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PERIODIC ORBITS BIFURCATING FROM A HOPF EQUILIBRIUM OF 2–DIMENSIONAL POLYNOMIAL KOLMOGOROV SYSTEMS OF ARBITRARY DEGREE

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Abstract. A Hopf equilibrium of a differential system in $\mathbb{R}^2$ is an equilibrium point whose linear part has eigenvalues $\pm\omega i$ with $\omega \neq 0$. We provide necessary and sufficient conditions for the existence of a limit cycle bifurcating from a Hopf equilibrium of 2–dimensional polynomial Kolmogorov systems of arbitrary degree. We provide an estimation of the bifurcating small limit cycle and also characterize the stability of this limit cycle.

1. Introduction and statements of the main results

A polynomial differential system

$$\dot{x} = \frac{dx}{dt} = P(x,y), \qquad \dot{y} = \frac{dy}{dt} = Q(x,y),$$

in $\mathbb{R}^2$ has degree n if the maximum of the degrees of the polynomials P and Q is n. A quadratic polynomial vector field $X = (P,Q)$ with x a factor of P and y a factor of Q is a *Lotka–Volterra system*. While an n–degree polynomial vector field $X = (P,Q)$ with x a factor of P and y a factor of Q is a *Kolmogorov system*.

Lotka–Volterra systems were initially considered independently by Alfred J. Lotka in 1925 [15] and by Vito Volterra in 1926 [20], as a model for studying the interactions between two species. Later on Kolmogorov [11] in 1936 extended these systems to arbitrary dimension and arbitrary degree, these kinds of systems are now called Kolmogorov systems.

Many natural phenomena can be modeled by the Kolmogorov systems such as the time evolution of conflicting species in biology [16], chemical reactions [10], hydrodynamics [5], economics [18], the coupling of waves in laser physics [12], the evolution of electrons, ions and neutral species in plasma physics [13], etc.

Here we study the polynomial Kolmogorov systems in the plane, i.e. differential systems of the form

$$\dot{x} = xf(x,y), \quad \dot{y} = yg(x,y), \tag{1}$$

where f and g are polynomials of degree larger than 1. In fact we are interested in the existence of limit cycles of Kolmogorov systems living in the positive quadrant of the plane, and consequently surrounding some equilibrium points (see for instance Theorem 1.31 of [8]) which are in the positive quadrant.

We recall that a *limit cycle* of the Kolmogorov system (1) is a periodic solution of system (1) isolated in the set of all periodic solutions of (1). In general to detect the existence of limit cycles is a difficult problem.

A *Hopf equilibrium* of a differential system in $\mathbb{R}^2$ is an equilibrium point whose linear part has eigenvalues $\pm\omega i$ with $\omega \neq 0$. Here a *Hopf bifurcation* means that some limit cycles bifurcate from a Hopf equilibrium when some parameter of the differential system varies, but in the literature not always a Hopf bifurcation has this meaning, see for instance [1].

Our objective is to provide necessary and sufficient conditions for the existence of a limit cycle bifurcating from a Hopf equilibrium of 2–dimensional polynomial Kolmogorov systems of arbitrary degree, using the averaging theory of second order. We also provide the stability of the small limit cycle which is born in the Hopf bifurcation, and an estimation of its size in function of the bifurcation parameter.

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We translate an equilibrium point (a, b) of the positive quadrant to the point $(1, 1)$ doing the rescaling $(x, y) \to (x/a, y/b)$, and we obtain that any Kolmogorov system in the plane of degree 2, i.e. any Lotka–Volterra system can be written as

$$\dot{x} = x(a_1(x-1) + a_2(y-1)), \quad \dot{y} = y(b_1(x-1) + b_2(y-1)). \tag{2}$$

This is a quadratic polynomial differential system with two invariant straight lines and by a Bautin's result [2] we know that such differential systems have no limit cycles (see also [7]).

A Kolmogorov system having an equilibrium in the positive quadrant can be written as

$$\begin{aligned} \dot{x} &= x\big(a_1(x-1) + a_2(y-1) + a_3(x-1)^2 + a_4(x-1)(y-1) + a_5(y-1)^2\big), \\ \dot{y} &= y\big(b_1(x-1) + b_2(y-1) + b_3(x-1)^2 + b_4(x-1)(y-1) + b_5(y-1)^2\big), \end{aligned} \tag{3}$$

