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DYNAMICS AND DARBOUX INTEGRABILITY OF THE D_2 POLYNOMIAL VECTOR FIELDS OF DEGREE 2 IN $\mathbb{R}^3$

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ABSTRACT. We characterize the Darboux integrability and the global dynamics of the 3-dimensional polynomial differential systems of degree 2 which are invariant under the D_2 symmetric group, which have been partially studied previously by several authors.

1. Introduction and statement of the results

Differential systems having some symmetries appear often in many applications, and consequently have been studied by several authors, see for instance [6, 8, 9, 12].

In this paper we shall study the dynamics of the 3-dimensional autonomous polynomial differential systems of degree 2 symmetric with respect to the group of symmetries generated by the following transformations of $\mathbb{R}^3$

$$(x,y,z) \mapsto (x,y,z), \quad (x,y,z) \mapsto (-x,-y,z),$$
$$(x,y,z) \mapsto (-x,y,-z), \quad (x,y,z) \mapsto (x,-y,-z).$$

In [11] it is proved that such 3-dimensional autonomous systems are

$$\dot{x} = ax + yz, \quad \dot{y} = by + xz, \quad \dot{z} = z + xy, \tag{1}$$

and

$$\dot{x} = ax + yz, \quad \dot{y} = by + xz, \quad \dot{z} = z - xy, \tag{2}$$

where a, b are non-zero parameters and the dot means derivative in t. According to [2] systems (1) and (2) are equivariant under the D_2 symmetry group.

Note that system (2) can be transformed into system (1) by the change

$$(x,y,z) \to (iX, iY, Z) \tag{3}$$

with $X, Y, Z \in \mathbb{R}$. Hence it is enough to study the integrability of system (1). Moreover, system (1) is invariant by the change

$$(a,b,x,y,z) \to (b,a,y,x,z). \tag{4}$$

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Katsios and Anastassiou in [11] studied the Darboux polynomials of systems (1) and (2), and also provided some few results on the dynamics of these systems, see also [1]. Here our first objective is to study the Darboux integrability of system (1), and consequently of system (2) too. After we describe the global dynamics of these systems taking into account their behavior near infinity.

A *first integral* is a nonconstant function $h\colon \mathbb{R}^3 \to \mathbb{R}$ that is constant on all solution curves $(x(t), y(t), z(t))$ of the system, that is $h(x(t), y(t), z(t)) =$ const. for all values of t for which the solution $(x(t), y(t), z(t))$ is defined on $\mathbb{R}^3$. We say that system is *completely integrable* if it has two independent first integrals.

One approach to obtain the first integrals of system (1) on a given class consists in using the Darboux theory of integrability (see [10]). The Darboux theory of integrability have been developed strongly these past years (see for instance [5, 7, 13, 15] and the references therein). In particular using this theory we charactarize all the values of the parameters for which systems (1) and (2) has a polynomial, a rational or a Darboux first integral (see the definitions below).

We now give a rigorous formulation of our results. We recall that a *Darboux polynomial* of systems (1) is a polynomial $f = f(x, y, z) \in \mathbb{C}[x, y, z] \setminus \mathbb{C}$ such that

$$(ax + yz)\frac{\partial f}{\partial x} + (by + xz)\frac{\partial f}{\partial y} + (z \pm xy)\frac{\partial f}{\partial z} = Kf, \tag{5}$$

for some polynomial

$$K = K(x, y, z) = a_0 x + a_1 y + a_2 z + a_3 \quad \text{with } a_i \in \mathbb{C}. \tag{6}$$

The polynomial K is called the *cofactor* of the Darboux polynomial f. We note that $f = 0$ is an *invariant algebraic surface* for the flow of systems (1) and (2): if an orbit of the system has a point on this surface then the whole orbit is contained in it. The polynomial K is called the cofactor of the Darboux polynomial f or of the invariant algebraic surface.

It was proved in [11] the following theorem.

Theorem 1. *The following statements hold.*

(i) *System* (1) *has invariant algebraic surfaces if and only if $a = 1$, or $b = 1$, or $b = a$. If $a = 1$ the irreducible invariant algebraic surface is $x^2 - z^2 = 0$ with cofactor* 2*. If $b = 1$ the irreducible invariant algebraic surface is $y^2 - z^2 = 0$ with cofactor* 2*, and if $b = a$ the irreducible invariant algebraic surface is $x^2 - y^2 = 0$ with cofactor $2a$.*

(ii) *System* (2) *has invariant algebraic surfaces or invariant straight lines if and only if $a = 1$, or $b = 1$, or $b = a$. If $a = 1$ the invariant straight line is $x^2 + z^2 = 0$ with cofactor* 2*. If $b = 1$ the invariant straight line* *is $y^2 + z^2 = 0$ with cofactor 2, and if $b = a$ the invariant algebraic surface is $x^2 - y^2 = 0$ with cofactor $2a$.*

Note that as stated above, statement (ii) can be obtained from statement (i) by simply using (3). Moreover, using (4) it is clear that the case $b = 1$ is obtained from the case $a = 1$.

Since the cofactors of the Darboux polynomials x^2-z^2, y^2-z^2 and x^2-y^2 for $a = b = 1$ are all equal to 2, from statement (i) of Theorem 8.7 of [7] it follows easily the next corollary.

Corollary 2. *If $b = a = 1$ then system* (1) *is completely integrable with the two first integrals*

$$(x^2 - z^2)/(y^2 - z^2) \quad \textit{and} \quad (x^2 - z^2)/(x^2 - y^2).$$

Since a polynomial first integral is a Darboux polynomial with zero cofactor, from Theorem 1 it follows the next result.

