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DYNAMICS OF THE FITZHUGH-NAGUMO SYSTEM HAVING INVARIANT ALGEBRAIC SURFACES

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ABSTRACT. In this paper we study the dynamics of the FitzHugh-Nagumo system $\dot{x} = z,\ \dot{y} = b\,(x - dy)\,,\ \dot{z} = x\,(x-1)\,(x-a) + y + cz$ having invariant algebraic surfaces. This system has four different types of invariant algebraic surfaces. The dynamics of the FitzHugh-Nagumo system having two of these classes of invariant algebraic surfaces have been characterized in [21]. Using the quasi-homogeneous directional blow up and the Poincaré compactification, we describe the dynamics of the FitzHugh-Nagumo system having the two remaining classes of invariant algebraic surfaces. Moreover for these FitzHugh-Nagumo systems we prove that they do not have limit cycles.

1. INTRODUCTION

The FitzHugh-Nagumo system is given by the partical differential system

$$u_t = u_{xx} - f\,(u) - v, \quad v_t = \varepsilon\,(u - \gamma v)\,, \tag{1}$$

where $f\,(u) = u\,(u-1)\,(u-a)$, and $0 < a < 1/2$, $\varepsilon > 0$, $\gamma > 0$ are parameters. We say that a bounded solution $(u, v)\,(x, t)$ of the FitzHugh-Nagumo system (1) with $x, t \in \mathbb{R}$ is a *travelling wave* if $(u, v)\,(x, t) = (u, v)\,(\xi)$, where $\xi = x + ct$ and c is the constant denoting the wave speed. Substituting $u = u\,(\xi)\,, v = v\,(\xi)$ into (1), we obtain the ordinary differential system

$$\begin{aligned} \dot{x} &= z = P\,(x, y, z)\,, \\ \dot{y} &= b\,(x - dy) = Q\,(x, y, z)\,, \\ \dot{z} &= x\,(x-1)\,(x-a) + y + cz = R\,(x, y, z)\,. \end{aligned} \tag{2}$$

Here the dot denotes derivative with respect to ξ, $x = u$, $y = v$, $z = \dot{u}$, $b = \varepsilon/c$ and $d = \gamma$, see for more details [11].

The FitzHugh-Nagumo system (1) is classical differential system introduced independently by FitzHugh [8] and Nagumo et al. [19]. It is an important model for describing the excitation of neural membranes and the propagation of nerve impulses along an axon. Besides its biological interest, the FitzHugh-Nagumo system has gained wide investigation from the mathematical point of view, such as the existence, uniqueness and stability of its traveling wave solutions, see for instance [2, 9, 12–14, 19], etc.

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In recent years the FitzHugh-Nagumo system (2) has been investigated from the points of view of its dynamics and integrability. The analytical integrability of the FitzHugh-Nagumo system (2) has been studied by Llibre and Valls in [17]. The Liouvillian integrability of the planar FitzHugh-Nagumo system can be found in [18]. A polynomial $f(x,y,z) \in \mathbb{C}[x,y,z]$ is called a *Darboux polynomial* for the FitzHugh-Nagumo system (2) if

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} + R\frac{\partial f}{\partial z} = kf,$$

for some polynomial $k(x,y,z) \in \mathbb{C}[x,y,z]$, which is called the *cofactor* of f. If f is a Darboux polynomial of system (2), then $f = 0$ is an *invariant algebraic surface* of system (2), because if an orbit of system (2) has a point on the surface $f = 0$ the whole orbit is contained in such a surface. This follows easily from the definition of Darboux polynomial.

The following result completes the classification of all the invariant algebraic surfaces of the FitzHugh-Nagumo system, see [22].

Theorem 1. *There are four types of generators for the invariant algebraic surfaces of the FitzHugh-Nagumo system* (2)*, and two polynomial first integrals when* $b = c = 0$*, see Table 1.*

Table 1: The generators of FitzHugh-Nagumo system (2)

Darboux polynomials	Cofactors	Parameters
$f_1 = \frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + \left(\frac{1}{9}c^2 - 1\right)x^2$	$\frac{4}{3}c$	$a = -1, bd = -c, c \neq 0,$ $b = \frac{2}{27}c^3 - \frac{1}{3}c$
$f_2 = \frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz + \left(\frac{1}{9}c^2 - 1\right)x^2 - \frac{1}{2}dy^2$	$\frac{4}{3}c$	$a = -1, bd = -\frac{2}{3}c, c \neq 0,$ $b = \frac{2}{27}c^3 - \frac{1}{3}c$
$f_3 = \frac{1}{2}x^4 - z^2 + 2xy + \frac{2}{3}cxz - \frac{2}{3}(a+1)x^3$ $+ \left(\frac{1}{9}c^2 + a\right)x^2 - \frac{2}{9}c(a+1)z - \frac{2}{3}(a+1)y$ $- \frac{2}{27}c^2(a+1)x$	$\frac{4}{3}c$	$a \neq -1, bd = -c, c \neq 0,$ $b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c,$ $2c^2 + 3a^2 - 12a + 3 = 0$
$f_4 = \frac{1}{2}x^4 - z^2 - \frac{1}{2}dy^2 + 2xy + \frac{2}{3}cxz - \frac{2}{3}(a+1)x^3$ $+ \left(\frac{1}{9}c^2 + a\right)x^2 - \frac{2}{9}c(a+1)z - \frac{1}{3}(a+1)y$ $- \frac{2}{27}c^2(a+1)x$	$\frac{4}{3}c$	$a \neq -1, bd = -\frac{2c}{3}, c \neq 0,$ $b = \frac{2}{27}c^3 - \frac{1}{9}a^2c + \frac{1}{9}ac - \frac{1}{9}c,$ $2c^2 + a^2 - 7a + 1 = 0$
$f_5 = y$	0	$b = c = 0$
$f_6 = \frac{1}{4}x^4 - \frac{1}{2}z^2 - \frac{1}{3}(a+1)x^3 + xy + \frac{1}{2}ax^2$	0	$b = c = 0$

A nonconstant function $H(x,y,z,t) : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{C}$ is called an *invariant* of the FitzHugh-Nagumo system (2) if $H(x,y,z,t)$ is a constant along each trajectory $(x(t), y(t), z(t))$ of system (2). If H is independent of the time t, it is a *first integral.* If H is of the form $f(x,y,z)\exp(-\sigma t)$ with $f \in \mathbb{R}[x,y,z]$ and σ a non-zero constant, then H is called a *Darboux invariant.* Note that $H = f(x,y,z)\exp(-\sigma t)$ is a Darboux invariant if and only if f is a Darboux polynomial with cofactor σ. From Theorem 1 we summarize the results on the integrability and on the existence of Darboux invariants for system (2) in the next theorem, for a proof see [22].