if its degree is 3, and it can be written as

$$\begin{aligned} \dot{x} &= x\big(a_1(x-1) + a_2(y-1) + a_3(x-1)^2 + a_4(x-1)(y-1) + a_5(y-1)^2 + \\ &\quad a_6(x-1)^3 + a_7(x-1)^2(y-1) + a_8(x-1)(y-1)^2 + a_9(y-1)^3 + h.o.t.\big), \\ \dot{y} &= y\big(b_1(x-1) + b_2(y-1) + b_3(x-1)^2 + b_4(x-1)(y-1) + b_5(y-1)^2 + \\ &\quad b_6(x-1)^3 + b_7(x-1)^2(y-1) + b_8(x-1)(y-1)^2 + b_9(y-1)^3 + h.o.t.\big), \end{aligned} \tag{4}$$

if its degree is larger than 3, where h.o.t. denotes higher order terms in $x-1$ and $y-1$, i.e. terms of order at least four in the variables $x-1$ and $y-1$.

Proposition 1. *Let ω be a positive real number and $b_1 \neq 0$. A Kolmogorov system* (3) *or* (4) *has a Hopf equilibrium at* $(1, 1)$ *with eigenvalues $\pm\omega i$ if and only if*

$$a_1 = -b_2, \; a_2 = -\frac{b_2^2 + \omega^2}{b_1}.$$

Proposition 1 is proved in section 3.

We define the quantities $A = -2a_3^2b_1b_2^3 - 2a_3^2b_1b_2\omega^2 + 3a_3a_4b_1^2b_2^2 + a_3a_4b_1^2\omega^2 - 2a_3a_5b_1^3b_2 + a_3b_1^2b_2^3 + a_3b_1^2b_2\omega^2 + a_3b_1b_2^4 + a_3b_1b_2^3b_4 + 3a_3b_1b_2^2\omega^2 + a_3b_1b_2b_4\omega^2 + 2a_3b_1\omega^4 - 2a_3b_2^4b_3 - 4a_3b_2^2b_3\omega^2 - 2a_3b_3\omega^4 - a_4^2b_1^3b_2 + a_4a_5b_1^4 - a_4b_1^3b_2^2 - a_4b_1^3b_2b_5 - a_4b_1^2b_2^3 - a_4b_1^2b_2\omega^2 + a_4b_1b_2^3b_3 + a_4b_1b_2b_3\omega^2 + a_5b_1^4b_2 + 2a_5b_1^4b_5 + a_5b_1^3b_2^2 - a_5b_1^3b_2b_4 + b_1^3b_2^2b_5 + 2b_1^3b_2b_5^2 + 2b_1^3b_5\omega^2 - b_1^2b_2^3b_4 + b_1^2b_2^3b_5 - 3b_1^2b_2^2b_4b_5 - b_1^2b_2b_4\omega^2 + b_1^2b_2b_5\omega^2 - b_1^2b_4b_5\omega^2 + b_1b_2^4b_3 - b_1b_2^4b_4 + 2b_1b_2^3b_3b_5 + b_1b_2^3b_4^2 + b_1b_2^2b_3\omega^2 - b_1b_2^2b_4\omega^2 + 2b_1b_2b_3b_5\omega^2 + b_1b_2b_4^2\omega^2 + b_2^5b_3 - b_2^4b_3b_4 + 2b_2^3b_3\omega^2 - 2b_2^2b_3b_4\omega^2 + b_2b_3\omega^4 - b_3b_4\omega^4$, and

$$C = A + b_1\omega^2\left(3a_6b_2^2 + 3a_6\omega^2 - 2\text{a}7b_1b_2 + a_8b_1^2 + 3b_1^2b_9 - 2b_1b_2b_8 + b_2^2b_7 + b_7\omega^2\right).$$

Theorem 2. *Consider the Kolmogorov systems* (3) *or* (4) *with*

$$a_1 = -b_2 + \varepsilon^2 B, \; a_2 = -\frac{b_2^2 + \omega^2}{b_1}, \tag{5}$$

with $\omega > 0$, $b_1 \neq 0$ and ε a small parameter (the bifurcation parameter).

(a) *System* (3) *has one limit cycle $(x(t, \varepsilon), y(t, \varepsilon))$ bifurcating from the Hopf equilibrium* $(1, 1)$ *when $\varepsilon = 0$ if and only if $b_1AB < 0$. This limit cycle is stable if $B > 0$ and unstable if $B < 0$. Moreover*

$$(x(0, \varepsilon), y(0, \varepsilon)) = \left(1 + O(\varepsilon^2), 1 - \varepsilon\frac{b_1\omega r^*}{b_2^2 + \omega^2} + O(\varepsilon^2)\right), \tag{6}$$

where

$$r^* = 2\omega\sqrt{-\frac{b_1B(b_2^2 + \omega^2)}{A}}.$$

(b) *System* (4) *has one limit cycle $(x(t, \varepsilon), y(t, \varepsilon))$ bifurcating from the Hopf equilibrium* $(1, 1)$ *when $\varepsilon = 0$ if and only if $b_1CB < 0$. This limit cycle is stable if $B > 0$ and unstable if $B < 0$. Moreover the equality* (6) *holds with*

$$r^* = 2\omega\sqrt{-\frac{b_1B(b_2^2 + \omega^2)}{C}}.$$

Theorem 2 is proved in section 3 using the averaging theory of second order. A brief summary of the averaging theory that we need for proving Theorem 2 is stated in section 2.