Corollary 3. *System* (1) *with $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0) \cup (1, 1)\}$ has no polynomial first integrals.*

An *exponential factor* of system (1) is a nonconstant function of the form $F = \exp(h/g)$ with $g, h \in \mathbb{C}[x, y, z]$ coprime polynomials, and the function F satisfies (5) for some polynomial K as in (6), which is also called the *cofactor* of the exponential factor.

Theorem 4. *System* (1) *with $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0) \cup (1, 1)\}$ has no exponential factors.*

The proof of Theorem 4 is given in section 2.

A first integral is called *Darboux* if it is of the form $f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q}$ with $f_1, \ldots, f_p$ Darboux polynomials and $F_1, \ldots, F_q$ exponential factors and $\lambda_j, \mu_j \in \mathbb{C}$ for all j and k.

Theorem 5. *System* (1) *with $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0) \cup (1, 1)\}$ has no Darboux first integrals.*

The proof of Theorem 5 is given in section 3.

Our second objective is to describe the global dynamics in the compactification in $\mathbb{R}^3$ in function of the parameters a, b. Roughly speaking the *Poincaré ball B* is obtained identifying $\mathbb{R}^3$ with the interior of the 3-dimensional ball of radius one centered at the origin of coordinates and extending analytically the flow of system (1) to the boundary $\mathbb{S}^2$ of B, i.e. the infinity of $\mathbb{R}^3$. In $\mathbb{R}^3$ we can go to infinity in as many directions as points of $\mathbb{S}^2$. Doing that we are able to study the behavior of systems (1) and (2) in a neighborhood of infinity and to describe completely the globally dynamics of such differential systems. We recall that the global description of the flow of a differential system in $\mathbb{R}^3$ in general is very difficult if not impossible. Here we can do it using the Poincaré compactification due to the existence of a Darboux invariant for system (1).

An *invariant* of system (1) on an open subset U of $\mathbb{R}^3$ is a nonconstant C^1 function I in the variables x, y, z and t such that $I(x(t), y(t), z(t), t)$ is constant on all solution curves $(x(t), y(t), z(t))$ of system (1) contained in U, i.e.

$$(ax + yz)\frac{\partial I}{\partial x} + (by + xz)\frac{\partial I}{\partial y} + (z \pm xy)\frac{\partial I}{\partial z} + \frac{\partial I}{\partial t} = 0,$$

for all $(x, y, z) \in U$. An invariant I is called a *Darboux invariant* if it can be written in the form

$$I(x, y, z, t) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} e^{s\,t},$$

where, for $i = 1, \dots p$, $f_i = 0$ are invariant algebraic surfaces of system (1), $\lambda_i \in \mathbb{C}$, and $s \in \mathbb{R} \setminus \{0\}$.

Let $\phi_p(t)$ be the solution of system (1) passing through the point $p \in \mathbb{R}^3$, defined on its maximal interval (α_p, ω_p) and such that $\phi_p(0) = p$. If $\omega_p = \infty$, we define the *ω-limit set* of p as

$$\omega(p) = \{q \in \mathbb{R}^3 : \exists\{t_n\} \text{ with } t_n = \infty \text{ and } \phi_p(t_n) = q \text{ when } n = \infty\}.$$

In the same way, if $\alpha_p = -\infty$, we define the *α-limit set* of p as

$$\alpha(p) = \{q \in \mathbb{R}^3 : \exists\{t_n\} \text{ with } t_n = -\infty \text{ and } \phi_p(t_n) = q \text{ when } n = \infty\}.$$

For more details on the ω– and α–limit sets see for instance section 1.4 of [7]. The existence of a Darboux invariant of system (1) provides information about the ω– and α–limit sets of all orbits of system (1). More precisely, there is the following result, for a proof see Proposition 5 of [14].

Proposition 6. *Let $\mathbb{S}^2$ be the infinity of the Poincaré sphere and $I(x, y, z, t) = f(x, y, z)e^{st}$ be a Darboux invariant of system (1). Let also $p \in \mathbb{R}^3$ and $\phi_p(t)$ be the solution of system (1) with maximal interval (α_p, ω_p) such that $\phi_p(0) = p$. Then*

(1) *Assume that $s > 0$. If $\omega_p = \infty$, then the $\omega(p)$ is contained in the closure $\overline{\{f(x, y, z) = 0\}}$ in the Poincaré ball of $\{f(x, y, z) = 0\}$, and if $\alpha_p = -\infty$ then $\alpha(p) \subset \overline{\{f(x, y, z) = 0\}} \cap \mathbb{S}^2$, being $\mathbb{S}^2$ the boundary of the Poincaré ball.*
(2) *Assume that $s < 0$. If $\alpha_p = -\infty$, then the $\alpha(p)$ is contained in $\overline{\{f(x, y, z) = 0\}}$, and if $\omega_p = \infty$ then $\omega(p) \subset \overline{\{f(x, y, z) = 0\}} \cap \mathbb{S}^2$.*

Now we study the global dynamics of systems (1) and (2).

Theorem 7. *The following holds for system (1) with $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0) \cup (1, 1)\}$.*

(a) *The dynamics at infinity are topologically equivalent to the one of Figure 1.*

(b) *When $b = a \neq 1$ the phase portraits in the Poincaré disc defined by the invariant planes $x = y$, $x = -y$, are topologically equivalent to the ones given in Figure 2 when $a > 0$ and Figure 3 when $a < 0$. Moreover, any orbit starting at a point outside these planes either approach these planes or approach the infinity. In particular such orbits end at infinity if $a > 0$, and start at infinity if $a < 0$*

(c) *When $a = 1$ with $b \neq 1$ the phase portraits in the Poincaré disc defined by the invariant planes $x = z$, $x = -z$, are topologically equivalent to the ones given in Figures 2 when $b > 0$ and Figure 3 when $b < 0$. Moreover, any orbit starting at a point outside these planes either approach these planes or approach the infinity. In particular such orbits end at infinity.*

The proof of Theorem 7 is given in section 4.