Theorem 2. *The FitzHugh-Nagumo system* (2) *has the Darboux invariants I and the first integrals H given in Table 2.*

Table 2: Darboux invariants and first integrals of system (2)

	Darboux polynomials	Parameters
(a)	$a=-1, bd=-c, c\neq 0,$ $b=\frac{2}{27}c^3-\frac{1}{3}c$	$I=\left(\frac{1}{2}x^4-z^2+2xy+\frac{2}{3}cxz+\left(\frac{1}{9}c^2-1\right)x^2\right)\exp\left(-\frac{4}{3}ct\right)$
(b)	$a=-1, bd=-\frac{2}{3}c, c\neq 0,$ $b=\frac{2}{27}c^3-\frac{1}{3}c$	$I=\left(\frac{1}{2}x^4-z^2+2xy+\frac{2}{3}cxz+\left(\frac{1}{9}c^2-1\right)x^2-\frac{1}{2}dy^2\right)\exp\left(-\frac{4}{3}ct\right)$
(c)	$a\neq -1, bd=-c, c\neq 0,$ $b=\frac{2}{27}c^3-\frac{1}{9}a^2c+\frac{1}{9}ac-\frac{1}{9}c,$ $2c^2+3a^2-12a+3=0$	$I=\left(\frac{1}{2}x^4-z^2+2xy+\frac{2}{3}cxz-\frac{2}{3}(a+1)x^3\right.$ $+\left(\frac{1}{9}c^2+a\right)x^2-\frac{2}{9}c(a+1)z-\frac{2}{3}(a+1)y$ $\left.-\frac{2}{27}c^2(a+1)x\right)\exp\left(-\frac{4}{3}ct\right)$
(d)	$a\neq -1, bd=-\frac{2c}{3}, c\neq 0,$ $b=\frac{2}{27}c^3-\frac{1}{9}a^2c+\frac{1}{9}ac-\frac{1}{9}c,$ $2c^2+a^2-7a+1=0$	$I=\left(\frac{1}{2}x^4-z^2-\frac{1}{2}dy^2+2xy+\frac{2}{3}cxz-\frac{2}{3}(a+1)x^3\right.$ $+\left(\frac{1}{9}c^2+a\right)x^2-\frac{2}{9}c(a+1)z-\frac{1}{3}(a+1)y$ $\left.-\frac{2}{27}c^2(a+1)x\right)\exp\left(-\frac{4}{3}ct\right)$
(e)	$b=c=0$	$H_1=y$ $H_2=\frac{1}{4}x^4-\frac{1}{2}z^2-\frac{1}{3}(a+1)x^3+xy+\frac{1}{2}ax^2$

In this paper we classify the dynamics of FitzHugh-Nagumo system (2) having an invariant algebraic surface. For parameter $c\neq 0$, we can assume $c>0$, because $c<0$ can be reduced to $c>0$ doing the change of variables $(x,y,z,t)\mapsto(x,y,-z,-t)$.

Since the FitzHugh-Nagumo system (2) is a polynomial differential system in $\mathbb{R}^3$ we can compactify it in order to study its dynamics in a neighborhood of the infinity doing the *Poincaré compactification*. Roughly speaking the Poincaré compactification consists in identify $\mathbb{R}^3$ with the interior of the closed ball $\mathbb{B}^3$ centered at the origin of $\mathbb{R}^3$ and radius one, called the *Poincaré ball*. Then the boundary of the ball $\mathbb{B}^3$, i.e. the sphere $\mathbb{S}^2$, is identified with the infinity of $\mathbb{R}^3$, and called the *Poincaré sphere*. In $\mathbb{R}^3$ we can go to infinity in as many directions as points has the sphere $\mathbb{S}^2$. Then the FitzHugh-Nagumo system (2) can be extended analytically to the Poincaré ball $\mathbb{B}^3$, For more details on the Poincaré compactification and the expression of the extended differential systems in the different local charts of $\mathbb{B}^3$, see for instance [5].

The following theorem is the first main result of this paper.

Theorem 3. *For all values of the parameters $a,b,c,d\in\mathbb{R}$ the phase portrait of the FitzHugh-Nagumo system* (2) *on the Poincaré sphere is topologically equivalent to the one of Figure 1. Then system* (2) *has the circle defined by the boundary of the plane $x=0$ at infinity filled up of singular points, and there are no additional singular points at the sphere of infinity in the Poincaré compactification.*

The dynamics of system (2) with the invariant algebraic surfaces $f_1=0$ and $f_2=0$ have been characterized in [21]. We only need to characterize the dynamics of system (2) with the invariant algebraic surfaces $f_3=0$ and $f_4=0$.

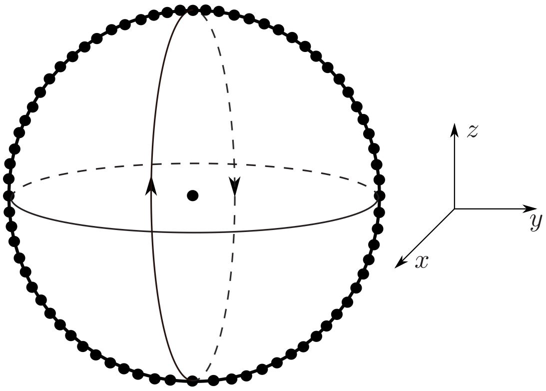


FIGURE 1. Phase portrait at infinity on the Poincaré ball of system (2).

