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Minimal set of periods for continuous self–maps of a bouquet of circles

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Let G_k be a bouquet of circles; i.e. the quotient space of the interval $[0,k]$ obtained by identifying all points of integer coordinates to a single point, called the branching point of G_k. Thus, G_1 is the circle, G_2 is the eight space and G_3 is the trefoil. Let $f : G_k \to G_k$ a continuous map such that for $k > 1$ the branching point is fixed.
If $\operatorname{Per}(f)$ denotes the set of periods of f, the minimal set of periods of f, denoted by $\operatorname{MPer}(f)$, is defined as $\bigcap_{g \simeq f} \operatorname{Per}(g)$ where $g : G_k \to G_k$ is homological to f.
The sets $\operatorname{MPer}(f)$ are well–known for circle maps. Here, we classify all the sets $\operatorname{MPer}(f)$ for self–maps of the eight space.

Keywords: periods, periodic orbits, eight space, continuous maps, set of periods.

1. Introduction and statement of the results

In dynamical systems it is often the case that topological information can be used to study qualitative or quantitative properties of the system. This work deals with the problem of determining the set of periods of the periodic orbits of a map given the homology class of the map.

A *finite graph* (simply a *graph*) G is a topological space formed by a finite set of points V (points of V are called *vertices*) and a finite set of open arcs (called *edges*) in such a way that each open arc is attached by its endpoints to vertices. An open arc is a subset of G homeomorphic to the open interval $(0,1)$. Note that a finite graph is compact, since it is the union of a finite number of compact subsets (the closed edges and the vertices). Notice that a closed edge is homeomorphic either to the closed interval [0,1], or to the circle. It may be either connected or disconnected, and it may have isolated vertices.

The *valence* of a vertex is the number of edges with the vertex as an endpoint (where the closed edges homeomorphic to a circle are counted twice). The vertices with valence 1 of a connected graph are *endpoints* of the graph and the vertices with valence larger than 2 are *branching points*.

Suppose that $f : G \to G$ is a continuous map, in what follows a *graph map*. A *fixed point* of f is a point x in G such that $f(x) = x$. We will call x a *periodic point of period n* if x is a fixed point of f^n but it is not fixed by any f^k for $1 \le k < n$. We denote by $\operatorname{Per}(f)$ the set of natural numbers corresponding to periods of the periodic points of f.

Let G be a connected graph and let f be a graph map. Then f induces endomorphisms $f_{*n} : H_n(G) \to H_n(G)$ (for n=0,1) on the integral homology groups of G, where $H_0(G) \approx \mathbb{Z}$ (because G is connected) and $H_1(G) \approx \mathbb{Z} \oplus \overset{k}{\ldots} \oplus \mathbb{Z}$ where k is the number of independent circuits or loops of G as elements of $H_1(G)$. A *circuit* of G is a subset of G homeomorphic to the circle. The endomorphisms f_{*0} and f_{*1} are represented by integer matrices. Furthermore, since G is connected f_{*0} is the identity.

The endomorphism f_{*1} will play a main role in our analysis of the minimal sets of periods for graph maps on G. *In what follows f_{*1} will be denoted by f_*.* For example, if $H_1(G) \approx \mathbb{Z} \oplus \mathbb{Z}$ and

$$f_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

this means that the graph G has two independent oriented circuits. Moreover, if the first circuit covers itself exactly a_1 times following the same orientation (not necessarily in a consecutive way) and exactly a_2 times following the converse orientation (not necessarily in a consecutive way), then $a = a_1 - a_2$. Similarly, if the first circuit covers the second one exactly b_1 times following the same orientation (not necessarily in a consecutive way) and exactly b_2 times following the converse orientation (not necessarily in a consecutive way), then $b = b_1 - b_2$. An analogous explanation can be given with the second independent circuit and with b and d instead of c and a, respectively.

Let G_k be a *bouquet of k circles*, that is, the quotient space of $[0, k]$ obtained by identifying all points of integer coordinates to a single point. Notice that G_1 is the *circle* and that G_2 is usually called the *eight space*. For the G_k graph we have $H_0(G_k) \approx \mathbb{Z}$, $H_1(G_k) \approx \mathbb{Z} \oplus \overset{k}{\ldots} \oplus \mathbb{Z}$, $f_{*0} \approx id$ and $f_{*1} = f_* = A$, where A is a $k \times k$ integral matrix. For more details on graph maps see Llibre [1991] or Llibre & Sá [1995].

Our main goal is to study the set $\mathrm{Per}(f)$ for graph maps. More explicitly, we want to provide a description of the minimal set of periods (see below) attained within the homology class of a given graph map. When the map $g : G \to G$ is homological to f (i.e. g induces the same endomorphisms than f on the homology groups of G), we shall write $g \simeq f$. We define the *minimal set of periods* of f to be the set

$$\mathrm{MPer}(f) = \bigcap_{g \simeq f} \mathrm{Per}(g).$$

From its definition $\mathrm{MPer}(f)$ is the maximal subset of periods contained in $\mathrm{Per}(g)$ for all $g \simeq f$.

Our main objective is to *characterize the minimal sets of periods* $\mathrm{MPer}(f)$ *for graph maps* $f : G_i \to G_i$ with the branching point a fixed point for $i = 2, 3$. So, always $1 \in \mathrm{MPer}(f)$. Even for circle maps $f : G_1 \to G_1$ the characterization of all minimal sets of periods $\mathrm{MPer}(f)$ is interesting and nontrivial, see Theorem A. This result was stated by Efremova [Efremova, 1978] and Block, Guckenheimer, Misiurewicz and Young [Block *et al.*, 1980] without giving a complete proof. As far as we know the first complete proof was given in Alsedà *et al.* [2000].

We denote by $\mathbb{N}$ the set of all natural numbers, and by k$\mathbb{N}$ the set $\{kl : l \in \mathbb{N}\}$.

Theorem A. *Let $f : G_1 \to G_1$ be a circle map such that the endomorphism induced by f on the first homology group is $f_* = (d)$ (i.e. d is the degree of f). Then the following statements hold.*

(a) If $d \notin \{-2, -1, 0, 1\}$, then $\mathrm{MPer}(f) = \mathbb{N}$.
(b) If $d = -2$, then $\mathrm{MPer}(f) = \mathbb{N} \setminus \{2\}$.
(c) If $d \in \{-1, 0\}$, then $\mathrm{MPer}(f) = \{1\}$.
(d) If $d = 1$, then $\mathrm{MPer}(f) = \emptyset$.

In the next theorem we characterize the minimal sets of periods for eight maps, i.e. for continuous maps $f : G_2 \to G_2$.

Theorem B. *Let $f : G_2 \to G_2$ be an eight map such that*

$$f_* = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Suppose that the branching point is a fixed point. Then the following statements hold.

(a) If $\{a,d\} \not\subset \{-2,-1,0,1\}$*, then* $\mathrm{MPer}(f)=\mathbb{N}$.
(b) If $-2 \in \{a,d\}$ *and* $\{a,d\} \subset \{-2,-1,0,1\}$*, then*

$$\mathrm{MPer}(f) = \begin{cases} \mathbb{N}\setminus\{2\} & \textit{if } bc=0, \\ \mathbb{N} & \textit{if } bc \neq 0. \end{cases}$$

(c) Assume that $\{a,d\} \subset \{-1,0,1\}$.
(c1) If $|a|+|d|=2$*, then*

$$\mathrm{MPer}(f) = \begin{cases} \{1\} & \textit{if } bc=0, \\ \mathbb{N}\setminus\{2\} & \textit{if } bc=1, \\ \mathbb{N} & \textit{if } bc=-1 \textit{ or } |bc|>1. \end{cases}$$

(c2) If $|a|+|d|=1$ *and*
(c21) $a=1,\ d=0$*, then*

$$\mathrm{MPer}(f) = \begin{cases} \{1\} & \textit{if } bc=0, \\ \mathbb{N}\setminus\{2\} & \textit{if } (b,c)\in R, \\ \mathbb{N} & \textit{otherwise;} \end{cases}$$

where R={(1,1),(-1,-1),(1,2),(-1,-2)}.
(c22) $a=0,\ d=1$*, then it follows (c21) interchanging b and c.*
(c23) $a=-1,\ d=0$*, then*

$$\mathrm{MPer}(f) = \begin{cases} \{1\} & \textit{if } bc=0, \\ \mathbb{N}\setminus\{2\} & \textit{if } (b,c)\in R, \\ \mathbb{N}\setminus\{3\} & \textit{if } bc=-1, \\ \mathbb{N} & \textit{otherwise.} \end{cases}$$

(c24) $a=0,\ d=-1$*, then it follows (c23) interchanging b and c.*
(c3) If $|a|+|d|=0$*, then*

$$\mathrm{MPer}(f) = \begin{cases} \{1\} & \textit{if } bc=0 \textit{ or } bc=1, \\ \{1,2\} & \textit{if } bc=-1, \\ \{1\}\cup(2\mathbb{N}\setminus\{2\}) & \textit{if } bc=2, \\ \{1\}\cup(2\mathbb{N}\setminus\{4\}) & \textit{if } bc=-2, \\ \{1\}\cup 2\mathbb{N} & \textit{if } |bc|>2. \end{cases}$$

We remark that Theorem B implies Theorem A if f has a fixed point, by choosing, for instance, $a=b=c=0$.

The study of the minimal set of periods of a homotopy class of maps instead of its homology class is the main objective of the fixed point theory, see for instance the books of Brown [Brown, 1971], Jiang [Jiang, 1983] and Kiang [Kiang, 1989]. Other extensions from circle maps to n–dimensional torus has been done in [Alsedà *et al.*, 1995] and [Jiang & Llibre, 1998], and from circle maps to transversal n–sphere maps in [Casasayas *et al.*, 1995]. Some different results on the periods of graph maps have been given in [Abdulla *et al.*, 2017; Alsedà *et al.*, 2005; Arai, 2016; Alsedà & Ruette, 2008; Bernhardt, 2006, 2011; Llibre, 1991; Llibre & Misiurewicz, 2006].

This work is organized as follows. How to obtain a given period for a graph map by using the notion of f–covering is described in Section 2. The proof of Theorem B is given in Section 3.

2. Periods and f–covering

Let $f: G \to G$ be a graph map and $x \in G$ a periodic point of period n. The set $\{x, f(x), \ldots, f^{n-1}(x)\}$ is called the *periodic orbit* of x.

A set $I \subset G$ will be called an *interval* if there is a homeomorphism $h: J \to I$ where J is $[0,1]$, $(0,1]$, $[0,1)$ or $(0,1)$. The set $h((0,1))$ will be called the *interior* of I. If $J=[0,1]$ the interval I will be called *closed*; if $J = (0, 1)$ it will be called *open*. Notice that it may happen that the above terminology does not coincide with the one used when we think about I as a subset of G (the same applies to the edges of G). For example, if $G = I = [0, 1]$ and h = identity, then for I regarded as a subset of the topological space G, I is both open and closed and the interior of I is I.

Let C_1 and C_2 be two circuits of G_k. A closed interval $I = [a, b]$ is *basic* if $I \subset C_i$, $f(I) = C_j$ where $\{i, j\} \subset \{1, 2, \ldots, k\}$, $f(a) = f(b) = p$, where p is the branching point of G_k, and there is no other closed interval $K \subsetneq I$ such that $f(K) = C_j$. If $f(C_i) = C_j$ and $f(K) \neq C_j$ for all closed interval K, $K \subset C_i$, then we also say that C_i is a basic interval. Let I and J be two basic intervals, $K \subset I$, $L \subset J$ two subintervals. If $L \neq C_j$, we say that K *f–covers* L, and we write $K \to L$, if there exists a closed subinterval M of K such that $f(M) = L$. If $L = J = C_j$, we say that $K = I$ f–covers L because either $f(K) = L$, or $K = I = C_i$ and $f(K) = L$, by the definition of basic intervals.

