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Period function of planar turning points

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Abstract. This paper is devoted to the study of the period function of planar generic and non-generic turning points. In the generic case (resp. non-generic) a non-degenerate (resp. degenerate) center disappears in the limit $\epsilon \to 0$, where $\epsilon \geq 0$ is the singular perturbation parameter. We show that, for each $\epsilon > 0$ and $\epsilon \sim 0$, the period function is monotonously increasing (resp. has exactly one minimum). The result is valid in an ϵ-uniform neighborhood of the turning points. We also solve a part of the conjecture about a uniform upper bound for the number of critical periods inside classical Liénard systems of fixed degree, formulated by De Maesschalck and Dumortier in 2007. We use singular perturbation theory and the family blow-up.

Keywords: critical periods, family blow-up, period function, slow-fast systems.

2020 Mathematics Subject Classification: 34E15, 34E17.

1 Introduction

We consider slow-fast polynomial Liénard equations of center type

$$X_{\epsilon,\eta} : \begin{cases} \dot{x} = y - \left(x^{2n} + \sum_{k=1}^{l} a_k x^{2n+2k} \right), \\ \dot{y} = \epsilon^{2n} \left(-x^{2n-1} + \sum_{k=1}^{m} b_k x^{2n+2k-1} \right), \end{cases} \tag{1.1}$$

where $l, m, n \geq 1$, $\eta := (a_1, \ldots, a_l, b_1, \ldots, b_m)$ is kept in a compact set K of $\mathbb{R}^{l+m}$ and $\epsilon \geq 0$ is the singular perturbation parameter kept small. System $X_{\epsilon,\eta}$ is invariant under the symmetry $(x,t) \to (-x,-t)$ and has a center at the origin for all $\epsilon > 0$, $\epsilon \sim 0$, and for all $\eta \in K$. The center is non-degenerate when $n = 1$ or nilpotent when $n > 1$. In the limit $\epsilon = 0$, we encounter drastic changes in the dynamics of (1.1): the system has a curve of singular points, given by $\{y = x^{2n} + \sum_{k=1}^{l} a_k x^{2n+2k}\}$, passing through the origin, and horizontal regular orbits (see Figure 1.1).

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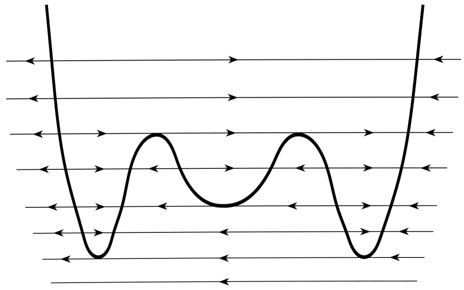

Figure 1.1: Dynamics of $X_{0,\eta}$ with 5 contact points.

A portion of the curve of singularities near the origin consists of the normally attracting part $\{x > 0\}$, the normally repelling part $\{x < 0\}$ and the contact point $(x, y) = (0, 0)$ between them. We call the contact point a turning point because closed orbits surrounding the center, for $\epsilon > 0$ small, pass from the attracting part to the repelling part of the curve of singularities. When $n = 1$, the turning point at the origin is generic (sometimes called simple). When $n > 1$, we deal with a non-generic or degenerate turning point.

The period function of a center assigns to each periodic orbit its minimal period. Isolated critical points of the period function are called critical periods (or critical periodic orbits) and are central in the qualitative study of the period function. One can note that critical periods do not depend on the parametrization of the set of periodic orbits used. Indeed, if $\{\gamma_s\}_{s\in(0,1)}$ is such a parametrization and $s \mapsto T(s)$ is the period of the periodic orbit γ_s, for any diffeomorphism $s \mapsto \xi = \xi(s)$, $\frac{d}{ds}T(\xi(s)) = \frac{d}{d\xi}T(\xi(s))\frac{d}{ds}\xi(s)$. Therefore the number of isolated zeros of $\frac{d}{ds}(T \circ \xi)$ and $\frac{d}{ds}T$ are the same.

The main purpose of this paper is to give a complete local study of the period function of $X_{\epsilon,\eta}$, near the center at the origin, in both the generic and non-generic case. The study is valid in a small fixed neighborhood of the turning point that is independent of (ϵ, η). Thus, the neighborhood does not shrink to the origin as $\epsilon \to 0$. In the generic (resp. non-generic) case, the period function of the center in $X_{\epsilon,\eta}$ is strictly monotonous increasing (resp. has exactly one critical period which is a minimum). More precisely, let us denote by $T(y; \epsilon)$ the period function of the center at the origin of system (1.1) with $\epsilon > 0$, $\epsilon \sim 0$, parametrized by the positive y-axis. Then we have:

Theorem 1.1. *Let $l, m \geq 1$ and $n = 1$ (resp. $n > 1$) be fixed. For any compact $K \subset \mathbb{R}^{l+m}$ there exist $\epsilon_0 > 0$ and $y_0 > 0$ small enough such that the period function $T(y; \epsilon)$ of the center of system (1.1) is strictly monotonous increasing (resp. has a global minimum) in the interval $]0, y_0]$, for all $\epsilon \in]0, \epsilon_0]$ and $\eta \in K$.*

We prove Theorem 1.1 in Section 3.4. To prove Theorem 1.1, we use a blow-up at the origin in the (x, y, ϵ)-space to desingularize system (1.1). Roughly speaking, after the blow-up we distinguish between "very small", "small" and "intermediate" closed orbits surrounding the center $(x, y) = (0, 0)$. The period function of the center of system (1.1) cannot be studied uniformly in these three regions and we have to use different techniques for each type of closed orbits. To treat the period function near the very small closed orbits (the ones closest to the center), we use Chicone and Jacobs [2], in the generic case, and generalized polar coordinates, in the non-generic case. The small closed orbits can be treated using the monotonicity criterion due to Schaaf [11], in the generic case, and a result due to Sabatini [10], in the non-generic case. The size of the very small and small closed orbits tends to zero as $\epsilon \to 0$. In order to obtain the result in an (ϵ, η)-uniform neighborhood of $(x, y) = (0, 0)$, the period function near the passage from the small closed orbits to large closed orbits of size $O(1)$ in the (x, y)-space has to be studied. In this passage, we encounter the so-called intermediate closed orbits. The period function near such intermediate closed orbits, in both the generic and non-generic case, will be studied using techniques from [6, 8], where small-amplitude limit cycles in an ϵ-uniform neighborhood of slow-fast Hopf points have been investigated (the slow-fast Hopf points correspond to the generic case). For more details we refer to Section 2.

We point out that Theorem 1.1 can be proved in a more general framework of smooth Liénard systems. More precisely, the same local result is true if we replace (1.1) with $\{\dot{x} = y - x^{2n} + O(x^{2n+2}), \dot{y} = \epsilon^{2n}\left(-x^{2n-1} + O(x^{2n+1})\right)\}$ where $O(x^{2n+2})$ (resp. $O(x^{2n+1})$) is an even (resp. odd) C^∞-perturbation term that may depend on parameters kept in a compact set. The proof in this more general setting is analogous to the proof for polynomial Liénard equations presented in this paper.

Theorem 1.1, in the generic case $n = 1$, can be used to solve a part of the following conjecture formulated in [4]: there exists a uniform upper bound on the number of critical periods of classical Liénard equations $\{\dot{x} = y - G(x), \dot{y} = -x\}$ where G is an even polynomial of degree $2N$, $N \geq 1$, and $G(0) = 0$. Moreover, this upper bound is conjectured to be $2N - 2$. Following Theorem 5 in [4], this can be reduced to the following problem: there exist a small $\epsilon_0 > 0$ and an integer $M > 0$ such that the slow-fast system

$$\begin{cases} \dot{x} = y - \left(x^{2N} + \sum_{k=1}^{N-1} c_{2k} x^{2k}\right), \\ \dot{y} = -\epsilon x, \end{cases} \tag{1.2}$$

has at most M critical periods, for all $\epsilon \in]0, \epsilon_0]$ and $(c_2, c_4, \ldots, c_{2(N-1)}) \in \mathbb{S}^{N-2}$. The following result covers the case where the curve of singularities of (1.2), at level $\epsilon = 0$, has only one contact point, the one at the origin $(x, y) = (0, 0)$.

Theorem 1.2. *Let $c_2^0 > 0$ be small and fixed and let $N \geq 1$ be a fixed integer. Denote by C the set of all values $(c_2, c_4, \ldots, c_{2(N-1)}) \in \mathbb{S}^{N-2}$ such that $c_2 \geq c_2^0$ and $\frac{G'(x)}{x} > 0$ for all $x \in \mathbb{R}$, where $G(x) = x^{2N} + \sum_{k=1}^{N-1} c_{2k} x^{2k}$. For any compact set $\tilde{C}$, with $\tilde{C} \subset C$, there exists a small $\epsilon_0 > 0$ such that system (1.2) has no critical periods for all $\epsilon \in]0, \epsilon_0]$ and $(c_2, c_4, \ldots, c_{2(N-1)}) \in \tilde{C}$.*

We prove Theorem 1.2 in Section 3.5. Note that keeping the parameter in a compact set $\tilde{C}$ ensures that the critical curve has no contact points other than the origin. The compact set

$$\tilde{C} = \{(c_2, c_4, \ldots, c_{2(N-1)}) \in \mathbb{S}^{N-2} \mid c_2 \geq c_2^0 \text{ and } c_i \geq 0 \text{ for } i = 4, \ldots, 2(N-1)\}$$

is always contained in the set C defined in Theorem 1.2. When $N = 1$, Theorem 1.2 implies that $\{\dot{x} = y - x^2, \dot{y} = -\epsilon x\}$ has no critical periods for all $\epsilon \in]0, \epsilon_0]$, for some small $\epsilon_0 > 0$. When $N = 2$, we have to deal with the slow-fast systems $\{\dot{x} = y - x^4 \pm x^2, \dot{y} = -\epsilon x\}$. From Theorem 1.2 follows that the system with the negative sign in front of x^2 has no critical periods. The system with the positive sign in front of x^2 is conjectured to have at most 2 critical periods (see [4]). As explained in [4], it is more difficult to deal with the part of the conjecture when the curve of singularities of (1.2) has more contact points.

When c_2 is uniformly nonzero, Theorem 1.1 in the generic case implies that system (1.2) has no critical periods in an ϵ-uniform neighborhood of the origin in the (x, y)-space. It suffices to notice that the change of coordinates $(x, y) \to (c_2 x, c_2 y)$ transforms (1.2) into (1.1). See Section 3.5.