Theorem 4. *For $c > 0$, the FitzHugh-Nagumo system (2) with the invariant algebraic surfaces $f_3 = 0$ and $f_4 = 0$ have the following dynamics.*

(a) The boundary at infinity of the surface $f_3 = 0$ at infinity is the great circle $\{x = 0\} \cap \{x^2 + y^2 + z^2 = 1\}$ filled up with singular points. The dynamics of system (2) on the invariant algebraic surface $f_3 = 0$ are characterized by Figure 4.

(b) The boundary at infinity of the surface $f_4 = 0$ at infinity is the great circle $\{x = 0\} \cap \{x^2 + y^2 + z^2 = 1\}$ filled up with singular points. The dynamics of system (2) on the invariant algebraic surface $f_4 = 0$ are characterized by either Figure 7 or 9.

The paper is organized as follows. We present some preliminary results in section 2. The proofs of Theorems 3 and 4 will be given in sections 3 and 4, respectively.

2. Preliminary results

In this section we recall some results that we shall need for proving our theorems.

Lemma 5. *Let $f(x, y, z) = 0$ be an invariant algebraic surface of the differential system (2) with the constant cofactor $k \neq 0$, and let $I = f(x, y, z)e^{-kt}$ be its associated Darboux invariant. Assume that $\phi(t) = (x(t), y(t), z(t))$ is a solution of system (2) not contained in the invariant surface $f(x, y, z) = 0$.*

(a) If $k > 0$ the ω-limit of $\phi(t)$ inside the Poincaré ball $\mathbb{B}^3$ is contained in the sphere of Poincaré $\mathbb{S}^2$ intersection with the closure of the surface $f(x, y, z) = 0$, and the α-limit of $\phi(t)$ inside the Poincaré ball $\mathbb{B}^3$ is contained in the closure of the surface $f(x, y, z) = 0$.

(b) If $k < 0$ then statement (a) holds interchanging the ω-limit by the α-limit.

The proof of Lemma 5 is given in Proposition 5 of [16]. In fact there it is proved for polynomial differential systems in $\mathbb{R}^2$, but that proof works for polynomial differential systems in $\mathbb{R}^n$. To describe the dynamics of the FitzHugh-Nagumo system (2) having an invariant algebraic surface, by Lemma 5, we only need to investigate it on the invariant surface.

Lemma 6. *Let $f(x,y,z) = 0$ be an algebraic invariant surface of degree m in $\mathbb{R}^3$. The extension of this surface to the boundary of the Poincaré ball is given by the equations*

$$w^m f\left(\frac{x}{w}, \frac{y}{w}, \frac{z}{w}\right) = 0, \quad w = 0.$$

A proof of Lemma 6 can be found in Lemma 2.1 of [15].

For studying the local phase portrait of the singular points of a two-dimensional differential systems whose linear part is identically zero we shall use the *quasi-homogeneous directional blow up* (or (α,β)-blow up) technique, see Chapter 3 of [7] or [1, 6] for more details. The quasi-homogeneous directional blow up coordinate change can be written as

positive x-direction: $(x,y) \mapsto (\bar{u}^\alpha, \bar{u}^\beta \bar{v})$, negative x-direction: $(x,y) \mapsto (-\bar{u}^\alpha, \bar{u}^\beta \bar{v})$,

positive y-direction: $(x,y) \mapsto (\bar{u}\bar{v}^\alpha, \bar{v}^\beta)$, negative y-direction: $(x,y) \mapsto (\bar{u}\bar{v}^\alpha, -\bar{v}^\beta)$,

where $(\alpha,\beta) \in \mathbb{N}^+ \times \mathbb{N}^+$. Using the *Newton diagram* we can determine the values of α and β, see [3] and [4] for more details.

A *phase portrait* of a differential system defined in an open set U of $\mathbb{R}^n$ is the decomposition of U as the union of all the orbits of the differential system.

We say that two phase portraits one defined on the open set U and the other defined in the open set V are *topologically equivalent* if there exists a homeomorphism $h : U \to V$ which send the orbits on U onto the orbits of V, preserving or reversing the orientation of all the orbits.

The *separatrices* of a differential system or vector field on a surface S are the singular points, the limit cycles and the separatrices of all its hyperbolic sectors. The set Σ all separatrices of a vector field on S is a closed set. The open connected components of $S \setminus \Sigma$ are called *canonical regions.* A *separatrix configuration* Σ^* of a vector field on S is the union of Σ with one orbit in each canonical region.

Let Σ_1^* and Σ_2^* two separatrices configurations of two vector fiels on the surface S. We say that Σ_1^* and Σ_2^* are *topologically equivalent* if there exists a homeomorphism $h : \Sigma_1^* \to \Sigma_2^*$ which send the orbits on Σ_1^* onto the orbits of Σ_2^*, preserving or reversing the orientation of all the orbits.

Neumann provided the following theorem in [20].

Theorem 7 (*Neumann's Theorem*)**.** *Two continuous flows on a surface S with isolated singular points are topologically equivalent if and only if their separatrix configurations are topologically equivalent.*

This theorem shows that the phase portrait of a differential system on a surface is determined by its separatrix configurations if all the singular points of the differential system are isolated.

The following result provides necessary and sufficient conditions in order that all the roots of a polynomial $g(z) \in \mathbb{R}[z]$ have negative real parts, see page 231 of [10].

Theorem 8 (Routh-Hurwitz Criterion)**.** *All roots of the real polynomial $g(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_{n-1} z + a_n$ $(a_0 > 0)$ have negative real parts if and only if*

$$\Delta_1 > 0, \Delta_2 > 0, \ldots, \Delta_n > 0,$$

where

$$\Delta_i = \det \begin{pmatrix} a_1 & a_3 & a_5 & \cdots & & \\ a_0 & a_2 & a_4 & \cdots & & \\ 0 & a_1 & a_3 & \cdots & & \\ 0 & a_0 & a_2 & a_4 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ & & & & & a_i \end{pmatrix} \quad (a_k = 0 \text{ if } k > n)$$

is the Hurwitz determinant of order i $(i = 1, 2, \cdots, n)$.

