Phi_L(U)$, and define $Y := \rho\,(\Phi_L)_\star(X)$, which is a complete vector field in $\mathbb{R}^{N+2}$ that leaves invariant $\nu =$ constant. By abuse of notation we also refer to the flow of this new vector field by ψ.

We define $E := (\Phi_L)^{-1}(B)$, which is an open neighbourhood of $(p_{\nu_0}, \nu_0) \in \mathbb{R}^{N+2}$. Since $Y|_B = (\Phi_L)_\star(X)$, we have that Φ_L is a conjugacy between the flows of $X|_E$ and $Y|_B$, that we shall denote by φ_E and ψ_B, respectively. We take open neighbourhoods $L^\pm$ of $\ell^\pm \times \{\nu_0\}$ in $\mathbb{R}^{N+2}$, see Figure 5, such that $L^+ \cap L^- \subset E$,

$$L^+ \cap \partial E \subset (\Phi_L)^{-1}(B \cap \{y = \delta\}) \text{ and } L^- \cap \partial E \subset (\Phi_L)^{-1}(B \cap \{x = \delta\}).$$

Similarly we take open neighbourhoods T^+ and T^- of $\{x = 0, y > 0\} \times \{\nu_0\}$ and $\{y = 0, x > 0\} \times \{\nu_0\}$ in $\mathbb{R}^{N+2}$, respectively, such that $T^+ \cap T^- \subset B$,

$$T^+ \cap \partial B \subset \{y = \delta\} \text{ and } T^- \cap \partial B \subset \{x = \delta\}.$$

Hence, in particular, $L^\pm \cap \partial E$ is a transversal section for X and $T^\pm \cap \partial B$ is a transversal section for Y. This will be a key point in the forthcoming reasoning.

Let us denote the endpoint of $\ell^\pm$ not being p_{ν_0} by $p_\pm$ and take $\tau_- < 0 < \tau_+$ such that $\varphi(\tau_\pm, (p_\pm, \nu_0)) \in E$. By the continuous dependence with respect to initial conditions there exists a neighbourhood $W_\pm$ of $(p_\pm, \nu_0)$ such that

(a) $\varphi(\tau_\pm, W_\pm) \subset E$.

(b) $\varphi(< 0, 2\tau_\pm >, W_\pm) \subset L^\pm$.

(c) $M_\pm := \varphi(< 0, \tau_\pm >, W_\pm)$ are disjoint.

(d) $\psi\Big(< 0, -2\tau_\pm >, \Phi_L\big(\varphi(< 0, 2\tau_\pm >, W_\pm) \cap E\big)\Big) \subset T^\pm$.

(Here $< a, b >$ stands for the smallest closed interval containing $a, b \in \mathbb{R}$.)

For a given $z \in V := M_+ \cup E \cup M_- \subset \mathbb{R}^{N+2}$ we define

$$\Phi(z) := \psi\big(- t_z, \Phi_L(\varphi(t_z, z)\big),$$

where t_z is any $t \in < 0, \tau_\pm >$ such that $\varphi(t, z) \in E$ for $z \in M_\pm$, whereas $t_z = 0$ for $z \in E$. We must prove that the function $\Phi\colon V \to \mathbb{R}^{N+2}$ is well defined, smooth and injective.

Let us show first that Φ is well defined. To this end we remark that $M_\pm \cap E \neq \emptyset$. Let us fix for instance any $z \in M_+$ and suppose that

$$\varphi(t_1, z) =: w_1 \in E \text{ and } \varphi(t_2, z) =: w_2 \in E \text{ with } t_2 > t_1 \geqslant 0.$$

Then we must show that $\psi(-t_1, \Phi_L(w_1)) = \psi(-t_2, \Phi_L(w_2))$. Note that $w_2 = \varphi(t_2 - t_1, w_1)$. Moreover, thanks to the inclusion in (b), the definition of M_+ in (c) and the transversality of X at $L^+ \cap \partial E$, it turns out that $\varphi(t, w_1) \in E$ for all $t \in [0, t_2 - t_1]$. Likewise, due to (d) and the transversality of Y at $T^+ \cap \partial B$, it follows that $\psi(-t, \Phi_L(\varphi(t, w_1)) \in B$ for all $t \in [0, t_2 - t_1]$. Accordingly, using also that Φ_L is a conjugacy between φ_E and ψ_B,

$$\begin{aligned}\psi\big(t_1 - t_2, \Phi_L(w_2)\big) = \psi\big(t_1 - t_2, \Phi_L(\varphi(t_2 - t_1, w_1)\big) &= \psi_B\big(t_1 - t_2, \Phi_L(\varphi_E(t_2 - t_1, w_1)\big) \\ &= \Phi_L(\varphi_E(t_1 - t_2, \varphi_E(t_2 - t_1, w_1)) = \Phi_L(w_1),\end{aligned}$$

that implies $\psi(-t_1, \Phi_L(w_1)) = \psi(-t_2, \Phi_L(w_2))$, as desired.