2 Blow-up and statement of results

2.1 Family blow-up at the origin in the (x, y, ϵ)-space

To be able to study the period function near the turning point, uniformly in (ϵ, η) with $\epsilon > 0$ small, we have to desingularize the system $X_{\epsilon,\eta}$ near $(x, y, \epsilon) = (0, 0, 0)$ using the so-called family blow-up. The family blow-up is the following "singular" coordinate change with $n \geq 1$:

$$\Psi : \mathbb{R}^+ \times \mathbb{S}^2_+ \to \mathbb{R}^3 : (r, (\bar{x}, \bar{y}, \bar{\epsilon})) \mapsto (x, y, \epsilon) = (r\bar{x}, r^{2n}\bar{y}, r\bar{\epsilon}),\ \bar{\epsilon} \geq 0.$$

We define the blown-up vector field as the pullback of $X_{\epsilon,\eta} + 0\frac{\partial}{\partial \epsilon}$ divided by r^{2n-1}: $\bar{X}_\eta := \frac{1}{r^{2n-1}}\Psi^*(X_{\epsilon,\eta} + 0\frac{\partial}{\partial \epsilon})$. To study the blown-up vector field $\bar{X}_\eta$ (or $r^{2n-1}\bar{X}_\eta$) near the blow-up locus $\{0\} \times \mathbb{S}^2_+$, it is convenient to use different charts with "rectified" coordinates, instead of the spherical coordinates. For our purposes, only the family chart $\{\bar{\epsilon} = 1\}$ and the phase directional chart $\{\bar{y} = 1\}$ are relevant for the study of the period function since all closed orbits near the center $(x, y) = (0, 0)$ are visible therein (see Figure 2.1).

In the family chart $\{\bar{\epsilon} = 1\}$, we have

$$(x, y, \epsilon) = (r\bar{x}, r^{2n}\bar{y}, r)$$

with $(\bar{x}, \bar{y})$ kept in an arbitrary but fixed compact set. In this chart, $r = \epsilon$ and system (1.1) becomes $X_F := \epsilon^{2n-1}\bar{X}_F$, where

$$\bar{X}_F : \begin{cases} \dot{\bar{x}} = \bar{y} - \left(\bar{x}^{2n} + \sum_{k=1}^{l} a_k \epsilon^{2k} \bar{x}^{2n+2k}\right), \\ \dot{\bar{y}} = -\bar{x}^{2n-1} + \sum_{k=1}^{m} b_k \epsilon^{2k} \bar{x}^{2n+2k-1}. \end{cases} \tag{2.1}$$

System (2.1) is invariant under the symmetry $(\bar{x}, t) \to (-\bar{x}, -t)$ and has a center at the origin, for all $\epsilon \geq 0$, $\epsilon \sim 0$ and $\eta \in K$. When $\epsilon = 0$, we are located on the blow-up locus and the vector field (2.1) becomes

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \bar{x}^{2n}, \\ \dot{\bar{y}} = -\bar{x}^{2n-1}. \end{cases} \tag{2.2}$$

A first integral of (2.2) is given by

$$H(\bar{x}, \bar{y}) = e^{-2n\bar{y}}\left(\bar{y} - \bar{x}^{2n} + \frac{1}{2n}\right). \tag{2.3}$$

Note that the invariant curve $\{\bar{y} = \bar{x}^{2n} - \frac{1}{2n}\}$ is the boundary of the period annulus (see Figure 2.1). The main advantage of the family blow-up is that the blown-up vector field (2.1) has no curves of singularities.

The $(\bar{x}, \bar{y})$-compact sets in which we will study system (2.1) (see Section 2.2) shrink to the origin in the (x, y)-space as $\epsilon \to 0$. To obtain (ϵ, η)-uniform results, we also have to study $X_{\epsilon,\eta}$ in the phase directional chart $\{\bar{y} = 1\}$. In the chart $\{\bar{y} = 1\}$, we deal with the coordinate change

$$(x, y, \epsilon) = (RX, R^{2n}, RE),$$

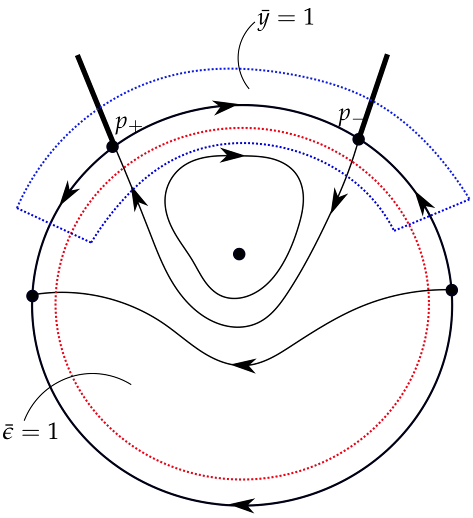

Figure 2.1: The family blow-up at the origin $(x, y, \epsilon) = (0, 0, 0)$ and dynamics on the blow-up locus. To study the period function of (1.1) in an (ϵ, η)-uniform neighborhood of $(x, y) = (0, 0)$, it suffices to use the charts $\{\bar{\epsilon} = 1\}$ and $\{\bar{y} = 1\}$.

where $R \geq 0$ and $E \geq 0$ are small and X is kept in any compact set. System (1.1) becomes $X_D := R^{2n-1}\bar{X}_D$, where

$$\bar{X}_D : \begin{cases} \dot{X} = 1 - \left(X^{2n} + \sum_{k=1}^{l} a_k R^{2k} X^{2n+2k} \right) + \frac{1}{2n} X E^{2n} F(X, R, \eta), \\ \dot{R} = -\frac{1}{2n} R E^{2n} F(X, R, \eta), \\ \dot{E} = \frac{1}{2n} E^{2n+1} F(X, R, \eta), \end{cases} \tag{2.4}$$

with $F(X, R, \eta) = X^{2n-1} - \sum_{k=1}^{m} b_k R^{2k} X^{2n+2k-1}$. For $R = E = 0$, the system has semi-hyperbolic singularities at $X = -1$ (denoted by p_+) and $X = 1$ (denoted by p_-). The singularity p_+ (resp. p_-) has the X-axis as unstable (resp. stable) manifold and a two-dimensional center manifold, transverse to the X-axis.

Using (2.1) and (2.4) we easily detect the singular polycycle Γ on the blow-up locus consisting of singularities p_+ and p_- and the regular orbits that are heteroclinic to them (see Figure 2.1). Note that $p_\pm$ are the end points of the regular curve $\{\bar{y} = \bar{x}^{2n} - \frac{1}{2n}\}$.

It is clear now that the study of the period function of the center in (1.1), in a small ϵ-uniform neighborhood of $(x, y) = (0, 0)$, can be divided into three parts: the study near the center $(\bar{x}, \bar{y}) = (0, 0)$ of (2.1), the study of the interior of the period annulus inside the family (2.1), away from $(\bar{x}, \bar{y}) = (0, 0)$ and Γ, and the study near Γ, combining systems (2.1) and (2.4). The results related to the first two parts (resp. the third part) are stated in Section 2.2 (resp. Section 2.3).

2.2 Statement of results inside the vector field $\bar{X}_F$

Let $l, m \geq 1$ be fixed. For the vector field $\bar{X}_F$ given in (2.1) we define by $T_F(\bar{y}; \epsilon)$ the period function of the center at the origin parametrized by the $\bar{y}$-axis. As we will see in Sections 3.1 and 3.2, the function $T_F(\bar{y}; \epsilon)$ is well defined in any compact interval $[\bar{y}_1, \bar{y}_2]$ and when the turning point is generic it can be extended analytically to $\bar{y} = 0$. We prove the following two results concerning the period function of system $\bar{X}_F$. The first one states the behaviour of the period function of the center of system (2.1) close to the equilibrium at the origin, whereas the second one is a global statement in the interior of the period annulus.

Theorem 2.1. *For any compact $K \subset \mathbb{R}^{l+m}$ there exist $\bar{y}_0 > 0$ small enough and $\epsilon_0 > 0$ small enough such that $\frac{d}{d\bar{y}}T_F(\bar{y};\epsilon) > 0$ (resp. $\frac{d}{d\bar{y}}T_F(\bar{y};\epsilon) < 0$) for $n = 1$ (resp. $n > 1$) for all $\bar{y} \in]0, \bar{y}_0]$, $\epsilon \in [0, \epsilon_0]$ and $\eta \in K$. Moreover, $T_F(\bar{y};\epsilon) \to 2\pi$ (resp. $+\infty$) as $\bar{y} \to 0^+$ for $n = 1$ (resp. $n > 1$).*

Theorem 2.2. *For any compact $K \subset \mathbb{R}^{l+m}$ and any $0 < \bar{y}_1 < \bar{y}_2 < +\infty$ there exists $\epsilon_0 > 0$ small enough such that $\frac{d}{d\bar{y}}T_F(\bar{y};\epsilon) > 0$ (resp. $\frac{d^2}{d\bar{y}^2}T_F(\bar{y};\epsilon) > 0$) for $n = 1$ (resp. $n > 1$) for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$, $\epsilon \in [0, \epsilon_0]$ and $\eta \in K$.*

We prove Theorem 2.1 in Section 3.1 and Theorem 2.2 in Section 3.2. A key fact in the proof of the previous results is that, when $\epsilon = 0$, the vector field (2.1) becomes (2.2) with a first integral given by (2.3). As we will see, the period function of (2.1) is an ϵ-perturbation of the period function of (2.2).

2.3 Statement of results near Γ

In both the generic and non-generic case, we have the following result about the period function of the center of the vector field $r^{2n-1}\bar{X}_\eta$, with $r > 0$, in an η-uniform neighborhood of Γ.

Theorem 2.3. *Let $l, m \geq 1$ be fixed. For any compact $K \subset \mathbb{R}^{l+m}$ there exists $\epsilon_0 > 0$ small enough such that the period function of the center of system $r^{2n-1}\bar{X}_\eta$, with $r > 0$, near the polycycle Γ is monotonous increasing for all $\epsilon \in]0, \epsilon_0]$ and $\eta \in K$.*

We prove Theorem 2.3 in Section 3.3. For a precise definition of a neighborhood of Γ in the family blow-up coordinates and the period function near Γ see Section 3.3.

3 Proof of Theorem 1.1–Theorem 2.3

First we prove Theorem 2.1, Theorem 2.2 and Theorem 2.3. Then we glue them together and prove Theorem 1.1 (see Section 3.4). Theorem 1.2 is proved in Section 3.5.

