Minimal set of periods
for continuous self–maps of a bouquet of circles

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Let $G_k$ be a bouquet of circles; i.e. the quotient space of the interval $[0,k]$ obtained by identifying all points of integer coordinates to a single point, called the branching point of $G_k$. Thus, $G_1$ is the circle, $G_2$ is the eight space and $G_3$ is the trefoil. Let $f : G_k \to G_k$ a continuous map such that for $k > 1$ the branching point is fixed.

If $\text{Per}(f)$ denotes the set of periods of $f$, the minimal set of periods of $f$, denoted by $\text{MPer}(f)$, is defined as $\bigcap_{g \simeq f} \text{Per}(g)$ where $g : G_k \to G_k$ is homological to $f$.

The sets $\text{MPer}(f)$ are well–known for circle maps. Here, we classify all the sets $\text{MPer}(f)$ for self–maps of the eight space.

Keywords: periods, periodic orbits, eight space, continuous maps, set of periods.

1. Introduction and statement of the results

In dynamical systems it is often the case that topological information can be used to study qualitative or quantitative properties of the system. This work deals with the problem of determining the set of periods of the periodic orbits of a map given the homology class of the map.

A finite graph (simply a graph) $G$ is a topological space formed by a finite set of points $V$ (points of $V$ are called vertices) and a finite set of open arcs (called edges) in such a way that each open arc is attached by its endpoints to vertices. An open arc is a subset of $G$ homeomorphic to the open interval $(0,1)$. Note that a finite graph is compact, since it is the union of a finite number of compact subsets (the closed edges and the vertices). Notice that a closed edge is homeomorphic either to the closed interval $[0,1]$, or to the circle. It may be either connected or disconnected, and it may have isolated vertices.

The valence of a vertex is the number of edges with the vertex as an endpoint (where the closed edges homeomorphic to a circle are counted twice). The vertices with valence 1 of a connected graph are endpoints of the graph and the vertices with valence larger than 2 are branching points.

Suppose that $f : G \to G$ is a continuous map, in what follows a graph map. A fixed point of $f$ is a point $x$ in $G$ such that $f(x) = x$. We will call $x$ a periodic point of period $n$ if $x$ is a fixed point of $f^n$ but it is not fixed by any $f^k$ for $1 \leq k < n$. We denote by $\text{Per}(f)$ the set of natural numbers corresponding to
periods of the periodic points of $f$.

Let $G$ be a connected graph and let $f$ be a graph map. Then $f$ induces endomorphisms $f_{n} : H_{n}(G) \rightarrow H_{n}(G)$ (for $n=0,1$) on the integral homology groups of $G$, where $H_{0}(G) \approx \mathbb{Z}$ (because $G$ is connected) and $H_{1}(G) \approx \mathbb{Z} \oplus \mathbb{Z}$ where $k$ is the number of independent circuits or loops of $G$ as elements of $H_{1}(G)$. A circuit of $G$ is a subset of $G$ homeomorphic to the circle. The endomorphisms $f_{0}$ and $f_{1}$ are represented by integer matrices. Furthermore, since $G$ is connected $f_{0}$ is the identity.

The endomorphism $f_{1}$ will play a main role in our analysis of the minimal sets of periods for graph maps on $G$. In what follows $f_{1}$ will be denoted by $f_{*}$. For example, if $H_{1}(G) \approx \mathbb{Z} \oplus \mathbb{Z}$ and

$$f_{*} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

this means that the graph $G$ has two independent oriented circuits. Moreover, if the first circuit covers itself exactly $a_{1}$ times following the same orientation (not necessarily in a consecutive way) and exactly $a_{2}$ times following the converse orientation (not necessarily in a consecutive way), then $a = a_{1} - a_{2}$. Similarly, if the first circuit covers the second one exactly $b_{1}$ times following the same orientation (not necessarily in a consecutive way) and exactly $b_{2}$ times following the converse orientation (not necessarily in a consecutive way), then $b = b_{1} - b_{2}$. An analogous explanation can be given with the second independent circuit and with $b$ and $d$ instead of $a$ and $c$, respectively.