Corollary 9. *All roots of the real polynomial $a_0 z^3 + a_1 z^2 + a_2 z + a_3$ $(a_0 > 0)$ have negative real parts if and only if*

$$\Delta_1 = a_1 > 0, \Delta_2 = a_1 a_2 - a_3 a_0 > 0, a_3 > 0.$$

Lemma 10. *If system (2) has the invariant algebraic surface $f_4 = 0$, then*

$$E_1 = \left(\frac{a+1}{3}, \frac{(a-2)(a+1)(2a-1)}{27}, 0 \right)$$

is singular point, which is a local repeller.

Proof. Note that the parameters a, b, c, d satisfy $c > 0$, $d = 9/\left((a-2)(2a-1)\right)$, $b = -2c/(3d) = -2(a-2)(2a-1)c/27$, $c^2 = -\left(a^2 - 7a + 1\right)/2$ and $\left(7 - 3\sqrt{5}\right)/2 < a < \left(7 + 3\sqrt{5}\right)/2$ (see Table 1). With these conditions system (2) becomes

$$\begin{aligned} \dot{x} &= z, \\ \dot{y} &= -\frac{2}{27}(a-2)(2a-1)cx + \frac{2}{3}cy, \\ \dot{z} &= x\,(x-1)\,(x-a) + y + cz. \end{aligned} \tag{3}$$

It is easy to check that E_1 is a singular point of system (3). After the time reversal system (3) can be rewritten as

$$\begin{aligned} \dot{x} &= -z, \\ \dot{y} &= \frac{2}{27}(a-2)(2a-1)cx - \frac{2}{3}cy, \\ \dot{z} &= -x\,(x-1)\,(x-a) - y - cz. \end{aligned} \tag{4}$$

The Jacobian matrix of system (4) at E_1 is

$$J = \begin{pmatrix} 0 & 0 & -1 \\ \frac{2}{27}(a-2)(2a-1)c & -\frac{2}{3}c & 0 \\ \frac{1}{3}\left(a^2 - a + 1\right) & -1 & -c \end{pmatrix}. \tag{5}$$

The characteristic equation of matrix J is $27\lambda^3 + 45c\lambda^2 + 9\left(a^2 - a + 2c^2 + 1\right)\lambda + 2(a+1)^2 c = 0$. Using the notations of Lemma 10 we have $45c > 0$, $2(a+1)^2 c > 0$ and $\Delta_2 = 27c\left(13a^2 - 19a + 30c^2 + 13\right) = -54\left(a^2 - 43a + 1\right)c$. Since $c > 0$ and $\left(7 - 3\sqrt{5}\right)/2 < a < \left(7 + 3\sqrt{5}\right)/2$, then $\Delta_2 > 0$. By Corollary 9 all th eigenvalues of E_1 have negative real parts. So the singular point E_1 of system (4) is a local attractor. This means that E_1 is a local repeller for system (2). This ends the proof. □

3. Proof of Theorem 3

Poincaré introduced his compactification for polynomial vector fields in $\mathbb{R}^2$. For the extension of the Poincaré compactification to polynomial vector fields in $\mathbb{R}^n$ see [5]. Using the results of [5] for the Poincaré compactification in $\mathbb{R}^3$ we get that in the local chart U_1 the Poincaré compactified system (2) becomes

$$\begin{aligned} \dot{u} &= w^2\left(b - bdu - vu\right), \\ \dot{v} &= 1 - (a+1)w + aw^2 + cvw^2 + uw^2 - v^2w^2, \\ \dot{w} &= -vw^3. \end{aligned} \tag{6}$$

On the invariant plane $w = 0$ (the infinity in U_1) system (6) reduces to

$$\dot{u} = 0, \quad \dot{v} = 1. \tag{7}$$

This system has no singular points.

In the local chart U_2 the Poincaré compactified system (2) writes

$$\begin{aligned} \dot{u} &= w^2\left(v + bdu - bu^2\right), \\ \dot{v} &= u^3 - (a+1)u^2w + w^2 + (a - bv)\,uw^2 + (bd+c)vw^2, \\ \dot{w} &= bw^3\left(d - u\right). \end{aligned} \tag{8}$$

On the invariant plane $w = 0$ (again the infinity in U_2) system (8) becomes

$$\dot{u} = 0, \quad \dot{v} = u^3. \tag{9}$$

The singular points of system (9) filled up the straight line $u = 0$.

Once we have studied the singular points on the local charts U_1 and U_2 the unique additional singular point at infinity can be the origin of the local chart U_3. Of course, at infinity all the diametrically points with respect to the origin of $\mathbb{B}^3$ of the singular points of the charts U_k for $k = 1, 2, 3$ are also infinite singular points.

In the local chart U_3 system (2) is given by

$$\begin{aligned} \dot{u} &= -u^4 + (a+1)u^3w + w^2 - au^2w^2 - (c+v)\,uw^2, \\ \dot{v} &= -u^3v + (a+1)u^2vw - v^2w^2 + (b - av)\,uw^2 - (bd+c)vw^2, \\ \dot{w} &= -u^3w + (1+a)u^2w^2 - cw^3 - auw^3 - vw^3. \end{aligned} \tag{10}$$

So the origin of U_3 is also a singular point.

In summary, taking into account the definition of the local charts U_k for $k = 1, 2, 3$ and their symmetric charts V_k for $k = 1, 2, 3$ (see [5], or in particular [15]) all the infinite singular points of the FitzHugh-Nagumo system (2) filled up the boundary of the plane $x = 0$ at infinity in the Poincaré sphere.

This and the flows on the local charts U_1 and V_1 completes the proof of Theorem 3.