Theorem 8. *The following holds for system (2) with $(a, b) \in \mathbb{R}^2 \setminus \{(0, 0) \cup (1, 1)\}$.*

(a) *The dynamics at infinity are topologically equivalent to the one of Figure 4.*

(b) *When $b = a \neq 1$ the phase portraits in the Poincaré disc defined by planes $x = y$, $x = -y$, are topologically equivalent to the ones given in Figure 3 when $a > 0$ and Figure 5 when $a < 0$. Moreover, any orbit starting at a point outside these planes either approach these planes or approach the infinity. In particular such orbits end at infinity if $a > 0$, and start at infinity if $a < 0$.*

(c) *When $a = 1$ with $b \neq 1$ the dynamics reduces to the straight line $x = z = 0$ containing a unique orbit solution of $\dot{y} = by$. In particular all such orbits end at infinity. Moreover if $b > 0$ they start either at the origin or perhaps at infinity at the endpoins of the y-axis, and if $b < 0$ they start at infinity at the end points of the y-axis except the ones which are on the 2-dimensional unstable manifold at the origin.*

The proof of Theorem 8 is given in section 5.

2. Proof of Theorem 4

Let $F = \exp(h/g)$ be an exponential factor of system (1). We use the well-known result that ensures that if $g = g(x, y, z)$ is not a constant polynomial, then $g = 0$ is an invariant algebraic surface (see [4, 5] and Proposition 8.6 of [7] for details).

We will also use the following result (see Proposition 8.4 of [7] for a proof):

Proposition 9. *Let f be a polynomial and $f = \prod_{j=1}^{n} f_j^{\alpha_j}$ its decomposition in irreducible factors in $\mathbb{C}[x, y, z]$. Then f is a Darboux polynomial if and* *only if all the f_j are Darboux polynomials. Moreover, if K and K_j are the cofactors of f and f_j respectively, then $K = \sum_{j=1}^{n} \alpha_j K_j$.*

From Theorem 1 and Proposition 9 we get

$$g = \begin{cases} f_1^{n_1} f_2^{n_2}, & f_1 = x+y,\ f_2 = x-y,\ n_1, n_2 \geq 0,\ \text{if } b = a \text{ with } a \neq 1, \\ f_3^{n_1} f_4^{n_2}, & f_3 = x+z,\ f_4 = x-z,\ n_1, n_2 \geq 0, \text{if } a = 1 \text{ with } b \neq 1, \\ \text{const.}, & \text{if } b \neq a \text{ and } a \neq 1. \end{cases}$$

We separate the proof of Theorem 4 into three different cases.

Case 1*: $b \neq a$ and $a \neq 1$* In this case any exponential factor F must be of the form $F = \exp(h)$, where $h = h(x,y,z)$ is a nonconstant polynomial satisfying

$$(ax+yz)\frac{\partial h}{\partial x} + (by+xz)\frac{\partial h}{\partial y} + (z+xy)\frac{\partial h}{\partial z} = K, \tag{7}$$

with K as in (6). The singular points of system (1) are $(0,0,0)$ and

$$(\sqrt{b}, \sqrt{a}, -\sqrt{ab}) \quad (-\sqrt{b}, -\sqrt{a}, -\sqrt{ab}), \quad (\sqrt{b}, -\sqrt{a}, \sqrt{ab}), \quad (-\sqrt{b}, \sqrt{a}, \sqrt{ab}).$$

By substituting them in (7) we get $a_0 = a_1 = a_2 = a_3 = 0$, i.e. $K = 0$. Hence h is a polynomial first integral of system (1), which is in contradiction with Corollary 3. So in this case there are no exponential factors.

Case 2*: $b = a$ and $a \neq 1$.* In this case we have that any exponential factor must be of the form $F = \exp(h/(f_1^{n_1} f_2^{n_2}))$ with $n_1, n_2 \geq 0$, and $h = h(x,y,z)$ a polynomial coprime with $f_1^{n_1} f_2^{n_2}$ satisfying

$$\begin{aligned} &(ax+yz)\frac{\partial h}{\partial x} + (ay+xz)\frac{\partial h}{\partial y} + (z+xy)\frac{\partial h}{\partial z} - ((n_1+n_2)a + (n_1-n_2)z)h \\ &= K f_1^{n_1} f_2^{n_2}, \end{aligned} \tag{8}$$

where K is as in (6). We consider two subcases.

Subcase 2.1*: $n_1 = n_2 = 0$.* The singular points of system (1) with $b = a$ are $(0,0,0)$ and

$$(\sqrt{a}, \sqrt{a}, -a) \quad (-\sqrt{a}, -\sqrt{a}, -a), \quad (\sqrt{a}, -\sqrt{a}, a), \quad (-\sqrt{a}, \sqrt{a}, a).$$

By substituting them into (8) we obtain $a_0 = a_1 = a_2 = a_3 = 0$, i.e. $K = 0$. Hence h is a polynomial first integral of system (1), which is in contradiction with Corollary 3. So in this case there are no exponential factors.