Let $G_{k}$ be a bouquet of $k$ circles, that is, the quotient space of $[0, k]$ obtained by identifying all points of integer coordinates to a single point. Notice that $G_{1}$ is the circle and that $G_{2}$ is usually called the eight space. For the $G_{k}$ graph we have $H_{0}(G_{k}) \approx \mathbb{Z}$, $H_{1}(G_{k}) \approx \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \mathbb{Z}$, $f_{0} = id$ and $f_{1} = f_{*} = A$, where $A$ is a $k \times k$ integral matrix. For more details on graph maps see Llibre [1991] or Llibre & Sá [1995].

Our main goal is to study the set $Per(f)$ for graph maps. More explicitly, we want to provide a description of the minimal set of periods (see below) attained within the homology class of a given graph map. When the map $g : G \rightarrow G$ is homological to $f$ (i.e. $g$ induces the same endomorphisms as $f$ on the homology groups of $G$), we shall write $g \simeq f$. We define the minimal set of periods of $f$ to be the set

$$MPer(f) = \bigcap_{g \simeq f} Per(g).$$

From its definition $MPer(f)$ is the maximal subset of periods contained in $Per(g)$ for all $g \simeq f$.

Our main objective is to characterize the minimal sets of periods $MPer(f)$ for graph maps $f : G_{i} \rightarrow G_{i}$ with the branching point a fixed point for $i = 2, 3$. So, always $1 \in MPer(f)$. Even for circle maps $f : G_{1} \rightarrow G_{1}$ the characterization of all minimal sets of periods $MPer(f)$ is interesting and nontrivial, see Theorem A. This result was stated by Efremova [Efremova, 1978] and Block, Guckenheimer, Misiurewicz and Young [Block et al., 1980] without giving a complete proof. As far as we know the first complete proof was given in Alsedá et al. [2000].

We denote by $\mathbb{N}$ the set of all natural numbers, and by $k\mathbb{N}$ the set $\{kl : l \in \mathbb{N}\}$.

Theorem A. Let $f : G_{1} \rightarrow G_{1}$ be a circle map such that the endomorphism induced by $f$ on the first homology group is $f_{*} = (d)$ (i.e. $d$ is the degree of $f$). Then the following statements hold.

(a) If $d \notin \{-2, -1, 0, 1\}$, then $MPer(f) = \mathbb{N}$.
(b) If $d = -2$, then $MPer(f) = \mathbb{N} \setminus \{2\}$.
(c) If $d \in \{-1, 0\}$, then $MPer(f) = \{1\}$.
(d) If $d = 1$, then $MPer(f) = \emptyset$.

In the next theorem we characterize the minimal sets of periods for eight maps, i.e. for continuous maps $f : G_{2} \rightarrow G_{2}$.

Theorem B. Let $f : G_{2} \rightarrow G_{2}$ be an eight map such that

$$f_{*} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$ 

Suppose that the branching point is a fixed point. Then the following statements hold.
(a) If \( \{a, d\} \not\subseteq \{-2, -1, 0, 1\} \), then \( \text{MPer}(f) = \mathbb{N} \).

(b) If \(-2 \in \{a, d\}\) and \(\{a, d\} \subseteq \{-2, -1, 0, 1\} \), then

\[
\text{MPer}(f) = \begin{cases} 
\mathbb{N} \setminus \{2\} & \text{if } bc = 0, \\
\mathbb{N} & \text{if } bc \neq 0.
\end{cases}
\]

(c) Assume that \( \{a, d\} \subseteq \{-1, 0, 1\} \).

(c1) If \( |a| + |d| = 2 \), then

\[
\text{MPer}(f) = \begin{cases} 
\{1\} & \text{if } bc = 0, \\
\mathbb{N} \setminus \{2\} & \text{if } bc = 1, \\
\mathbb{N} & \text{if } bc = -1 \text{ or } |bc| > 1.
\end{cases}
\]

(c2) If \( |a| + |d| = 1 \) and

(c21) \( a = 1, d = 0 \), then

\[
\text{MPer}(f) = \begin{cases} 
\{1\} & \text{if } bc = 0, \\
\mathbb{N} \setminus \{2\} & \text{if } (b, c) \in R, \\
\mathbb{N} & \text{otherwise};
\end{cases}
\]

where \( R = \{(1, 1), (-1, -1), (1, 2), (-1, -2)\} \).

(c22) \( a = 0, d = 1 \), then it follows (c21) interchanging \( b \) and \( c \).