Note that Theorem 2 provides sufficient conditions in order that one limit cycle bifurcates from a Hopf equilibrium point localized in the positive quadrant of a planar Kolmogorov system of arbitrary degree, and that (6) gives an estimation of the size of the small limit cycle which bifurcates from the equilibrium $(1,1)$ in function of the bifurcation parameter ε.

2. The averaging theory for periodic orbits

The method of averaging started with the classical works of Lagrange and Laplace who provided an intuitive justification of this theory.

The first formalization of the averaging theory is due to Fatou [9] in 1928. Important contributions to this theory were made by Krylov and Bogoliubov [4] in the 1930s and Bogoliubov [3] in 1945. The averaging theory of first order for studying periodic orbits can be found in [19].

Now we shall present the basic results from averaging theory that we need for proving the results of this paper.

The next theorem provides a first and the second order approximation for the periodic solutions of a periodic differential system, for the proof see Theorems 11.5 and 11.6 of Verhulst [19], and Buica and Llibre [6].

Consider the differential equation

$$\dot{x}(t) = \varepsilon F_1(t,x) + \varepsilon^2 F_2(t,x) + \varepsilon^3 R(t,x,\varepsilon), \tag{7}$$

where $F_1, F_2 : \mathbb{R} \times D \to \mathbb{R}^n, R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ are continuous functions, T-periodic in the first variable, and D is an open subset of $\mathbb{R}^n$. Assume that the following hypotheses (i) and (ii) hold. Assume:

(i) F_1, F_2, R are locally Lipschitz with respect to x, $F_1(t,.) \in C^1(D)$ for all $t \in \mathbb{R}$, and R is differentiable with respect to ε. We define the averaging functions of first and second order f_1, f_2: $D \longrightarrow \mathbb{R}^n$ as

$$\begin{aligned} f_1(z) = {} & \frac{1}{T}\int_0^T F_1(s,z)ds, \\ f_2(z) = {} & \frac{1}{T}\int_0^T \left[D_z F_1(s,z) \int_0^s F_1(t,z)dt + F_2(s,z) \right] ds. \end{aligned} \tag{8}$$

(ii) For $V \subset D$ an open and bounded set and for each $\varepsilon \in (-\varepsilon_f, \varepsilon_f) \setminus \{0\}$, there exists $a \in V$ such that $f_1(a) + \varepsilon f_2(a) = 0$ and $d_B(f_1 + \varepsilon f_2, V, a) \neq 0$.

Then for $|\varepsilon| > 0$ sufficiently small there exists a T-periodic solution $x(t,\varepsilon)$ of the system (7) such that $x(0,\varepsilon) \to a$ when $\varepsilon \to 0$.

Here $d_B(f_1 + \varepsilon f_2, V, 0)$ denotes the Brouwer degree of the function $f_1 + \varepsilon f_2$ in the neighborhood V of zero. It is known that if the function $f_1 + \varepsilon f_2$ is C^1 then it is sufficient to check that the determinant of the Jacobian matrix $D(f_1 + \varepsilon f_2(a_\varepsilon))$ is non-zero in order to have that $d_B(f_1 + \varepsilon f_2, V, 0) \neq 0$, for more details see [14].

On the other hand if one of the real parts of the eigenvalues of the Jacobian matrix $D(f_1 + \varepsilon f_2)(a_\varepsilon)$ is positive the periodic solution $x(t;\varepsilon)$ is unstable. If all the real parts of the eigenvalues of this matrix are negative the periodic solution is locally stable. For a proof see Theorem 11.6 of [19].

For a general information on the averaging theory see for instance the books [17, 19].