Lemma 1. *Suppose that I_1, I_2, …, I_n are intervals such that $I_1 \to I_2 \to \ldots \to I_n \to I_1$ with I_1 different from a circuit. Then there is a fixed point z of f^n such that $z \in I_1$, $f(z) \in I_2$, …, $f^{n-1}(z) \in I_n$.*

Proof. Since $I_n \to I_1$, and I_1 is not a circuit, there is a closed interval $J_n \subset I_n$ such that $f(J_n) = I_1$. Similarly, there are closed intervals or circuits $J_1, \ldots, J_{n-1}$ such that for each $k = 1, \ldots, n-1$, $J_k \subset I_k$ and $f(J_k) = J_{k+1}$. It follows that $f^n(J_1) = I_1$ and since $J_1 \subset I_1$ and I_1 is not a circuit, by Bolzano's Theorem f^n has a fixed point $z \in J_1$. Clearly, $z \in I_1$, $f(z) \in I_2$, …, $f^{n-1}(z) \in I_n$. ■

A sequence of the form $I_1 \to I_2 \to \ldots \to I_n \to I_1$ is called a *loop of length n*. Let $I_1 \to I_2 \to \ldots \to I_n \to I_1$ and $J_1 \to J_2 \to \ldots \to J_m \to J_1$ be two loops such that $I_1 = J_1$. We define the *concatenation* of these two loops as the loop $I_1 \to I_2 \to \ldots \to I_n \to I_1 \to J_2 \to \ldots \to J_m \to I_1$. We say that a loop is a m–repetition, $m \geq 2$, of a given loop if it is the concatenation of that loop with itself m times. We say that a loop is *non–repetitive* if it is not a m–repetition of any of its subloops with $m \geq 2$.

In what follows a *G_k–map* f is a continuous map $f : G_k \to G_k$ such that $f(p) = p$.

Proposition 1. *Let f be a G_k–map. Suppose that f has two intervals I_1 and I_2 such that Int$(I_1) \cap$ Int$(I_2) = \emptyset$ and $I_1 \cap I_2$ has no fixed points. If f has the subgraph $\circlearrowleft I_1 \rightleftarrows I_2 \circlearrowright$, then* $\mathrm{Per}(f) = \mathbb{N}$.

Proof. Clearly, since $p \notin I_1 \cap I_2$, at least one of the intervals, I_1 and I_2, is not a circuit. Without loss of generality we assume that I_1 is not a circuit. We consider the non–repetitive loop $I_1 \to I_2 \to I_1 \to \ldots \to I_1$ of length $n \geq 2$. Since $\mathrm{Int}(I_1) \cap \mathrm{Int}(I_2) = \emptyset$ and $I_1 \cap I_2$ has no fixed points, by Lemma 1 there is a periodic point z of f with period $n \geq 2$. That is, $\mathrm{Per}(f) = \mathbb{N}$. ■

In what follows when we say "we have two intervals A and B" we are really saying that we have two different intervals A and B. We remark that if we have two basic intervals I_1 and I_2 such that $p \notin I_1 \cap I_2$, then they satisfy the assumptions of Proposition 1.

Proposition 2. *Let f be a G_k–map. Suppose that f has three intervals I_1, I_2 and I_3 such that Int$(I_i) \cap$ Int$(I_j) = \emptyset$ for all $i \neq j$ and $I_i \cap I_j$ has no fixed points for some $i \neq j$. If f has the subgraph $\circlearrowleft I_1 \to I_2 \to I_3 \to I_1$ then* $\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2\}$. *Moreover, if $I_2 \cap I_3 = \emptyset$ and $I_3 \to I_2$, then* $2 \in \mathrm{Per}(f)$.

Proof. We consider the non–repetitive loop $I_1 \to I_2 \to I_3 \to I_1 \to \ldots \to I_1$ of length $n \geq 3$. Since $\mathrm{Int}(I_i) \cap \mathrm{Int}(I_j) = \emptyset$ for all $i \neq j$ and $I_i \cap I_j$ has no fixed points for some $i \neq j$, by Lemma 1 there is a periodic point z of f with period $n \geq 3$. Therefore, $\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2\}$.

We suppose now that $I_2 \cap I_3 = \emptyset$ and $I_3 \to I_2$. We consider the non–repetitive loop $I_2 \to I_3 \to I_2$ of length 2. By Lemma 1 there is a periodic point z of f with period 2. ■

We remark that if we have three basic intervals I_1, I_2 and I_3 such that $p \notin I_i$ for some $i \in \{1, 2, 3\}$, then we are in the assumptions of Proposition 2.

3. The eight

In this section we shall prove Theorem B. The two circuits of G_2 are denoted by C_1 and C_2. If f_* is given as in Theorem B, we consider that the circuit C_1 covers itself $|a|$ times and it covers C_2 $|c|$ times. Similarly for the circuit C_2.

Proof. [Proof of Statement (a) of Theorem B] Suppose that $\{a,d\} \not\subset \{-2,-1,0,1\}$.

Case 1: Assume that $\{\mathbf{a},\mathbf{d}\} \not\subset \{-\mathbf{2},-\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2}\}$. Without loss of generality, we may assume that $|a| \geq 3$. From the graph of f (see for instance Figure 1), it is clear that there are two basic intervals I_1 and I_2, in C_1, such that $p \notin I_1 \cap I_2$ and f has the subgraph of Proposition 1, so $\mathrm{Per}(f) = \mathbb{N}$. That is, $\mathrm{MPer}(f) = \mathbb{N}$.

(a) $a=3,\ d=-1$ (b) $a=3,\ d=-1$ (c) $a=-3,\ d=0$

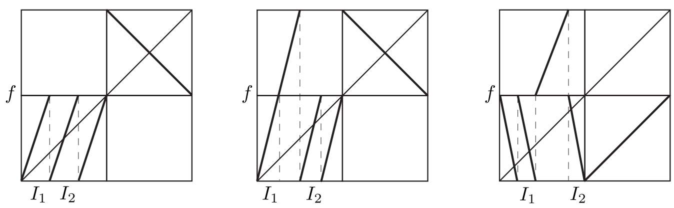


Fig. 1. Examples of maps with $\{a,d\} \not\subset \{-2,-1,0,1,2\}$.

Case 2: Suppose that $\mathbf{2} \in \{\mathbf{a},\mathbf{d}\}$ and $\{\mathbf{a},\mathbf{d}\} \subset \{-\mathbf{2},-\mathbf{1},\mathbf{0},\mathbf{1},\mathbf{2}\}$. Without loss of generality, we may assume that $a = 2$.

Since $a = 2$ this means that f has at least two basic intervals I_1 and I_2 in C_1 such that f has the subgraph of Proposition 1. If $p \notin I_1 \cap I_2$, then, by Proposition 1 $\mathrm{Per}(f) = \mathbb{N}$. But not always I_1 and I_2 satisfy that $p \notin I_1 \cap I_2$. In this case let p and a_0 be the endpoints of I_1, b_0 and p the endpoints of I_2 (see for instance Figure 2).

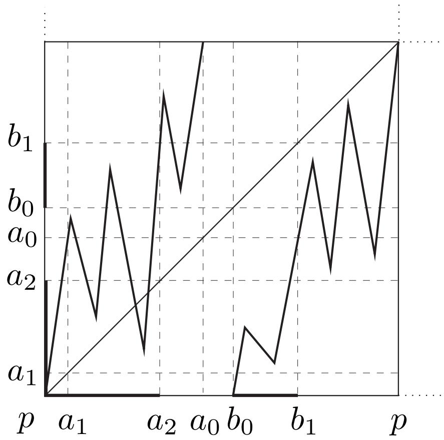
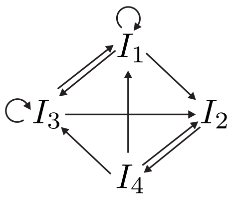


Fig. 2. $I_{1_1} = [p, a_1]$, $I_{2_1} = [b_0, b_1]$ and $I_{1_3} = [a_1, a_2]$.

We establish an ordering in the intervals I_1 and I_2 in such a way that p is the smallest element of I_1 and the greatest of I_2. Set $I_1 = [p, a_0]$ and $I_2 = [b_0, p]$. Notice that we may have $a_0 = b_0$. Consider the subset $(f|I_1)^{-1}(a_0)$ of C_1. Let a_1 be the infimum of the points in $(f|I_1)^{-1}(a_0)$. Consider the subset $(f|I_2)^{-1}(a_0)$ of C_1 and choose b_1 to be the infimum of the points in $(f|I_2)^{-1}(a_0)$. Set $I_{1_1} = [p, a_1]$, $I_{1_2} = [a_1, a_0]$ and $I_{2_1} = [b_0, b_1]$. Now we take the interval $I_{1_3} = [a_1, a_2]$ where a_2 denotes the infimum of the points in the subset $(f|I_{1_2})^{-1}(b_1)$ of C_1. Then f has the subgraph $\circlearrowleft I_{1_1} \to I_{1_3} \rightleftarrows I_{2_1} \to I_{1_1}$. Since $I_{2_1} \cap I_{1_3} = \emptyset$, by Proposition 2, $n \in \mathrm{Per}(f)$, for all $n \geq 1$. Therefore, $\mathrm{MPer}(f) = \mathbb{N}$. This proves Statement (a). ■

Proof. [Proof of Statement (b) of Theorem B] Suppose that $-\mathbf{2} \in \{\mathbf{a},\mathbf{d}\}$ and $\{\mathbf{a},\mathbf{d}\} \subset \{-\mathbf{2},-\mathbf{1},\mathbf{0},\mathbf{1}\}$. Without loss of generality, we may assume that $\mathbf{a} = -\mathbf{2}$.

First we suppose that $\mathbf{bc} \neq \mathbf{0}$. We always have four basic intervals I_1, I_2, I_3 and I_4, I_1, I_2, $I_3 \subset C_1$ and $I_4 \subset C_2$, such that either $p \notin I_1 \cap I_3$ or $I_2 \cap I_4 = \emptyset$ and f has the subgraph



(see for instance Figure 3).

(a) $a{=}-2, d{=}\ 0, bc{=}\ 1$ (b) $a{=}-2, d{=}\ 1, bc{=}\ 1$ (c) $a{=}-2, d{=}-1, bc{=}-1$

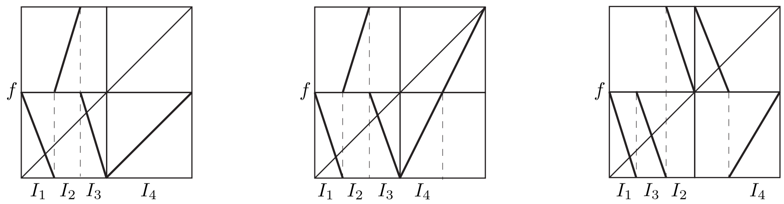


Fig. 3. Examples of maps with $a = -2$ and $bc \neq 0$.

If $p \notin I_1 \cap I_3$, by Proposition 1, $\mathrm{Per}(f) = \mathbb{N}$. If $I_2 \cap I_4 = \emptyset$, by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$. Therefore, if $bc \neq 0$, $\mathrm{MPer}(f) = \mathbb{N}$.

We suppose now that $\mathbf{bc} = \mathbf{0}$. As it can be deduced from the examples of Figure 4, $2 \notin \mathrm{MPer}(f)$.

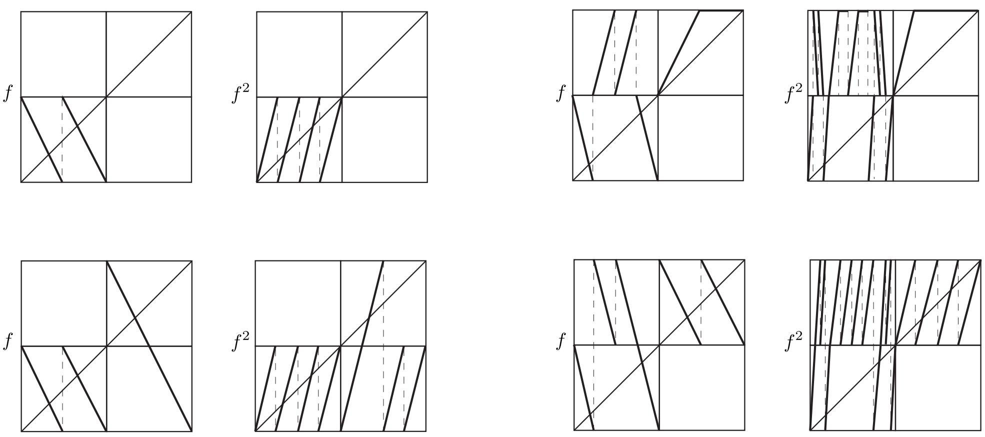


Fig. 4. Examples of maps with $a = -2$, $d \in \{-2, -1, 0, 1\}$, $bc = 0$ and $2 \notin \mathrm{Per}(f)$.