(c23) \( a = -1, d = 0 \), then

\[
\text{MPer}(f) = \begin{cases} 
\{1\} & \text{if } bc = 0, \\
\mathbb{N} \setminus \{2\} & \text{if } (b, c) \in R, \\
\mathbb{N} \setminus \{3\} & \text{if } bc = -1, \\
\mathbb{N} & \text{otherwise}.
\end{cases}
\]

(c24) \( a = 0, d = -1 \), then it follows (c23) interchanging \( b \) and \( c \).

(c3) If \( |a| + |d| = 0 \), then

\[
\text{MPer}(f) = \begin{cases} 
\{1\} & \text{if } bc = 0 \text{ or } bc = 1, \\
\{1, 2\} & \text{if } bc = -1, \\
\{1\} \cup (2\mathbb{N} \setminus \{2\}) & \text{if } bc = 2, \\
\{1\} \cup (2\mathbb{N} \setminus \{4\}) & \text{if } bc = -2, \\
\{1\} \cup 2\mathbb{N} & \text{if } |bc| > 2.
\end{cases}
\]

We remark that Theorem B implies Theorem A if \( f \) has a fixed point, by choosing, for instance, \( a = b = c = 0 \).

The study of the minimal set of periods of a homotopy class of maps instead of its homology class is the main objective of the fixed point theory, see for instance the books of Brown [Brown, 1971], Jiang [Jiang, 1983] and Kiang [Kiang, 1989]. Other extensions from circle maps to \( n \)-dimensional torus has been done in [Alsedà et al., 1995] and [Jiang & Llibre, 1998], and from circle maps to transversal \( n \)-sphere maps in [Casasayas et al., 1995]. Some different results on the periods of graph maps have been given in [Abdulla et al., 2017; Alsedà et al., 2005; Arai, 2016; Alsedà & Juette, 2008; Bernhardt, 2006, 2011; Llibre, 1991; Llibre & Misiurewicz, 2006].

This work is organized as follows. How to obtain a given period for a graph map by using the notion of \( f \)-covering is described in Section 2. The proof of Theorem B is given in Section 3.

2. Periods and \( f \)-covering

Let \( f : G \to G \) be a graph map and \( x \in G \) a periodic point of period \( n \). The set \( \{x, f(x), \ldots, f^{n-1}(x)\} \) is called the periodic orbit of \( x \).

A set \( I \subset G \) will be called an interval if there is a homeomorphism \( h : J \to I \) where \( J \) is \([0, 1], (0, 1], [0, 1) \) or \((0, 1)\). The set \( h((0, 1)) \) will be called the interior of \( I \). If \( J = [0, 1] \) the interval \( I \) will be called
closed; if \( J = (0 , 1) \) it will be called open. Notice that it may happen that the above terminology does not coincide with the one used when we think about \( I \) as a subset of \( G \) (the same applies to the edges of \( G \)). For example, if \( G = I = [0 , 1] \) and \( h = \text{identity} \), then for \( I \) regarded as a subset of the topological space \( G \), \( I \) is both open and closed and the interior of \( I \).

Let \( C_1 \) and \( C_2 \) be two circuits of \( G_k \). A closed interval \( I = [a , b] \) is basic if \( I \subset C_1 \), \( f(I) = C_j \) where \( \{i , j\} \subset \{1 , 2 , \ldots , k\} \), \( f(a) = f(b) = p \), where \( p \) is the branching point of \( G_k \), and there is no other closed interval \( K \subset I \) such that \( f(K) = C_j \). If \( f(C_1) = C_1 \) and \( f(K) \neq C_j \) for all closed interval \( K , K \subset C_1 \), then we also say that \( C_1 \) is a basic interval. Let \( I \) and \( J \) be two basic intervals, \( K \subset I \), \( L \subset J \) two subintervals. If \( L \neq C_j \), we say that \( K \) \( f \)-covers \( L \), and we write \( K \rightarrow L \), if there exists a closed subinterval \( M \) of \( K \) such that \( f(M) = L \). If \( L = J = C_j \), we say that \( K \) \( f \)-covers \( L \) because either \( f(K) = L \), or \( K = I = C_i \) and \( f(K) = L \), by the definition of basic intervals.

**Lemma 1.** Suppose that \( I_1, I_2, \ldots , I_n \) are intervals such that \( I_1 \rightarrow I_2 \rightarrow \ldots \rightarrow I_n \rightarrow I_1 \) with \( I_1 \) different from a circuit. Then there is a fixed point \( z \) of \( f^n \) such that \( z \in I_1, f(z) \in I_2, \ldots , f^{n-1}(z) \in I_n \).