3.1 Proof of Theorem 2.1

Let us start considering the case $n = 1$. We define by $\mathcal{T}_F(\bar{x};\epsilon)$ the period function of system (2.1) parametrized by the $\bar{x}$-axis. Notice that, since $n = 1$, the center at the origin is non-degenerate and therefore the period function can be extended analytically to $\bar{x} = 0$. For $\epsilon \geq 0$ small system (2.1) is an analytic perturbation of the quadratic system

$$\begin{cases} \dot{\bar{x}} = \bar{y} - \bar{x}^2, \\ \dot{\bar{y}} = -\bar{x}. \end{cases} \tag{3.1}$$

Therefore we can consider the Taylor's series development at $\epsilon = 0$ of $\mathcal{T}_F(\bar{x};\epsilon)$,

$$\mathcal{T}_F(\bar{x};\epsilon) = \mathcal{T}_0(\bar{x}) + O(\epsilon),$$

where $\mathcal{T}_0(\bar{x})$ is the period function of system (3.1) parametrized by the $\bar{x}$-axis. In particular, if $\frac{d}{d\bar{x}}\mathcal{T}_0(\bar{x}) > 0$ then $\frac{d}{d\bar{x}}\mathcal{T}_F(\bar{x};\epsilon) > 0$ for every $\epsilon \geq 0$ small enough. In consequence, the assertion concerning $n = 1$ in Theorem 2.1 will follow once we show that the period function $\mathcal{T}_0(\bar{x})$ of the quadratic system (3.1) is monotonous increasing near the origin.

To do so we use Chicone and Jacobs [2] result on quadratic centers to deduce that, in a neighborhood of the origin,

$$\mathcal{T}_0(\bar{x}) = 2\pi + p_2(\lambda)\bar{x}^2 + O(\bar{x}^3),$$

where $p_2(\lambda) = \frac{\pi}{12}(16\lambda_2^2 + 8\lambda_2\lambda_5 + \lambda_5^2 + 18\lambda_3^2 - 12\lambda_3\lambda_6 + 9\lambda_3\lambda_4 + 10\lambda_6^2 - \lambda_4\lambda_6 + \lambda_4^2)$, $\lambda = (\lambda_i)_{i=2}^6$, and λ_i stand for the coefficients of the Bautin's normal form for quadratic systems

$$\begin{cases} \dot{x} = -y - \lambda_3 x^2 + (2\lambda_2 + \lambda_5)xy + \lambda_6 y^2, \\ \dot{y} = x + \lambda_2 x^2 + (2\lambda_3 + \lambda_4)xy - \lambda_2 y^2. \end{cases}$$

In our case system (3.1) can be brought to the Bautin's normal form with the change of variable $\{\bar{y} \mapsto -\bar{y}\}$ and corresponds to the parameters $\lambda_2 = \lambda_5 = \lambda_6 = 0$, $\lambda_3 = 1$ and $\lambda_4 = -2$. Consequently, for system (3.1) the period function near the origin can be written as

$$\mathcal{T}_0(\bar{x}) = 2\pi + \frac{\pi}{3}\bar{x}^2 + O(\bar{x}^3).$$

This fact, together with the discussion at the beginning of the section, shows that there exist $\epsilon_0, \bar{x}_0 > 0$ small such that $\frac{d}{d\bar{x}}\mathcal{T}_F(\bar{x};\epsilon) > 0$ for $\bar{x} \in]0, \bar{x}_0]$ and $\epsilon \in [0, \epsilon_0]$. Since monotonicity is unaltered by parametrization, this finishes the proof of Theorem 2.1 for the case $n = 1$.

For $n > 1$ the center at the origin becomes degenerate and Chicone–Jacobs procedure do not apply. With the aim of studying the period function of system (2.1) near the origin $(\bar{x}, \bar{y}) = (0, 0)$ for $n > 1$ we consider the change to generalized polar coordinates

$$(\bar{x}, \bar{y}) = (r\cos\theta, r^n \sin\theta)$$

with $r \geq 0$ and $\theta \in \mathbb{T}$. After this change system (2.1) is written as

$$\begin{cases} \dot{r} = \dfrac{r^n}{\cos^2\theta + n\sin^2\theta}\big(\cos\theta\sin\theta - \cos^{2n-1}\theta\sin\theta + O(r)\big), \\ \dot{\theta} = \dfrac{r^{n-1}}{\cos^2\theta + n\sin^2\theta}\big(-n\sin^2\theta - \cos^{2n}\theta + O(r)\big). \end{cases}$$

We note that terms with ϵ small are inside $O(r)$ so the forthcoming arguments are uniform with respect to $\epsilon \in [0, \epsilon_0]$.

For $r > 0$ small enough we have $\dot{\theta} < 0$. Therefore we can parametrize the orbits near the origin by $\varphi := -\theta$. We denote by $\mathscr{T}_F(s;\epsilon)$ the period of the solution $r(\varphi, s)$ and for the sake of simplicity we write $f(\varphi) := \cos^2\varphi + n\sin^2\varphi$, $\alpha(\varphi) := \cos^{2n-1}\varphi\sin\varphi - \cos\varphi\sin\varphi$, $\beta(\varphi) := n\sin^2\varphi + \cos^{2n}\varphi$. Note that $\beta(\varphi) > 0$. Due to the symmetry of system (2.1) the function $\mathscr{T}_F(s;\epsilon)$ writes

$$\mathscr{T}_F(s;\epsilon) = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{d\varphi}{\dot{\varphi}} = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{r(\varphi,s)^{n-1}\big(\beta(\varphi) + O(r(\varphi,s))\big)}.$$

Moreover,

$$\frac{\frac{d}{d\varphi}r(\varphi,s)}{r(\varphi,s)} = \frac{\alpha(\varphi)}{\beta(\varphi)} + O(r(\varphi,s)) = \frac{\alpha(\varphi)}{\beta(\varphi)} + O(s),$$

where in the second equality we use $r(\varphi, s) = O(s)$. Therefore,

$$r(\varphi, s) = r(0, s)e^{\int_0^\varphi \left(\frac{\alpha(\phi)}{\beta(\phi)} + O(s)\right)d\phi} = s\left(e^{\int_0^\varphi \frac{\alpha(\phi)}{\beta(\phi)}d\phi} + O(s)\right).$$

We denote $\rho(\varphi) := e^{\int_0^\varphi \frac{\alpha(\phi)}{\beta(\phi)}d\phi} > 0$. Substituting the previous equality in the expression of $\mathscr{T}_F(s;\epsilon)$ and taking into account that $O(r(\varphi, s)) = O(s)$ we get

$$\begin{aligned}
\mathscr{T}_F(s;\epsilon) &= \frac{2}{s^{n-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{(\rho(\varphi) + O(s))^{n-1}(\beta(\varphi) + O(s))} \\
&= \frac{2}{s^{n-1}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{f(\varphi)}{\rho(\varphi)^{n-1}\beta(\varphi)} + O(s)\right) d\varphi \\
&= \frac{2}{s^{n-1}} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{\rho(\varphi)^{n-1}\beta(\varphi)} + O(s)\right).
\end{aligned}$$

Since f, ρ and β are positive, the last equality shows that $\mathscr{T}_F(s;\epsilon) \to +\infty$ as $s \to 0^+$ for $n > 1$. Moreover,

$$\begin{aligned}
\frac{d}{ds}\mathscr{T}_F(s;\epsilon) &= -\frac{2(n-1)}{s^n} \left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{\rho(\varphi)^{n-1}\beta(\varphi)} + O(s)\right) + \frac{2}{s^{n-1}}O(1) \\
&= \frac{1}{s^n} \left(-2(n-1) \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{f(\varphi)d\varphi}{\rho(\varphi)^{n-1}\beta(\varphi)} + O(s)\right).
\end{aligned}$$

The last equality shows that $\frac{d}{ds}\mathscr{T}_F(s;\epsilon) \to -\infty$ as $s \to 0^+$. This ends the proof of Theorem 2.1 for $n > 1$.

Remark 3.1. We could also use the following generalized polar coordinates

$$(\bar{x}, \bar{y}) = (r\rho_1(\theta), r^n\rho_2(\theta))$$

where $(\rho_1(\theta), \rho_2(\theta))$ is the solution of $\{\dot{x} = -y, \dot{y} = x^{2n-1}\}$ with initial condition $(x(0), y(0)) = (1, 0)$. Using this coordinate change the above expressions become simpler (e.g. $\beta(\varphi) = 1$ for all φ, with $\varphi = -\theta$).

3.2 Proof of Theorem 2.2

In order to study the global behaviour of the period function of system (2.1) uniformly on $\epsilon \geq 0$ small in a compact set inside the period annulus it is enough to study the period function of the system (2.2), that is when $\epsilon = 0$. We denote by $T_0(\bar{y})$ the period function of system (2.2) parametrized by the positive $\bar{y}$-axis, and we consider $\bar{y}$ inside an arbitrary compact interval $[\bar{y}_1, \bar{y}_2]$ with $0 < \bar{y}_1 < \bar{y}_2 < +\infty$. By continuity with respect to the small parameter ϵ of system (2.1), taking ϵ small enough the $\bar{y}$-axis is also transversal to all orbits of (2.1), which are also periodic for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$. We can then define $T_F(\bar{y};\epsilon)$ as the period function of system (2.1) parametrized by the same $\bar{y}$ as T_0. The function $T_F(\bar{y};\epsilon)$ is analytic for $\epsilon \geq 0$, $\epsilon \sim 0$, and so we can consider its Taylor's series development at $\epsilon = 0$,

$$T_F(\bar{y};\epsilon) = T_0(\bar{y}) + O(\epsilon).$$

Then, since the center of system (2.2) is not isochronous, properties of the period function of system for $\epsilon = 0$ are reflected for $\epsilon \geq 0$ small enough. In particular, $\frac{d}{d\bar{y}}T_0(\bar{y}) > 0$ and $\frac{d^2}{d\bar{y}^2}T_0(\bar{y}) > 0$ for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ will imply $\frac{d}{d\bar{y}}T_F(\bar{y};\epsilon) > 0$ and $\frac{d^2}{d\bar{y}^2}T_F(\bar{y};\epsilon) > 0$ for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ and $\epsilon \geq 0$ small, respectively. For this reason, Theorem 2.2 is a consequence of the following result.

Proposition 3.2. *The period function of the center (2.2) is strictly monotone increasing for $n = 1$ and it is strictly convex for $n > 1$.*

The first part of the proof of Proposition 3.2 relies on the application of the following monotonicity criterion due to Schaaf [11].