Since $a = -2$, this means that f has at least two basic intervals I_1 and I_2 in C_1 such that f has the subgraph of Proposition 1. If $p \notin I_1 \cap I_2$ then by Proposition 1 $\mathrm{Per}(f) = \mathbb{N}$. But not always $p \notin I_1 \cap I_2$. In this case let p and a_0 be the endpoints of I_1, b_0 and p the endpoints of I_2 (see for instance Figure 5). We consider an ordering in the intervals I_1 and I_2 in such a way that p is the smallest element of I_1 and the greatest of I_2. Write $I_1 = [p, a_0]$ and $I_2 = [b_0, p]$. Notice that we may have $a_0 = b_0$. Consider the subsets $(f|I_1)^{-1}(a_0)$ and $(f|I_2)^{-1}(a_0)$ of C_1. Let a_1 be the infimum of the points in $(f|I_1)^{-1}(a_0)$ and b_1 the infimum of the points in $(f|I_2)^{-1}(a_0)$. Set $I_{1_1} = [p, a_1]$, $I_{1_2} = [a_1, a_0]$ and $I_{2_1} = [b_1, p]$. Then f has the subgraph $\circlearrowleft I_{1_2} \to I_{1_1} \to I_{2_1} \to I_{1_2}$. Since we are in the assumptions of Proposition 2, $n \in \mathrm{Per}(f)$, for all $n \neq 2$. Therefore, $\mathrm{MPer}(f) = \mathbb{N} \setminus \{2\}$. This proves Statement (b). ■

Proof. [Proof of Statement (c1) of Theorem B] Suppose that $\{\mathbf{a}, \mathbf{d}\} \subset \{-\mathbf{1}, \mathbf{0}, \mathbf{1}\}$ and $|\mathbf{a}| + |\mathbf{d}| = \mathbf{2}$. We consider first the case $\mathbf{bc} = \mathbf{0}$. Without loss of generality, we may assume that $c = 0$. From the examples

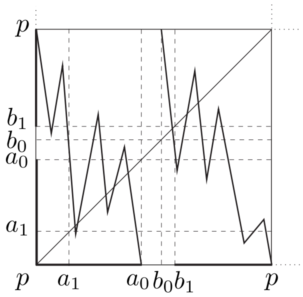


Fig. 5. $I_{1_1} = [p, a_1]$, $I_{1_2} = [a_1, a_0]$ and $I_{2_1} = [b_1, p]$.

of Figure 6 it is clear that $n \notin \mathrm{MPer}(f)$ for any $n \in \mathbb{N}$ larger than 1, so $\mathrm{MPer}(f) = \{1\}$ since the branching is fixed.

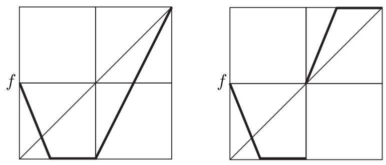


Fig. 6. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $bc = 0$.

We assume now that $|\mathbf{bc}| > \mathbf{1}$. From the graph of f (see for instance Figure 7) it is easy to see that we always have three basic intervals I_1, I_2 and I_3, with I_1, $I_2 \subset C_1$ and $I_3 \subset C_2$, such that $p \notin I_i$ for some $i \in \{1, 2, 3\}$ and f has the subgraph of Proposition 2, so $\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2\}$. Now we will prove that $2 \in \mathrm{MPer}(f)$.

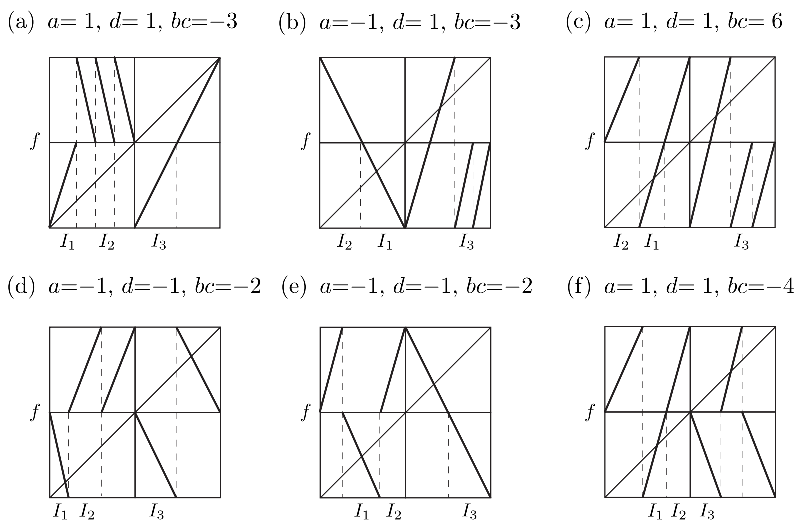


Fig. 7. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $|bc| > 1$.

If $\{\mathbf{b}, \mathbf{c}\} \not\subset \{-\mathbf{2}, -\mathbf{1}, \mathbf{1}, \mathbf{2}\}$, that is, if either $|b| \geq 3$ or $|c| \geq 3$, we can choose I_2 in one circuit and I_3 in the other circuit in such a way that $I_2 \cap I_3 = \emptyset$ (see (a), (b) and (c) of Figure 7) and $I_3 \to I_2$. By Proposition 2, $2 \in \mathrm{Per}(f)$. If $\{\mathbf{b}, \mathbf{c}\} \subset \{-\mathbf{2}, -\mathbf{1}, \mathbf{1}, \mathbf{2}\}$ in general there do not exist two basic intervals I_i and I_j, $I_i \neq I_j$, such that $p \notin I_i \cap I_j$ and $I_i \rightleftarrows I_j$ (see (e) and (f) of Figure 7). If they exist then by Lemma 1 considering the non–repetitive loop $I_i \to I_j \to I_i$ there is a periodic point z of f with period 2. If they do not exist, we shall find two intervals with empty intersection such that one f–covers the other.

We suppose first that $|bc| = 2$. We may assume, without loss of generality, that $|b| = 1$ and $|c| = 2$. We know that f has five basic intervals, I_1, I_2, I_3, I_4 and I_5, the first three in C_1 and the other two in C_2, such that $f(I_2) = f(I_3) = f(I_5) = C_2$ and $f(I_1) = f(I_4) = C_1$. Let p and a_0 be the endpoints of I_2, a_0 and a_1 the endpoints of I_1, a_1 and p the endpoints of I_3 (see for instance Figure 8).

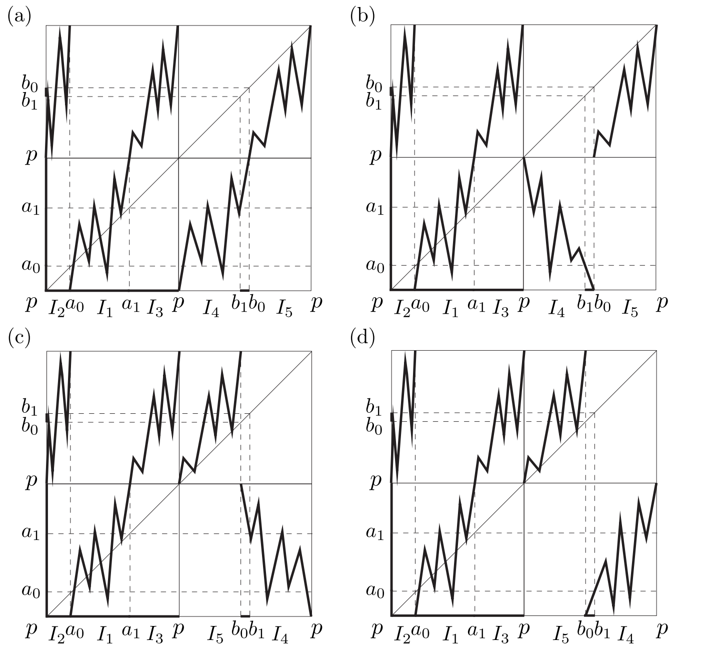


Fig. 8. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $|bc| = 2$.

We consider an ordering in the intervals I_1, I_2 and I_3 in such a way that p is the smallest element of I_2 and the greatest of I_3. Set $I_2 = [p, a_0]$, $I_1 = [a_0, a_1]$ and $I_3 = [b_0, p]$. We have two possibilities for the interval I_4: either $I_4 = [p, b_0]$ or $I_4 = [b_0, p]$. If $I_4 = [p, b_0]$ and $b = 1$ let b_1 be the supremum of the points in $(f|I_4)^{-1}(a_1)$ and $I_{4_2} = [b_1, b_0]$. We have $I_{4_2} \rightleftarrows I_3$ and $I_{4_2} \cap I_3 = \emptyset$, so, by Lemma 1, $2 \in \mathrm{Per}(f)$. If $I_4 = [p, b_0]$ and $b = -1$ set $b_1 = \sup\{(f|I_4)^{-1}(a_0)\}$ and $I_{4_2} = [b_1, b_0]$. Then $I_{4_2} \rightleftarrows I_2$ and $I_{4_2} \cap I_2 = \emptyset$ so, by Lemma 1, $2 \in \mathrm{Per}(f)$. If $I_4 = [b_0, p]$ and $b = 1$ write $b_1 = \inf\{(f|I_4)^{-1}(a_0)\}$ and $I_{4_1} = [b_0, b_1]$. Then $I_{4_1} \rightleftarrows I_2$ and $I_{4_1} \cap I_2 = \emptyset$, so, by Lemma 1, $2 \in \mathrm{Per}(f)$. If $I_4 = [b_0, p]$ and $b = -1$ take $b_1 = \inf\{(f|I_4)^{-1}(a_1)\}$ and $I_{4_1} = [b_0, b_1]$. Then $I_{4_1} \rightleftarrows I_3$ and $I_{4_1} \cap I_3 = \emptyset$, so, by Lemma 1, $2 \in \mathrm{Per}(f)$.

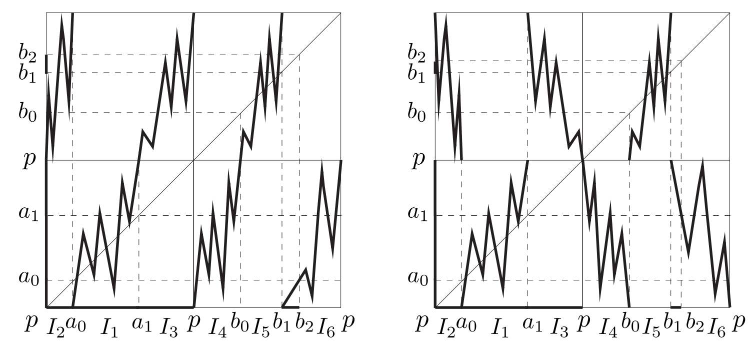


Fig. 9. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $|bc| = 4$.

Suppose now that $|bc| = 4$. We know that f has six basic intervals, I_1, I_2, I_3, I_4, I_5 and I_6, the first three in C_1 and the other three in C_2, such that $f(I_2) = f(I_3) = f(I_5) = C_2$ and $f(I_1) = f(I_4) = f(I_6) = C_1$ (see for instance Figure 9). Using the same ordering as above set $I_2 = [p, a_0]$, $I_1 = [a_0, a_1]$, $I_3 = [b_0, p]$,

$I_4 = [p, b_0]$, $I_5 = [b_0, b_1]$ and $I_6 = [b_1, p]$. If $b = 2$ set $b_2 = \inf\{(f|I_6)^{-1}(a_0)\}$ and $I_{6_1} = [b_1, b_2]$. Then $I_{6_1} \rightleftarrows I_2$ and $I_{6_1} \cap I_2 = \emptyset$ so, by Lemma 1, $2 \in \mathrm{Per}(f)$. If $b = -2$ write $b_2 = \inf\{(f|I_6)^{-1}(a_1)\}$ and $I_{6_1} = [b_1, b_2]$. Then $I_{6_1} \rightleftarrows I_3$ and $I_{6_1} \cap I_3 = \emptyset$ so, by Lemma 1, $2 \in \mathrm{Per}(f)$. Therefore, if $|bc| > 1$, $\mathrm{MPer}(f) = \mathbb{N}$.

We suppose that $|\mathbf{bc}| = \mathbf{1}$. We assume that $\mathbf{b} = \mathbf{c} = \mathbf{1}$. As it can be seen from the examples (a), (c) and (e) of Figure 10, $2 \notin \mathrm{MPer}(f)$. Now we will prove that $\mathrm{Per}(f) = \mathbb{N} \setminus \{2\}$.