Let us turn next to the smoothness of Φ. To this end we observe that we can take $t_z = \tau_\pm$ for any $z \in M_\pm$, which implies that Φ is smooth on $M_+ \cup M_-$ because $\varphi(\tau_\pm, \cdot)$, $\psi(-\tau_\pm, \cdot)$ and Φ_L are smooth. The smoothness on E is clear since $\Phi|_E = \Phi_L$ by definition.

With regard to the injectivity we take $z_1, z_2 \in V$ with $z_1 \neq z_2$ and we claim that then $\Phi(z_1) \neq \Phi(z_2)$. There are four cases to consider:

1. $z_1, z_2 \in E$. The claim is obvious in this case because $\Phi|_E = \Phi_L$.
2. $z_1, z_2 \in M_\pm$. Since we can take $t_{z_1} = t_{z_2} = \tau_\pm$, the claim follows from the injectivity of $\varphi(\tau_\pm, \cdot)$, $\psi(-\tau_\pm, \cdot)$ and Φ_L.
3. $z_1 \in M_\pm \setminus E$ and $z_2 \in E$. Due to $\Phi(z_2) \in B$, it suffices to show that
$$\Phi(M_\pm \setminus E) \cap B = \emptyset. \tag{34}$$
We take $z \in M_\pm \setminus E$, so that $\varphi(t_z, z) \in L^\pm \cap E$ with $t_z \in < 0, \tau_\pm >$ and $w := \Phi_L(\varphi(t_z, z)) \in T^\pm \cap B$. By contradiction, if $\Phi(z) = \psi(-t_z, w) \in B$ then, due to (d) and the transversality of Y at $T^\pm \cap \partial B$, the compact set $K = \{\psi(-t, w) : t \in < 0, t_z >\}$ is inside B. However $C = \{\varphi(t, z) : t \in < 0, t_z >\}$ intersects ∂E and it is easy to verify that $\Phi(C) \subset K$, which contradicts $\Phi(\partial E) \subset \partial B$.
4. $z_1 \in M_+ \setminus E$ and $z_2 \in M_- \setminus E$. We have $\Phi(M_\pm) \subset T^\pm$ and thus, from (34), $\Phi(M_\pm \setminus E) \subset T^\pm \setminus B$. On account of $T^+ \cap T^- \subset B$, the claim follows in this case as well.

Finally we remark that, by construction, if $z = (p, \nu) \in \mathbb{R}^2 \times \mathbb{R}^N$ then $\Phi(z) = (\phi_\nu(p), \nu)$. This completes the proof. ∎

Proof of Corollary D. Consider the given $\nu_0 \in U \subset \mathbb{R}^N$ and let us fix that the outer boundary Π_ν has k hyperbolic saddles, $p_\nu^1, p_\nu^2, \ldots, p_\nu^k$, that for convenience we label according to the sense of the flow with the first one not being in ℓ_∞. (Here we use the assumption that Π_ν is a persistent polycycle, see Definition 1.11). Denote the hyperbolicity ratio of p_ν^i by $\lambda_i(\nu)$ and set $\lambda_0^i := \lambda_i(\nu_0)$. Let us suppose first that $k \geqslant 2$. (We postpone the case $k = 1$ because the proof is slightly different.)

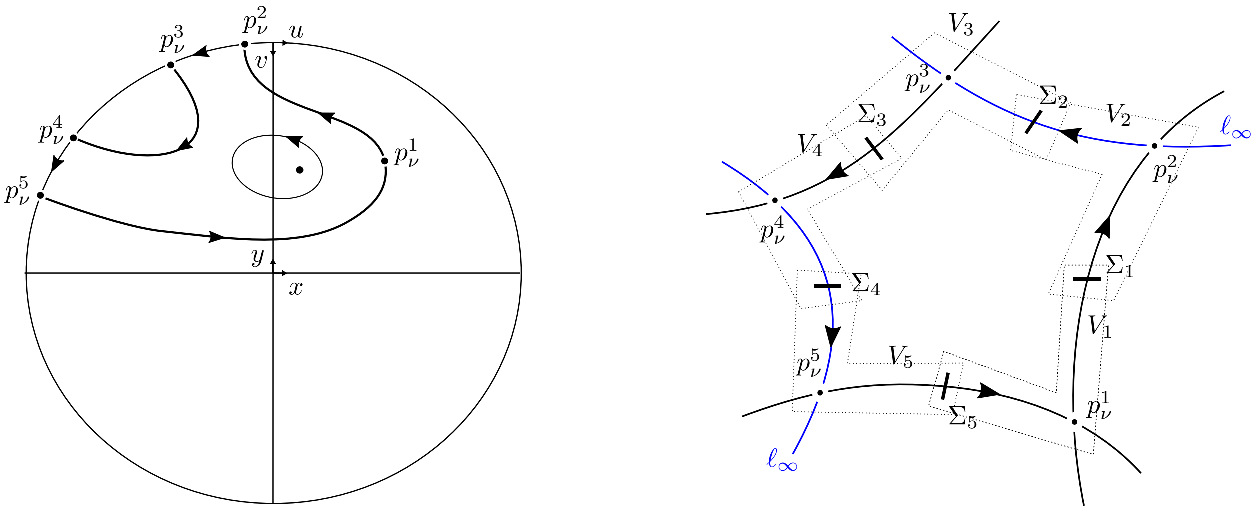

Figure 6: Following the proof of Corollary D, polycycle Π_ν for $k = 5$ in the Poincaré disc (left) and a ν-slice of the sections Σ_i and sets V_i (right).