Theorem 3.3. *Consider a Hamiltonian system of the form $\dot{u} = -v$, $\dot{v} = g(u)$, where function g satisfy the following assumptions:*

1. *$g : \mathbb{R} \to \mathbb{R}$ is three times continuously differentiable with $g(0) = 0$ and $g'(0) > 0$.*
2. *For all $u \in \mathbb{R}$ where $g'(u) > 0$: $\big(5(g'')^2 - 3g'g'''\big)(u) > 0$.*
3. *If $g'(u) = 0$ then $g(u)g''(u) < 0$.*

Then the origin is a center and the period function is strictly increasing in the whole period annulus.

One of the key elements to prove the second part of Proposition 3.2 is to show that at most one critical period can exist in the interior of the period annulus. To do so we use the following result due to Sabatini [10]. For the sake of shortness in the statement, we define the following operator for smooth functions defined on an interval I:

$$\mathcal{K}[g] := \frac{3g^2g''^2 - 3gg'^2g'' - g^2g'g'''}{g'^4}.$$

Theorem 3.4. *Consider a Hamiltonian of the form $H(u,v) = G(u) + F(v)$, where $G(u) = \alpha u^{2k} + o(u^{2k}) \in C^\infty(I_G)$, $F(v) = \beta v^{2\ell} + o(v^{2\ell}) \in C^\infty(I_F)$, $0 \in I_G \cap I_F$, $0 < k, \ell \in \mathbb{N}$, $\alpha, \beta > 0$. Here I_G and I_F denote the maximal interval of definition of G and F, respectively. Then the origin is a center and if*

$$\mu_{s2} := 4\left(1 + 2\frac{GG''}{G'^2}\frac{FF''}{F'^2} + \mathcal{K}[G] + \mathcal{K}[F]\right) > 0$$

then the period function is strictly convex in the whole period annulus.

Proof of Proposition 3.2. The change of variables $\{u = \ln(1 + 2n(\bar{y} - \bar{x}^{2n})), v = \bar{x}\}$ transforms (2.2) into the Hamiltonian system with separable variables

$$\begin{cases} \dot{u} = -2nv^{2n-1}, \\ \dot{v} = V_n'(u), \end{cases} \tag{3.2}$$

where $V_n(u) = \frac{1}{2n}(e^u - u - 1)$. We notice that both periodic functions of system (3.2) and (2.2) are the same through the change of variable. We shall prove the results for (3.2).

Let us prove the first assertion of the statement. To do so, we apply Schaaf's criterion to system (3.2) with $n = 1$. After a positive constant rescaling of time and taking $g = V_1'$ we have that the assumptions in Theorem 3.3 are fulfilled since $g'(u) = V_1''(u) = \frac{1}{2}e^u > 0$ for all $u \in \mathbb{R}$ and $\big(5(g'')^2 - 3g'g'''\big)(u) = \frac{1}{2}e^{2u} > 0$ for all $u \in \mathbb{R}$. Therefore the period function of system (3.2) is strictly increasing and so $\frac{d}{d\bar{y}}T_0(\bar{y}) > 0$ for all $\bar{y} > 0$. This proves the assertion concerning $n = 1$.

Let us consider $n > 1$. With the aim of applying Theorem 3.4 we denote $G(u) = V_n(u) = \frac{1}{2n}(e^u - u - 1)$ and $F(v) = v^{2n}$. Clearly the first part of the assumptions of the theorem are fulfilled since $V_n(u) = \frac{1}{4n}u^2 + o(u^2)$. We claim that $\mu_{s2} \geq \frac{1}{n^2} > 0$ for all $n \geq 2$. After showing the inequality, the result follows by direct application of Theorem 3.4.

Using the expressions of F and G we have that

$$\hat{\mu}_{s2}(u,n) := \mu_{s2}(u,n) - \frac{1}{n^2} = \frac{4e^u}{n(e^u-1)^4}\eta(u,n),$$

where $\eta(u,n) = nu^2 + (1+3n)u + 2n + 1 + (2nu^2 - 2u - 3n - 3)e^u + ((1-3n)u+3)e^{2u} + (n-1)e^{3u}$. A direct computation shows that

$$\frac{d}{dn}\hat{\mu}_{s2}(u) = \frac{4(e^u-u-1)e^u}{n^2(e^u-1)^2} > 0$$

for all $u \in \mathbb{R}$. Therefore to prove the claim it is enough to show that $\eta(u,2) \geq 0$.

We perform a derivation-division procedure with respect to e^u achieving the following equality:

$$e^{-u}\frac{d^3}{du^3}\left(e^{-u}\frac{d^3}{du^3}\eta(u,2)\right) = 216e^u - 40u - 156.$$

The previous expression has exactly two simple negative zeros. Indeed, its derivative is zero only at $u = \ln(5/27)$, the image at $u = 0$ is positive and the limits $u \to \pm\infty$ are both $+\infty$. A sequence of simple arguments of continuity, number of zeros of the derivative, the values at $u = 0$ and the values of the limits at $\pm\infty$ yields to show that $\eta(u,2) \geq 0$ for all $u \in \mathbb{R}$. This finishes the proof of the claim. □

3.3 Proof of Theorem 2.3

We define a section $\Sigma_1 \subset \{X = 0\}$ parametrized by $(R_1, E_1) \in [0, R_1^0] \times [0, E_1^0]$ for some small $R_1^0, E_1^0 > 0$. The section Σ_1 is defined using the coordinates (X, R, E) of (2.4) (we write (R_1, E_1) instead of (R, E) to avoid confusion later). Similarly, we define $\Sigma_4 \subset \{\bar{x} = 0\}$ parametrized by $(\bar{y}, \epsilon)$, with $\epsilon \in [0, R_1^0 E_1^0]$, where $(\bar{x}, \bar{y}, \epsilon)$ are the coordinates of (2.1). The sections Σ_1, Σ_4 are transverse to the blown-up vector field $\bar{X}_\eta$ and located near the polycycle Γ (see Figure 3.1).

Since system (2.1) (resp. (2.4)) is invariant under the symmetry $(\bar{x}, t) \to (-\bar{x}, -t)$ (resp. $(X, t) \to (-X, -t)$), it suffices to study the time spent between Σ_1 and Σ_4, i.e. the half time period function of $r^{2n-1}\bar{X}_\eta$, denoted by H. Our goal is to prove that $\mathcal{L}H > 0$ on Σ_1 (for $R_1^0, E_1^0 > 0$ small enough but fixed), with $\epsilon > 0$, where $\mathcal{L}$ is the Lie-derivative along the vector field $R\frac{\partial}{\partial R} - E\frac{\partial}{\partial E}$ (see Section 3.3.5). This implies that $r^{2n-1}\bar{X}_\eta$ $(r > 0)$ has no critical periods near Γ and the period function is monotonous increasing there. When $\epsilon = 0$, system (1.1) has no center.

We aligned up H in three parts: the time $H_{1,2}$ spent between Σ_1 and Σ_2 (Section 3.3.2), the time $H_{2,3}$ spent between Σ_2 and Σ_3, near the semi-hyperbolic singularity p_- (Section 3.3.1), and the time $H_{3,4}$ between Σ_3 and Σ_4 (Section 3.3.3). In Section 3.3.4 we glue the local results together. Section 3.3.5 is devoted to the study of the Lie-derivative $\mathcal{L}H$.

3.3.1 The study of $H_{2,3}$

In this section we study the time $H_{2,3}$ inside the family X_D, i.e. $\bar{X}_D$ multiplied by R^{2n-1}. First, we bring $\widetilde{X}_D := F(X, R, \eta)^{-1}\bar{X}_D$, locally near $p_- = (1, 0, 0)$, to a normal form which simplifies the study of $H_{2,3}$ (transverse sections $\Sigma_{2,3}$ will be defined in the normal form coordinates). Since p_- is partially hyperbolic for all $\eta \in K$, there exists a C^k η-family of center manifolds at p_-, given as a graph of $X = 1 + \psi(R, E, \eta)$ with $\psi(0, 0, \eta) \equiv 0$. Following [5] in the generic

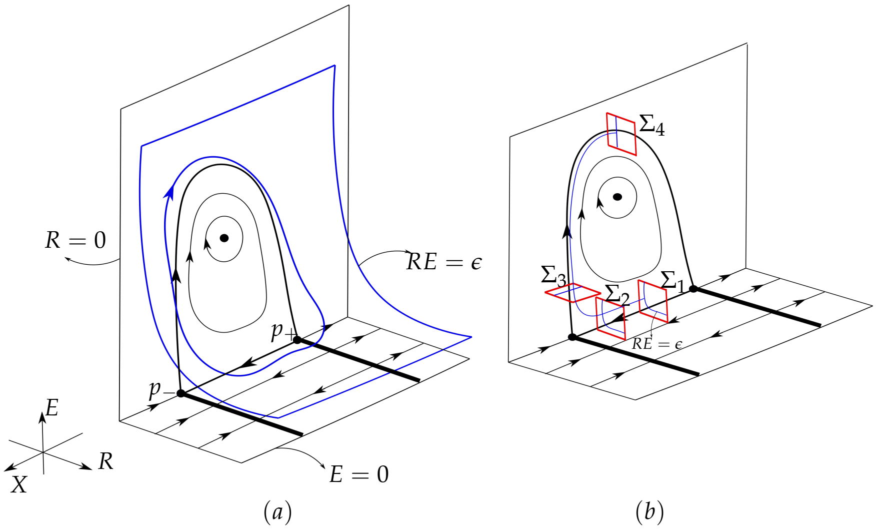

Figure 3.1: (a) Closed orbits near the polycycle Γ, inside $RE = \epsilon$, for a fixed $\epsilon > 0$. Γ is located on the blow-up locus $\{r = 0\}$ (it corresponds to $\{R = 0\}$ in the phase directional chart). The center is visible in the family chart. (b) The study of the time spent inside $\{x \geq 0\}$ is divided into three parts: $\Sigma_1 \to \Sigma_2$, $\Sigma_2 \to \Sigma_3$ and $\Sigma_3 \to \Sigma_4$. In the first two parts, we use the vector field X_D, and in the last part we use X_F.

case or [3] in the non-generic case, an η-family of center manifolds can be chosen to be C^∞ (i.e. ψ can be C^∞). We fix such ψ.