4. Proof of Theorem 4

4.1. **System (1) with the invariant algebraic surface $f_3 = 0$.** In this case system (2) has the invariant algebraic surface

$$f_3 = \frac{x^4}{2} - \frac{2}{3}(a+1)x^3 + \left(a + \frac{c^2}{9}\right)x^2 + \frac{2cxz}{3} + 2xy - z^2 - \frac{2}{27}(a+1)c^2x - \frac{2}{9}(a+1)cz - \frac{2}{3}(a+1)y = 0.$$

Since $2c^2 + 3a^2 - 12a + 3 = 0$, then $2 - \sqrt{3} < a < 2 + \sqrt{3}$. By Lemma 6 the boundary of this surface in the Poincaré sphere $\mathbb{S}^2$, i.e. at the infinity of $\mathbb{R}^3$, is described by the equations

$$\frac{x^4}{2} - \frac{2}{27}(a+1)\left(c^2x + 3cz + 9y\right)w^3 + \frac{1}{9}\left(9ax^2 + c^2x^2 + 6cxz + 18xy - 9z^2\right)w^2 - \frac{2}{3}(a+1)x^3w = 0,$$

and $w = 0$, that is, by $x = 0$ and $w = 0$. This means that the boundary of this surface at infinity is the circle filled up with infinite singular points.

Define

$$V_1 = \{(x,y,z) : f_3(x,y,z) = 0, 3x - a - 1 = 0\},$$
$$V_2 = \{(x,y,z) : f_3(x,y,z) = 0, 3x - a - 1 \neq 0\}.$$

The set V_1 reduces to $162z^2 + (a+1)^2(2c^2+3a^2-12a+3) = 0$, i.e. to the straight line intersection of the planes $3x - a - 1 = 0$ and $z = 0$, because $2c^2 + 3a^2 - 12a + 3 = 0$. Therefore system (2) restricted to the straight line V_1 becomes

$$\dot{y} = b\left(\frac{a+1}{3} - dy\right).$$

In the surface V_2 we have

$$y(x,z) = \frac{36(a+1)x^3 - 27x^4 - 6\left(9a + c^2\right)x^2 + 4cx(ac + c - 9z) + 6z(2(a+1)c + 9z)}{36(3x - a - 1)},$$

and system (2) restricted to this surface is

$$\dot{x} = z,$$
$$\dot{z} = \frac{x(3x - 2a - 2)\left(27x^2 - 18x - 18ax + 18a - 2c^2\right) + 24c(3x - a - 1)z + 54z^2}{36(3x - a - 1)}.$$

Under the condition $2c^2 + 3a^2 - 12a + 3 = 0$ the above system reduces to

$$\begin{aligned} &\dot{x} = z, \\ &\dot{z} = \frac{1}{12}\left(x(3x - a - 1)(3x - 2 - 2a) + 8cz + \frac{18z^2}{3x - a - 1}\right). \end{aligned} \tag{11}$$

After a rescaling of the time variable $d\xi = 12\,(3x - a - 1)\,d\tau$, system (11) becomes

$$\begin{aligned} &x' = 12z(3x - a - 1), \\ &y' = x(3x - a - 1)^2(3x - 2a - 2) + 8c(3x - a - 1)z + 18z^2, \end{aligned} \tag{12}$$

where now the prime denotes derivative with respect to τ.

Using the Poincaré compactification in $\mathbb{R}^2$ (see for instance Chapter 5 of [7]) we investigate the infinite singular points of system (12). Doing the change to the local chart U_1 given by $x = 1/v, y = u/v$ and rescaling the time $ds = v^4 d\tau$ system (12) can be written as

(13)

$$u' = 27 - 2(a+1)\left((a+1)^2 + 4cu - 6u^2\right)v^3 + 3\left(5(a+1)^2 + 8cu - 6u^2\right)v^2 - 36(a+1)v,$$
$$v' = 12uv^3(av + v - 3).$$

On the u-axis system (13) has no singular points.

Now we write system (12) in the local chart U_2 using the change $x = u/v, y = 1/v, ds = v^4 d\tau$ to get

(14)
$$\dot{u} = 2(a+1)\left((a+1)^2u^2 + 4cu - 6\right)v^3 + 3\left(6 - 5(a+1)^2u^2 - 8cu\right)uv^2 + 36(a+1)u^4v - 27u^5,$$
$$\dot{v} = 2(a+1)\left(4c + (a+1)^2u\right)v^4 - 3\left(5(a+1)^2u^2 + 8cu + 6\right)v^3 + 36(a+1)u^3v^2 - 27u^4v.$$

The linear part of the origin of system (14) is identically zero. We use the quasi-homogeneous blow up $(u, v) \mapsto \left(\bar{u}, \bar{u}^2\bar{v}\right)$ in the positive u-direction. After division by $\bar{u}^4$, we obtain

(15)
$$\dot{\bar{u}} = -3\bar{u}\left(8c\bar{u}\bar{v}^2 - 6\bar{v}^2 + 9\right),$$
$$\dot{\bar{v}} = 27\bar{v} - 54\bar{v}^3 + 12\left(2c\bar{v} + 3a + 3\right)\bar{u}\bar{v}^2 + (a+1)\left(8c\bar{v} - 15(a+1)\right)\bar{u}^2\bar{v}^3 + 2(a+1)^3\bar{u}^3\bar{v}^4.$$

On $\{\bar{u} = 0\}$ system (15) has a saddle at $(0, 0)$ and two stable nodes at $\left(0, \pm 1/\sqrt{2}\right)$.

Consider the blow-up $(u, v) \mapsto \left(-\bar{u}, \bar{u}^2\bar{v}\right)$ in the negative u-direction. After cancelling a common factor $\bar{u}^4$, we have

(16)
$$\dot{\bar{u}} = 3\bar{u}\left(8c\bar{u}\bar{v}^2 + 6\bar{v}^2 - 9\right),$$
$$\dot{\bar{v}} = 27\bar{v} - 54\bar{v}^3 - 12\left(2c\bar{v} + 3a + 3\right)\bar{u}\bar{v}^2 + (a+1)\left(8c\bar{v} - 15(a+1)\right)\bar{u}^2\bar{v}^3 - 2(a+1)^3\bar{u}^3\bar{v}^4.$$

On the line $\bar{u} = 0$, system (16) has a saddle at $(0, 0)$ and two stable nodes at $\left(0, \pm 1/\sqrt{2}\right)$.