Subcase 2.2*: $n_1 > 0$ or $n_2 > 0$* We first consider the case $n_2 > 0$. Assume first that $h = c \in \mathbb{C} \setminus \{0\}$. Then from (8) we have

$$-c((n_1+n_2)a + (n_1-n_2)z) = K(x+y)^{n_1}(x-y)^{n_2}. \tag{9}$$

Since $n_2 > 0$ and the left-hand side of (9) is not divisible by $x - y$, we get a contradiction.

Assume now that h is not a constant. Now we set $y = x$ and denote by $\bar{h} = h(x, x, z)$ (the restriction of h to $y = x$). Then $\bar{h}$ is a polynomial different from zero, because h and f_2 are coprime. The polynomial $\bar{h}$ satisfies (8) restricted to $y = x$ (that is to $f_2 = 0$) and so

$$x(a+z)\frac{\partial \bar{h}}{\partial x} + (z + x^2)\frac{\partial \bar{h}}{\partial z} = ((n_1 + n_2)a + (n_1 - n_2)z)\bar{h}. \tag{10}$$

Equation (10) implies that $\bar{h}$ is a Darbox polynomial of system

$$\dot{x} = x(a+z), \quad \dot{z} = z + x^2,$$

with cofactor $(n_1+n_2)a+(n_1-n_2)z$. We will show that this is not possible.

Let $n \geq 1$ be the degree of $\bar{h}$. We expand $\bar{h} = \bar{h}(x, y) = \sum_{i=0}^{n} \bar{h}_i(x, y)$ where $\bar{h}_i$ is a homogeneous polynomial of degree i. Then it follows from (10) that $\bar{h}_n$ satisfies

$$xz\frac{\partial \bar{h}_n}{\partial x} + x^2\frac{\partial \bar{h}_n}{\partial z} = (n_1 - n_2)z\bar{h}_n. \tag{11}$$

The general solution of (11) is $\bar{h}_n = x^{n_1-n_2}\bar{H}_n(z^2 - x^2)$ where $\bar{H}_n$ is an arbitrary C^1 function, but since $\bar{h}_n$ must be a homogeneous polynomial of degree n, we have that without loss of generality we can take

$$\bar{h}_n = x^{n_1-n_2}(z^2 - x^2)^{(n-n_1+n_2)/2},$$

with $n - n_1 + n_2$ even.

Computing the terms of degree n in (10) we get

$$ax\frac{\partial \bar{h}_n}{\partial x} + xz\frac{\partial \bar{h}_{n-1}}{\partial x} + z\frac{\partial \bar{h}_n}{\partial z} + x^2\frac{\partial \bar{h}_{n-1}}{\partial z} = (n_1 + n_2)a\bar{h}_n + (n_1 - n_2)z\bar{h}_{n-1}.$$

Solving this partial differential equation we obtain

$$\begin{aligned}\bar{h}_{n-1} = x^{n_1-n_2}\bar{H}_{n-1}(z^2 - x^2) \mp x^{n_1-n_2}(z^2 - x^2)^{\frac{n-n_1+n_2}{2}-1}\Bigg((a-1)(n + n_2 - n_1)z \\ + (n + n_2 - 2an_2 - n_1)\sqrt{z^2 - x^2}\log\frac{x}{z^2 - x^2 + z\sqrt{z^2 - x^2}}\Bigg),\end{aligned}$$

where $\bar{H}_{n-1}$ is an arbitrary C^1 function, but since $\bar{h}_{n-1}$ must be a homogeneous polynomial of degree $n-1$ and $n - n_1 + n_2$ is even, then we get

$$\bar{h}_{n-1} = \pm 2a(a-1)n_2 z x^{n-2an_2}(z^2 - x^2)^{an_2-1},$$

with $n_1 = n + n_2 - 2an_2$ being an integer.

Computing the terms of degree $n-1$ in (10) we obtain

$$ax\frac{\partial \bar{h}_{n-1}}{\partial x} + xz\frac{\partial \bar{h}_{n-2}}{\partial x} + z\frac{\partial \bar{h}_{n-1}}{\partial z} + x^2\frac{\partial \bar{h}_{n-2}}{\partial z} = (n+2n_2-2an_2)a\bar{h}_{n-1} + (n-2an_2)z\bar{h}_{n-2}.$$

Solving this partial differential equation we have

$$\begin{aligned}\bar{h}_{n-2} = \mp x^{n-2an_2} 2an_2(a-1)(z^2 - x^2)^{an_2}((a-1)(an_2 - 1)x^2 + (z^2 - x^2)\log x) \\ + x^{n-2an_2}\bar{H}_{n-2}(z^2 - x^2),\end{aligned}$$

where $\bar{H}_{n-2}$ is an arbitrary C^1 function, but since $\bar{h}_{n-2}$ must be a homogeneous polynomial of degree $n-2$ and $a \notin \{0,1\}$ we must have $n_2 = 0$, in contradiction because $n_2 > 0$. This contradiction shows that if $n_2 > 0$ there are no exponential factors.

The case $n_1 > 0$ is treated analogously obtaining the theorem in this case.

Case 3*: $a = 1$ and $b \neq a$* In this case, proceeding in a similar manner as in Case 2 working with f_3 and f_4 instead of f_1 and f_2 we also reach to a contradiction and so there are no exponential factors also in this case. This concludes the proof of Theorem 4.

3. Proof of Theorem 5

Let G be a Darboux first integral of system (1).

If $b \neq a$ and $a \neq 1$, by Theorems 1 and 4 there are no Darboux first integrals.