3. Proofs

Proof of Proposition 1. The characteristic polynomial of the linear part of the Lotka-Volterra system (3) at the equilibrium point $(1,1)$ is

$$p(\lambda) \quad = \quad \lambda^2 - (a_1 + b_2)\lambda + a_1 b_2 - a_2 b_1.$$

Imposing that $p(\lambda) = \lambda^2 + \omega^2$ in order to have a Hopf equilibrium, we obtain the system

$$a_1 + b_2 = 0, \quad a_1 b_2 - a_2 b_1 = \omega^2.$$

Solving this system we get the family of zero–Hopf equilibria described in Proposition 1. This completes the proof. ∎

Proof of Theorem 2. We shall prove that a periodic orbit bifurcates from the zero–Hopf equilibrium point $(1,1)$ of the Kolmogorov system (3) for the parameters of system (3) given in the statement of Proposition 1.

We perturb system (3) with the parameters given in (5). We translate the equilibrium point $(1,1)$ to the origin of coordinates doing the change of variables $x = X+1, y = Y+1$. Then system (3) becomes

$$\begin{aligned} \dot{X} &= \frac{1}{b_1}(1+X)\left((B\varepsilon^2 - b_2)b_1X - (b_2^2+\omega^2)Y + a_3b_1X^2 + a_4b_1XY + a_5b_1Y^2\right), \\ \dot{Y} &= (1+Y)\left(b_1X + b_2Y + b_3X^2 + b_4XY + b_5Y^2\right). \end{aligned} \tag{9}$$

In order to simplify the application of the averaging theory, for computing the Hopf bifurcation we write the linear part of system (9) with $\varepsilon = 0$ at the equilibrium point $(0,0)$ into its real Jordan normal form, i.e. into the form

$$\begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}.$$

Then doing the change of variables

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\dfrac{b_2}{\omega} & -\dfrac{b_2^2+\omega^2}{b_1\omega} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix},$$

whose inverse is

$$\begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{b_1\omega}{b_2^2+\omega^2} & -\dfrac{b_1b_2}{b_2^2+\omega^2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

The differential system (9) in the variables (u,v) writes

$$\begin{aligned} \dot{u} = &-\frac{1}{b_1\omega(b_2^2+\omega^2)^2}\Big((a_5b_1^3b_2^3 - a_4b_1^2b_2^4 + a_3b_1b_2^5 + b_2^6b_3 - b_1b_2^5b_4 + b_1^2b_2^4b_5)v^2 + (a_5b_1^3b_2^3 \\ &-a_4b_1^2b_2^4 + a_3b_1b_2^5 - b_1b_2^5b_3 + b_1^2b_2^4b_4 - b_1^3b_2^3b_5)v^3 + (2a_5b_1^3b_2^2 - a_4b_1^2b_2^3 + b_1^2b_2^4 + b_1b_2^5 \\ &-b_1b_2^4b_4 + 2b_1^2b_2^3b_5)uv\omega + (2a_5b_1^3b_2^2 - a_4b_1^2b_2^3 - b_1b_2^4b_3 + 2b_1^2b_2^3b_4 - 3b_1^3b_2^2b_5)uv^2\omega + \\ &(a_5b_1^3b_2 + b_1^2b_2^3 + b_1^2b_2^2b_5)u^2\omega^2 + b_1b_2^4v\omega^2 + (a_5b_1^3b_2 + b_1^2b_2^2b_4 - 3b_1^3b_2b_5)u^2v\omega^2 + \\ &(-a_4b_1^2b_2^2 + 2a_3b_1b_2^3 - b_1^2b_2^3 + 3b_2^4b_3 - 2b_1b_2^3b_4 + b_1^2b_2^2b_5)v^2\omega^2 + (-a_4b_1^2b_2^2 + 2a_3b_1b_2^3 \\ &-2b_1b_2^3b_3 + b_1^2b_2^2b_4)v^3\omega^2 - b_1^3b_5u^3\omega^3 + (-a_4b_1^2b_2 + 2b_1b_2^3 - 2b_1b_2^2b_4 + 2b_1^2b_2b_5)uv\omega^3 \\ &+(2b_1^2b_2b_4 - a_4b_1^2b_2 - 2b_1b_2^2b_3)uv^2\omega^3 + (b_1^2b_2 + b_1^2b_5)u^2\omega^4 + 2b_1b_2^2v\omega^4 + b_1^2b_4u^2v\omega^4 \\ &+(a_3b_1b_2 - b_1^2b_2 + 3b_2^2b_3 - b_1b_2b_4)v^2\omega^4 + (a_3b_1b_2 - b_1b_2b_3)v^3\omega^4 + (b_1b_2 - b_1^2 - \\ &b_1b_4)uv\omega^5 - b_1b_3uv^2\omega^5 + b_1v\omega^6 + b_3v^2\omega^6\Big) - \varepsilon^2\frac{1}{\omega}Bb_2v(1+v), \\ \dot{v} = &\frac{(1+v)}{(b_2^2+\omega^2)^2}\Big((a_5b_1^2b_2^2 - a_4b_1b_2^3 + a_3b_2^4)v^2 + b_2^4u\omega + (2a_5b_1^2b_2 - a_4b_1b_2^2)uv\omega + \omega^2 \\ &a_5b_1^2u^2 + (-a_4b_1b_2 + 2a_3b_2^2)v^2\omega^2 + 2b_2^2u\omega^3 - a_4b_1uv\omega^3 + a_3v^2\omega^4 + u\omega^5\Big) \\ &+\varepsilon^2Bv(1+v). \end{aligned} \tag{10}$$