We perform blow-up $(u, v) \mapsto \left(\bar{u}\bar{v}, \bar{v}^2\right)$ in the positive v-direction as well as $(u, v) \mapsto \left(\bar{u}\bar{v}, -\bar{v}^2\right)$ in the negative v-direction. After division by $\bar{v}^4$, we obtain respectively

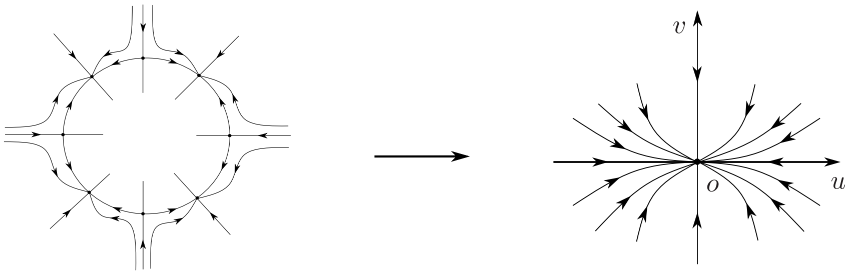
(17)
$$\dot{\bar{u}} = 27\bar{u} - \left((a+1)^3\bar{v}^2 + 12c\right)\bar{u}^2\bar{v} - 4(a+1)c\bar{v}^2\bar{u} - 18(a+1)\bar{u}^4\bar{v} + \frac{15}{2}(a+1)^2\bar{u}^3\bar{v}^2 - \frac{27}{2}\bar{u}^5,$$
$$\dot{\bar{v}} = -9\bar{v} + (a+1)\left(4c - \frac{15}{2}(a+1)\bar{u}^2\right)\bar{v}^3 + 6\left(3(a+1)\bar{u}^2 - 2c\right)\bar{u}\bar{v}^2 + (a+1)^3\bar{u}\bar{v}^4 - \frac{27}{2}\bar{u}^4\bar{v},$$

and

(18)
$$\dot{\bar{u}} = 27\bar{u} + \left((a+1)^3\bar{v}^2 - 12c\right)\bar{v}\bar{u}^2 + 4(a+1)c\bar{v}^2\bar{u} + 18(a+1)\bar{u}^4\bar{v} + \frac{15}{2}(a+1)^2\bar{u}^3\bar{v}^2 - \frac{27}{2}\bar{u}^5,$$
$$\dot{\bar{v}} = -9\bar{v} - (a+1)\left(\frac{15}{2}(a+1)\bar{u}^2 + 4c\right)\bar{v}^3 - 6\left(3(a+1)\bar{u}^2 + 2c\right)\bar{u}\bar{v}^2 - (a+1)^3\bar{u}\bar{v}^4 - \frac{27}{2}\bar{u}^4\bar{v}.$$

The origins of systems (17) and (18) are saddles. The blow up quasi-homogeneous procedure and the local phase portrait of system (14) at the origin are described in Figure 2.

System (12) has three singular points: the origin, $e_1 = (2(a+1)/3, 0)$ and $e_2 = ((a+1)/3, 0)$. The origin is a saddle because its eigenvalues are $-2(a+1)\left(2c \pm \sqrt{4c^2 + 6(a+1)^2}\right)$. The eigenvalues of e_1 are $2(a+1)\left(2c \pm \sqrt{4c^2 + 6(a+1)^2}\right)$, so e_1 is a saddle. The linear part of e_2 is identically zero. Moving the singular point



(1) Blow up. (2) Local phase portrait.

FIGURE 2. The local phase portrait of system (14) at the origin.

e_2 to the origin through the change $(x, z) \mapsto (x + (a+1)/3, z)$, system (12) can be written as

$$\dot{x} = 36xz, \quad \dot{z} = 3\left(9x^4 - (a+1)^2x^2 + 8cxy + 6y^2\right). \tag{19}$$

Applying $(1,1)$-type quasi-homogeneous directional blow up, we get that the origin of system (19) has the local phase portrait given in Figure 3.

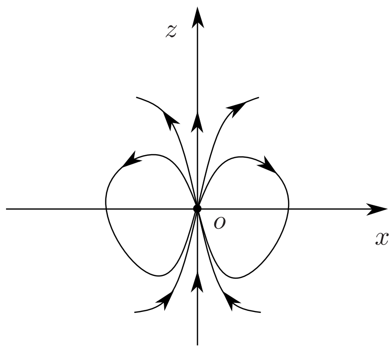


FIGURE 3. Local phase portrait of system (19) at the origin.

In summary on the invariant surface $f_3 = 0$ we have the phase portrait in the Poincaré disc of system (11) in Figure 4.

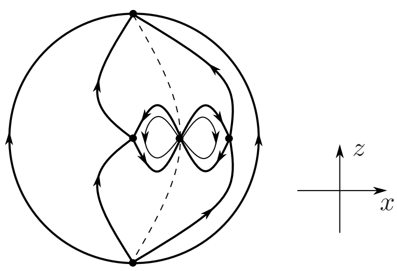


FIGURE 4. Phase portrait for system (11). The dashed straight line is $3x - a - 1 = 0$, where the system is not defined.

4.2. **System (1) with the invariant algebraic surface $f_4 = 0$.** System (1) has the invariant algebraic surface

$$\begin{aligned} f_4 = \; & \tfrac{x^4}{2} - \tfrac{2}{3}(a+1)x^3 + \left(a + \tfrac{c^2}{9}\right)x^2 + \tfrac{2cxz}{3} - \tfrac{dy^2}{2} + 2xy - z^2 - \tfrac{2}{27}(a+1)c^2x \\ & -\tfrac{2}{9}(a+1)cz - \tfrac{1}{3}(a+1)y = 0. \end{aligned}$$

Using Lemma 6 the boundary of this surface at the Poincaré sphere of the infinity is given by the intersection of the surfaces

$$\begin{aligned} & \tfrac{x^4}{2} - \tfrac{2}{3}(a+1)x^3w + w^2\left(ax^2 + \tfrac{c^2x^2}{9} + \tfrac{2cxz}{3} - \tfrac{dy^2}{2} + 2xy - z^2\right) \\ & -\tfrac{1}{27}(a+1)w^3\left(2c^2x + 6cz + 9y\right) = 0, \end{aligned}$$

and $w = 0$, that is $x = 0$ and $w = 0$. So the boundary of this surface at infinity is again the circle of the infinity filled up with singular points.