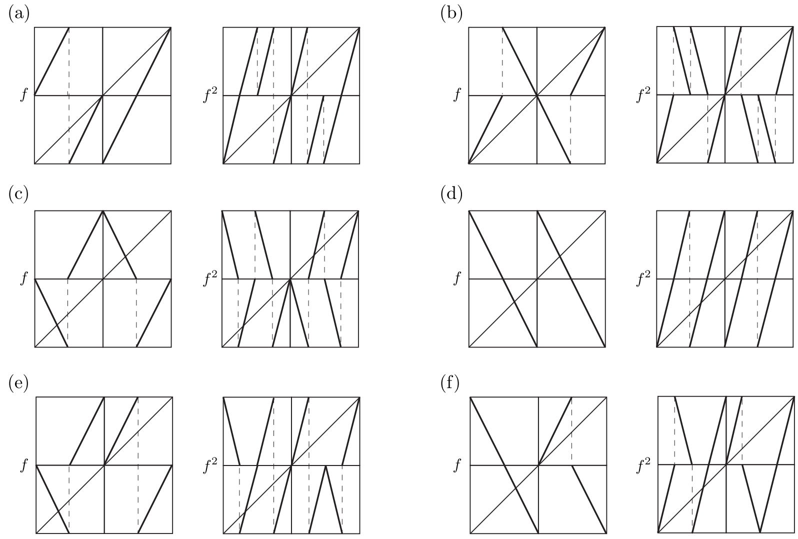


Fig. 10. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$, $bc = 1$ and $2 \notin \mathrm{Per}(f)$.

We know that f has four basic intervals, I_1, I_2, I_3 and I_4, the first two in C_1 and the other two in C_2, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = f(I_4) = C_2$. We have four possibilities for these intervals. Let $a_0 \in I_1 \cap I_2$ and $b_0 \in I_3 \cap I_4$ (see for instance Figure 11). First, we take the interval I_3 to be $[p, b_0]$. Set $I_{3_2} = [b_1, b_0]$ where $b_1 = \sup\{(f|I_3)^{-1}(a_0)\}$. If $I_1 = [p, a_0]$ then f has the subgraph $\circlearrowright I_4 \to I_{3_2} \to I_2 \to I_4$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N} \setminus \{2\}$. If $I_1 = [a_0, p]$ then f has the subgraph $\circlearrowright I_1 \to I_2 \to I_{3_2} \to I_1$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N} \setminus \{2\}$.

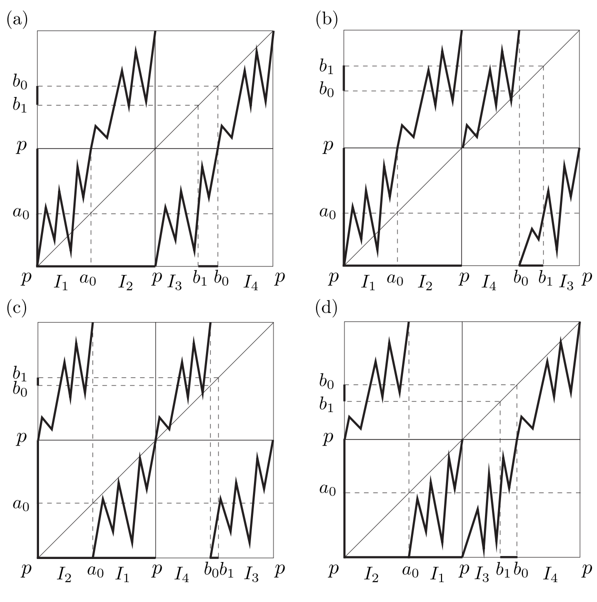


Fig. 11. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $b = c = 1$.

Now we take the interval I_3 to be $[b_0, p]$. Set $b_1 = \inf\{(f|I_3)^{-1}(a_0)\}$ and $I_{3_1} = [b_0, b_1]$. If $I_1 = [p, a_0]$ then f has the subgraph $\circlearrowleft I_1 \to I_2 \to I_{3_1} \to I_1$ and by Proposition 2, $\text{Per}(f) = \mathbb{N} \setminus \{2\}$. If $I_1 = [a_0, p]$ then f has the subgraph $\circlearrowleft I_4 \to I_{3_1} \to I_2 \to I_4$ and by Proposition 2, $\text{Per}(f) = \mathbb{N} \setminus \{2\}$. Therefore, if $|a| = |d| = 1$ and $b = c = 1$ then $\text{MPer}(f) = \mathbb{N} \setminus \{2\}$.

We assume now that $\mathbf{b} = \mathbf{c} = \mathbf{-1}$. As it can be seen from the examples (b), (d) and (f) of Figure 10, $2 \notin \text{MPer}(f)$. Now we will prove that $\text{Per}(f) = \mathbb{N} \setminus \{2\}$.

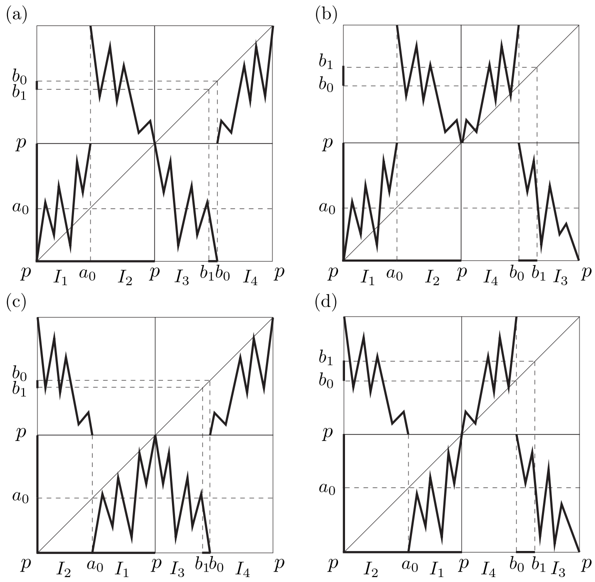


Fig. 12. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $b = c = -1$.

We know that f has four basic intervals, I_1, I_2, I_3 and I_4, the first two in C_1 and the other two in C_2, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = f(I_4) = C_2$. We have four possibilities for these intervals. Let $a_0 \in I_1 \cap I_2$ and $b_0 \in I_3 \cap I_4$ (see for instance Figure 12). First we take I_3 to be the interval $[p, b_0]$. Consider $b_1 = \sup\{(f|I_3)^{-1}(a_0)\}$ and $I_{3_2} = [b_1, b_0]$. If $I_1 = [p, a_0]$ then f has the subgraph $\circlearrowleft I_1 \to I_2 \to I_{3_2} \to I_1$ and by Proposition 2, $\text{Per}(f) = \mathbb{N} \setminus \{2\}$. If $I_1 = [a_0, p]$ then f has the subgraph $\circlearrowleft I_4 \to I_{3_2} \to I_2 \to I_4$ and by Proposition 2, $\text{Per}(f) = \mathbb{N} \setminus \{2\}$.

If $I_3 = [b_0, p]$ consider $b_1 = \inf\{(f|I_3)^{-1}(a_0)\}$ and $I_{3_1} = [b_0, b_1]$. If $I_1 = [p, a_0]$ then f has the subgraph $\circlearrowleft I_4 \to I_{3_1} \to I_2 \to I_4$ and by Proposition 2, $\text{Per}(f) = \mathbb{N} \setminus \{2\}$. If $I_1 = [a_0, p]$ then f has the subgraph $\circlearrowleft I_1 \to I_2 \to I_{3_1} \to I_1$ and by Proposition 2, $\text{Per}(f) = \mathbb{N} \setminus \{2\}$. Therefore, if $b = c = -1$ then $\text{MPer}(f) = \mathbb{N} \setminus \{2\}$. Hence, if $|a| + |d| = 2$ and $bc = 1$, $\text{MPer}(f) = \mathbb{N} \setminus \{2\}$.

We consider now case $\mathbf{b} = -\mathbf{1}$ and $\mathbf{c} = \mathbf{1}$. We know that f has four basic intervals, I_1, I_2, I_3 and I_4, the first two in C_1 and the other two in C_2, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = f(I_4) = C_2$. We have four possibilities for these intervals. Let $a_0 \in I_1 \cap I_2$ and $b_0 \in I_3 \cap I_4$ (see for instance Figure 13). We suppose first that $I_2 = [a_0, p]$. If $I_3 = [p, b_0]$ choose $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and set $I_{2_1} = [a_0, a_1]$. Then f has the subgraph $\circlearrowleft I_1 \to I_{2_1} \rightleftarrows I_3 \to I_1$ with $I_3 \cap I_{2_1} = \emptyset$ and by Proposition 2, $\text{Per}(f) = \mathbb{N}$. If $I_3 = [b_0, p]$ denote $b_1 = \inf\{(f|I_3)^{-1}(a_0)\}$ and $I_{3_1} = [b_0, b_1]$. Then f has the subgraph $\circlearrowleft I_4 \to I_{3_1} \rightleftarrows I_2 \to I_4$ with $I_2 \cap I_{3_1} = \emptyset$ and by Proposition 2, $\text{Per}(f) = \mathbb{N}$.

We consider now $I_2 = [p, a_0]$. If $I_3 = [p, b_0]$ set $b_1 = \sup\{(f|I_3)^{-1}(a_0)\}$ and $I_{3_2} = [b_1, b_0]$. Then f has the subgraph $\circlearrowleft I_4 \to I_{3_2} \rightleftarrows I_2 \to I_4$ with $I_2 \cap I_{3_2} = \emptyset$ and by Proposition 2, $\text{Per}(f) = \mathbb{N}$. If $I_3 = [b_0, p]$ write $a_1 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_1, a_0]$. Then f has the subgraph $\circlearrowleft I_1 \to I_{2_2} \rightleftarrows I_3 \to I_1$ with $I_3 \cap I_{2_2} = \emptyset$ and by Proposition 2, $\text{Per}(f) = \mathbb{N}$. Therefore, if $b = -1$ and $c = 1$, then $\text{MPer}(f) = \mathbb{N}$.

We consider now case $\mathbf{b} = \mathbf{1}$ and $\mathbf{c} = -\mathbf{1}$. We know that f has four basic intervals, I_1, I_2, I_3 and I_4, the first two in C_1 and the other two in C_2, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = f(I_4) = C_2$. We have again four possibilities for these intervals. Let $a_0 \in I_1 \cap I_2$ and $b_0 \in I_3 \cap I_4$ (see for instance Figure 14). We take the interval I_2 to be $[a_0, p]$. If $I_3 = [p, b_0]$ define $b_1 = \sup\{(f|I_3)^{-1}(a_0)\}$ and $I_{3_2} = [b_1, b_0]$. It follows that

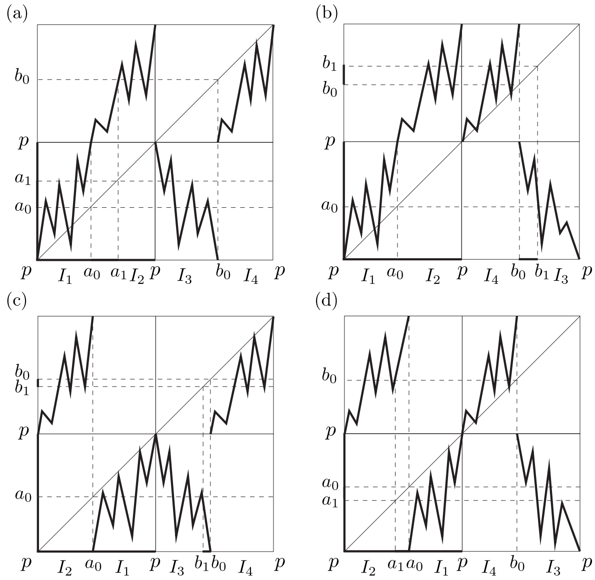


Fig. 13. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$, $b = -1$ and $c = 1$.