Doing the rescaling of the variables $(u,v) = (\varepsilon U, \varepsilon V)$ system (10) in the new variables (U,V) writes

$$
\begin{aligned}
\dot{U} = \ & -V\omega - \varepsilon \frac{1}{b_1\omega(b_2^2+\omega^2)^2}\Big((a_5 b_1^3 b_2^3 - a_4 b_1^2 b_2^4 + a_3 b_1 b_2^5 + b_2^6 b_3 - b_1 b_2^5 b_4 + b_1^2 b_2^4 b_5)V^2 \\
& +(2a_5 b_1^3 b_2^2 - a_4 b_1^2 b_2^3 + b_1^2 b_2^4 + b_1 b_2^5 - b_1 b_2^4 b_4 + 2b_1^2 b_2^3 b_5)UV\omega + (a_5 b_1^3 b_2 + b_1^2 b_2^3 + \\
& b_1^2 b_2^2 b_5)U^2\omega^2 + (-a_4 b_1^2 b_2^2 + 2a_3 b_1 b_2^3 - b_1^2 b_2^3 + 3b_2^4 b_3 - 2b_1 b_2^3 b_4 + b_1^2 b_2^2 b_5)V^2\omega^2 + \\
& (-a_4 b_1^2 b_2 + 2b_1 b_2^3 - 2b_1 b_2^2 b_4 + 2b_1^2 b_2 b_5)UV\omega^3 + (b_1^2 b_2 + b_1^2 b_5)U^2\omega^4 + (a_3 b_1 b_2 - \\
& b_1^2 b_2 + 3b_2^2 b_3 - b_1 b_2 b_4)V^2\omega^4 + (-b_1^2 + b_1 b_2 - b_1 b_4)UV\omega^5 + b_3 V^2\omega^6\Big) + \varepsilon^2 \frac{1}{\omega(b_2^2+\omega^2)^2} \\
& \Big(-Bb_2^5 V + (-a_5 b_1^2 b_2^3 + a_4 b_1 b_2^4 - a_3 b_2^5 + b_2^5 b_3 - b_1 b_2^4 b_4 + b_1^2 b_2^3 b_5)V^3 - (2a_5 b_1^2 b_2^2 \\
& -a_4 b_1 b_2^3 - b_2^4 b_3 + 2b_1 b_2^3 b_4 - 3b_1^2 b_2^2 b_5)UV^2\omega - 2Bb_2^3 V\omega^2 - (a_5 b_1^2 b_2 + b_1 b_2^2 b_4 - \\
& 3b_1^2 b_2 b_5)U^2 V\omega^2 + (a_4 b_1 b_2^2 - 2a_3 b_2^3 + 2b_2^3 b_3 - b_1 b_2^2 b_4)V^3\omega^2 + b_1^2 b_5 U^3\omega^3 + (a_4 b_1 b_2 \\
& +2b_2^2 b_3 - 2b_1 b_2 b_4)UV^2\omega^3 - Bb_2 V\omega^4 - b_1 b_4 U^2 V\omega^4 + (b_2 b_3 - a_3 b_2)V^3\omega^4 + b_3 UV^2\omega^5\Big) \\
& +O(\varepsilon^3), \\
\dot{V} = \ & U\omega + \varepsilon \frac{1}{(b_2^2+\omega^2)^2}\Big((a_5 b_1^2 b_2^2 - a_4 b_1 b_2^3 + a_3 b_2^4)V^2 + (2a_5 b_1^2 b_2 - a_4 b_1 b_2^2 + b_2^4)UV\omega \\
& +a_5 b_1^2 U^2\omega^2 + (-a_4 b_1 b_2 + 2a_3 b_2^2)V^2\omega^2 + (-a_4 b_1 + 2b_2^2)UV\omega^3 + a_3 V^2\omega^4 + UV\omega^5\Big) \\
& +\varepsilon^2 \frac{1}{(b_2^2+\omega^2)^2} V\Big(Bb_2^4 + (a_5 b_1^2 b_2^2 - a_4 b_1 b_2^3 + a_3 b_2^4)V^2 + (2a_5 b_1^2 b_2 - a_4 b_1 b_2^2)UV\omega \\
& +2Bb_2^2\omega^2 + a_5 b_1^2 U^2\omega^2 + (-a_4 b_1 b_2 + 2a_3 b_2^2)V^2\omega^2 - a_4 b_1 UV\omega^3 + B\omega^4 + a_3 V^2\omega^4) + O(\varepsilon^3).
\end{aligned}
\tag{11}
$$