From the invariant algebraic surface $f_4 = 0$ we get

$$z^{\pm}(x,y) = \frac{1}{18}\left(2c(3x - a - 1) \pm \sqrt{2\eta}\right),$$

where

(20)
$$\begin{aligned} \eta(x,y) = \; & 2c^2\left(18x^2 - 12(a+1)x + (a+1)^2\right) + 27\big(\left(3x^2 - 4(a+1)x + 6a\right)x^2 \\ & -2(a - 6x + 1)y - 3dy^2\big). \end{aligned}$$

System (2) restricted to the surface $f_4 = 0$ becomes

(21)
$$\dot{x} = z^{\pm}(x,y), \quad \dot{y} = b\,(x - dy)\,.$$

Let (x_0, y_0) be a singular point of system (21). Then $x_0 = dy_0$ and $z^{\pm}(x_0, y_0) = 0$. We note that $f_4\,(x_0, y_0, z^{\pm}(x_0, y_0)) = f_4\,(dy_0, y_0, 0) = 0$, that is,

(22)
$$\frac{1}{54}\left(27d^4y_0^3 - 36(a+1)d^3y_0^2 + 3d\left(2d\left(9a + c^2\right) + 27\right)y_0 - 2\left(2c^2d + 9\right)(a+1)\right)y_0 = 0.$$

The singular points of system (21) are characterized by equation (22). The parameters a, b, c, d satisfy $c > 0$, $d = 9/\left(2a^2 - 5a + 2\right)$, $b = -2c/(3d) = -2(a-2)(2a-1)c/27$, $c^2 = -\left(a^2 - 7a + 1\right)/2$ and $\left(7 - 3\sqrt{5}\right)/2 < a < \left(7 + 3\sqrt{5}\right)/2$ (see Table 1). Equation (22) can be reduced to

$$y_0\left(27y_0 - 4a^3 + 6a^2 + 6a - 4\right)\left(27y_0 - 2a^3 + 3a^2 + 3a - 2\right)^2 = 0.$$

The solutions of this equation are

$$y_0 = 0, y_1 = \frac{1}{27}(a-2)(a+1)(2a-1) \text{ and } y_2 = \frac{2}{27}(a-2)(a+1)(2a-1).$$

Let $e_0 = (0,0)$, $e_1 = (dy_1, y_1)$ and $e_2 = (dy_2, y_2)$, that is,

$$e_1 = \left(\frac{a+1}{3}, \frac{(a-2)(a+1)(2a-1)}{27}\right) \text{ and } e_2 = \left(\frac{2\,(a+1)}{3}, \frac{2(a-2)(a+1)(2a-1)}{27}\right).$$

Doing the adequate computations, we have

(23)
$$z^{+}\left(e_0\right) = z^{\pm}\left(e_1\right) = z^{-}\left(e_2\right) = 0.$$

Consider the subsystems of system (21)

$$\dot{x} = z^{+}(x,y) = \frac{1}{18}\left(2c(3x-a-1)+\sqrt{2\eta}\right), \quad \dot{y} = b(x-dy). \tag{24}$$

From equation (23), system (24) has two finite singular points e_0 and e_1. The Jacobian matrix of system (24) is

$$J(x,y) = \begin{pmatrix} \frac{1}{18}\left(6c+\frac{\partial_x \eta}{\sqrt{2\eta}}\right) & \frac{\partial_y \eta}{18\sqrt{2\eta}} \\ b & -bd \end{pmatrix}. \tag{25}$$

Since $\det(J(e_0)) = -(a+1)^2/9 < 0$, e_0 is a saddle. The Jacobian matrix $J(x,y)$ at the singular point e_1 is not defined at e_1 because of $\eta(e_1) = 0$. This means that system (24) is not analytic at e_1. In order to get local phase portrait of system (24) at e_1, we go back to investigate system (2). From Lemma 10 it follows that the restriction of the singular point E_1 to the surface $f_4 = 0$ becomes the singular point e_1 of system (24), which is repeller.

Assume that system (24) has a limit cycle. Then the limit cycle surrounding the singular point e_1 of system (24) must intersect the line $x = (a+1)/3$, see Figure 5. The flow of system (24) over the line $x = (a+1)/3$ are given by

$$\dot{x}\Big|_{x=\frac{a+1}{3}} = \frac{\sqrt{2\eta}}{18}\Big|_{x=\frac{a+1}{3}} = \frac{1}{9}\sqrt{-\frac{(27y-2a^3+3a^2+3a-2)^2}{2(a-2)(2a-1)}} \geq 0,$$

which is in contradiction with the limit cycle turning clockwise in positive time. Thus, system (24) has no limit cycles.

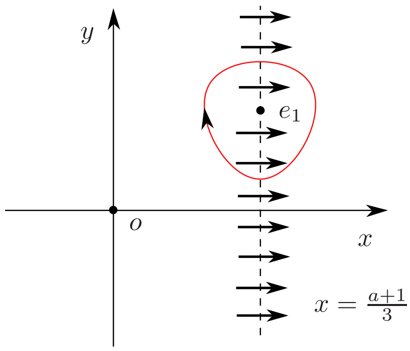


FIGURE 5. Flow of system (24) over the straight line $x = (a+1)/3$.

Taking the Poincaré transformation $x = 1/v$, $y = u/v$ and rescaling the time $dt = v^2 d\tau$, system (24) becomes

$$\begin{aligned} \dot{u} &= bv + u\left(\frac{c}{9}(a+1)v^2 + \frac{c}{3}v - \frac{\sqrt{2\delta}}{18}\right), \\ \dot{v} &= -\frac{\sqrt{2\delta}}{18}v - \frac{c}{3}v^2 + \frac{c}{9}(a+1)v^3, \end{aligned} \tag{26}$$

where

$$\delta = 2(a+1)^2c^2v^4 - 6(a+1)\left(4c^2+9u\right)v^3 + 9\left(18a+4c^2-9u(du-4)\right)v^2 - 108(a+1)v + 81.$$

On the u-axis the origin of system (26) is the unique singular point which is a stable node.