If $b = a \notin \{0,1\}$ then it follows from Theorems 1 and 4 that $G = cf_1^{\lambda_1} f_2^{\lambda_2}$ with $c, \lambda_1, \lambda_2 \in \mathbb{C}$. Imposing that G is a first integral from statement (i) of Proposition 8.7 of [7] it must hold

$$a(\lambda_1 + \lambda_2) + z(\lambda_2 - \lambda_1) = 0.$$

This implies $\lambda_1 = \lambda_2 = 0$, so in this case a Darboux first integral does not exist.

Finally, if $a = 1$ with $b \neq 1$, it follows from Theorems 1 and 4 that $G = cf_3^{\lambda_3} f_4^{\lambda_4}$ with $c, \lambda_3, \lambda_4 \in \mathbb{C}$. Imposing that G is a first integral it must hold

$$\lambda_3 + \lambda_4 + y(\lambda_4 - \lambda_3) = 0.$$

This implies $\lambda_3 = \lambda_4 = 0$, so in this case a Darboux first integral does not exist. This concludes the proof of Theorem 5.

4. Proof of Theorem 7

Now we present the analysis of the flow of system (1) at infinity using the Poincaré compactification of the system in $\mathbb{R}^3$, for more details on the Poincaré compactification see [6]. The expression of the Poincaré compactification $p(X)$ of system (1) in the local chart U_1 is given by

$$\begin{aligned} \dot{z}_1 &= z_2 + (b-a)z_1 z_3 - z_1^2 z_2, \\ \dot{z}_2 &= z_1 + (1-a)z_2 z_3 - z_1 z_2^2, \\ \dot{z}_3 &= -z_3(a z_3 + z_1 z_2). \end{aligned} \tag{12}$$

System (12) has the five singular points

$$(-1,-1,0), \quad (1,-1,0), \quad (0,0,0), \quad (-1,1,0), \quad (1,1,0).$$

The eigenvalues of the Jacobian matrix at $(-1,-1,0)$ and $(1,-1,0)$ are $-2,-2,-1$ and $2,2,1$, respectively. Moreover, the eigenvalues of the Jacobian matrix at $(0,0,0)$ are $-1,1,0$, and $(0,0,0)$ restricted to infinity (that is to $z_3=0$) it is a saddle. Finally, the eigenvalues of the Jacobian matrix at $(-1,1,0)$ and $(1,1,0)$ are $-2,-2,-1$ and $2,2,1$, respectively.

The expression of the Poincaré compactification $p(X)$ of system (1) in the local chart U_2 is given by

$$\begin{aligned}\dot z_1 &= z_2+(a-b)z_1z_3-z_1^2z_2,\\ \dot z_2 &= z_1+(1-b)z_2z_3-z_1z_2^2,\\ \dot z_3 &= -z_3(bz_3+z_1z_2).\end{aligned} \tag{13}$$

In the local chart U_2 we only need to study the singular points satisfying $z_1=z_3=0$, because the other infinite singular points in this chart were already studied in the local chart U_1. System (13) coincides with system (12) changing b by a. So its origin is such that the eigenvalues of its associated Jacobian matrix are $-1,1,0$, and restricted to infinity is a saddle.

Finally, the expression of the Poincaré compactification $p(X)$ of system (2) in the local chart U_3 is

$$\begin{aligned}\dot z_1 &= z_2+(a-1)z_1z_3-z_1^2z_2,\\ \dot z_2 &= z_1+(b-1)z_2z_3-z_1z_2^2,\\ \dot z_3 &= -z_3(z_3+z_1z_2).\end{aligned}$$

In the local chart U_3 we only need to study the origin. We note that the origin is a singular point whose eigenvalues of its associated Jacobian matrix are $-1,1,0$, and $(0,0,0)$ restricted to infinity is a saddle.

The flow in the local charts V_i $(i=1,2,3)$ are the same as the flows in the local charts U_i $(i=1,2,3)$ reserving in an appropriate way the direction of the time.

Taking into account that the systems in the local charts U_1 and U_2 restricted to $z_3=0$ are

$$\begin{aligned}\dot z_1 &= z_2(1-z_1)(1+z_1),\\ \dot z_2 &= z_1(1-z_2)(1+z_2),\end{aligned}$$

we have that the straight lines $z_1=\pm1$ and $z_2=\pm1$ are invariant, and so the local phase portrait on the Poincaré sphere at infinity is topologically equivalent to the one drawn in Figure 1. This completes the proof of statement (a).

Consider the case $b=a\neq1$. We first study the dynamics on $x=y$. We have the system

$$\dot x = x(a+z), \quad \dot z = z+x^2. \tag{14}$$

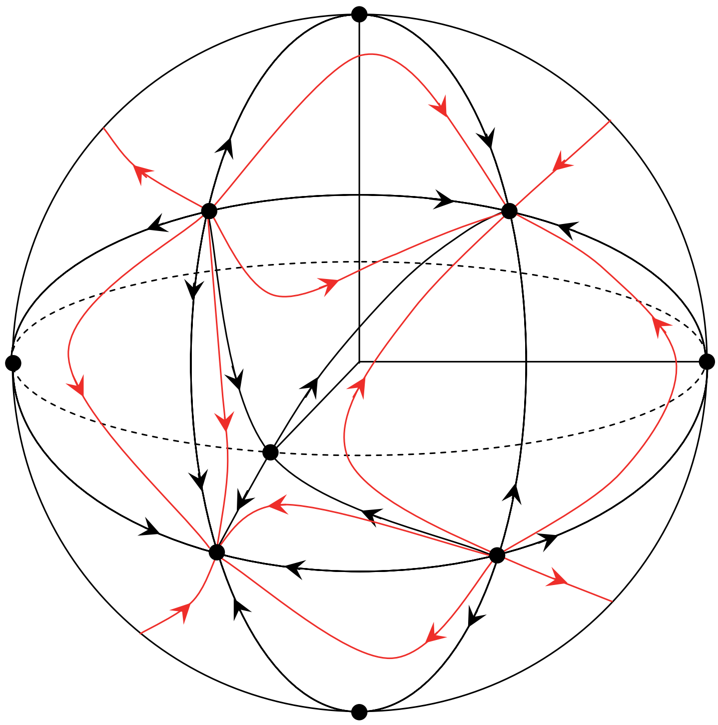

FIGURE 1. The phase portrait on the sphere $\mathbb{S}^2$ of the infinity. There are only drawn the orbits contained in the closure of the local chart U_1.