Using the coordinate change $Z = X - (1 + \psi(R, E, \eta))$, the fixed family of center manifolds becomes $\{Z = 0\}$ and the vector field $\widetilde{X}_D$ changes to

$$\begin{cases} \dot{Z} = -\big(\Phi(R, E, \eta) + O(Z)\big)Z, \\ \dot{R} = -\dfrac{1}{2n} RE^{2n}, \\ \dot{E} = \dfrac{1}{2n} E^{2n+1}, \end{cases} \tag{3.3}$$

where Φ is a smooth function with $\Phi(0, 0, \eta) = 2n$. We used the fact that the family of center manifolds is invariant for $\widetilde{X}_D$. Now, we can normally linearize the vector field (3.3) using Theorem 1.1 of [7].

Theorem 3.5. *There is a smooth family $\Pi_\eta : (Z, R, E) \to (\bar{Z}, R, E)$ of local diffeomorphisms, defined in an η-uniform neighborhood of the origin in the (Z, R, E)-space, which brings (3.3) into the normally linearized vector field*

$$\hat{X}_D : \begin{cases} \dot{Z} = -\Phi(R, E, \eta)Z, \\ \dot{R} = -\dfrac{1}{2n} RE^{2n}, \\ \dot{E} = \dfrac{1}{2n} E^{2n+1}, \end{cases} \tag{3.4}$$

where Φ is defined in (3.3) and where we denote $\bar{Z}$ again by Z. The diffeomorphisms Π_η preserve $\{RE = \text{const}\}$: $\Pi_\eta(Z, R, E) = (Z(1 + Z\pi_\eta(Z, R, E)), R, E)$ with a smooth family π_η.

Remark 3.6. The coordinate change in the normal linearization theorem from [7] is C^∞-smooth and preserves the parameter η and the leaves of the foliation $\{RE = \text{const}\}$ (the center variables R, E are preserved). In [6], this normal linearization theorem has been used in the generic case (see also Remark 1.2 in [7]). In the same way we apply it to the non-generic case. We point out that we could also use C^k center manifolds and the normal linearization theorem of [1] with a C^k-coordinate change that preserves η and $\{RE = \text{const}\}$. The size of the domain of the coordinate change may tend to zero as $k \to \infty$. The finite smoothness is not a problem in our proof.

We conclude that, in the normal form coordinates (Z, R, E) of (3.4), X_D can be written as

$$R^{2n-1}\kappa(Z, R, E, \eta)\hat{X}_D, \tag{3.5}$$

where $\kappa(Z, R, E, \eta) = F(Z(1 + O(Z)) + 1 + \psi, R, \eta)$ and $\kappa(0, 0, 0, \eta) = 1$.

In the normal form coordinates, we define $\Sigma_2 \subset \{Z = -Z_0\}$, parametrized by $(R_2, E_2) \in [0, R_2^0] \times [0, E_2^0]$ for some small constants $Z_0, R_2^0, E_2^0 > 0$, and $\Sigma_3 \subset \{E = E_3\}$, parametrized by (Z, R) with $Z \sim 0$ and $R \in [0, R_3]$ for some small constants $R_3, E_3 > 0$. All the constants are chosen such that the transverse sections $\Sigma_{2,3}$ are located in the domain of Π_η^{-1} and such that the passage w.r.t. $\hat{X}_D$ between Σ_2 and Σ_3 is well-defined.

We can now find the time $H_{2,3}(R_2, E_2)$ in (3.5), spent between Σ_2 and Σ_3. Note that the orbit of $\hat{X}_D$ (or (3.5) with $R > 0$) with the initial point $(R_2, E_2) > (0, 0)$ on Σ_2 has the form

$$\left(Z(E, R_2, E_2), \frac{R_2E_2}{E}, E\right)$$

with $Z(E, R_2, E_2) = -Z_0 \exp\left(-2n \int_{E_2}^{E} \frac{\Phi(\frac{R_2E_2}{s}, s, \eta)}{s^{2n+1}} ds\right)$. Using this, the time $H_{2,3}$ can be written as

$$H_{2,3}(R_2, E_2) = \frac{2n}{(R_2E_2)^{2n-1}} \int_{E_2}^{E_3} \frac{dE}{E^2\kappa(Z(E, R_2, E_2), \frac{R_2E_2}{E}, E, \eta)}. \tag{3.6}$$

Since $|Z(E, R_2, E_2)| \leq Z_0$ for $E \geq E_2$ and κ is positive and bounded for $(Z, R, E) \sim (0, 0, 0)$ and $\eta \in K$, it is clear that (3.6) tends to $+\infty$ as $\epsilon = R_2E_2 \to 0$, uniformly in η. (Note that the integral in (3.6) is of order $O(\frac{1}{E_2})$.) We will use the expression (3.6) in Section 3.3.4.

We conclude this section with a result about the transition map of $\hat{X}_D$ between Σ_2 and Σ_3.

Proposition 3.7. *There is a C^∞-function J in $(R_2, E_2, E_2^2 \ln E_2, \eta)$ such that the transition map $(R_2, E_2) \to (Z, R)$ along the trajectories of (3.4) between Σ_2 and Σ_3 is given by $R = \frac{R_2E_2}{E_3}$ and*

$$Z = -Z_0 \exp\left(-\frac{1}{E_2^{2n}} J(R_2, E_2, E_2^2 \ln E_2, \eta)\right)$$

with $J(0, 0, 0, \eta) = 2n$.

Proof. When $n = 1$, the proof of the proposition can be found in [6] (Proposition 4.9). The proof of the case "$n > 1$" is analogous to the proof of the case "$n = 1$". □

Proposition 3.7 implies that the transition map between Σ_2 and Σ_3 is C^∞-smooth in (R_2, E_2, η). This will be used in the gluing process in Section 3.3.4.

3.3.2 The study of $H_{1,2}$

In this section we deal with the time $H_{1,2}$, spent between Σ_1 and Σ_2, inside the vector field X_D. The smooth sections $\Sigma_{1,2}$ are defined above. Note that the system (2.4) has no singularities between Σ_1 and Σ_2 (since the section Σ_2 is located uniformly away from the singularity p_-, the X-component of (2.4) is strictly positive between Σ_1 and Σ_2, for all $(R,E) \sim (0,0)$ and for all η kept in the compact set K). Since X_D is (2.4), multiplied by R^{2n-1}, it can be seen that

$$H_{1,2}(R_1,E_1) = \frac{1}{R_1^{2n-1}} I_1\left(R_1,E_1,\eta\right), \tag{3.7}$$

where I_1 is a strictly positive C^∞-function. We conclude this section with

Proposition 3.8. *There exists a C^∞-function $J(R_1,E_1,\eta)$ such that the transition map $(R_1,E_1) \to (R_2,E_2)$ along the trajectories of (2.4) between Σ_1 and Σ_2 is given by*

$$(R_2,E_2) = \left(R_1(1+E_1^{2n}J(R_1,E_1,\eta)), E_1(1+E_1^{2n}J(R_1,E_1,\eta))^{-1}\right).$$

Proof. In the generic case ($n=1$), the proof of the proposition is given in [6, Proposition 5.1]. The proof of the non-generic case ($n>1$) is analogous to the proof of the generic case. □

We use (3.7) and Proposition 3.8 in Section 3.3.4.

3.3.3 The study of $H_{3,4}$

In this section we deal with the time $H_{3,4}$, spent between Σ_3 and Σ_4, inside the vector field X_F (X_F is equal to (2.1), multiplied by a constant $\epsilon^{2n-1} = (RE)^{2n-1}$). The smooth sections $\Sigma_{3,4}$ are defined above. If we parametrize Σ_3 with $(\bar{x},\epsilon)$ ($(\bar{x},\bar{y},\epsilon)$ are the coordinates of (2.1)), then we can write $H_{3,4}$ as

$$H_{3,4}(\bar{x},\epsilon) = \frac{1}{\epsilon^{2n-1}} I_3\left(\bar{x},\epsilon,\eta\right), \tag{3.8}$$

where I_3 is a strictly positive C^∞-function. This follows from the fact that the vector field (2.1) is regular along Γ on the blow-up locus, between Σ_3 and Σ_4 (see Figure 3.1).

3.3.4 The study of H

In this section we glue together the local results obtained in Sections 3.3.1–3.3.3 and find an expression for the half time period function H. We know that

$$H(R_1,E_1) = H_{1,2}(R_1,E_1) + H_{2,3}(R_2,E_2) + H_{3,4}(\bar{x},\epsilon),$$

where the orbit of $r^{2n-1}\bar{X}_\eta$ ($\bar{X}_\eta$ is the blown-up vector field defined in Section 2) with the initial point $(R_1,E_1) \in \Sigma_1$ intersects section Σ_2 at the point (R_2,E_2) and section Σ_3 at the point $(\bar{x},\epsilon)$. From (3.6) follows that

$$H_{2,3}(R_2,E_2) = \frac{2n}{(R_1E_1)^{2n-1}} \int_{E_2}^{E_3} \frac{dE}{E^2\kappa(Z(E,R_2,E_2), \frac{R_1E_1}{E}, E, \eta)}, \tag{3.9}$$

where R_2 and E_2 are the C^∞-functions of (R_1,E_1,η) given in Proposition 3.8. Here we used that $\epsilon = R_1E_1 = R_2E_2$. Now, we want express $H_{3,4}(\bar{x},\epsilon)$ in terms of (R_1,E_1). Let us recall that the constant $E_3 > 0$ comes from the definition of Σ_3. Using $\bar{x} = \frac{X}{E_3}$ and $X = Z(1+O(Z)) +$

$1+\psi(\frac{R_1E_1}{E_3}, E_3, \eta)$ on Σ_3 ($O(Z)$ is a C^∞-function, see Section 3.3.1), and the fact that Z is a C^∞-function in (R_1, E_1, η) (we combine Proposition 3.7 and Proposition 3.8), we see that $\bar{x}$ is a C^∞-function of (R_1, E_1, η) and $\epsilon = R_1E_1$. This and (3.8) imply that

$$H_{3,4}(\bar{x}, \epsilon) = \frac{1}{(R_1E_1)^{2n-1}}\tilde{I}_3\left(R_1, E_1, \eta\right) \tag{3.10}$$

where $\tilde{I}_3$ is a strictly positive C^∞-function. Combining (3.7), (3.9) and (3.10), we finally get

$$H(R_1, E_1) = \frac{2n}{(R_1E_1)^{2n-1}}\left(\int_{E_2}^{E_3} \frac{dE}{E^2\kappa(Z(E, R_2, E_2), \frac{R_1E_1}{E}, E, \eta)} + I(R_1, E_1, \eta)\right), \tag{3.11}$$

where I is a C^∞-function (thus, bounded). Note that the $H_{2,3}$-contribution is dominant in (3.11) and that $H(R_1, E_1)$ tends to $+\infty$ as $\epsilon = R_1E_1 \to 0$, uniformly in η. We know that $R_2 = R_1\left(1 + o(1)\right)$ and $E_2 = E_1\left(1 + o(1)\right)$ where the $o(1)$-terms are C^∞-functions of (R_1, E_1, η), equal to 0 when $E_1 = 0$. In Section 3.3.5 we show that the Lie-derivative of the integral in (3.11) is of order $O(\frac{1}{E_1})$.