Doing the Poincaré transformation $x = u/v$, $y = 1/v$ with the scaling $dt = v^2 d\tau$, we obtain

$$\begin{aligned} \dot{u} &= \frac{\sqrt{2\delta_1}}{18} - \frac{c}{9}(a+1)v^2 - \frac{c}{3}uv - bu^2 v, \\ \dot{v} &= -\left(\frac{2}{3}c + bu\right) v^2, \end{aligned} \tag{27}$$

where

$$\begin{aligned} \delta_1 = \quad & 81u^4 - 108(a+1)u^3 v + 9\left(2\left(9au + 2c^2 u + 18\right)u - 9d\right)v^2 - 6(a+1)\left(4c^2 u + 9\right)v^3 \\ & + 2(a+1)^2 c^2 v^4. \end{aligned}$$

The origin of system (27) is degenerate. With the help of $(1, 2)$-type quasi-homogeneous directional blow up, we obtain that the local phase portrait of system (27) at origin is topologically equivalent to Figure 6.

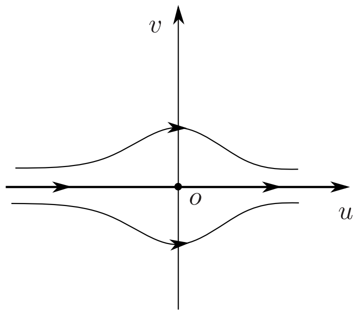


FIGURE 6. Local phase portrait of system (27) at the origin.

Based on the above discussions the phase portrait of system (24) in the Poincaré disc is described in Figure 7.

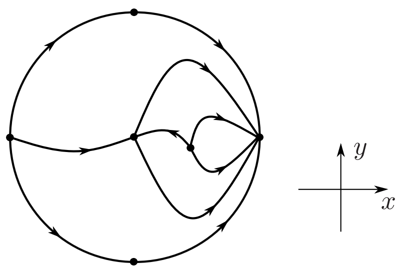


FIGURE 7. Phase portrait for system (24).

For subsystems of system (21)

$$\dot{x} = z^-(x, y) = \frac{1}{18}\left(2c(3x - a - 1) - \sqrt{2\eta}\right), \quad \dot{y} = b(x - dy), \tag{28}$$

it has two singular points e_2 and e_1. The Jacobian matrix of system (24) is

$$J\left(x,y\right)=\begin{pmatrix} \frac{1}{18}\left(6c-\frac{\partial_x\eta}{\sqrt{2\eta}}\right) & -\frac{\partial_y\eta}{18\sqrt{2\eta}} \\ b & -bd \end{pmatrix}. \tag{29}$$

Since $\det\left(J\left(e_2\right)\right)=-(a+1)^2/9<0$, e_2 is a saddle. System (29) is also not analytic at e_1. By Lemma 10, the singular point e_1 is repeller. In the same way as system, system (29) has no limit cycles.

Using the Poincaré transformation $x=1/v$, $y=u/v$ and rescaling the time $dt=v^2d\tau$, system (28) becomes

$$\begin{aligned} \dot{u}&=bv+u\left(\frac{c}{9}(a+1)v^2+\frac{c}{3}v+\frac{\sqrt{2\delta}}{18}\right),\\ \dot{v}&=\frac{\sqrt{2\delta}}{18}v-\frac{c}{3}v^2+\frac{c}{9}(a+1)v^3, \end{aligned} \tag{30}$$

with

$$\delta=2(a+1)^2c^2v^4-6(a+1)\left(4c^2+9u\right)v^3+9\left(18a+4c^2-9u(du-4)\right)v^2-108(a+1)v+81.$$

On the u-axis, the origin is the unique singular point of system (30), which is an unstable node.

Making the Poincaré transformation $x=u/v$, $y=1/v$ with the scaling $dt=v^2d\tau$, system (28) can be written as

$$\begin{aligned} \dot{u}&=-\frac{\sqrt{2\delta_1}}{18}-\frac{c}{9}(a+1)v^2-\frac{c}{3}uv-bu^2v,\\ \dot{v}&=-\left(\frac{2}{3}c+bu\right)v^2, \end{aligned} \tag{31}$$

with

$$\begin{aligned} \delta_1=\ & 81u^4-108(a+1)u^3v+9\left(2\left(9au+2c^2u+18\right)u-9d\right)v^2-6(a+1)\left(4c^2u+9\right)v^3\\ & +2(a+1)^2c^2v^4. \end{aligned}$$

The origin of system (31) is degenerate. Doing $(1,2)$-type quasi-homogeneous directional blow up, the local phase portrait of system (31) at origin is topologically equivalent to Figure 8.

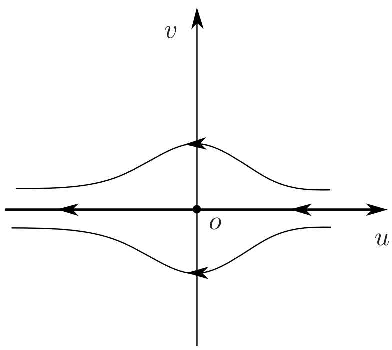


FIGURE 8. Local phase portrait of system (31) at the origin.

So the phase portrait of system (28) is given in Figure 9. From Theorem 4 we know that the phase portraits of systems (24) and (28) are topologically equivalent. In fact the phase portraits of systems (24) and (28) describe the dynamics of system (2) on the surface $f_4 = 0$ projected onto the (x, y)-plane.

This completes the proof of Theorem 4.

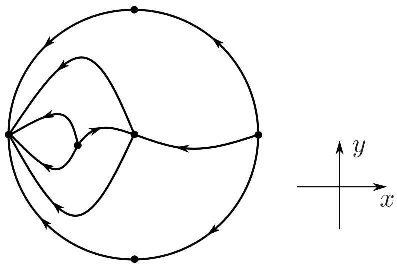


FIGURE 9. Phase portrait for system (28).

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