**Proof.** Since \( I_n \rightarrow I_1 \), and \( I_1 \) is not a circuit, there is a closed interval \( J_n \subset I_n \) such that \( f(J_n) = I_1 \). Similarly, there are closed intervals or circuits \( J_1, \ldots , J_{n-1} \) such that for each \( k = 1 , \ldots , n-1 \), \( J_k \subset I_k \), and \( f(J_k) = J_{k+1} \). It follows that \( f^n(J_1) = I_1 \) and since \( J_1 \subset I_1 \) and \( I_1 \) is not a circuit, by Bolzano’s Theorem \( f^n \) has a fixed point \( z \in J_1 \). Clearly, \( z \in I_1, f(z) \in I_2, \ldots , f^{n-1}(z) \in I_n \).

A sequence of the form \( I_1 \rightarrow I_2 \rightarrow \ldots \rightarrow I_n \rightarrow I_1 \) is called a loop of length \( n \). Let \( I_1 \rightarrow I_2 \rightarrow \ldots \rightarrow I_n \rightarrow I_1 \) and \( J_1 \rightarrow J_2 \rightarrow \ldots \rightarrow J_m \rightarrow J_1 \) be two loops such that \( I_1 = J_1 \). We define the concatenation of these two loops as the loop \( I_1 \rightarrow I_2 \rightarrow \ldots \rightarrow I_n \rightarrow I_1 \rightarrow J_2 \rightarrow \ldots \rightarrow J_m \rightarrow J_1 \). We say that a loop is a \( m \)-repetition if it is not a \( m \)-repetition of any of its subloops with \( m \geq 2 \).

In what follows a \( G_k \)-map \( f \) is a continuous map \( f : G_k \rightarrow G_k \) such that \( f(p) = p \).

**Proposition 1.** Let \( f \) be a \( G_k \)-map. Suppose that \( f \) has two intervals \( I_1 \) and \( I_2 \) such that \( \text{Int}(I_1) \cap \text{Int}(I_2) = \emptyset \) and \( I_1 \cap I_2 \) has no fixed points. If \( f \) has the subgraph \( \odot \) then \( \text{Per}(f) = \mathbb{N} \).

**Proof.** Clearly, since \( p \notin I_1 \cap I_2 \), at least one of the intervals, \( I_1 \) and \( I_2 \), is not a circuit. Without loss of generality we assume that \( I_1 \) is not a circuit. We consider the non–repetitive loop \( I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow \ldots \rightarrow I_1 \) of length \( n \geq 2 \). Since \( \text{Int}(I_1) \cap \text{Int}(I_2) = \emptyset \) and \( I_1 \cap I_2 \) has no fixed points, by Lemma 1 there is a periodic point \( z \) of \( f \) with period \( n \geq 2 \). That is, \( \text{Per}(f) = \mathbb{N} \).

In what follows when we say “we have two intervals \( A \) and \( B \)” we are really saying that we have two different intervals \( A \) and \( B \). We remark that if we have two basic intervals \( I_1 \) and \( I_2 \) such that \( p \notin I_1 \cap I_2 \), then they satisfy the assumptions of Proposition 1.

**Proposition 2.** Let \( f \) be a \( G_k \)-map. Suppose that \( f \) has three intervals \( I_1, I_2 \) and \( I_3 \) such that \( \text{Int}(I_1) \cap \text{Int}(I_j) = \emptyset \) for all \( i \neq j \) and \( I_i \cap I_j \) has no fixed points for some \( i \neq j \). If \( f \) has the subgraph \( \odot \) then \( \text{Per}(f) > \mathbb{N} \setminus \{2\} \). Moreover, if \( I_2 \cap I_3 = \emptyset \) and \( I_3 \rightarrow I_2 \), then \( 2 \in \text{Per}(f) \).

**Proof.** We consider the non–repetitive loop \( I_1 \rightarrow I_2 \rightarrow I_3 \rightarrow I_1 \) of length \( n \geq 3 \). Since \( \text{Int}(I_i) \cap \text{Int}(I_j) = \emptyset \) for all \( i \neq j \) and \( I_i \cap I_j \) has no fixed points for some \( i \neq j \), by Lemma 1 there is a periodic point \( z \) of \( f \) with period \( n \geq 3 \). Therefore, \( \text{Per}(f) > \mathbb{N} \setminus \{2\} \).