We must pass to polar coordinates for introducing the angle θ which later on we shall take as the new independent variable in order that the differential system (11) becomes periodic in θ and we can apply the averaging theory, where (r,θ) are defined by $U = r\cos\theta$ and $V = r\sin\theta$, and we obtain

$$
\begin{aligned}
\dot{r} = \ & -\varepsilon \frac{1}{b_1\omega(b_2^2+\omega^2)^2} r^2 \Big(b_1^2\omega^2(a_5 b_1 b_2 + (b_2+b_5)(b_2^2+\omega^2))\cos^3\theta - b_1\omega(a_5 b_1^2(-2b_2^2+\omega^2) \\
& +(b_2^2+\omega^2)(a_4 b_1 b2 - (b_2 - b_4)(b_2^2+\omega^2) + b_1(-b_2^2 - 2b_2 b_5 + \omega^2)))\cos^2\theta\sin(\theta) + (a_5 b_1^3 b_2 \\
& (b_2^2 - 2\omega^2) + (b_2^2+\omega^2)(b_2^4 b_3 - b_1 b_2^3 b_4 + b_1^2 b_2^2 b_5 - b_1^2 b_2\omega^2 - b_1 b_2^2\omega^2 + 2b_2^2 b_3\omega^2 - b_1 b_2 b_4\omega^2 \\
& -b_1\omega^4 + b_3\omega^4 + a_4 b_1^2(-b_2^2+\omega^2) + a_3 b_1 b_2(b_2^2+\omega^2)))\cos\theta\sin\theta^2 - b_1\omega(a_5 b_1^2 b_2^2 + \\
& (b_2^2+\omega^2)(-a_4 b_1 b_2 + a_3(b_2^2+\omega^2)))\sin^3\theta\Big) + \varepsilon^2 \frac{1}{\omega(b_2^2+\omega^2)^2}\Big(b_1^2 b_5 r^2\omega^3\cos(\theta)^4 - b_1 r^2 \\
& \omega^2(a_5 b_1 b_2 + b_4 b_2^2 - 3b_1 b_2 b_5 + b_4\omega^2)\cos^3\theta\sin\theta + r^2\omega(b_2^4 b_3 - 2b_1 b_2^3 b_4 + 3b_1^2 b_2^2 b_5 + 2b_2^2 \\
& b_3\omega^2 - 2b_1 b_2 b_4\omega^2 + b_3\omega^4 + a_5 b_1^2(-2b_2^2+\omega^2) + a_4 b_1 b_2(b_2^2+\omega^2))\cos^2\theta\sin^2\theta + \omega\sin^2\theta \\
& (B(b_2^2+\omega^2)^2 + r^2(a_5 b_1^2 b_2^2 + (b_2^2+\omega^2)(-a_4 b_1 b_2 + a_3(b_2^2+\omega^2)))\sin^2\theta) - \cos\theta\sin\theta(Bb_2 \\
& (b_2^2+\omega^2)^2 + r^2(a_5 b_1^2 b_2(b_2^2 - 2\omega^2) + a_4 b_1(-b_2^4+\omega^4) + b_2(-b_2^4 b_3 + b_1 b_2^3 b_4 + b_1 b_2 b_4\omega^2 - b_3\omega^4 \\
& +a_3(b_2^2+\omega^2)^2 - b_2^2(b_1^2 b_5 + 2b_3\omega^2)))\sin^2\theta)\Big) + O(\varepsilon^3), \\
\dot{\theta} = \ & \omega + \varepsilon \frac{1}{b_1\omega(b_2^2+\omega^2)^2} r\Big(a_5 b_1^3\omega^3\cos^3\theta + b_1\omega^2(3a_5 b_1^2 b_2 - (a_4 b_1 - b_1 b_2 - b_2^2 - b_1 b_5 - \omega^2)(b_2^2 + \\
& \omega^2))\cos^2\theta\sin\theta + b_1\omega(3a_5 b_1^2 b_2^2 + (b_2^2+\omega^2)(-2a_4 b_1 b_2 + b_1 b_2^2 + b_2^3 - b_2^2 b_4 + 2b_1 b_2 b_5 - b_1\omega^2 \\
& +b_2\omega^2 - b_4\omega^2 + a_3(b_2^2+\omega^2)))\cos\theta\sin^2\theta + (a_5 b_1^3 b_2^3 + (b_2^2+\omega^2)(-a_4 b_1^2 b_2^2 + b_2^4 b_3 - b_1 b_2^3 b_4 \\
& +b_1^2 b_2^2 b_5 - b_1^2 b_2\omega^2 + 2b_2^2 b_3\omega^2 - b_1 b_2 b_4\omega^2 + b_3\omega^4 + a_3 b_1 b_2(b_2^2+\omega^2)))\sin^3\theta\Big) + \varepsilon^2 \frac{1}{2\omega(b_2^2+\omega^2)^2} \\
& \sin\theta\Big(\omega\cos\theta + b_2\sin\theta)((b_2^2+\omega^2)(2B(b_2^2+\omega^2) + r^2(a_5 b_1^2 - a_4 b_1 b_2 + a_3 b_2^2 - b_2^2 b_3 + b_1 b_2 b_4 \\
& -b_1^2 b_5 + a_3\omega^2 - b_3\omega^2)) - r^2(a_3 b_2^4 - b_2^4 b_3 + b_1 b_2^3 b_4 - b_1^2 b_2^2 b_5 + 2a_3 b_2^2\omega^2 - 2b_2^2 b_3\omega^2 + b_1 b_2 b_4\omega^2 + \\
& b_1^2 b_5\omega^2 + a_3\omega^4 - b_3\omega^4 + a_5 b_1^2(b_2^2 - \omega^2) - a_4 b_1 b_2(b_2^2+\omega^2))\cos(2\theta) + b_1 r^2\omega(2a_5 b_1 b_2 + b_2^2 b_4 - \\
& 2b_1 b_2 b_5 + b_4\omega^2 - a_4(b_2^2+\omega^2))\sin(2\theta)\Big) + O(\varepsilon^3).
\end{aligned}
\tag{12}
$$