Since by assumption the infinite line ℓ_∞ is invariant for the flow of $p(X_\nu)$, if $p_\nu^i \in \ell_\infty$ then exactly one of its separatrices is contained in ℓ_∞. (This easily implies that Π_ν has an even number of singularities at infinity, assembled in pairs, see Figure 6.) For each $i = 1, 2, \dots, k$ we place a transversal section Σ_i in the heteroclinic connection between p_ν^i and p_ν^{i+1}. In addition we denote the Dulac map and the Dulac time of X_ν from Σ_{i-1} to Σ_i by $D_i(\,\cdot\,;\nu)$ and $T_i(\,\cdot\,;\nu)$, respectively. Thus, setting $\hat{D}_i := D_i \circ D_{i-1} \circ \dots \circ D_1$, the return time from $\Sigma_0 := \Sigma_k$ to Σ_0 is given by

$$T(s) = T_1(s) + \big(T_2 \circ \hat{D}_1\big)(s) + \ldots + \big(T_k \circ \hat{D}_{k-1}\big)(s). \tag{35}$$

In order to study each Dulac time it is more convenient to compactify the vector field X_ν by means of a projective change of coordinates rather than to use the Poincaré compactification $p(X_\nu)$. To this end we note that, since the center is not global, there exists a straight line $\ell_a := \{\alpha_1 x + \alpha_2 y = \beta\}$ not intersecting $\mathscr{P}_\nu$ for all $\nu \approx \nu_0$ and such that $p_\nu^i \notin \ell_a \cap \ell_\infty$ for all $i = 1, 2, \dots, k$. Without lost of generality, by means of a rotation and a translation, we can assume that $\ell_a = \{y = 0\}$. That being said we perform the projective change of coordinates $\{u = \frac{x}{y}, v = \frac{1}{y}\}$, which brings $X_\nu(x, y)$ to

$$\hat{X}_\nu(u, v) = \frac{1}{v^{d-1}}\left(\hat{P}(u, v; \nu)\partial_u + v\hat{Q}(u, v; \nu)\partial_v\right).$$

Here $d \geqslant 2$ is the degree of X_ν, whereas $\hat{P}$ and $\hat{Q}$ are polynomials with the coefficients depending $\mathscr{C}^\infty$ on ν. (Let us remark that X_ν and $\hat{X}_\nu$ are conjugated, which is essential to study the return time, whereas X_ν and $p(X_\nu)$ are only equivalent.) Note that in doing so the infinite line is mapped to $\{v = 0\}$. On account of this, by applying Lemma 4.3, for each $i = 1, 2, \dots, k$ there exists a neighbourhood V_i of $(p_{\nu_0}^i, \nu_0)$ and a $\mathscr{C}^\infty$ diffeomorphism $\Phi_i \colon V_i \to \Phi_i(V_k)$ with $\Phi_i(x, y, \nu) = (\phi_\nu^i(x, y), \nu)$ such that $(\phi_\nu^i)_\star(\hat{X}_\nu|_{V_i}) = \bar{X}_\nu^i$, where

$$\bar{X}_\nu^i(u, v) = \frac{1}{v^{\kappa_i}}\big(u\bar{P}_i(u, v; \nu)\partial_u + v\bar{Q}_i(u, v; \nu)\partial_y\big) \text{ with } \kappa_i = \begin{cases} d-1 & \text{if } p_\nu^i \in \ell_\infty, \\ 0 & \text{if } p_\nu^i \notin \ell_\infty, \end{cases}$$

and $\bar{P}_i, \bar{Q}_i \in \mathscr{C}^\infty(\Phi_i(V_i))$. We choose $\varepsilon > 0$ small enough so that $2\varepsilon < \min(1, \lambda_0^i)$ for all $i = 1, 2, \dots, k$, which in particular implies that $\mathscr{B}^0_{\lambda_0^i, 2\varepsilon} = \{(0, 0)\}$. Then, for each $i = 1, 2, \dots, k$,

- we apply Theorem B to the family $\{\bar{X}_\nu^i,\, \nu\approx\nu_0\}$ with $L=\lambda_0^i+2\varepsilon$ to conclude that the Dulac map D_i of the hyperbolic saddle p_ν^i from Σ_{i-1} to Σ_i can be written as

$$D_i(s;\nu)=\Delta_{00}^i s^{\lambda_i(\nu)}+\mathcal{F}^\infty_{\lambda_0^i+2\varepsilon}(\nu_0)=s^{\lambda_i(\nu)}\big(\Delta_{00}^i+\mathcal{F}^\infty_\varepsilon(\nu_0)\big),$$

where $\Delta_{00}^i(\nu)>0$. (In the second equality above we use Lemmas A.2 and A.3.) Note in particular that $D_i\in\mathcal{F}^\infty_{\lambda_0^i-\varepsilon}(\nu_0)\subset\mathcal{F}^\infty_\varepsilon(\nu_0)$ due to $\lambda_0^i>2\varepsilon$.