We suppose now that \( I_2 \cap I_3 = \emptyset \) and \( I_3 \rightarrow I_2 \). We consider the non–repetitive loop \( I_2 \rightarrow I_3 \rightarrow I_2 \) of length 2. By Lemma 1 there is a periodic point \( z \) of \( f \) with period 2.

We remark that if we have three basic intervals \( I_1, I_2 \) and \( I_3 \) such that \( p \notin I_i \) for some \( i \in \{1, 2, 3\} \), then we are in the assumptions of Proposition 2.
3. The eight

In this section we shall prove Theorem B. The two circuits of \(G_2\) are denoted by \(C_1\) and \(C_2\). If \(f_*\) is given as in Theorem B, we consider that the circuit \(C_1\) covers itself \(|a|\) times and it covers \(C_2\) \(|c|\) times. Similarly for the circuit \(C_2\).

**Proof.** [Proof of Statement (a) of Theorem B] Suppose that \(\{a, d\} \not\subset \{-2, -1, 0, 1\}\).

Case 1: Assume that \(\{a, d\} \not\subset \{-2, -1, 0, 1\}\). Without loss of generality, we may assume that \(|a| \geq 3\). From the graph of \(f\) (see for instance Figure 1), it is clear that there are two basic intervals \(I_1\) and \(I_2\), in \(C_1\), such that \(p \not\in I_1 \cap I_2\) and \(f\) has the subgraph of Proposition 1, so \(\text{Per}(f) = \mathbb{N}\). That is, \(\text{MPer}(f) = \mathbb{N}\).

![Fig. 1. Examples of maps with \(\{a, d\} \not\subset \{-2, -1, 0, 1\}\).](image)

**Case 2:** Suppose that \(2 \in \{a, d\}\) and \(\{a, d\} \subset \{-2, -1, 0, 1\}\). Without loss of generality, we may assume that \(a = 2\).

Since \(a = 2\) this means that \(f\) has at least two basic intervals \(I_1\) and \(I_2\) in \(C_1\) such that \(f\) has the subgraph of Proposition 1. If \(p \not\in I_1 \cap I_2\), then, by Proposition 1 \(\text{Per}(f) = \mathbb{N}\). But not always \(I_1\) and \(I_2\) satisfy that \(p \not\in I_1 \cap I_2\). In this case let \(p\) and \(a_0\) be the endpoints of \(I_1\), \(b_0\) and \(p\) the endpoints of \(I_2\) (see for instance Figure 2).

![Fig. 2. \(I_1 = [p, a_1], I_2 = [b_0, b_1]\) and \(I_1 = [a_1, a_2]\).](image)

We establish an ordering in the intervals \(I_1\) and \(I_2\) in such a way that \(p\) is the smallest element of \(I_1\) and the greatest of \(I_2\). Set \(I_1 = [p, a_0]\) and \(I_2 = [b_0, p]\). Notice that we may have \(a_0 = b_0\). Consider the subset \((f|I_1)^{-1}(a_0)\) of \(C_1\). Let \(a_1\) be the infimum of the points in \((f|I_1)^{-1}(a_0)\). Consider the subset \((f|I_2)^{-1}(a_0)\) of \(C_1\) and choose \(b_1\) to be the infimum of the points in \((f|I_2)^{-1}(a_0)\). Set \(I_{11} = [p, a_1]\), \(I_{21} = [a_1, a_0]\) and \(I_{22} = [b_0, b_1]\). Now we take the interval \(I_{13} = [a_1, a_2]\) where \(a_2\) denotes the infimum of the points in the subset \((f|I_{13})^{-1}(b_1)\) of \(C_1\). Then \(f\) has the subgraph \(\cap I_{11} \rightarrow I_{13} \rightarrow I_2 \rightarrow I_{12}\). Since \(I_{22} \cap I_{12} = \emptyset\), by Proposition 2, \(n \in \text{Per}(f)\), for all \(n \geq 1\). Therefore, \(\text{MPer}(f) = \mathbb{N}\). This proves Statement (a). ■

**Proof.** [Proof of Statement (b) of Theorem B] Suppose that \(-2 \in \{a, d\}\) and \(\{a, d\} \subset \{-2, -1, 0, 1\}\). Without loss of generality, we may assume that \(a = -2\).
First we suppose that $bc \neq 0$. We always have four basic intervals $I_1$, $I_2$, $I_3$ and $I_4$, $I_1$, $I_2$, $I_3 \subset C_1$ and $I_4 \subset C_2$, such that either $p \notin I_1 \cap I_3$ or $I_2 \cap I_4 = \emptyset$ and $f$ has the subgraph

![Diagram](see for instance Figure 3).