We take θ as the new independent variable and system (12) becomes into the normal form for applying the averaging theory

$$
\begin{aligned}
\dot{r} = \ & -\varepsilon\frac{1}{b_1\omega^2(b_2^2+\omega^2)^2}r^2\Big(b_1^2\omega^2(a_5b_1b_2+(b_2+b_5)(b_2^2+\omega^2))\cos^3\theta-b_1\omega(a_5b_1^2(-2b_2^2+\omega^2)\\
&+(b_2^2+\omega^2)(a_4b_1b2-(b_2-b_4)(b_2^2+\omega^2)+b_1(-b_2^2-2b_2b_5+\omega^2)))\cos^2\theta\sin\theta+(a_5b_1^3b_2\\
&(b_2^2-2\omega^2)+(b_2^2+\omega^2)(b_2^4b_3-b_1b_2^3b_4+b_1^2b_2^2b_5-b_1^2b_2\omega^2-b_1b_2^2\omega^2+2b_2^2b_3\omega^2-b_1b_2b_4\omega^2\\
&-b_1\omega^4+b_3\omega^4+a_4b_1^2(-b_2^2+\omega^2)+a_3b_1b_2(b_2^2+\omega^2)))\cos\theta\sin^2\theta-b_1\omega(a_5b_1^2b_2^2+(b_2^2+\\
&\omega^2)(-a_4b_1b_2+a_3(b_2^2+\omega^2)))\sin^3\theta\Big)+\varepsilon^2\Big(\frac{1}{b_1^2\omega^4(b_2^2+\omega^2)^4)}r^3\big(b_1^2\omega^2(a_5b_1b_2+(b_2+b_5)\\
&(b_2^2+\omega^2))\cos^3\theta-b_1\omega(a_5b_1^2(-2b_2^2+\omega^2)+(b_2^2+\omega^2)(a_4b_1b_2-(b_2-b_4)(b_2^2+\omega^2)+b_1\\
&(-b_2^2-2b_2b_5+\omega^2)))\cos^2\theta\sin\theta+(a_5b_1^3b_2(b_2^2-2\omega^2)+(b_2^2+\omega^2)(b_2^4b_3-b_1b_2^3b_4+b_1^2b_2^2b_5\\
&-b_1^2b_2\omega^2-b_1b_2^2\omega^2+2b_2^2b_3\omega^2-b_1b_2b_4\omega^2-b_1\omega^4+b_3\omega^4+a_4b_1^2(-b_2^2+\omega^2)+a_3b_1b_2(b_2^2+\\
&\omega^2)))\cos\theta\sin^2\theta-b_1\omega(a_5b_1^2b_2^2+(b_2^2+\omega^2)(-a_4b_1b_2+a_3(b_2^2+\omega^2)))\sin^3\theta)(a_5b_1^3\omega^3\\
&\cos^3\theta+b_1\omega^2(3a_5b_1^2b_2-(a_4b_1-b_1b_2-b_2^2-b_1b_5-\omega^2)(b_2^2+\omega^2))\cos^2\theta\sin\theta+b_1\omega\\
&(3a_5b_1^2b_2^2+(b_2^2+\omega^2)(-2a_4b_1b_2+b_1b_2^2+b_2^3-b_2^2b_4+2b_1b_2b_5-b_1\omega^2+b_2\omega^2-b_4\omega^2+a_3\\
&(b_2^2+\omega^2)))\cos\theta\sin^2\theta+(a_5b_1^3b_2^3+(b_2^2+\omega^2)(-a_4b_1^2b_2^2+b_2^4b_3-b_1b_2^3b_4+b_1^2b_2^2b_5-b_1^2b_2\\
&\omega^2+2b_2^2b_3\omega^2-b_1b_2b_4\omega^2+b_3\omega^4+a_3b_1b_2(b_2^2+\omega^2)))\sin^3\theta\big)+\frac{1}{\omega^2(b_2^2+\omega^2)^2}r\big(b_1^2b_5r^2\omega^3\\
&\cos^4\theta-b_1r^2\omega^2(a_5b_1b_2+b_2^2b_4-3b_1b_2b_5+b_4\omega^2)\cos^3\theta\sin\theta+r^2\omega(b_2^4b_3-2b_1b_2^3b_4+\\