The unique singular points are $(0,0)$ and $(\pm\sqrt{a}, -a)$ that exist only when $a > 0$. The origin is a saddle if $a < 0$ and an unstable node if $a > 0$. When $a > 0$, the singular points $(\pm\sqrt{a}, -a)$ are both saddles.

The expression of the Poincaré compactification of system (14) in the local chart U_1 is

$$\dot{u} = 1 - u^2 + (1-a)uv, \quad \dot{z}_2 = -v(u + av). \tag{15}$$

System (23) restricted to $v = 0$ has the two infinite singular points $(\pm 1, 0)$. Moreover, $(1,0)$ is a stable node and $(-1,0)$ is an unstable node.

On the local chart U_2 system (14) becomes

$$\dot{u} = u + (a-1)uv + u^3, \quad \dot{v} = -v(v + u^2).$$

The origin is a singular point which is semihyperbolic. Using Theorem 2.19 in [7] we conclude that it is a saddle-node. Using that the straight line $x = 0$ is invariant it follows easily that the phase portraits in the Poincaré disc of system (14) are given in Figure 2 for $a > 0$ and in Figure 3 for $a < 0$.

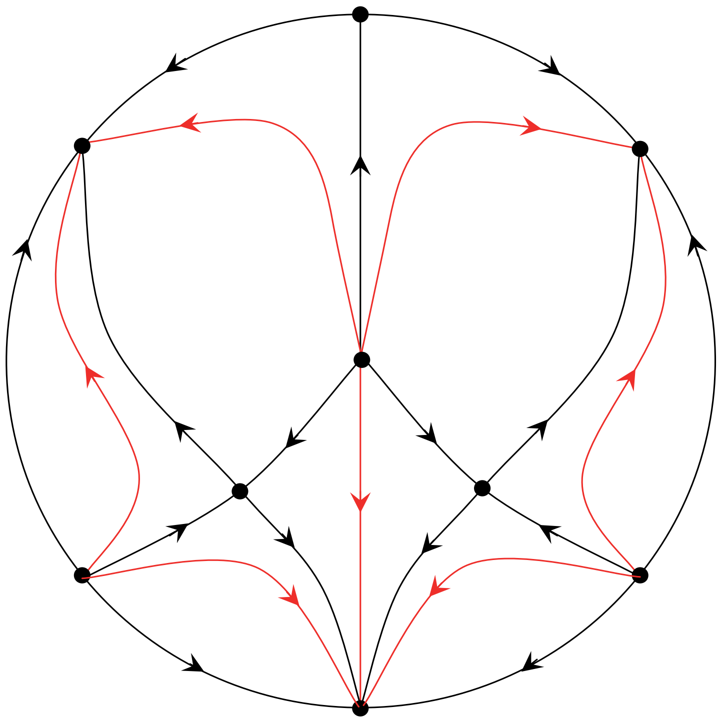

FIGURE 2. Phase portrait of system (14) for $a > 0$.

Now we study the dynamics on $x = -y$. We have the system

$$\dot{x} = x(a - z), \quad \dot{z} = z - x^2. \tag{16}$$

Introducing the change of variables $(X, Z) = (x, -z)$ we get that system (16) becomes (14), and so the phase portraits in the Poincaré disc of system (16) can be obtained from Figures 2 and 3 doing a symmetry with respect to the x-axis.

We recall that if an invariant algebraic surface $f(x, y, z) = 0$ has a constant cofactor $\kappa \in \mathbb{R}$, then $fe^{-\kappa t}$ is a Darboux invariant, see for a proof statement (vi) of Theorem 8.7 of [7]. Therefore, since system (1) with $b = a \neq 1$ has the invariant algebraic surface $x^2 - y^2 = 0$ with cofactor $2a$ it has the Darboux invariant $(x^2 - y^2)e^{-2at}$, and by Proposition 6 all the orbits outside the surface $x^2 - y^2 = 0$ has ω-limit at infinity if $a > 0$, and α-limit at infinity if $a < 0$. This completes the proof of the second part of statement (b).

Consider the case $a = 1$ with $b \neq 1$. We first study the dynamics on $z = x$. We have the system