(a) $a = -2$, $d = 0$, $bc = 1$  
(b) $a = -2$, $d = 1$, $bc = 1$  
(c) $a = -2$, $d = -1$, $bc = -1$

(see for instance Figure 3).

If $p \notin I_1 \cap I_3$, by Proposition 1, $\text{Per}(f) = \mathbb{N}$. If $I_2 \cap I_4 = \emptyset$, by Proposition 2, $\text{Per}(f) = \mathbb{N}$. Therefore, if $bc \neq 0$, $\text{MPer}(f) = \mathbb{N}$.

We suppose now that $bc = 0$. As it can be deduced from the examples of Figure 4, $2 \notin \text{MPer}(f)$.

![Diagrams](Fig. 4. Examples of maps with $a = -2$, $d \in \{-2, -1, 0, 1\}$, $bc = 0$ and $2 \notin \text{Per}(f)$.)

Since $a = -2$, this means that $f$ has at least two basic intervals $I_1$ and $I_2$ in $C_1$ such that $f$ has the subgraph of Proposition 1. If $p \notin I_1 \cap I_2$ then by Proposition 1 $\text{Per}(f) = \mathbb{N}$. But not always $p \notin I_1 \cap I_2$. In this case let $p$ and $a_0$ be the endpoints of $I_1$, $b_0$ and $p$ the endpoints of $I_2$ (see for instance Figure 5). We consider an ordering in the intervals $I_1$ and $I_2$ in such a way that $p$ is the smallest element of $I_1$ and the greatest of $I_2$. Write $I_1 = [p, a_0]$ and $I_2 = [b_0, p]$. Notice that we may have $a_0 = b_0$. Consider the subsets $(f|I_1)^{-1}(a_0)$ and $(f|I_2)^{-1}(a_0)$ of $C_1$. Let $a_1$ be the infimum of the points in $(f|I_1)^{-1}(a_0)$ and $b_1$ the infimum of the points in $(f|I_2)^{-1}(a_0)$. Set $I_{11} = [p, a_1]$, $I_{12} = [a_1, a_0]$ and $I_{21} = [b_1, p]$. Then $f$ has the subgraph $\bigcirc I_{12} \rightarrow I_{11} \rightarrow I_{21} \rightarrow I_{12}$. Since we are in the assumptions of Proposition 2, $n \in \text{Per}(f)$, for all $n \neq 2$. Therefore, $\text{MPer}(f) = \mathbb{N} \setminus \{2\}$. This proves Statement (b).

**Proof.** [Proof of Statement (c1) of Theorem B] Suppose that $\{a, d\} \subset \{-1, 0, 1\}$ and $|a| + |d| = 2$. We consider first the case $bc = 0$. Without loss of generality, we may assume that $c = 0$. From the examples

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of Figure 6 it is clear that \( n \notin \text{MPer}(f) \) for any \( n \in \mathbb{N} \) larger than 1, so \( \text{MPer}(f) = \{1\} \) since the branching is fixed.

We assume now that \(|bc| > 1\). From the graph of \( f \) (see for instance Figure 7) it is easy to see that we always have three basic intervals \( I_1, I_2 \) and \( I_3 \), with \( I_1, I_2 \subset C_1 \) and \( I_3 \subset C_2 \), such that \( p \notin I_i \) for some \( i \in \{1, 2, 3\} \) and \( f \) has the subgraph of Proposition 2, so \( \text{Per}(f) \supset \mathbb{N} \setminus \{2\} \). Now we will prove that \( 2 \in \text{MPer}(f) \).