3.3.5 Lie-derivative of H

When we fix any value of (ϵ, η), with $\epsilon > 0$ small, H is 1-variable function defined on interval $\{(R_1, E_1) \in \Sigma_1 \mid R_1E_1 = \epsilon\}$ (see Figure 3.1(b)). To study critical periods of H on such intervals, we define the Lie-derivative of H along the vector field $R_1\frac{\partial}{\partial R_1} - E_1\frac{\partial}{\partial E_1}$ (it is tangent to the intervals and without singularities there):

$$\mathcal{L}H := R_1\frac{\partial H}{\partial R_1} - E_1\frac{\partial H}{\partial E_1}.$$

It can be easily seen that the Lie-derivative of a C^∞-function in (R_1, E_1, η) (e.g. $\widetilde{I}$ in (3.11)) is a C^∞-function in (R_1, E_1, η), equal to zero when $(R_1, E_1) = (0, 0)$. We also have $\mathcal{L}\left(R_1E_1\right) = 0$ and $\mathcal{L}(R_1^{l_1}E_1^{l_2}) = (l_1 - l_2)R_1^{l_1}E_1^{l_2}$ for $l_1, l_2 \in \mathbb{Z}$. For more details about the Lie-derivative we refer the reader to [6,9].

The Lie-derivative of the time (3.11) can be written as

$$\begin{aligned}(\mathcal{L}H)(R_1, E_1) = \frac{2n}{(R_1E_1)^{2n-1}}&\left(\frac{1 + o(1)}{E_1\kappa(-Z_0, R_1(1 + o(1)), E_1(1 + o(1)), \eta)}\right.\\ &\left.+\int_{E_2}^{E_3} \frac{-\frac{\partial\kappa}{\partial Z}(Z(E, R_2, E_2), \frac{R_1E_1}{E}, E, \eta)}{E^2\left(\kappa(Z(E, R_2, E_2), \frac{R_1E_1}{E}, E, \eta)\right)^2}(\mathcal{L}Z)(E, R_1, E_1)dE + o(1)\right),\end{aligned} \tag{3.12}$$

where $o(1)$-terms are C^∞-functions of (R_1, E_1, η), equal to zero when $(R_1, E_1) = (0, 0)$, and $\mathcal{L}Z$ is given by

$$(\mathcal{L}Z)(E, R_1, E_1) = \frac{2nZ_0\left(\Phi(R_1, E_1, \eta) + o(1)\right)}{E_1^{2n}}\exp\left(-2n\int_{E_2}^{E} \frac{\Phi(\frac{R_1E_1}{s}, s, \eta)}{s^{2n+1}}ds\right), \tag{3.13}$$

where $o(1)$-terms have the same property as above. We show that the first term in (3.12) is dominant.

Since κ is uniformly positive near the origin ($\kappa(0,0,0,\eta)=1$) and $\frac{\partial\kappa}{\partial Z}$ and Φ are bounded, we find an upper bound for the integral in (3.12):

$$\left|\int_{E_1(1+o(1))}^{E_3}\right| \leq \frac{\alpha Z_0}{E_1^{2n}}\int_{E_1(1+o(1))}^{E_3}\frac{1}{E^2}\exp\left(-2n\int_{E_1(1+o(1))}^{E}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right)dE \tag{3.14}$$

for some constant $\alpha>0$ independent of Z_0. (We write $E_2=E_1(1+o(1))$.) For $E_1>0$ and $E_1\sim 0$, we aligned up the integral on the right-hand side of (3.14) in two parts:

$$\int_{E_1(1+o(1))}^{E_3}=\int_{E_1(1+o(1))}^{2E_1}+\int_{2E_1}^{E_3}.$$

We denote the first integral by J_1 and the second by J_2. We make in J_1 the change of variable $E=E_1\tau$, getting

$$\begin{aligned}
J_1 &= \frac{1}{E_1}\int_{1+o(1)}^{2}\frac{1}{\tau^2}\exp\left(-2n\int_{E_1(1+o(1))}^{E_1\tau}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right)d\tau\\
&= \frac{1}{E_1}\int_{1+o(1)}^{2}\frac{1}{\tau^2}\exp\left(-\frac{2n}{E_1^{2n}}\int_{1+o(1)}^{\tau}\frac{\Phi(\frac{R_1}{u},E_1u,\eta)}{u^{2n+1}}du\right)d\tau\\
&\leq \frac{1}{E_1}\int_{1+o(1)}^{2}\frac{1}{\tau^2}\exp\left(-\frac{\beta}{E_1^{2n}}\left(\tau-1-o(1)\right)\right)d\tau\\
&\leq \gamma E_1^{2n-1}
\end{aligned}\tag{3.15}$$

for some constants $\beta,\gamma>0$ independent of Z_0. In the second step we used the change of variable $s=E_1u$ and in the third step we used the fact that the integrand function in $\int_{1+o(1)}^{\tau}$ is uniformly positive ($\Phi(0,0,\eta)=2n$). In the last step the term $\frac{1}{\tau^2}$ is bounded on the segment $[1+o(1),2]$ and the integral of the exponential function is bounded by E_1^{2n}, multiplied by a positive constant. Note also that the $o(1)$-terms in the last step are equal.

Concerning the integral J_2 we get

$$\begin{aligned}
J_2 &= \int_{2E_1}^{E_3}\frac{1}{E^2}\exp\left(-2n\int_{E_1(1+o(1))}^{E}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right)dE\\
&\leq \int_{2E_1}^{E_3}\frac{1}{E^2}\exp\left(-2n\int_{E_1(1+o(1))}^{2E_1}\frac{\Phi(\frac{R_1E_1}{s},s,\eta)}{s^{2n+1}}ds\right)dE\\
&= \int_{2E_1}^{E_3}\frac{1}{E^2}\exp\left(-\frac{2n}{E_1^{2n}}\int_{(1+o(1))}^{2}\frac{\Phi(\frac{R_1}{u},E_1u,\eta)}{u^{2n+1}}du\right)dE\\
&\leq \exp\left(-\frac{\beta}{E_1^{2n}}\right)\int_{2E_1}^{E_3}\frac{1}{E^2}dE\leq\frac{\gamma}{E_1}\exp\left(-\frac{\beta}{E_1^{2n}}\right)
\end{aligned}\tag{3.16}$$

for some new constants $\beta,\gamma>0$. Finally, combining inequalities (3.15) and (3.16) we obtain

$$\left|\int_{E_1(1+o(1))}^{E_3}\right|\leq\frac{\alpha Z_0}{E_1^{2n}}(J_1+J_2)\leq\frac{\alpha_1Z_0}{E_1}+\frac{\alpha_2}{E_1^{2n+1}}\exp\left(-\frac{\beta}{E_1^{2n}}\right)$$

for some constants $\alpha_1,\alpha_2,\beta>0$. It is clear now that the first term in (3.12) is the leading term since $\frac{1}{\kappa}>\alpha_1Z_0$ ($Z_0>0$ is as small as we want but fixed).

We conclude that there are no critical periods for any fixed level $\epsilon > 0$ on Σ_1 with $R_1^0, E_1^0 > 0$ small enough and fixed, uniformly in η. The Lie-derivative $\mathcal{L}H$ tends to $+\infty$ as $\epsilon \to 0$, uniformly in η. Since $\mathcal{L}H > 0$ and $2n\bar{y}\frac{\partial}{\partial \bar{y}} + 0\frac{\partial}{\partial \epsilon} = R_1\frac{\partial}{\partial R_1} - E_1\frac{\partial}{\partial E_1}$ the period function is monotonous increasing (as large $\bar{y}$ increases, i.e. as we go away from the center $(\bar{x}, \bar{y}) = (0, 0)$, the period function increases). This completes the proof of Theorem 2.3.

3.4 Proof of Theorem 1.1

Let $n \geq 1$ and $T(y;\epsilon)$ be the period function of the center at the origin of system (1.1) with $\epsilon > 0$, $\epsilon \sim 0$, parametrized by the positive y-axis, with $y \sim 0$. We have the following relation between the (x, y, ϵ)-coordinates, the family directional coordinates and the phase directional coordinates defined in Section 2.1:

$$x = \epsilon\bar{x} = RX, \quad y = \epsilon^{2n}\bar{y} = R^{2n}, \quad \epsilon = RE.$$

Note that the positive y-axis is given by $\{x = 0\}$. In the family chart (resp. the phase directional chart), it corresponds to $\{\bar{x} = 0\}$ (resp. $\{X = 0\}$).

For each $\epsilon > 0$ and $\epsilon \sim 0$, we consider T in the following intervals: $]0, \epsilon^{2n}\bar{y}_0]$, $[\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ and $[\epsilon^{2n}\bar{y}_3, y_0]$ where $\bar{y}_0, \bar{y}_1, y_0 > 0$ are small and independent of ϵ and $\bar{y}_2, \bar{y}_3 > 0$ are large and independent of ϵ. For $\bar{y}_0, y_0$ small and $\bar{y}_3$ large, it suffices to decrease $\bar{y}_1$ and increase $\bar{y}_2$ to cover the interval $]0, y_0]$. In the interval $]0, \epsilon^{2n}\bar{y}_0]$ (resp. $[\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ and $[\epsilon^{2n}\bar{y}_3, y_0]$) we use Theorem 2.1 (resp. Theorem 2.2 and Theorem 2.3). Let us recall that the results of Theorem 2.3 are valid in a section $\Sigma_1 \subset \{X = 0\}$ parametrized by $(R, E) \in]0, R_1^0] \times]0, E_1^0]$ where $R_1^0, E_1^0 > 0$ are small enough and fixed (see Section 3.3). The interval $]0, R_1^0] \times \{E_1^0\}$ corresponds to $y = \epsilon^{2n}(E_1^0)^{-2n}$ and $\epsilon \in]0, R_1^0E_1^0]$ (we denote $(E_1^0)^{-2n}$ by $\bar{y}_3$). The interval $\{R_1^0\} \times]0, E_1^0]$ is given by $y = (R_1^0)^{2n}$ (we denote $(R_1^0)^{2n}$ by y_0) and $\epsilon \in]0, R_1^0E_1^0]$. Theorem 2.1 is valid for $\bar{y} \in]0, \bar{y}_0]$ and $\epsilon \in]0, \epsilon_0]$ where $\bar{y}_0, \epsilon_0 > 0$ are small enough. In the (y, ϵ)-coordinates, it corresponds to $y \in]0, \epsilon^{2n}\bar{y}_0]$ and $\epsilon \in]0, \epsilon_0]$. Finally, for any small $\bar{y}_1 > 0$ and any large $\bar{y}_2 > 0$, Theorem 2.2 is valid for $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ and $\epsilon \in]0, \epsilon_1]$ where $\epsilon_1 > 0$ is small enough. It corresponds to $y \in [\epsilon^{2n}\bar{y}_1, \epsilon^{2n}\bar{y}_2]$ and $\epsilon \in]0, \epsilon_1]$ in the original coordinates.