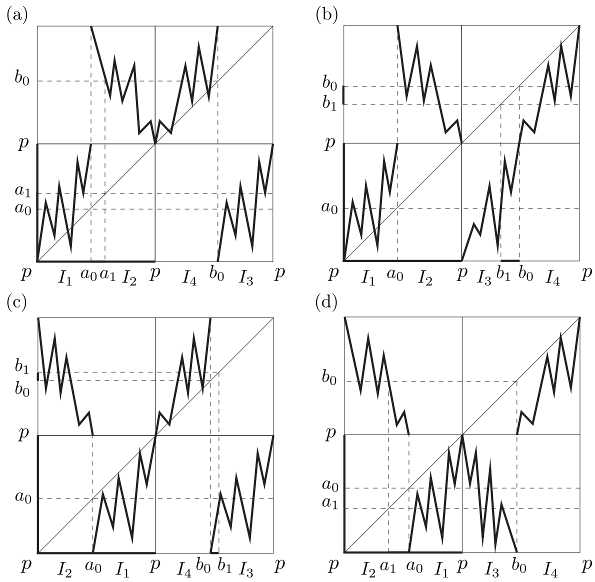


Fig. 14. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$, $b = 1$ and $c = -1$.

f has the subgraph $\circlearrowleft I_4 \to I_{3_2} \rightleftarrows I_2 \to I_4$ with $I_2 \cap I_{3_2} = \emptyset$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$. If $I_3 = [b_0, p]$ consider $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_1} = [a_0, a_1]$. Then f has the subgraph $\circlearrowleft I_1 \to I_{2_1} \rightleftarrows I_3 \to I_1$ with $I_3 \cap I_{2_1} = \emptyset$ and we get by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$.

Suppose that $I_2 = [p, a_0]$. If $I_3 = [p, b_0]$ set $a_1 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_1, a_0]$. Then f has the subgraph $\circlearrowleft I_1 \to I_{2_2} \rightleftarrows I_3 \to I_1$ with $I_3 \cap I_{2_2} = \emptyset$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$. If $I_3 = [b_0, p]$ consider $b_1 = \inf\{(f|I_3)^{-1}(a_0)\}$ and $I_{3_1} = [b_0, b_1]$. Then f has the subgraph $\circlearrowleft I_4 \to I_{3_1} \rightleftarrows I_2 \to I_4$ with $I_2 \cap I_{3_1} = \emptyset$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$. Therefore, if $b = 1$ and $c = -1$, then $\mathrm{MPer}(f) = \mathbb{N}$. Hence, if $|a| + |d| = 2$ and $bc = -1$ then $\mathrm{MPer}(f) = \mathbb{N}$. This completes the proof of Statement (c1). ■

Proof. [Proof of Statement (c21) of Theorem B] We assume now that $\mathbf{a} = \mathbf{1}$ and $\mathbf{d} = \mathbf{0}$. If $\mathbf{bc} = \mathbf{0}$ then $\mathrm{MPer}(f) = \{1\}$ as it can be deduced from the examples of Figure 15. We suppose that b and c are such that $|\mathbf{bc}| > \mathbf{1}$ and $(\mathbf{b}, \mathbf{c}) \notin \{(\mathbf{2}, \mathbf{1}), (\mathbf{2}, -\mathbf{1}), (-\mathbf{2}, \mathbf{1}), (-\mathbf{2}, -\mathbf{1})\}$. From the graph of f (see for instance Figure 16) it follows that there are three basic intervals I_1, I_2 and I_3, $I_1, I_2 \subset C_1$, $I_3 \subset C_2$, such that either $p \notin I_1 \cap I_2$

or $p \notin I_1 \cap I_3$ and f has the subgraph of Proposition 2, so $\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2\}$.

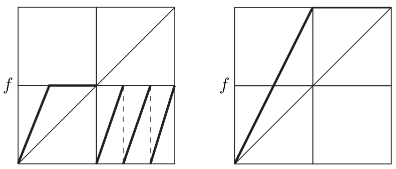


Fig. 15. Examples of maps with $a = 1$, $d = 0$ and $bc = 0$.

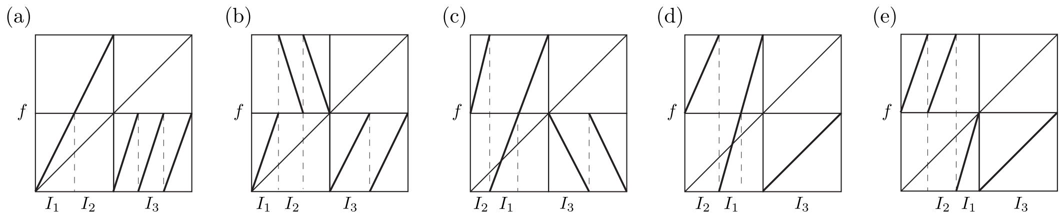


Fig. 16. Examples of maps with $a = 1$, $d = 0$, $(b, c) \notin \{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$ and $|bc| > 1$.

If $\{\mathbf{b}, \mathbf{c}\} \not\subset \{-\mathbf{2}, -\mathbf{1}, \mathbf{1}, \mathbf{2}\}$ then we can choose I_2 and I_3 such that $I_2 \cap I_3 = \emptyset$ and by Proposition 2, $2 \in \mathrm{Per}(f)$. If $\{\mathbf{b}, \mathbf{c}\} \subset \{-\mathbf{2}, -\mathbf{1}, \mathbf{1}, \mathbf{2}\}$ in general there do not exist two basic intervals I_i and I_j, $I_i \neq I_j$, such that $p \notin I_i \cap I_j$ and $I_i \rightleftarrows I_j$. If they exist then by Lemma 1 considering the non–repetitive loop $I_i \to I_j \to I_i$ there is a periodic point z of f with period 2. If they do not exist (see for instance (c) and (d) of Figure 16) and $(b, c) \in \{(1, 2), (-1, -2)\}$, $2 \notin \mathrm{Per}(f)$ as we can see from the examples of Figure 17. Now we will prove that if $(b, c) \in \{(1, -2), (-1, 2)\}$ or $|b| = |c| = 2$ then $2 \in \mathrm{Per}(f)$.

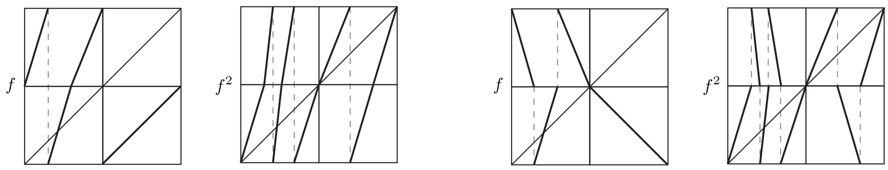


Fig. 17. Examples of maps with $a = 1$, $d = 0$, $(b, c) \in \{(1, 2), (-1, -2)\}$ and $2 \notin \mathrm{Per}(f)$.

We suppose first that $(b, c) \in \{(1, -2), (-1, 2)\}$. We know that f has four basic intervals, I_1, I_2, I_3 and I_4, the first three in C_1 and $I_4 = C_2$, such that $f(I_1) = f(I_4) = C_1$ and $f(I_2) = f(I_3) = C_2$. Let p and a_0 be the endpoints of I_2, a_0 and a_1 the endpoints of I_1, a_1 and p the endpoints of I_3 (see for instance Figure 18).

We consider an ordering in the intervals I_1, I_2 and I_3 in such a way that p is the smallest element of I_2 and the greatest of I_3. Under these assumptions set $I_2 = [p, a_0]$, $I_1 = [a_0, a_1]$ and $I_3 = [a_1, p]$. Define

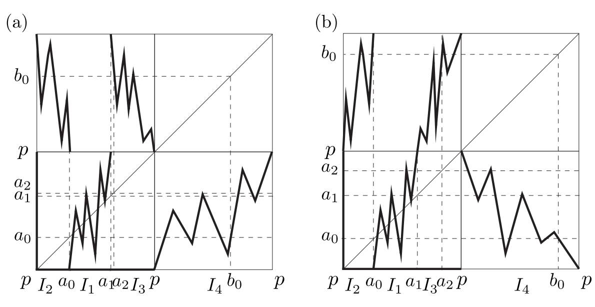


Fig. 18. Examples of maps with $a = 1$, $d = 0$ and $(b, c) \in \{(1, -2), (-1, 2)\}$.

$b_0 = \sup\{(f|I_4)^{-1}(a_0)\}$, $I_{4_1} = [p, b_0]$ and $I_{4_2} = [b_0, p]$. Set $a_2 = \inf\{(f|I_3)^{-1}(b_0)\}$ and $I_{3_1} = [a_1, a_2]$. If $(b, c) = (1, -2)$ we have $I_{4_2} \rightleftarrows I_{3_1}$ and $I_{4_2} \cap I_{3_1} = \emptyset$. If $(b, c) = (-1, 2)$ we get $I_{4_1} \rightleftarrows I_{3_1}$ and $I_{4_1} \cap I_{3_1} = \emptyset$. So, by Lemma 1, $2 \in \mathrm{Per}(f)$.

Suppose now that $|b| = |c| = 2$. We know that f has five basic intervals, I_1, I_2, I_3, I_4 and I_5, the first three in C_1 and the other two in C_2, such that $f(I_2) = f(I_3) = C_2$ and $f(I_1) = f(I_4) = f(I_5) = C_1$. Taking an ordering similar to the previous case define the intervals $I_2 = [p, a_0]$, $I_1 = [a_0, a_1]$, $I_3 = [a_1, p]$, $I_4 = [p, b_0]$ and $I_5 = [b_0, p]$ (see for instance Figure 19). Set $a_2 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_2, a_0]$. If $c = 2$ we have $I_{2_2} \rightleftarrows I_5$ and $I_{2_2} \cap I_5 = \emptyset$. If $c = -2$ we have $I_{2_2} \rightleftarrows I_4$ and $I_{2_2} \cap I_4 = \emptyset$. So, by Lemma 1, $2 \in \mathrm{Per}(f)$. Therefore, if $|bc| > 1$ and $(b, c) \notin \{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$ we have $\mathrm{MPer}(f) = \mathbb{N} \setminus \{2\}$ if $(b, c) \in \{(1, 2), (-1, -2)\}$ and $\mathrm{MPer}(f) = \mathbb{N}$ otherwise.

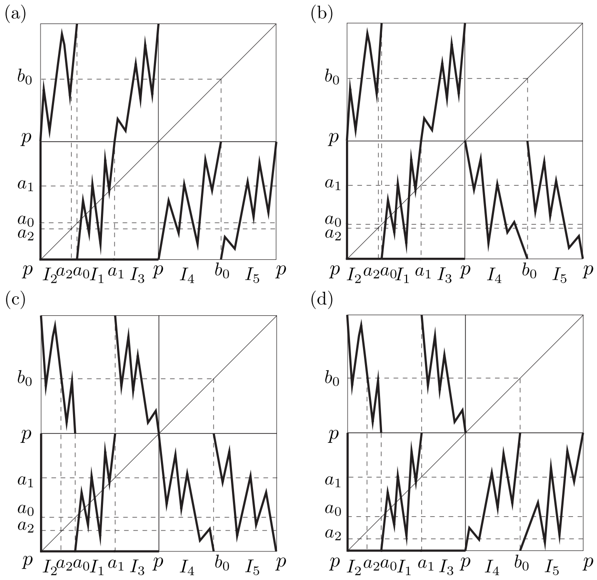


Fig. 19. Examples of maps with $a = 1$, $d = 0$ and $|b| = |c| = 2$.

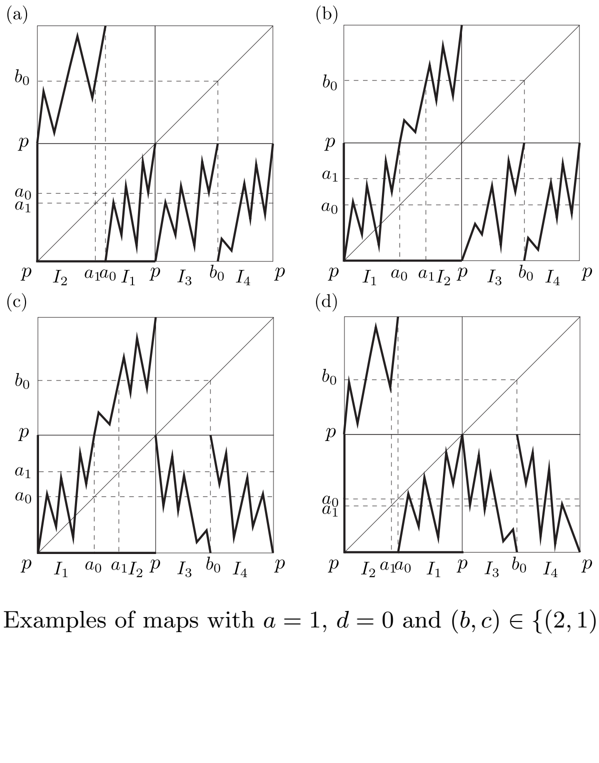


Fig. 20. Examples of maps with $a = 1$, $d = 0$ and $(b, c) \in \{(2, 1), (-2, 1)\}$.