If \( \{b, c\} \not\subset \{-2, -1, 1, 2\} \), that is, if either \( |b| \geq 3 \) or \( |c| \geq 3 \), we can choose \( I_2 \) in one circuit and \( I_3 \) in the other circuit in such a way that \( I_2 \cap I_3 = \emptyset \) (see (a), (b) and (c) of Figure 7) and \( I_3 \to I_2 \). By Proposition 2, \( 2 \in \text{Per}(f) \). If \( \{b, c\} \subset \{-2, -1, 1, 2\} \) in general there do not exist two basic intervals \( I_i \) and \( I_j, I_i \neq I_j \), such that \( p \notin I_i \cap I_j \) and \( I_i \leftrightarrow I_j \) (see (e) and (f) of Figure 7). If they exist then by Lemma 1 considering the non–repetitive loop \( I_i \to I_j \to I_i \) there is a periodic point \( z \) of \( f \) with period 2. If they do not exist, we shall find two intervals with empty intersection such that one \( f \)–covers the other.
We suppose first that $|bc| = 2$. We may assume, without loss of generality, that $|b| = 1$ and $|c| = 2$. We know that $f$ has five basic intervals, $I_1$, $I_2$, $I_3$, $I_4$ and $I_5$, the first three in $C_1$ and the other two in $C_2$, such that $f(I_2) = f(I_3) = f(I_5) = C_2$ and $f(I_1) = f(I_4) = f(I_6) = C_1$. Let $p$ and $a_0$ be the endpoints of $I_2$, $a_0$ and $a_1$ the endpoints of $I_1$, $a_1$ and $p$ the endpoints of $I_3$ (see for instance Figure 8).

![Figure 8](image1)

Fig. 8. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $|bc| = 2$.

We consider an ordering in the intervals $I_1$, $I_2$ and $I_3$ in such a way that $p$ is the smallest element of $I_2$ and the greatest of $I_3$. Set $I_2 = [p, a_0]$, $I_1 = [a_0, a_1]$ and $I_3 = [b_0, p]$. We have two possibilities for the interval $I_4$: either $I_4 = [p, b_0]$ or $I_4 = [b_0, p]$. If $I_4 = [p, b_0]$ and $b = 1$ let $b_1$ be the supremum of the points in $(f|I_4)^{-1}(a_1)$ and $I_{b_2} = [b_1, b_0]$. We have $I_{a_1} = I_3$ and $I_{a_2} \cap I_3 = \emptyset$, so, by Lemma 1, $2 \in \text{Per}(f)$. If $I_4 = [p, b_0]$ and $b = -1$ set $b_1 = \sup\{(f|I_4)^{-1}(a_0)\}$ and $I_{b_2} = [b_1, b_0]$. Then $I_{a_2} = I_2$ and $I_{a_3} \cap I_2 = \emptyset$, so, by Lemma 1, $2 \in \text{Per}(f)$. If $I_4 = [b_0, p]$ and $b = 1$ write $b_1 = \inf\{(f|I_4)^{-1}(a_0)\}$ and $I_{a_1} = [b_0, b_1]$. Then $I_{a_1} = I_3$ and $I_{a_4} \cap I_3 = \emptyset$, so, by Lemma 1, $2 \in \text{Per}(f)$.

![Figure 9](image2)

Fig. 9. Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $|bc| = 4$.

Suppose now that $|bc| = 4$. We know that $f$ has six basic intervals, $I_1$, $I_2$, $I_3$, $I_4$, $I_5$ and $I_6$, the first three in $C_1$ and the other three in $C_2$, such that $f(I_2) = f(I_3) = f(I_5) = C_2$ and $f(I_1) = f(I_4) = f(I_6) = C_1$ (see for instance Figure 9). Using the same ordering as above set $I_2 = [p, a_0]$, $I_1 = [a_0, a_1]$, $I_3 = [b_0, p]$,
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$I_4 = [p, b_0]$, $I_5 = [b_0, b_1]$ and $I_6 = [b_1, p]$. If $b = 2$ set $b_2 = \inf\{(f|I_6)^{-1}(a_0)\}$ and $I_6 = [b_1, b_2]$. Then $I_6 \leftrightarrow I_2$ and $I_6 \cap I_2 = \emptyset$, so by Lemma 1, $2 \in \operatorname{Per}(f)$. If $b = -2$ write $b_2 = \inf\{(f|I_6)^{-1}(a_1)\}$ and $I_6 = [b_1, b_2]$. Then $I_6 \leftrightarrow I_3$ and $I_6 \cap I_3 = \emptyset$, so by Lemma 1, $2 \in \operatorname{Per}(f)$. Therefore, if $|bc| > 1$, $\operatorname{MPer}(f) = \mathbb{N}$.