Note that the notion of critical period is independent of the chosen coordinates and the chosen transverse section (for example, if we work with the polar coordinates (r, θ) instead of $(\bar{x}, \bar{y})$, we have the same number of critical periods, counting multiplicity).

We consider two cases: $n = 1$ and $n > 1$. Suppose first that $n = 1$. Following Theorem 2.1, we have that $\frac{\partial T}{\partial y}(y;\epsilon) > 0$ for all $y \in]0, \epsilon^2\bar{y}_0]$ and $\epsilon \in]0, \epsilon_0]$. Indeed, we know that $T(y;\epsilon) = \frac{1}{\epsilon}T_F(\frac{y}{\epsilon^2};\epsilon)$ where $T_F(\bar{y};\epsilon)$ is the period function of the center of (2.1), parametrized by the positive $\bar{y}$-axis. Now, it suffices to see that

$$\frac{\partial T}{\partial y}(y;\epsilon) = \frac{1}{\epsilon^3}\frac{\partial T_F}{\partial \bar{y}}\left(\frac{y}{\epsilon^2};\epsilon\right) \tag{3.17}$$

and that $\frac{\partial T_F}{\partial \bar{y}}(\bar{y};\epsilon) > 0$ for all $\bar{y} \in]0, \bar{y}_0]$ and $\epsilon \in]0, \epsilon_0]$ (Theorem 2.1). On the other hand, we know that $T(y;\epsilon) = T_D(\sqrt{y}, \frac{\epsilon}{\sqrt{y}})$ where $T_D(R, E)$ is the period function of $r\bar{X}_\eta$ near the polycycle Γ ($\bar{X}_\eta$ is the blown-up vector field). Note that

$$\frac{\partial T}{\partial y}(y;\epsilon) = \frac{1}{2y}(\mathcal{L}T_D)\left(\sqrt{y}, \frac{\epsilon}{\sqrt{y}}\right)$$

and that $\mathcal{L}T_D > 0$ for all $(R, E) \in]0, R_1^0] \times]0, E_1^0]$ (see Theorem 2.3). Thus, $\frac{\partial T}{\partial y}(y;\epsilon) > 0$ for all $y \in [\epsilon^2\bar{y}_3, y_0]$ and $\epsilon > 0$ small. Finally, by taking $\bar{y}_1 < \bar{y}_0$ and $\bar{y}_2 > \bar{y}_3$, we have that

$\frac{\partial T_F}{\partial \bar{y}}(\bar{y};\epsilon) > 0$ for all $\bar{y} \in [\bar{y}_1, \bar{y}_2]$ and $\epsilon > 0$ small (see Theorem 2.2) and thus $\frac{\partial T}{\partial y}(y;\epsilon) > 0$ for all $y \in [\epsilon^2\bar{y}_1, \epsilon^2\bar{y}_2]$ and $\epsilon > 0$ small (see (3.17)). This ends the proof of Theorem 1.1 in the generic case.

Suppose now that $n > 1$. The study of the non-generic case is similar to the study of the generic case. We have $\frac{\partial T}{\partial y}(y;\epsilon) < 0$ for all $y \in]0, \epsilon^2\bar{y}_0]$ and for all $\epsilon > 0$ small (see Theorem 2.1), and $\frac{\partial T}{\partial y}(y;\epsilon) > 0$ for all $y \in [\epsilon^2\bar{y}_3, y_0]$ and $\epsilon > 0$ small (see Theorem 2.3). Using Theorem 2.2 we find that $\frac{\partial^2 T}{\partial y^2}(y;\epsilon) > 0$ for all $y \in [\epsilon^2\bar{y}_1, \epsilon^2\bar{y}_2]$ and $\epsilon > 0$ small. This implies that at most one critical period can exist in $]0, y_0]$. Since $\frac{\partial T}{\partial y}$ goes from $-$ to $+$, we conclude that precisely one critical period exists in $]0, y_0]$. This completes the proof of Theorem 1.1 in the non-generic case.

3.5 Proof of Theorem 1.2

We consider system (1.2) with $N \geq 1$ and denote $c := (c_2, c_4, \ldots, c_{2(N-1)}) \in \mathbb{S}^{N-2}$ (when $N = 1$, we don't have the parameter c). When $N \geq 2$, we assume that $c_2 \geq c_2^0$ for some arbitrarily small and fixed $c_2^0 > 0$. Let C and G be as defined in Theorem 1.2 and let $\tilde{C}$ be an arbitrary and fixed compact subset of C. Let $c \in \tilde{C}$. We replace ϵ in (1.2) by ϵ^2. It is clear that, if we can prove the result in a small interval in the new ϵ-space, then we have proved it in a small interval in the old ϵ-space.

If we apply the scaling $(x, y) = \left(\frac{\tilde{x}}{c_2}, \frac{\tilde{y}}{c_2}\right)$ to (1.2), we get

$$\begin{cases} \dot{x} = y - \left(x^2 + \sum_{k=2}^{N-1} \bar{c}_{2k} x^{2k} + \bar{c}_{2N} x^{2N}\right), \\ \dot{y} = -\epsilon^2 x, \end{cases} \tag{3.18}$$

where $\bar{c}_{2k} = c_{2k} c_2^{1-2k}$, for $k = 2, \ldots, N-1$, and $\bar{c}_{2N} = c_2^{1-2N}$. (We use the old notation (x, y) instead of $(\tilde{x}, \tilde{y})$ for the sake of simplicity.) Since c is kept in the compact set $\tilde{C}$, it is clear that $\bar{c} = (\bar{c}_4, \ldots, \bar{c}_{2N})$ is also contained in a compact set, denoted by $\bar{C}$, and that

$$\frac{\bar{G}'(x)}{x} > 0, \tag{3.19}$$

for all $x \in \mathbb{R}$ and $\bar{c} \in \bar{C}$, where $\bar{G}$ denotes the polynomial in x in the first equation of (3.18). Note that $\bar{G}(x) = c_2 G(\frac{x}{c_2})$ and that system (3.18) is of type (1.1) with $n = 1$.

It suffices to show that there exists $\epsilon_0 > 0$ small such that system (3.18) has no critical periods for all $\epsilon \in]0, \epsilon_0]$ and $\bar{c} \in \bar{C}$. Let $T(y;\epsilon)$ be the period function of the center at the origin of system (3.18) with $\epsilon > 0$ and $\epsilon \sim 0$, parametrized by the positive y-axis. In the rest of this section we prove that $\frac{d}{dy}T(y;\epsilon) > 0$ on $\{y > 0\}$, for all $\epsilon \in]0, \epsilon_0]$ and $\bar{c} \in \bar{C}$, for some $\epsilon_0 > 0$. This will imply that there are no critical periods uniformly in $\epsilon \sim 0$. We study the period function T in the following intervals: $]0, y_0]$, $[\rho, \frac{1}{\rho}]$ and $[y_1, \infty[$, where $y_0 > 0$ is small enough, $y_1 > 0$ is large enough and $\rho > 0$ is arbitrarily small (see Figure 3.2). When we find y_0 and y_1, we decrease ρ (i.e., increase the segment $[\rho, \frac{1}{\rho}]$) to cover the entire $\{y > 0\}$.

Following Theorem 1.1, there exist $\epsilon_0 > 0$ and $y_0 > 0$ such that $\frac{d}{dy}T(y;\epsilon) > 0$ for all $y \in]0, y_0]$ and $(\epsilon, \bar{c}) \in]0, \epsilon_0] \times \bar{C}$.

Consider now the period function T in the segment $[\rho, \frac{1}{\rho}]$, for any small and fixed $\rho > 0$. The reduced flow (sometimes called the slow system) of (3.18) along the critical curve $\{y =$

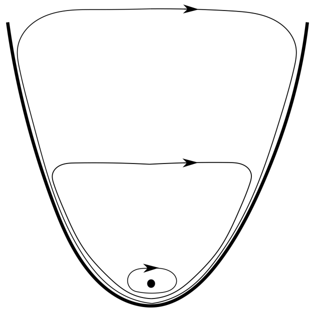

Figure 3.2: Dynamics of (3.18). The curve of singularities, at level $\epsilon = 0$, with indication of small-amplitude, detectable and large closed orbits, for $\epsilon > 0$ and $\epsilon \sim 0$.

$\bar{G}(x)\}$, away from the contact point $(x, y) = (0, 0)$, is given by

$$x' = -\frac{x}{\bar{G}'(x)} \text{ (or } y' = -x).$$

Note that the reduced flow is well-defined and uniformly negative for all x kept in large compact sets and $\bar{c} \in \bar{C}$. Here we use (3.19). The orbit through the point $y \in [\rho, \frac{1}{\rho}]$ is attracted to the curve of singularities $\{x > 0\}$, follows the reduced flow directed towards the turning point at the origin and then goes back to the point y due to the symmetry. This implies that T is well-defined for $y \in [\rho, \frac{1}{\rho}]$. Following [3, Theorem 2.1] or [4], the period function T and its derivative, restricted to the segment $[\rho, \frac{1}{\rho}]$, are given by

$$T(y; \epsilon) = 2\frac{1}{\epsilon^2}\big(T_0(y) + o(1)\big) \text{ and } \frac{d}{dy}T(y; \epsilon) = 2\frac{1}{\epsilon^2}\left(\frac{d}{dy}T_0(y) + o(1)\right),$$

with $\epsilon > 0$ small enough, where $T_0(y)$ is the transition time (at level $\epsilon = 0$) of the reduced flow along the attracting part of the curve of singularities between the ω-limit of the point $y \in [\rho, \frac{1}{\rho}]$ and the turning point. Using the expressions for the reduced flow we have for $y \in [\rho, \frac{1}{\rho}]$

$$T_0(y) = -\int_y^0 \frac{d\tilde{y}}{\tilde{x}} = -\int_y^0 \frac{d\tilde{y}}{g(\tilde{y})},$$

where $\tilde{x} = g(\tilde{y})$ represents the attracting part of the critical curve, i.e. $\tilde{y} = \bar{G}(g(\tilde{y}))$. Finally, we get

$$\frac{d}{dy}T_0(y) = \frac{1}{g(y)} > 0$$

for all $y \in [\rho, \frac{1}{\rho}]$ and $\bar{c} \in \bar{C}$. We conclude that $\frac{d}{dy}T(y; \epsilon) > 0$ for all $y \in [\rho, \frac{1}{\rho}]$, $\bar{c} \in \bar{C}$ and $\epsilon \in]0, \epsilon_0]$ for some small $\epsilon_0 > 0$. We point out that we are allowed to use the results of [3] because the reduced flow has no singularities.