We assume that $|\mathbf{bc}| > \mathbf{1}$ and $(\mathbf{b}, \mathbf{c}) \in \{(\mathbf{2}, \mathbf{1}), (\mathbf{2}, -\mathbf{1}), (-\mathbf{2}, \mathbf{1}), (-\mathbf{2}, -\mathbf{1})\}$. We know that f has four basic intervals, I_1, I_2, I_3 and I_4, the first two in C_1 and the others in C_2, such that $f(I_1) = f(I_3) = f(I_4) = C_1$ and $f(I_2) = C_2$. Let p and a_0 be the endpoints of I_1 and I_2, and b_0 and p the endpoints of I_3 and I_4 (see for instance Figures 20 and 21). For each pair (b, c) we have two possibilities for the intervals I_1 and I_2. If $(b, c) \in \{(2, 1), (-2, 1)\}$ and $I_2 = [a_0, p]$ write $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_1} = [a_0, a_1]$. Then f has the subgraph $\circlearrowright I_1 \to I_{2_1} \rightleftarrows I_3 \to I_1$ with $I_3 \cap I_{2_1} = \emptyset$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$. If $I_2 = [p, a_0]$ consider $a_1 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_1, a_0]$. Then f has the subgraph $\circlearrowright I_1 \to I_{2_2} \rightleftarrows I_4 \to I_1$ with $I_4 \cap I_{2_2} = \emptyset$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$.

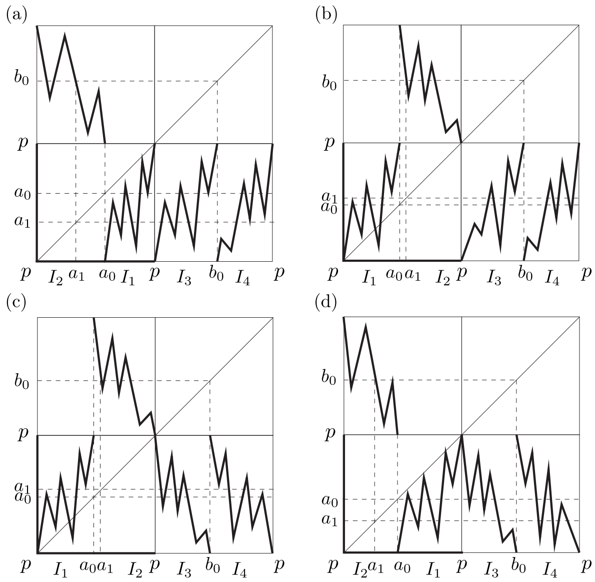


Fig. 21. Examples of maps with $a = 1$, $d = 0$ and $(b, c) \in \{(-2, -1), (2, -1)\}$.

If $(b, c) \in \{(-2, -1), (2, -1)\}$ and $I_2 = [a_0, p]$ set $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_1} = [a_0, a_1]$. Then f has the subgraph $\circlearrowright I_1 \to I_{2_1} \rightleftarrows I_4 \to I_1$ with $I_4 \cap I_{2_1} = \emptyset$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$. If $I_2 = [p, a_0]$ consider $a_1 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_1, a_0]$. Then f has the subgraph $\circlearrowright I_1 \to I_{2_2} \rightleftarrows I_3 \to I_1$ with $I_3 \cap I_{2_2} = \emptyset$ and by Proposition 2, $\mathrm{Per}(f) = \mathbb{N}$. Therefore, if $|\mathbf{bc}| > \mathbf{1}$ and $(\mathbf{b}, \mathbf{c}) \in \{(\mathbf{2}, \mathbf{1}), (\mathbf{2}, -\mathbf{1}), (-\mathbf{2}, \mathbf{1}), (-\mathbf{2}, -\mathbf{1})\}$, $\mathrm{MPer}(f) = \mathbb{N}$.

We consider the case $|\mathbf{bc}| = \mathbf{1}$. First assume that $\mathbf{bc} = \mathbf{1}$. As we can see from the examples of Figure 22, $2 \notin \mathrm{MPer}(f)$. Now we will prove that $\mathrm{Per}(f) = \mathbb{N} \setminus \{2\}$.

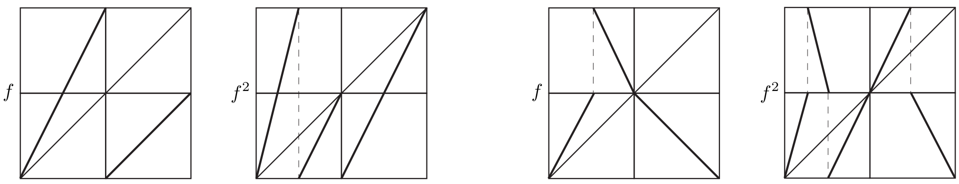


Fig. 22. Examples of maps with $a = 1$, $d = 0$, $bc = 1$ and $2 \notin \mathrm{Per}(f)$.

We know that f has three basic intervals, I_1, I_2 and I_3, the first two in C_1 and $I_3 = C_2$, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = C_2$. We have two possibilities for the intervals I_1 and I_2: either p is the smallest element of I_1 and the greatest of I_2 or p is the smallest element of I_2 and the greatest of I_1 (see for instance Figure 23). In the assumption that $b = c = 1$, if $I_1 = [p, a_0]$, write $b_0 = \inf\{(f|I_3)^{-1}(a_0)\}$, $I_{3_1} = [p, b_0]$, $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_1} = [a_0, a_1]$. Then f has the subgraph $\circlearrowright I_1 \to I_{2_1} \to I_{3_1} \to I_1$ and by Proposition 2, $\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2\}$. If $I_1 = [a_0, p]$, define $b_0 = \sup\{(f|I_3)^{-1}(a_0)\}$, $I_{3_2} = [b_0, p]$,

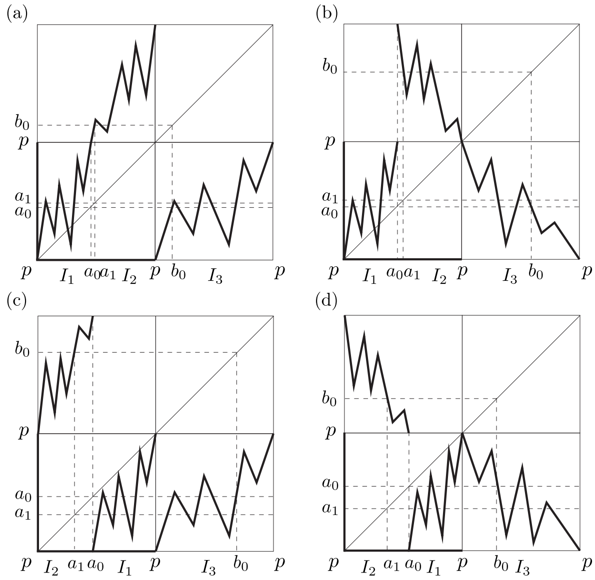


Fig. 23. Examples of maps with $a = 1$, $d = 0$ and $bc = 1$.

$a_1 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_1, a_0]$. Then f has the subgraph $\circlearrowright I_1 \to I_{2_2} \to I_{3_2} \to I_1$ and by Proposition 2, $\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2\}$.

If $b = c = -1$ we consider first the case $I_1 = [p, a_0]$. Set $b_0 = \sup\{(f|I_3)^{-1}(a_0)\}$, $I_{3_2} = [b_0, p]$, $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_1} = [a_0, a_1]$. Then f has the subgraph $\circlearrowright I_1 \to I_{2_1} \to I_{3_2} \to I_1$ and by Proposition 2, $\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2\}$. If $I_1 = [a_0, p]$, write $b_0 = \inf\{(f|I_3)^{-1}(a_0)\}$, $I_{3_1} = [p, b_0]$, $a_1 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_1, a_0]$. Then f has the subgraph $\circlearrowright I_1 \to I_{2_2} \to I_{3_1} \to I_1$ and by Proposition 2, $\mathrm{Per}(f) \supset \mathbb{N} \setminus \{2\}$. Therefore, if $a = 1$, $d = 0$ and $bc = 1$, $\mathrm{MPer}(f) = \mathbb{N} \setminus \{2\}$.

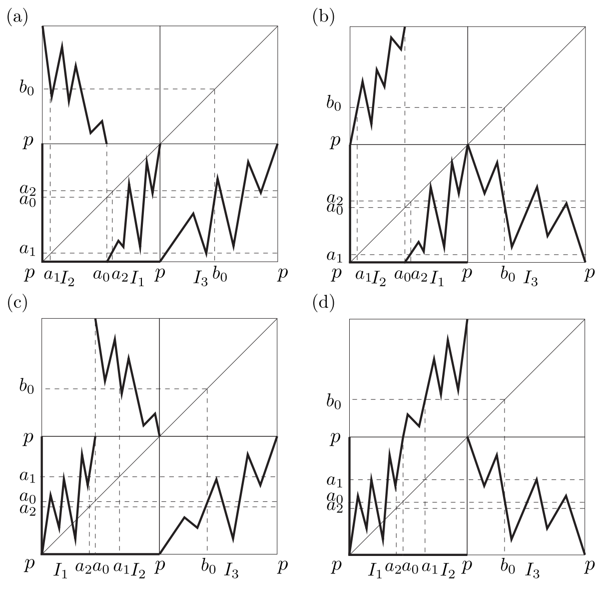
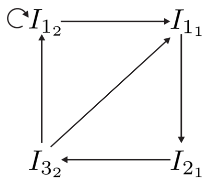


Fig. 24. Examples of maps with $a = 1$, $d = 0$ and $bc = -1$.

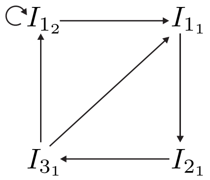
Assume now that $\mathbf{bc = -1}$. We know that f has three basic intervals, I_1, I_2 and I_3, the first two in C_1 and $I_3 = C_2$, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = C_2$. We have two possibilities for the intervals I_1 and I_2: either p is the smallest element of I_1 and the greatest of I_2 or p is the smallest element of I_2 and the greatest of I_1 (see for instance Figure 24). Define $b_0 = \inf\{(f|I_3)^{-1}(a_0)\}$, $I_{3_1} = [p, b_0]$, $I_{3_2} = [b_0, p]$ and

$a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$.

If $I_1 = [a_0, p]$ let $I_{2_1} = [p, a_1]$ and $I_{2_2} = [a_1, a_0]$. Consider $a_2 = \inf\{(f|I_1)^{-1}(a_1)\}$. We write $I_{1_1} = [a_0, a_2]$ and $I_{1_2} = [a_2, p]$. If $b = 1$ and $c = -1$ (see (a) of Figure 24) f has the subgraph

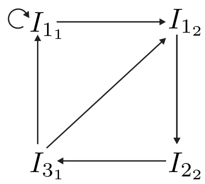


We consider the non–repetitive loops $I_{1_1} \to I_{2_1} \to I_{3_2} \to I_{1_1}$ and $I_{1_2} \to I_{1_1} \to I_{2_1} \to I_{3_2} \to I_{1_2} \to \ldots \to I_{1_2}$ of lengths 3 and $n \geq 4$, respectively. From the first loop and by Lemma 1 there is a periodic point z of f with period 3; from the second loop and by Lemma 1 there is a periodic point z of f with period $n \geq 4$. Moreover, $I_{3_1} \rightleftarrows I_{2_2}$ and $I_{3_1} \cap I_{2_2} = \emptyset$, so, by Lemma 1, $2 \in \mathrm{Per}(f)$. Hence, $\mathrm{Per}(f) = \mathbb{N}$. If $b = -1$ and $c = 1$ (see (b) of Figure 24) f has the subgraph

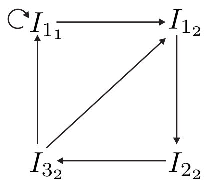


Now from the non–repetitive loops $I_{1_1} \to I_{2_1} \to I_{3_1} \to I_{1_1}$ and $I_{1_2} \to I_{1_1} \to I_{2_1} \to I_{3_1} \to I_{1_2} \to \ldots \to I_{1_2}$ of lengths 3 and $n \geq 4$, respectively, and $I_{3_2} \rightleftarrows I_{2_2}$ and $I_{3_2} \cap I_{2_2} = \emptyset$, it follows that $\mathrm{Per}(f) = \mathbb{N}$.