$$\dot{x} = x(1 + y), \quad \dot{y} = by + x^2. \tag{17}$$

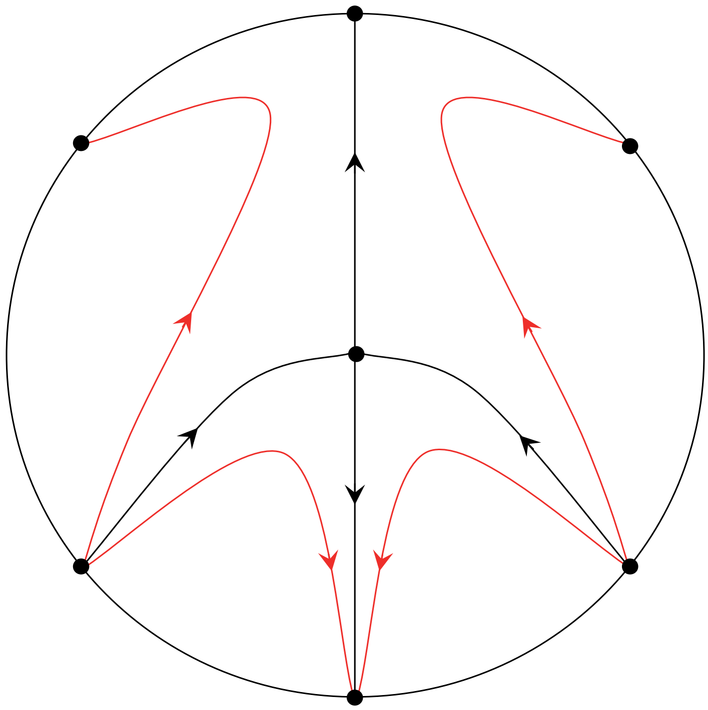

FIGURE 3. Phase portrait of system (14) for $a < 0$.

Introducing the reparameterization of time $(X, Z, T) \to (x/b, y/b, bt)$ we get that (17) becomes (14) with $a = 1/b \in \mathbb{R}\setminus\{1\}$. Therefore the phase portraits in the Poincaré disc of system (17) are topologically equivalent to the ones in Figures 2 for $b > 0$ and 3 for $b < 0$.

Now we study the dynamics on $z = -x$. We have the system

$$\dot{x} = x(1 - y), \quad \dot{y} = by - x^2. \tag{18}$$

Introducing the reparameterization of time $(X, Z, T) \to (x/b, -y/b, bt)$ we get that (18) becomes (14) with $a = 1/b \in \mathbb{R} \setminus \{1\}$. Therefore the phase portraits in the Poincaré disc of system (18) are topologically equivalent to the ones in Figures 2 for $b > 0$ and 3 for $b < 0$.

The second part of statement (c) is proved in a similar way to the second part of the proof of statement (b). This proof of statement (c) is completed.

5. Proof of Theorem 8

Now we present the analysis of the flow of system (2) at infinity using the Poincaré compactification of the system in $\mathbb{R}^3$. The expression of the

Poincaré compactification $p(X)$ of system (2) in the local chart U_1 is given by

$$\begin{aligned} \dot z_1 &= z_2 + (b-a)z_1z_3 - z_1^2z_2,\\ \dot z_2 &= -z_1 + (1-a)z_2z_3 - z_1z_2^2,\\ \dot z_3 &= -z_3(az_3 + z_1z_2). \end{aligned} \tag{19}$$

System (19) has the origin as its unique singular point. The eigenvalues of the Jacobian matrix at $(0,0,0)$ are $i,-i,0$. Moreover, restricted to the infinity (that is to $z_3=0$) it is a center because the Jacobian matrix at $(0,0)$ has eigenvalues $\pm i$ and system (19) restricted to $z_3=0$ has the rational first integral $(1+z_2^2)/(1-z_1^2)$ defined in a neighborhood of the origin.

The expression of the Poincaré compactification $p(X)$ of system (2) in the local chart U_2 is

$$\begin{aligned} \dot z_1 &= z_2 + (a-b)z_1z_3 - z_1^2z_2,\\ \dot z_2 &= -z_1 + (1-b)z_2z_3 - z_1z_2^2,\\ \dot z_3 &= -z_3(bz_3 + z_1z_2). \end{aligned} \tag{20}$$

System (20) coincides with system (19) interchanging b by a. So its unique infinite singular point is the origin that restricted to $z_3=0$ is a center.

Finally, the expression of the Poincaré compactification $p(X)$ of system (2) in the local chart U_3 is

$$\begin{aligned} \dot z_1 &= z_2 + (a-1)z_1z_3 + z_1^2z_2,\\ \dot z_2 &= z_1 + (b-1)z_2z_3 + z_1z_2^2,\\ \dot z_3 &= z_3(-z_3 + z_1z_2). \end{aligned} \tag{21}$$

The origin of system (21) restricted to $z_3=0$ is a singular point whose eigenvalues of its associated Jacobian matrix are $-1,1,0$ and restricted to infinity is a saddle.

The flow in the local charts V_i $(i=1,2,3)$ are the same as the flows in the local charts U_i $(i=1,2,3)$ reserving in an appropriate way the direction of the time.

Taking into account the eigenvectors and eigenvalues of the saddle at the origin of U_3, the local phase portrait in the Poincaré sphere at infinity is topologically equivalent to the one drawn in Figure 4. This completes the proof of statement (a).

We consider the case in which $b=a\neq 1$. In this case system (2) has two invariant algebraic surfaces $x-y=0$ and $x+y=0$. We will study the dynamics in both of them.