&3b_1^2b_2^2b_5+2b_2^2b_3\omega^2-2b_1b_2b_4\omega^2+b_3\omega^4+a_5b_1^2(-2b_2^2+\omega^2)+a_4b_1b_2(b_2^2+\omega^2))\cos^2\theta\sin^2\theta\\
&+\omega\sin^2\theta(B(b_2^2+\omega^2)^2+r^2(a_5b_1^2b_2^2+(b_2^2+\omega^2)(-a_4b_1b_2+a_3(b_2^2+\omega^2)))\sin^2\theta)-\cos\theta\\
&\sin\theta(Bb_2(b_2^2+\omega^2)^2+r^2(a_5b_1^2b_2(b_2^2-2\omega^2)+a_4b_1(-b_2^4+\omega^4)+b_2(-b_2^4b_3+b_1b_2^3b_4+b_1b_2b_4\\
&\omega^2-b_3\omega^4+a_3(b_2^2+\omega^2)^2-b_2^2(b_1^2b_5+2b_3\omega^2)))\sin^2\theta)^2)\Big)+O(\varepsilon^3).
\end{aligned}
\tag{13}
$$

Here the prime denotes derivative with respect to θ. We shall apply the averaging theory described in section 2 to the differential system (13). Using the notation of section 2 we have $t=\theta$, $T=2\pi$, $\mathbf{x}=r$. It is immediate to check that system (13) satisfies all the assumptions for applying the averaging theory. Now we compute the integrals (8), and we obtain

$$
\begin{aligned}
f_1(r) = \ & 0,\\
f_2(r) = \ & \frac{1}{8b_1\omega^3(b_2^2+\omega^2)}r(4b_1B\omega^2(b_2^2+\omega^2)+Ar^2).
\end{aligned}
$$

The system $f_2(r)=0$ has a unique positive solution

$$
r^*=2\omega\sqrt{-\frac{b_1B(b_2^2+\omega^2)}{A}}.
$$

The derivative of $f_2(r)$ at r^* is $Ar^*/(4b_1\omega^3(b_2^2+\omega^2)$. Therefore from section 2 we obtain that the small limit cycle is stable if $B>0$, and unstable if $B<0$. Going back through the changes of variables we obtain the expression (6). This completes the proof of statement (a) of the theorem.

By following for system (4) the same steps done for the system (3) we find that the averaged functions f_1 and f_2 are

$$
\begin{aligned}
f_1(r) = \ & 0,\\
f_2(r) = \ & \frac{1}{8b_1\omega^3(b_2^2+\omega^2)}r(4b_1B\omega^2(b_2^2+\omega^2)+Cr^2).
\end{aligned}
$$

The equation $f_2(r)=0$ has a unique positive solution

$$
r^*=2\omega\sqrt{-\frac{b_1B(b_2^2+\omega^2)}{C}}.
$$

Since the derivative of $f_2(r)$ at r^* is $Cr^*/(4b_1\omega^3(b_2^2+\omega^2)$, the statement (b) follows in a similar way to statement (a). □

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