If $I_1 = [p, a_0]$ let $I_{2_1} = [a_0, a_1]$, $I_{2_2} = [a_1, p]$. Define $a_2 = \sup\{(f|I_1)^{-1}(a_1)\}$, $I_{1_1} = [a_0, a_2]$ and $I_{1_2} = [a_2, p]$. If $b = 1$ and $c = -1$ (see (c) of Figure 24) f has the subgraph



Again from the non–repetitive loops $I_{1_2} \to I_{2_2} \to I_{3_1} \to I_{1_2}$ and $I_{1_1} \to I_{1_2} \to I_{2_2} \to I_{3_1} \to I_{1_1} \to \ldots \to I_{1_1}$ of lengths 3 and $n \geq 4$, respectively, $I_{3_2} \rightleftarrows I_{2_1}$ and $I_{3_2} \cap I_{2_1} = \emptyset$, $\mathrm{Per}(f) = \mathbb{N}$. If $b = -1$ and $c = 1$ (see (d) of Figure 24) f has the subgraph



We consider the non–repetitive loops $I_{1_2} \to I_{2_2} \to I_{3_2} \to I_{1_2}$ and $I_{1_1} \to I_{1_2} \to I_{2_2} \to I_{3_2} \to I_{1_1} \to \ldots \to I_{1_1}$ of lengths 3 and $n \geq 4$, respectively, $I_{3_1} \rightleftarrows I_{2_1}$ and $I_{3_1} \cap I_{2_1} = \emptyset$. We obtain that $\mathrm{Per}(f) = \mathbb{N}$. Therefore, if $a = 1$, $d = 0$ and $bc = -1$, $\mathrm{MPer}(f) = \mathbb{N}$. This completes the proof of Statement (c21). ∎

Proof. [Proof of Statement (c22) of Theorem B] If $\mathbf{a} = \mathbf{0}$ and $\mathbf{d} = \mathbf{1}$, by using the same kind of arguments that in the case $a = 1$ and $d = 0$, and interchanging b and c, we obtain Statement (c22). ∎

Proof. [Proof of Statement (c23) of Theorem B] We suppose that $\mathbf{a} = -\mathbf{1}$ and $\mathbf{d} = \mathbf{0}$. If $\mathbf{bc} = \mathbf{0}$ then $\mathrm{MPer}(f) = \{1\}$ as it can be seen from the examples of Figure 25. The cases in which $\mathrm{MPer}(f)$ is either $\mathbb{N} \setminus \{2\}$ or $\mathbb{N}$ can be proved following exactly the same kind of arguments that in the proof of Statement (c21).

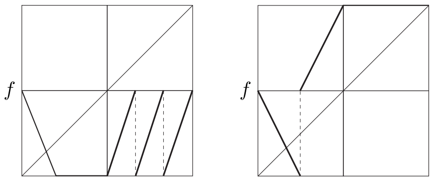


Fig. 25. Examples of maps with $a = -1$, $d = 0$ and $bc = 0$.

Assume now that $\mathbf{bc} = \mathbf{-1}$. From the examples of Figure 26 we can see that $3 \notin \mathrm{MPer}(f)$.

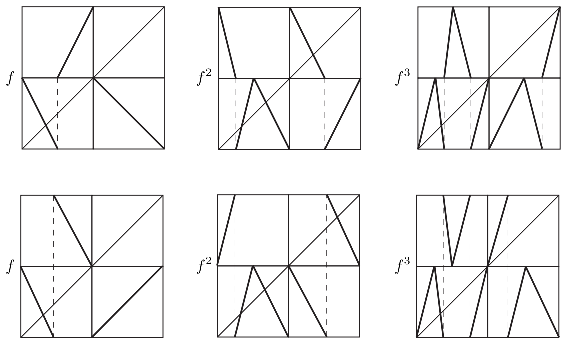


Fig. 26. Examples of maps with $a = -1$, $d = 0$, $bc = -1$ and $3 \notin \mathrm{Per}(f)$.

We know that f has three basic intervals, I_1, I_2 and I_3, the first two in C_1 and $I_3 = C_2$, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = C_2$. We have two possibilities for the intervals I_1 and I_2: either p is the smallest element of I_1 and the greatest of I_2 or p is the smallest element of I_2 and the greatest of I_1 (see for instance Figure 27). Denote $b_0 = \inf\{(f|I_3)^{-1}(a_0)\}$, $I_{3_1} = [p, b_0]$ $I_{3_2} = [b_0, p]$ and $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$.

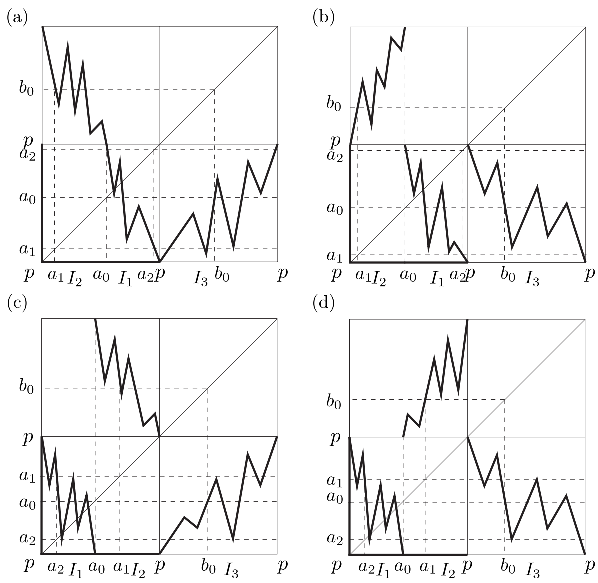


Fig. 27. Examples of maps with $a = 1$, $d = 0$ and $bc = -1$.

If $I_1 = [a_0, p]$ let $I_{2_1} = [p, a_1]$ and $I_{2_2} = [a_1, a_0]$. Consider $a_2 = \inf\{(f|I_1)^{-1}(a_1)\}$. Write $I_{1_1} = [a_0, a_2]$ and $I_{1_2} = [a_2, p]$. If $b = 1$ and $c = -1$ (see (a) of Figure 27) f has the subgraph $\circlearrowleft I_{1_1} \to I_{1_2} \to I_{2_1} \to I_{3_2} \to I_{1_1}$. We consider the non–repetitive loop $I_{1_1} \to I_{1_2} \to I_{2_1} \to I_{3_2} \to I_{1_1} \to \ldots \to I_{1_1}$ of length $n \geq 4$. By Lemma 1 there is a periodic point z of f with period $n \geq 4$. Moreover, $I_{3_1} \rightleftarrows I_{2_2}$ and

$I_{3_1} \cap I_{2_2} = \emptyset$, so, by Lemma 1, $2 \in \text{Per}(f)$. Hence, $\text{Per}(f) = \mathbb{N} \setminus \{3\}$. If $b = -1$ and $c = 1$ (see (b) of Figure 27) f has the subgraph $\circlearrowleft I_{1_1} \to I_{1_2} \to I_{2_1} \to I_{3_1} \to I_{1_1}$. We consider the non–repetitive loop $I_{1_1} \to I_{1_2} \to I_{2_1} \to I_{3_1} \to I_{1_1} \to \ldots \to I_{1_1}$ of length $n \geq 4$. By Lemma 1 there is a periodic point z of f with period $n \geq 4$. Moreover, $I_{3_2} \rightleftarrows I_{2_2}$ and $I_{3_2} \cap I_{2_2} = \emptyset$, so, by Lemma 1, $2 \in \text{Per}(f)$. Hence, $\text{Per}(f) = \mathbb{N} \setminus \{3\}$.

If $I_1 = [p, a_0]$ let $I_{2_1} = [a_0, a_1]$ and $I_{2_2} = [a_1, p]$. Consider $a_2 = \sup\{(f|I_1)^{-1}(a_1)\}$. Write $I_{1_1} = [p, a_2]$ and $I_{1_2} = [a_2, a_0]$. If $b = 1$ and $c = -1$ (see (c) of Figure 27) f has the subgraph $\circlearrowleft I_{1_2} \to I_{1_1} \to I_{2_2} \to I_{3_1} \to I_{1_2}$. From the non–repetitive loop $I_{1_2} \to I_{1_1} \to I_{2_2} \to I_{3_1} \to I_{1_2} \to \ldots \to I_{1_2}$ of length $n \geq 4$, $I_{3_2} \rightleftarrows I_{2_1}$ and $I_{3_2} \cap I_{2_1} = \emptyset$, we obtain that $\text{Per}(f) = \mathbb{N} \setminus \{3\}$. If $b = -1$ and $c = 1$ (see (d) of Figure 27) f has the subgraph $\circlearrowleft I_{1_2} \to I_{1_1} \to I_{2_2} \to I_{3_2} \to I_{1_2}$. Using the non–repetitive loop $I_{1_2} \to I_{1_1} \to I_{2_2} \to I_{3_2} \to I_{1_2} \to \ldots \to I_{1_2}$ of length $n \geq 4$, $I_{3_1} \rightleftarrows I_{2_1}$ and $I_{3_1} \cap I_{2_1} = \emptyset$, we get that $\text{Per}(f) = \mathbb{N} \setminus \{3\}$. Therefore, if $a = -1$, $d = 0$ and $bc = -1$, $\text{MPer}(f) = \mathbb{N} \setminus \{3\}$. This completes the proof of Statement (c23). ■

Proof. [Proof of Statement (c24) of Theorem B] If $\mathbf{a} = \mathbf{0}$ and $\mathbf{d} = -\mathbf{1}$, by using the same kind of arguments that in the case $a = -1$ and $d = 0$, and interchanging b and c, we obtain Statement (c24). ■

Proof. [Proof of Statement (c3) of Theorem B] We suppose that $\mathbf{a} = \mathbf{d} = \mathbf{0}$. If $\mathbf{bc} = \mathbf{0}$ or $\mathbf{bc} = \mathbf{1}$ we can deduce from the examples of Figure 28 that $\text{MPer}(f) = \{1\}$.

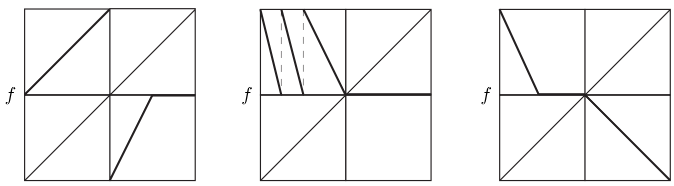


Fig. 28. Examples of maps with $a = d = 0$ and either $bc = 0$ or $bc = 1$.

If $\mathbf{bc} = -\mathbf{1}$ then $\text{MPer}(f) = \{1, 2\}$ (see for instance Figure 29).

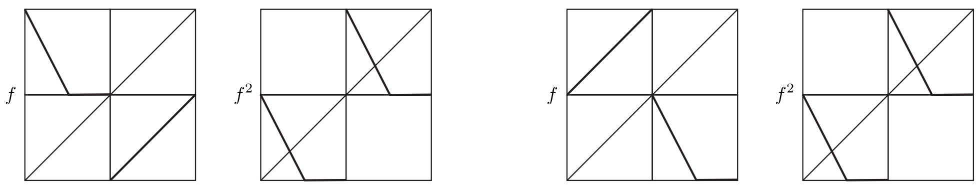
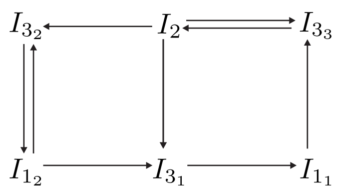


Fig. 29. Examples of maps with $a = d = 0$ and $bc = -1$.

We assume now that $|\mathbf{bc}| = \mathbf{2}$. Since $a = d = 0$ we may assume without loss of generality that $|b| = 1$ and $|c| = 2$. We consider first case $\mathbf{bc} = -\mathbf{2}$. Clearly, $\{1, 2\} \subset \text{Per}(f)$, no other odd number belongs to $\text{MPer}(f)$ and $4 \notin \text{MPer}(f)$ as it can be deduced from Figure 30. Now we will prove that $n \in \text{Per}(f)$ for any n even larger than 4.