- we apply Theorem A with $L=\varepsilon$ and $n=(n_1,n_2)=(0,\kappa_i)$ to show that the Dulac time T_i from Σ_{i-1} to Σ_i writes as

$$T_i(s;\nu)=T_0^i\log s+T_{00}^i+\mathcal{F}^\infty_\varepsilon(\nu_0),\tag{36}$$

where $T_0^i(\nu)=\frac{-1}{\bar{P}_i(0,0;\nu)}>0$ if $p_\nu^i\notin\ell_\infty$ and, due to $d\geqslant 2$, $T_0^i(\nu)\equiv 0$ if $p_\nu^i\in\ell_\infty$.

Since $D_1\in\mathcal{F}^\infty_\varepsilon(\nu_0)$, an easy application of Lemma A.2 shows that

$$\begin{aligned}\hat{D}_2(s)&=\big(s^{\lambda_1}(\Delta_{00}^1+\mathcal{F}^\infty_\varepsilon(\nu_0))\big)^{\lambda_2}\left(\Delta_{00}^2+\mathcal{F}^\infty_{\varepsilon^2}(\nu_0)\right)=\Delta_{00}^2(\Delta_{00}^1)^{\lambda_2}s^{\lambda_1\lambda_2}(1+\mathcal{F}^\infty_\varepsilon(\nu_0))^{\lambda_2}(1+\mathcal{F}^\infty_{\varepsilon^2}(\nu_0))\\&=\Delta_{00}^2(\Delta_{00}^1)^{\lambda_2}s^{\lambda_1\lambda_2}(1+\mathcal{F}^\infty_\varepsilon(\nu_0))(1+\mathcal{F}^\infty_{\varepsilon^2}(\nu_0))=s^{\lambda_1\lambda_2}\left(\Delta_{00}^2(\Delta_{00}^1)^{\lambda_2}+\mathcal{F}^\infty_{\varepsilon^2}(\nu_0)\right),\end{aligned}$$

where in the third equality we use that the map $(s,\nu)\mapsto(1+s)^{\lambda_2(\nu)}-1$ belongs to $\mathcal{F}_1^\infty(\nu_0)$. Similarly,

$$\hat{D}_i(s)=s^{\lambda_1\cdots\lambda_i}(\hat{\Delta}_{00}^i+\mathcal{F}^\infty_{\varepsilon^i}(\nu_0))\text{ for some }\hat{\Delta}_{00}^i>0,$$

so that $\hat{D}_i\in\mathcal{F}^\infty_{\varepsilon^i}(\nu_0)$. Hence, by applying Lemma A.2 again, from (36) we get that if $i\in\{1,2,\dots,k\}$ then

$$\big(T_i\circ\hat{D}_{i-1}\big)(s)=\hat{T}_i\log s+T_{00}^i+T_0^i\log\hat{\Delta}_{00}^i+\mathcal{F}^\infty_{\varepsilon^{i+1}}(\nu_0)\text{ with }\hat{T}_i:=T_0^i\textstyle\prod_{\ell=1}^{i-1}\lambda_\ell\geqslant 0$$

and where we set $\hat{D}_0=\mathrm{Id}$ for the sake of convenience. This equality, together with (35), shows that

$$T(s;\nu)=\bar{T}_0(\nu)\log s+\bar{T}_{00}(\nu)+\mathcal{F}^\infty_{\varepsilon^{k+1}}(\nu_0)\text{ with }\bar{T}_0:=\textstyle\sum_{i=1}^k\hat{T}_i.$$

Observe that, and this is the key point, $\bar{T}_0(\nu)\geqslant\hat{T}_1(\nu)=T_0^1(\nu)>0$ due to $p_\nu^1\notin\ell_\infty$ by construction. Hence, from (f) and (g) in Lemma A.2,

$$s\partial_sT(s;\nu)=\bar{T}_0(\nu)+s\mathcal{F}^\infty_{\varepsilon^{k+1}-1}(\nu_0)=\bar{T}_0(\nu)+\mathcal{F}^\infty_{\varepsilon^{k+1}}(\nu_0)\to\bar{T}_0(\nu_0)\neq 0\text{ as }(s,\nu)\to(0,\nu_0).$$