It remains to show that $\frac{d}{dy}T(y; \epsilon) > 0$ for $y \in [y_1, \infty[$, $\bar{c} \in \bar{C}$ and $\epsilon \in]0, \epsilon_0]$ for $y_1 > 0$ large enough and $\epsilon_0 > 0$ small enough. To investigate the period function when $y \to \infty$, we apply the coordinate change $(x, y) = \left(\frac{\tilde{x}}{q}, \frac{1}{q^{2N}}\right)$ to (3.18), where $q > 0$ is small and $\tilde{x}$ is kept in a compact set. In the new coordinates (3.18) becomes $\frac{1}{q^{2N-1}}X_\infty$ where the vector field X_∞ is given by

$$\begin{cases} \dot{\tilde{x}} = 1 - \left(q^{2N-2}\tilde{x}^2 + \sum_{k=2}^{N-1} \bar{c}_{2k} q^{2N-2k} \tilde{x}^{2k} + \bar{c}_{2N}\tilde{x}^{2N}\right) + \frac{1}{2N}\epsilon^2 q^{4N-2}\tilde{x}^2, \\ \dot{q} = \frac{1}{2N}\epsilon^2 q^{4N-1}\tilde{x}. \end{cases} \tag{3.20}$$

On the line $\{q = 0\}$ (it represents infinity in the (x, y)-phase space), system (3.20) has two semi-hyperbolic singularities $\tilde{x} = \pm\left(\frac{1}{\bar{c}_{2N}}\right)^{\frac{1}{2N}}$ (resp. $\tilde{x} = \pm 1$) when $N \geq 2$ (resp. $N = 1$). Note that $\bar{c}_{2N}$ is uniformly positive and bounded. It suffices to look at the positive sign. When $\epsilon = 0$, we have the curve of (semi-hyperbolic) singularities $\tilde{x} = \left(\frac{1}{\bar{c}_{2N}}\right)^{\frac{1}{2N}} + O(q)$ (resp. $\tilde{x} = 1$). The reduced flow is given by

$$q' = \frac{1}{2N} q^{4N-1}\left(\left(\frac{1}{\bar{c}_{2N}}\right)^{\frac{1}{2N}} + O(q)\right) \quad \left(\text{resp. } q' = \frac{1}{2}q^3\right).$$

Using a Takens normal form for C^k-equivalence (see e.g. [5]), system X_∞ near the semi-hyperbolic singularity on the line $\{q = 0\}$ is C^k-equivalent to

$$\begin{cases} \dot{\hat{x}} = -\hat{x}, \\ \dot{\hat{q}} = \epsilon^2 \hat{q}^{4N-1} h(\hat{q}, \epsilon, \bar{c}), \end{cases} \tag{3.21}$$

where h is a positive C^k-function. We denote system (3.21) by $\hat{X}_\infty$. We conclude that in the normal form coordinates $(\hat{x}, \hat{q})$ the vector field $\frac{1}{q^{2N-1}} X_\infty$ can be written as

$$\frac{1}{\hat{q}^{2N-1}\hat{h}(\hat{x}, \hat{q}, \epsilon, \bar{c})}\hat{X}_\infty, \tag{3.22}$$

where $\hat{h}$ is a positive C^k-function. We choose two transverse sections $\Sigma_- \subset \{\hat{x} = \hat{x}_0\}$, parametrized by $\hat{q}$, and $\Sigma_+ \subset \{\hat{q} = \hat{q}_0\}$, parametrized by $\hat{x}$, for some small and fixed $\hat{x}_0, \hat{q}_0 > 0$. We compute the time of (3.22) spent between Σ_- and Σ_+, near $(\hat{x}, \hat{q}) = (0, 0)$. The orbit of (3.21) or (3.22) starting at $\hat{q}_1 \in \Sigma_-$, with $\hat{q}_1 > 0$, is given by

$$\hat{x}(\hat{q}, \hat{q}_1) = \hat{x}_0 \exp\left(-\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}} \frac{dz}{z^{4N-1}h(z, \epsilon, \bar{c})}\right).$$

Now is the time spent by the orbit given by

$$\mathcal{T}(\hat{q}_1; \epsilon) = \frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_0} \frac{\bar{h}(\hat{x}(z, \hat{q}_1), z, \epsilon, \bar{c})dz}{z^{2N}}$$

with a positive C^k-function $\bar{h}$. The derivative is given by

$$\frac{d}{d\hat{q}_1}\mathcal{T}(\hat{q}_1; \epsilon) = -\frac{\bar{h}(\hat{x}_0, \hat{q}_1, \epsilon, \bar{c})}{\epsilon^2 \hat{q}_1^{2N}} + \frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_0} \frac{\frac{\partial \bar{h}}{\partial \hat{x}}(\hat{x}(z, \hat{q}_1), z, \epsilon, \bar{c})\frac{\partial \hat{x}}{\partial \hat{q}_1}(z, \hat{q}_1)}{z^{2N}}dz. \tag{3.23}$$

Now, we proceed exactly as in Section 3.3.5. The first term in (3.23) tends to $-\infty$ as $\epsilon^2\hat{q}_1^{2N} \to 0$ and we show that it is a dominant term. We have

$$\left|\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_0}\right| \leq \frac{\alpha \hat{x}_0}{\epsilon^4 \hat{q}_1^{4N-1}}\int_{\hat{q}_1}^{\hat{q}_0} \frac{\exp\left(-\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{z}\frac{ds}{s^{4N-1}h(s, \epsilon, \bar{c})}\right)}{z^{2N}}dz \tag{3.24}$$

with a positive constant α. We used the fact that $\frac{\partial \bar{h}}{\partial \hat{x}}$ is bounded and h is uniformly positive. For the $[\hat{q}_1, 2\hat{q}_1]$-part of the integral on the right hand side of (3.24), we get

$$\begin{aligned}
\int_{\hat{q}_1}^{2\hat{q}_1} &= \frac{1}{\hat{q}_1^{2N-1}} \int_1^2 \frac{\exp\left(-\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{\hat{q}_1\tilde{z}} \frac{ds}{s^{4N-1}h(s,\epsilon,\bar{c})}\right)}{\tilde{z}^{2N}} d\tilde{z} \\
&= \frac{1}{\hat{q}_1^{2N-1}} \int_1^2 \frac{\exp\left(-\frac{1}{\epsilon^2 \hat{q}_1^{4N-2}}\int_1^{\tilde{z}} \frac{d\tilde{s}}{\tilde{s}^{4N-1}h(\hat{q}_1\tilde{s},\epsilon,\bar{c})}\right)}{\tilde{z}^{2N}} d\tilde{z} \\
&\leq \frac{1}{\hat{q}_1^{2N-1}} \int_1^2 \frac{\exp\left(-\frac{\beta(\tilde{z}-1)}{\epsilon^2 \hat{q}_1^{4N-2}}\right)}{\tilde{z}^{2N}} d\tilde{z} \\
&\leq \gamma \epsilon^2 \hat{q}_1^{2N-1},
\end{aligned} \tag{3.25}$$

where $\beta, \gamma > 0$ are constants. (See Section 3.3.5 for each step.) On the other hand, we have

$$\begin{aligned}
\int_{2\hat{q}_1}^{\hat{q}_0} &\leq \int_{2\hat{q}_1}^{\hat{q}_0} \frac{\exp\left(-\frac{1}{\epsilon^2}\int_{\hat{q}_1}^{2\hat{q}_1} \frac{ds}{s^{4N-1}h(s,\epsilon,\bar{c})}\right)}{z^{2N}} dz \\
&= \int_{2\hat{q}_1}^{\hat{q}_0} \frac{\exp\left(-\frac{1}{\epsilon^2 \hat{q}_1^{4N-2}}\int_1^{2} \frac{d\tilde{s}}{\tilde{s}^{4N-1}h(\hat{q}_1\tilde{s},\epsilon,\bar{c})}\right)}{z^{2N}} dz \\
&\leq \exp\left(-\frac{\beta}{\epsilon^2 \hat{q}_1^{4N-2}}\right) \int_{2\hat{q}_1}^{\hat{q}_0} \frac{dz}{z^{2N}} \\
&\leq \frac{\gamma}{\hat{q}_1^{2N-1}} \exp\left(-\frac{\beta}{\epsilon^2 \hat{q}_1^{4N-2}}\right)
\end{aligned} \tag{3.26}$$

for some new constants $\beta, \gamma > 0$. Combining (3.24), (3.25) and (3.26) we finally have

$$\left| \frac{1}{\epsilon^2} \int_{\hat{q}_1}^{\hat{q}_0} \right| \leq \frac{\alpha_1 \hat{x}_0}{\epsilon^2 \hat{q}_1^{2N}} + \frac{\alpha_2}{\epsilon^4 \hat{q}_1^{6N-2}} \exp\left(-\frac{\beta}{\epsilon^2 \hat{q}_1^{4N-2}}\right)$$

for positive constants $\alpha_1, \alpha_2, \beta$. Now, it suffices to notice that $\hat{x}_0 > 0$ can be arbitrarily small but fixed.

The time of $\frac{1}{q^{2N-1}} X_\infty$ spent between $\{\tilde{x} = 0\}$ and Σ_- is of order $O(q^{2N-1})$ (X_∞ is regular in this region). Following [3], the time spent between Σ_- and the turning point and its derivative are of order $O(\frac{1}{\epsilon^2})$. This implies that the contribution (3.23) is dominant. Thus, $\frac{d}{dy} T(y; \epsilon) > 0$ for large y. This ends the proof of Theorem 1.2.

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