We first study the dynamics on $x=y$. We have the system

$$\dot x = x(a+z), \quad \dot z = z - x^2. \tag{22}$$

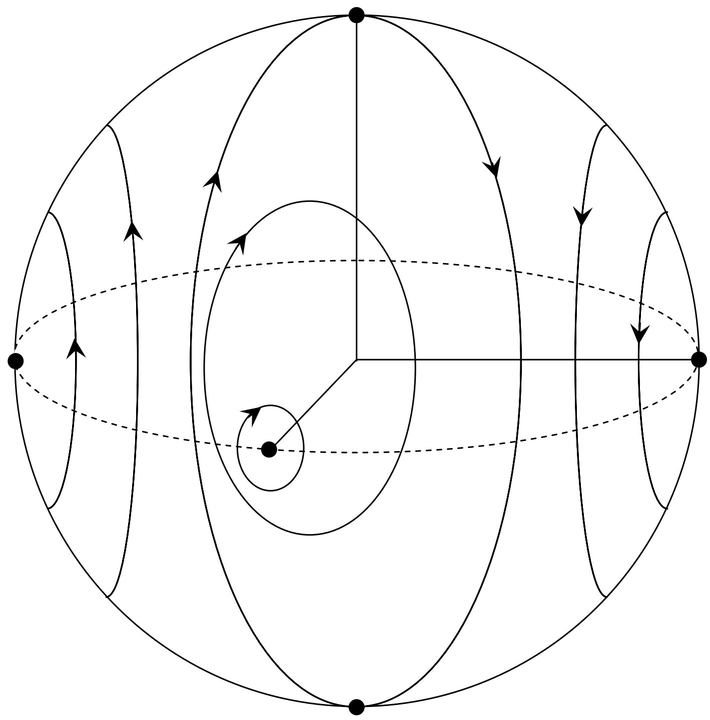

FIGURE 4. The phase portrait on the sphere $\mathbb{S}^2$ of the infinity. There are only drawn the orbits contained in the closure of the local chart U_1.

The unique singular points are $(0,0)$ and $(\pm\sqrt{-a}, -a)$ that exist only when $a < 0$. The origin is a saddle if $a < 0$ and an unstable node if $a > 0$. When $a < 0$, the singular points $(\pm\sqrt{-a}, -a)$ are both unstable nodi when $1 + 8a \geq 0$ and unstable foci when $1 + 8a < 0$.

The expression of the Poincaré compactification of system (22) in the local chart U_1 is

$$\dot{u} = -1 - u^2 + (1 - a)uv, \quad \dot{z}_2 = -v(u + av). \tag{23}$$

System (23) restricted to $v = 0$ has no singular points. On the local chart U_2 system (22) becomes

$$\dot{u} = u + u^3 + (a - 1)uv, \quad \dot{v} = v(-v + u^2).$$

The origin is a singular point which is semihyperbolic. Using Theorem 2.19 in [7] we conclude that it is a saddle-node.

Combining the above analysis in the finite part and at each local chart at infinity and taking into account that the straight line $x = 0$ is invariant we obtain for $a > 0$ (resp. $a < 0$) the phase portrait of Figure 3 doing a symmetry with respect to the x-axis (resp. the phase portrait of Figure 5).

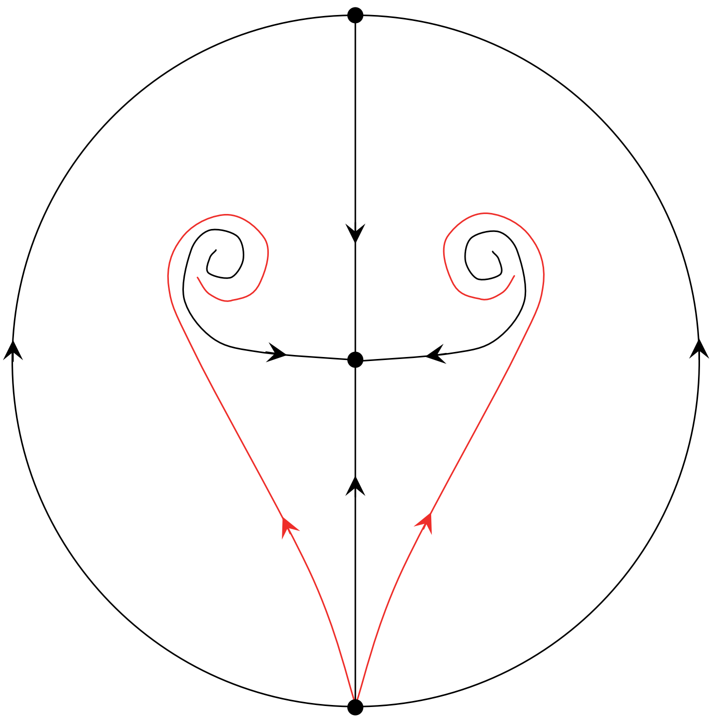

FIGURE 5. Phase portrait of system (24) for $a < 0$.

Now we study the dynamics on $x = -y$. We have the system

$$\dot{x} = x(a - z), \quad \dot{z} = z + x^2. \tag{24}$$

Introducing the change of variables $(X, Z) = (x, -z)$ we get that system (24) becomes system (14), and so the phase portraits in the Poincaré disc of system (24) are the ones of Figure 3 for $a > 0$ and the ones of Figure 5 doing a symmetry with respect to the x-axis for $a < 0$.

The second part of the proof of statement (b) of the theorem follows exactly as the second part of the proof of statement (b) of Theorem 7. The proof of statement (b) is completed.

Now we note that when $a = 1$ with $b \neq 1$ the dynamics of system (2) on $x^2 + z^2 = 0$ reduces to the straight line $x = z = 0$ containing a unique orbit solution of $\dot{y} = by$, $b \neq 1$. The eigenvalues at the origin are $1, b, 1$. Assume first that $b > 0$. Then the origin is a repeller. By Proposition 6, and since system (2) has the Darboux invariant $(x^2 + z^2)e^{-2t}$ all the orbits outside the y-axis end at infinity, and start either at the origin or eventually at the endpoints of the y-axis. Assume now that $b < 0$. Then the origin has a one-dimensional stable manifold and a 2-dimensional unstable manifold. As before all the orbits outside the y-axis end at infinity. But by Proposition 6 all these orbits start at infinity at the endpoints of the y-axis except the ones which are on the two-dimensional unstable manifold at the origin. This completes the proof of statement (c).

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