Therefore we can assert the existence of some $\delta>0$ such that if $s\in(0,\delta)$ and $\|\nu-\nu_0\|\leqslant\delta$ then $\partial_sT(s;\nu)\neq 0$. Consequently $\mathcal{Z}_0(T'(\,\cdot\,;\nu),\nu_0)=0$ and we claim that this implies $\mathrm{Crit}\big((\Pi_{\nu_0},X_{\nu_0}),X_\nu\big)=0$. By contradiction, if the criticality is not zero then there exists a sequence $\{\gamma_{\nu_i}\}_{i\in\mathbb{N}}$, where each γ_{ν_i} is a critical periodic orbit of X_{ν_i}, such that $\nu_i\to\nu_0$ and $d_H(\gamma_{\nu_i},\Pi_{\nu_0})\to 0$ as $i\to+\infty$. Then, since Π_ν varies continuously at ν_0 and

$$d_H(\gamma_{\nu_i},\Pi_{\nu_i})\leqslant d_H(\gamma_{\nu_i},\Pi_{\nu_0})+d_H(\Pi_{\nu_i},\Pi_{\nu_0}),$$

we have $d_H(\gamma_{\nu_i},\Pi_{\nu_i})\to 0$ as $i\to+\infty$, which contradicts that $\mathcal{Z}_0(T'(\,\cdot\,;\nu),\nu_0)=0$. (Let us remark that this last implication is not true without the assumption that Π_ν varies continuously at ν_0 because the parametrization $\tau(s;\nu)$ of the transversal section Σ_0 is taken such that $\tau(0;\nu)\in\Pi_\nu$ for all ν.) This proves the claim and so the result for the case $k\geqslant 2$ is true.

Let us consider finally the case $k=1$, i.e., assume that Π_ν is a (finite) saddle loop. In this case we place two transversal sections Σ_1 and Σ_2 in the heteroclinic connection. By applying Lemma 4.3 and Theorem A exactly as we did before we can assert that the Dulac time from Σ_1 to Σ_2 is $T_D(s;\nu)=T_0\log s+T_{00}+\mathcal{F}^\infty_\varepsilon(\nu_0)$ with $T_0=T_0(\nu)>0$. Then the return time from Σ_1 to Σ_1 is given by $T=T_D+T_R$, where $T_R(s;\nu)$ is the regular time of $-X_\nu$ from Σ_1 to Σ_2, which is a $\mathscr{C}^\infty$ function in a neighbourhood of $(s,\nu)=(0,\nu_0)$ by Lemma 2.4. Hence $s\partial_sT(s;\nu)=T_0(\nu)+\mathcal{F}^\infty_\varepsilon(\nu_0)+s\partial_sT_R(s;\nu)$ and $\lim_{(s,\nu)\to(0,\nu_0)}s\partial_sT(s;\nu)=T_0(\nu_0)\neq 0$. Then, exactly as before, this shows the validity of the result for $k=1$ as well and completes the proof. ■

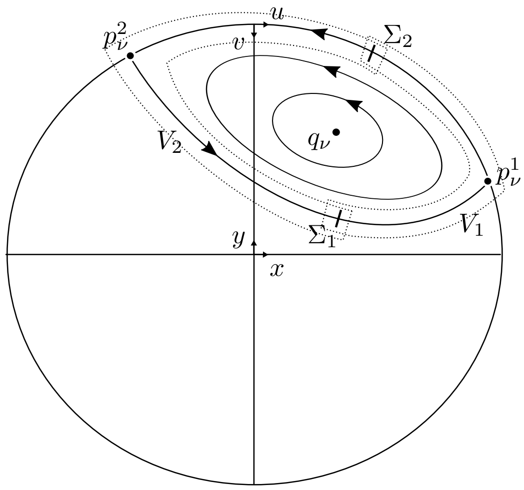

Figure 7: Following the proof of Corollary E, phase portrait of X_ν in the Poincaré disc with the shape of the period annulus of the center at q_ν and the placement of the transversal sections in the outer boundary Π_ν. (To be more precise V_1 and V_2 are in fact open sets in $\mathbb{RP}^2 \times U$.)

Proof of Corollary E. Note first of all that we can assume that both hyperbolic saddles are at the infinite line ℓ_∞, because otherwise the result follows by applying Corollary D.

For each $\nu \in U$, let $\mathscr{P}_\nu$ be the period annulus of the center q_ν and denote the two saddles at its outer boundary Π_ν by p_ν^1 and p_ν^2. (Here we use the assumption that Π_ν is a persistent polycycle, see Definition 1.11). Let us also denote the respective hyperbolicity ratios by $\lambda_1(\nu)$ and $\lambda_2(\nu)$. It is well known that $\lambda_1(\nu)\lambda_2(\nu) = 1$ for all $\nu \in U$, otherwise the return map defined near the polycycle Π_ν cannot be the identity. Moreover the center is not global since the infinite line ℓ_∞ is invariant for the flow of $p(X_\nu)$ by hypothesis and the two singularities in Π_ν are hyperbolic saddles. Thus there exists a straight line $\ell_a := \{\alpha_1 x + \alpha_2 y = \beta\}$ not intersecting $\mathscr{P}_\nu$ for all $\nu \approx \nu_0$ and such that $p_\nu^k \notin \ell_a \cap \ell_\infty$ for $k = 1, 2$. Without lost of generality, by means of a rotation and a translation, we can assume that $\ell_a = \{y = 0\}$, so that the shape of the period annulus $\mathscr{P}_\nu$ in the Poincaré disc is as we draw in Figure 7. We place a transversal section Σ_1 in the finite heteroclinic connection between the saddles and denote by $T_R(\,\cdot\,;\nu)$ the corresponding return time for X_ν. Note that T_R is a parametrization of the period function of the center q_ν near the outer boundary of $\mathscr{P}_\nu$. In order to study T_R we take an auxiliary transversal section Σ_2 in ℓ_∞ so that $T = T_1 + T_2$, where T_1 is the Dulac time of X_ν from Σ_1 to Σ_2 and T_2 is the Dulac time of $-X_\nu$ from Σ_1 to Σ_2. Our goal is to apply Theorem A to obtain the asymptotic development of T_1 and T_2. With this aim in view we perform firstly the projective change of coordinates $\{u = \frac{x}{y}, v = \frac{1}{y}\}$, that brings $X_\nu(x, y)$ to