We know that f has three basic intervals, I_1, I_2 and I_3, the first two in C_1 and $I_3 = C_2$, such that $f(I_1) = f(I_2) = C_2$ and $f(I_3) = C_1$ (see for instance Figure 31). Consider $b_0 = \inf\{(f|I_3)^{-1}(a_0)\}$, $a_1 = \inf\{(f|I_1)^{-1}(b_0)\}$, $b_1 = \inf\{(f|I_3)^{-1}(a_1)\}$. Set $I_{1_1} = [p, a_1]$, $I_{1_2} = [a_1, a_0]$, I_{3_1} the interval with endpoints b_1 and p, I_{3_2} the interval with endpoints b_1 and b_0, and I_{3_3} the interval with endpoints b_0 and p. Then f has the subgraph



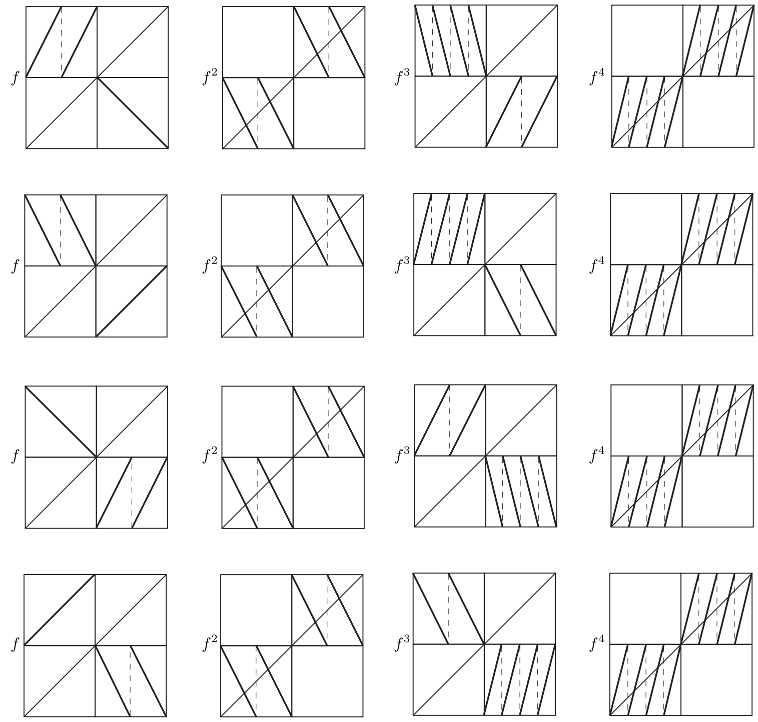


Fig. 30. Examples of maps with $a = d = 0$, $bc = -2$ and $4 \notin \text{Per}(f)$.

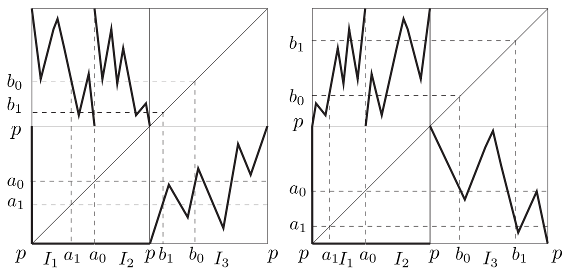


Fig. 31. Examples of maps with $a = d = 0$ and $bc = -2$.

We consider the non–repetitive loops $I_{3_2} \to I_{1_2} \to I_{3_2}$ and $I_2 \to I_{3_2} \to I_{1_2} \to I_{3_1} \to I_{1_1} \to I_{3_3} \to I_2 \to \ldots \to I_{3_3} \to I_2$ of lengths 2 and n even, $n \geq 6$, respectively. We have $I_{3_2} \cap I_{1_2} = \emptyset$, so, from the first loop and by Lemma 1 there is a periodic point z of f with period 2; from the second loop and by Lemma 1 there is a periodic point z of f with period n even $n \geq 6$. Therefore, if $bc = -2$ then $\text{MPer}(f) = \{1\} \cup (2\mathbb{N} \setminus \{4\})$.

We suppose that $\mathbf{bc} = \mathbf{2}$. No odd number other than 1 belongs to $\text{MPer}(f)$, as it can be seen from the examples of Figure 32. Also from Figure 32 we can deduce that $2 \notin \text{MPer}(f)$. Now we will prove that $n \in \text{Per}(f)$ for any n even larger than 2.

We know that f has three basic intervals, I_1, I_2 and I_3, the first two in C_1 and $I_3 = C_2$, such that $f(I_1) = f(I_2) = C_2$ and $f(I_3) = C_1$ (see for instance Figure 33). Denote $b_0 = \inf\{(f|I_3)^{-1}(a_0)\}$, $a_1 = \inf\{(f|I_1)^{-1}(b_0)\}$ and $b_1 = \inf\{(f|I_3)^{-1}(a_1)\}$. Write $I_{1_1} = [p, a_1]$, $I_{1_2} = [a_1, a_0]$, I_{3_2} the interval with endpoints b_1 and b_0, and I_{3_3} the interval with endpoints b_0 and p. Then f has the subgraph $I_{3_2} \to I_{1_2} \to I_{3_3} \rightleftarrows I_2 \to I_{3_2}$. We take the non–repetitive loop $I_2 \to I_{3_2} \to I_{1_2} \to I_{3_3} \to I_2 \to \ldots \to I_{3_3} \to I_2$ of length n even, $n \geq 4$. By Lemma 1 there is a periodic point z of f with period n even $n \geq 4$. Therefore, if $bc = 2$ then $\text{MPer}(f) = \{1\} \cup (2\mathbb{N} \setminus \{2\})$.

We consider now case $|\mathbf{bc}| > \mathbf{2}$. We must separate case $|b| = |c| = 2$ from the others. If $|\mathbf{b}| > \mathbf{2}$ or $|\mathbf{c}| > \mathbf{2}$ then there are three basic intervals I_1, I_2 and I_3 such that $I_2 \cap I_3 = \emptyset$ and $I_1 \rightleftarrows I_3 \rightleftarrows I_2$ (see for instance Figure 34). By Lemma 1 the non–repetitive loop $I_2 \to I_3 \to I_2$ gives a periodic point z of f with

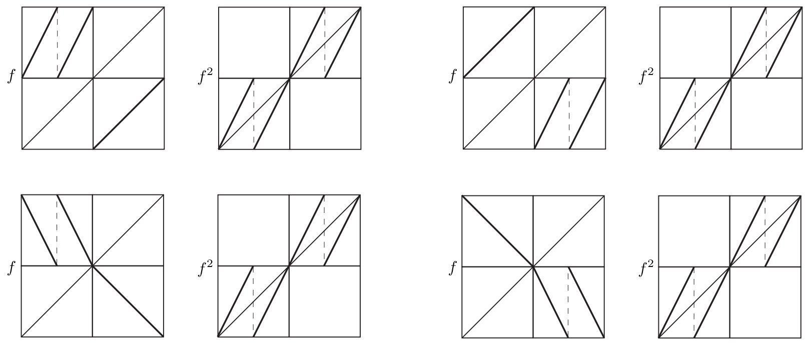


Fig. 32. Examples of maps with $a = d = 0$, $bc = 2$ and $2 \notin \mathrm{Per}(f)$.

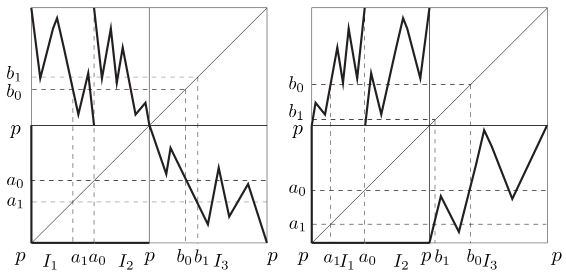


Fig. 33. Examples of maps with $a = d = 0$ and $bc = 2$.

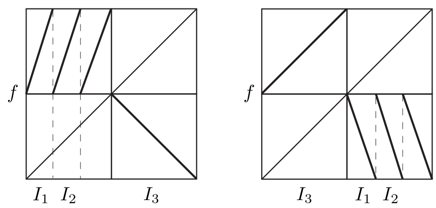


Fig. 34. Examples of maps with $a = d = 0$ and either $|b| > 2$ or $|c| > 2$.

period 2, and the non–repetitive loop $I_1 \to I_3 \to I_2 \to I_3 \to \ldots \to I_2 \to I_3 \to I_1$ of length n even larger than 2 gives a periodic point z of f with period n even. No odd number other than 1 belongs to $\mathrm{MPer}(f)$. Therefore, if $|b| > 2$ or $|c| > 2$, then $\mathrm{MPer}(f) = \{1\} \cup 2\mathbb{N}$.

We suppose that $|\mathbf{b}| = |\mathbf{c}| = \mathbf{2}$. Clearly, no odd number other than 1 belongs to $\mathrm{MPer}(f)$. Now we will prove that $n \in \mathrm{Per}(f)$ for any n even.

We know that f has four basic intervals, I_1, I_2, I_3 and I_4, the first two in C_1 and the others in C_2, such that $f(I_1) = f(I_2) = C_2$ and $f(I_3) = f(I_4) = C_1$ (see for instance Figure 35). Consider $b_1 = \inf\{(f|I_3)^{-1}(a_0)\}$ and $a_1 = \inf\{(f|I_1)^{-1}(b_1)\}$. Denote $I_{1_1} = [p, a_1]$, $I_{1_2} = [a_1, a_0]$, $I_2 = [a_0, p]$, $I_{3_1} = [p, b_1]$, $I_{3_2} = [b_1, b_0]$ and $I_4 = [b_0, p]$. If $(b, c) \in \{(2, 2), (-2, 2)\}$ then f has the subgraph $I_2 \rightleftarrows I_4 \rightleftarrows I_{1_2}$. We take the non–repetitive loops $I_4 \to I_{1_2} \to I_4$ and $I_2 \to I_4 \to I_{1_2} \to I_4 \to \ldots \to I_{1_2} \to I_4 \to I_2$, of lengths 2 and n even larger than 2, respectively. By Lemma 1 the first loop gives a periodic point z of f with period 2, and the second loop gives a periodic point z of f with period n even larger than 2. Hence, if $(b, c) \in \{(2, 2), (-2, 2)\}$, $\mathrm{Per}(f) = \{1\} \cup 2\mathbb{N}$.

If $(b, c) = (-2, -2)$ then f has the subgraph $I_4 \rightleftarrows I_{1_1} \rightleftarrows I_{3_2}$. We consider the non–repetitive loops $I_{3_2} \to I_{1_1} \to I_{3_2}$ and $I_4 \to I_{1_1} \to I_{3_2} \to I_{1_1} \to \ldots \to I_{3_2} \to I_{1_1} \to I_4$, of lengths 2 and n even larger than 2, respectively. By Lemma 1 the first loop gives a periodic point z of f with period 2, and the second loop gives a periodic point z of f with period n even larger than 2. Hence, if $(b, c) = (-2, -2)$, $\mathrm{Per}(f) = \{1\} \cup 2\mathbb{N}$.

If $(b, c) = (2, -2)$ then f has the subgraph $I_4 \rightleftarrows I_2 \rightleftarrows I_{3_2}$. We consider the non–repetitive loops $I_2 \to I_{3_2} \to I_2$ and $I_4 \to I_2 \to I_{3_2} \to I_2 \to \ldots \to I_{3_2} \to I_2 \to I_4$, of lengths 2 and n even larger than 2, respectively. By Lemma 1 the first loop gives a periodic point z of f with period 2, and the second loop

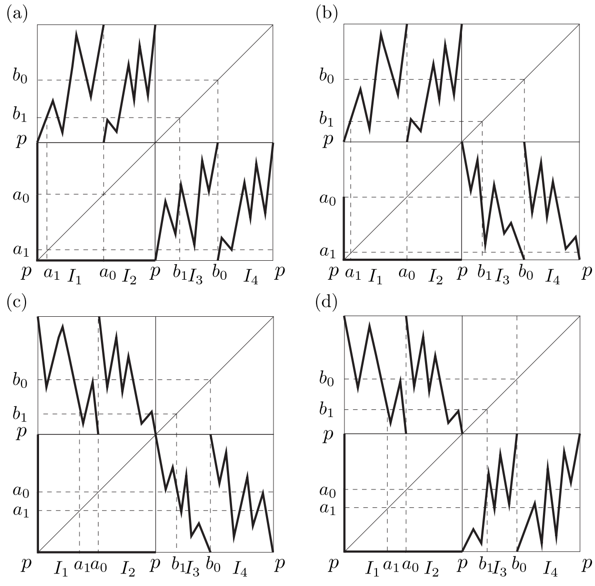


Fig. 35. Examples of maps with $a = d = 0$ and $|b| = |c| = 2$.

gives a periodic point z of f with period n even larger than 2. Hence, if $(b, c) = (-2, -2)$, $\text{Per}(f) = \{1\} \cup 2\mathbb{N}$. Therefore, if $|b| = |c| = 2$ then $\text{MPer}(f) = \{1\} \cup 2\mathbb{N}$. This completes the proof of Statement (c3). ■

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