$$\hat{X}_\nu(u, v) = \frac{1}{v^{d-1}} \left(\hat{P}(u, v; \nu)\partial_u + v\hat{Q}(u, v; \nu)\partial_v \right).$$

Here $d \geqslant 2$ is the degree of X_ν, whereas $\hat{P}$ and $\hat{Q}$ are polynomials with the coefficients depending $\mathscr{C}^\infty$ on ν. By an abuse of notation we still denote the two hyperbolic saddles of $\hat{X}_\nu$ at $v = 0$ coming from the two vertices of Π_ν at ℓ_∞ by p_ν^1 and p_ν^2. Secondly, by Lemma 4.3, for $k = 1, 2$ there exists a neighbourhood V_k of $(p_{\nu_0}^k, \nu_0)$ and a $\mathscr{C}^\infty$ diffeomorphism $\Phi_k\colon V_k \to \Phi_k(V_k)$ with $\Phi_k(x, y, \nu) = (\phi_\nu^k(x, y), \nu)$ such that $(\phi_\nu^k)_\star(\hat{X}_\nu|_{V_k}) = \bar{X}_\nu^k$, where

$$\bar{X}_\nu^k(u, v) = \frac{1}{v^{d-1}} \left(u\bar{P}_k(u, v; \nu)\partial_u + v\bar{Q}_k(u, v; \nu)\partial_y \right)$$

with $\bar{P}_k, \bar{Q}_k \in \mathscr{C}^\infty(\Phi_k(V_k))$. Note in addition that each $\bar{X}_\nu^k$ has a hyperbolic saddle at the origin with hyperbolicity ratio $\lambda_k(\nu)$. It is important to observe at this point that T_2 is the Dulac map of $-\bar{X}_\nu^2$ and that, for this vector field, the hyperbolicity ratio of the saddle is $1/\lambda_2(\nu) = \lambda_1(\nu)$. Therefore, and this is crucial, we will apply Theorem A to two *different* families of vector fields that have saddles with *equal* hyperbolicity ratio $\lambda(\nu) := \lambda_1(\nu) = 1/\lambda_2(\nu)$. In doing so for $\bar{X}_\nu^1$, setting $\lambda_0 := \lambda(\nu_0)$ and taking $n = (0, d-1)$, we have a well defined formal series

$$\hat{T}_1(s;\nu) := T_0^1(\nu)\log s + \begin{cases} \displaystyle\sum_{(i,j)\in\Lambda_n} T_{ij}^1(\nu)s^{i+\lambda(\nu)j} & \text{if } \lambda_0 \notin \mathbb{Q}, \\ \displaystyle\sum_{(i,j)\in\Lambda_n} \boldsymbol{T}_{ij}^1\big(\omega_\alpha(s);\nu\big)s^{i+\lambda(\nu)j} & \text{if } \lambda_0 \in \mathbb{Q}, \end{cases}$$

which is asymptotic to $T_1(s;\nu)$ as $(s,\nu) \to (0,\nu_0)$. Note that $T_0^1 \equiv 0$ because $n \neq (0,0)$ due to $d-1>0$ (otherwise the given vector field X_ν would be linear). Similarly, by applying Theorem A to $-\bar{X}_\nu^2$ we have a well defined formal series

$$\hat{T}_2(s;\nu) := T_0^2(\nu)\log s + \begin{cases} \displaystyle\sum_{(i,j)\in\Lambda_n} T_{ij}^2(\nu)s^{i+\lambda(\nu)j} & \text{if } \lambda_0 \notin \mathbb{Q}, \\ \displaystyle\sum_{(i,j)\in\Lambda_n} \boldsymbol{T}_{ij}^2\big(\omega_\alpha(s);\nu\big)s^{i+\lambda(\nu)j} & \text{if } \lambda_0 \in \mathbb{Q}, \end{cases}$$

which is asymptotic to $T_2(s;\nu)$ as $(s,\nu) \to (0,\nu_0)$ and where $T_0^2 \equiv 0$ again. Note that if $\lambda_0 \in \mathbb{Q}$ then

$$\boldsymbol{T}_{ij}^k(w;\nu) = \sum_{r\in\mathscr{A}_{ij\lambda_0}^n} T_{i-rp,j+rq}^k(\nu)(1+\alpha w)^r \text{ for } k=1,2,$$

where $\alpha(\nu) = p - \lambda(\nu)q$ and $\lambda_0 = p/q$ with $\gcd(p,q)=1$. Thus, setting $T_{ij}^3 := T_{ij}^1 + T_{ij}^2$ and $\boldsymbol{T}_{ij}^3 := \boldsymbol{T}_{ij}^1 + \boldsymbol{T}_{ij}^2$,

$$\hat{T}_R(s;\nu) := \hat{T}_1(s;\nu) + \hat{T}_2(s;\nu) = \begin{cases} \displaystyle\sum_{(i,j)\in\Lambda_n} T_{ij}^3(\nu)s^{i+\lambda(\nu)j} & \text{if } \lambda_0 \notin \mathbb{Q}, \\ \displaystyle\sum_{(i,j)\in\Lambda_n} \boldsymbol{T}_{ij}^3\big(\omega_\alpha(s);\nu\big)s^{i+\lambda(\nu)j} & \text{if } \lambda_0 \in \mathbb{Q}, \end{cases}$$

is a well defined formal series that is asymptotic to $T_R(s;\nu) = T_1(s;\nu) + T_2(s;\nu)$ as $(s,\nu) \to (0,\nu_0)$ in the sense established by Theorem A. We use at this point the assumption that the center q_{ν_0} is non-isochronous. On account of this, by applying the result of Saavedra and Mardešić in [17] we can assert that the formal series $\hat{T}_R(s;\nu_0)$ is not constant. This implies that ℓ_{ν_0}, computed as explained in Definition 1.8 with respect to the formal series $\hat{T}_R(s;\nu)$, is finite (i.e, $\ell_{\nu_0} \in \mathbb{N}$). Then the application of Theorem C (see Remark 3.5) shows that

$$\mathcal{Z}_0(T_R'(\,\cdot\,;\nu),\nu_0) \leqslant \ell_{\nu_0} < +\infty$$

and, since Π_ν varies continuously at ν_0, $\operatorname{Crit}\big((\Pi_{\nu_0}, X_{\nu_0}), X_\nu\big) \leqslant \ell_{\nu_0}$. This shows the validity of the result in case that the period annulus is unbounded and completes the proof because, as we already mentioned, the bounded case follows by Corollary D. ■

We conclude this section by pointing out that, even in an unfolding of polynomial centers, the fact that the outer boundary of its period annulus is a hyperbolic polycycle varying continuously does not imply its persistence as required in Corollaries D and E. Indeed, let us consider the 1-parametric family of quadratic differential systems

$$X_\nu \quad \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + (\nu-2)x^2 + 2y^2, \end{cases}$$

which has a center at the origin for all $\nu \in \mathbb{R}$. Figure 8 displays its phase portrait in the Poincaré disc for $\nu \approx 0$. (The reader is referred to [15] for the complete bifurcation diagram of the phase portrait of the

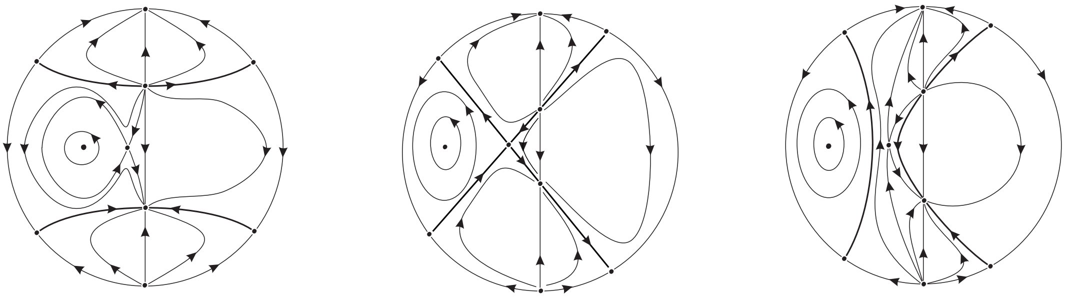

Figure 8: Phase portrait of the unfolding $\{X_\nu; \nu \approx 0\}$ in the Poincaré disc, where the origin is shifted to the left for convenience.

quadratic centers.) One can verify that the outer boundary Π_ν of its period annulus varies continuously at $\nu_0 = 0$. Following the notation in Definition 1.11, the hyperbolic polycycle Γ is the triangle with an edge at infinity and it occurs that both separatrix connections with the finite saddle are broken for $\nu \neq 0$. The outer boundary becomes a saddle loop for $\nu < 0$ and an unbounded 2-cycle for $\nu > 0$. Our goal for further research is to develop tools to study the criticality of this type of unfolding.

A Results about the class $\mathcal{F}_L^K(W)$

In this appendix, for reader's convenience, we collect some technical results from [19] about the class of functions $\mathcal{F}_L^K(W)$ that we use in the present paper.

Lemma A.1. *Let U be an open set of $\mathbb{R}^N$, $K \in \mathbb{Z}_{\geq 0}$ and $g(s;\mu) \in \mathscr{C}^K_{s>0}(U)$ such that, for some $W \subset U$ and $L \in \mathbb{R}$, $g(s;\mu) \in \mathcal{F}_L^K(W)$. If $L > K$ then g extends to a $\mathscr{C}^K$-function $\hat{g}$, defined in some open neighbourhood of $\{0\} \times W$ in $\mathbb{R}^{N+1}$, and satisfying $\partial^\nu \hat{g}(0;\mu) = 0$ for all $\mu \in W$ and $\nu \in \mathbb{Z}^{N+1}_{\geq 0}$ with $|\nu| \leqslant K$.*

Lemma A.2. *Let U and U' be open sets of $\mathbb{R}^N$ and $\mathbb{R}^{N'}$ respectively and consider $W \subset U$ and $W' \subset U'$. Then the following holds:*

(a) $\mathcal{F}_L^K(W) \subset \mathcal{F}_L^K(\hat{W})$ *for any* $\hat{W} \subset W$ *and* $\bigcap_n \mathcal{F}_L^K(W_n) = \mathcal{F}_L^K\left(\bigcup_n W_n\right)$.

(b) $\mathcal{F}_L^K(W) \subset \mathcal{F}_L^K(W \times W')$.

(c) $\mathscr{C}^K(U) \subset \mathcal{E}^K(U) \subset \mathcal{F}_0^K(W)$.

(d) *If* $K \geqslant K'$ *and* $L \geqslant L'$ *then* $\mathcal{F}_L^K(W) \subset \mathcal{F}_{L'}^{K'}(W)$.

(e) $\mathcal{F}_L^K(W)$ *is closed under addition.*

(f) *If* $f \in \mathcal{F}_L^K(W)$ *and* $\nu \in \mathbb{Z}^{N+1}_{\geq 0}$ *with* $|\nu| \leqslant K$ *then* $\partial^\nu f \in \mathcal{F}_{L-\nu_0}^{K-|\nu|}(W)$.

(g) $\mathcal{F}_L^K(W) \cdot \mathcal{F}_{L'}^K(W) \subset \mathcal{F}_{L+L'}^K(W)$.

(h) *Assume that* $\phi\colon U' \to U$ *is a* $\mathscr{C}^K$ *function with* $\phi(W') \subset W$ *and let us take* $g \in \mathcal{F}_{L'}^K(W')$ *with* $L' > 0$ *and verifying* $g(s;\eta) > 0$ *for all* $\eta \in W'$ *and* $s > 0$ *small enough. Consider also any* $f \in \mathcal{F}_L^K(W)$. *Then* $h(s;\eta) := f(g(s;\eta);\phi(\eta))$ *is a well-defined function that belongs to* $\mathcal{F}_{LL'}^K(W')$.

Next result gathers some interesting properties of the Ecalle-Roussarie compensator. In the statement we use the notation $x^+ := \max(x, 0)$ and $x^- := \max(-x, 0)$ for, respectively, the positive and negative part of a given $x \in \mathbb{R}$. Note in particular that then $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

Lemma A.3. *The following assertions hold:*

(*a*) *For each compact set $I \subset \mathbb{R}$ and $\nu \in \mathbb{Z}^2_{\geq 0}$ there exists a constant $C > 0$ such that*

$$|\partial^\nu \omega(s; \alpha)| \leqslant C s^{-\alpha^+ - \nu_0} |\log s|^{|\nu|+1} \text{for all } \alpha \in I \text{ and } s \in (0, 1/e).$$

Moreover $\lim_{s\to 0^+} \frac{1}{\omega(s;\alpha)} = \alpha^-$ uniformly on $\alpha \in \mathbb{R}$ so that, in particular, $\lim_{(s,\alpha)\to(0^+,0)} \frac{1}{\omega(s;\alpha)} = 0$.

(*b*) *For each $\varepsilon > 0$, $(s, \alpha) \mapsto \omega(s; \alpha)$ belongs to $\mathcal{F}^\infty_{-\varepsilon}(\{\alpha < \varepsilon\})$ and $(s; \alpha) \mapsto \frac{1}{\omega(s;\alpha)}$ belongs to $\mathcal{F}^\infty_{-\varepsilon}(\mathbb{R})$.*

(*c*) *For each $L \in \mathbb{R}$ and $\ell \in \mathbb{Z}$, $(s, \alpha, \beta) \mapsto s^\beta \omega^\ell(s; \alpha)$ belongs to $\mathcal{F}^\infty_L(\{(\alpha, \beta) \in \mathbb{R}^2 \,;\, \beta > L + \ell^+\alpha^+\})$.*

(*d*) *If $p(z; \mu) \in \mathscr{C}^K(U)[z, z^{-1}]$, where U is some open set of $\mathbb{R}^N$, then the function $(s, \alpha, \beta, \mu) \mapsto s^\beta p(\omega(s; \alpha); \mu)$ belongs to $\mathcal{F}^K_L(\{(\alpha, \beta, \mu) \in \mathbb{R}^2 \times U \,;\, \alpha = 0, \beta > L\})$.*

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