We suppose that $|bc| = 1$. We assume that $b = c = 1$. As it can be seen from the examples (a), (c) and (e) of Figure 10, $2 \notin \operatorname{MPer}(f)$. Now we will prove that $\operatorname{Per}(f) = \mathbb{N} \setminus \{2\}$.

**Fig. 10.** Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$, $bc = 1$ and $2 \notin \operatorname{Per}(f)$.

We know that $f$ has four basic intervals, $I_1$, $I_2$, $I_3$ and $I_4$, the first two in $C_1$ and the other two in $C_2$, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = f(I_4) = C_2$. We have four possibilities for these intervals. Let $a_0 \in I_1 \cap I_2$ and $b_0 \in I_3 \cap I_4$ (see for instance Figure 11). First, we take the interval $I_3$ to be $[p, a_0]$. Set $I_3' = [b_1, b_0]$ where $b_1 = \sup\{(f|I_3)^{-1}(a_0)\}$. If $I_1 = [p, a_0]$ then $f$ has the subgraph $\circlearrowright I_4 \rightarrow I_3' \rightarrow I_2 \rightarrow I_4$ and by Proposition 2, $\operatorname{Per}(f) = \mathbb{N} \setminus \{2\}$. If $I_1 = [a_0, p]$ then $f$ has the subgraph $\circlearrowright I_1 \rightarrow I_2 \rightarrow I_3' \rightarrow I_1$ and by Proposition 2, $\operatorname{Per}(f) = \mathbb{N} \setminus \{2\}$.

**Fig. 11.** Examples of maps with $\{a, d\} \subset \{-1, 0, 1\}$, $|a| + |d| = 2$ and $b = c = 1$. 
Now we take the interval \( I_3 \) to be \([b_0, p]\). Set \( b_1 = \inf\{(f|I_3)^{-1}(a_0)\} \) and \( I_{3_1} = [b_0, b_1] \). If \( I_1 = [p, a_0] \) then \( f \) has the subgraph \( \ominus I_1 \to I_2 \to I_3_1 \to I_1 \) and by Proposition 2, \( \text{Per}(f) = N \setminus \{2\} \). If \( I_1 = [a_0, p] \) then \( f \) has the subgraph \( \ominus I_4 \to I_3_1 \to I_2 \to I_4 \) and by Proposition 2, \( \text{Per}(f) = N \setminus \{2\} \). Therefore, if \( |a| = |d| = 1 \) and \( b = c = 1 \) then \( \text{MPer}(f) = N \setminus \{2\} \).

We assume now that \( b = c = -1 \). As it can be seen from the examples (b), (d) and (f) of Figure 10, \( 2 \not\in \text{MPer}(f) \). Now we will prove that \( \text{Per}(f) = N \setminus \{2\} \).

![Fig. 12. Examples of maps with \( \{a, d\} \subset \{-1, 0, 1\} \), \(|a| + |d| = 2 \) and \( b = c = -1 \).](image)

We know that \( f \) has four basic intervals, \( I_1, I_2, I_3 \) and \( I_4 \), the first two in \( C_1 \) and the other two in \( C_2 \), such that \( f(I_1) = f(I_3) = C_1 \) and \( f(I_2) = f(I_4) = C_2 \). We have four possibilities for these intervals. Let \( a_0 \in I_1 \cap I_2 \) and \( b_0 \in I_3 \cap I_4 \) (see for instance Figure 12). First we take \( I_3 \) to be the interval \([p, b_0]\). Consider \( b_1 = \sup\{(f|I_3)^{-1}(a_0)\} \) and \( I_{3_2} = [b_1, b_0] \). If \( I_1 = [p, a_0] \) then \( f \) has the subgraph \( \ominus I_1 \to I_2 \to I_3_2 \to I_1 \) and by Proposition 2, \( \text{Per}(f) = N \setminus \{2\} \). If \( I_1 = [a_0, p] \) then \( f \) has the subgraph \( \ominus I_4 \to I_3_2 \to I_2 \to I_4 \) and by Proposition 2, \( \text{Per}(f) = N \setminus \{2\} \). Therefore, if \( b = c = -1 \) then \( \text{MPer}(f) = N \setminus \{2\} \).

We consider now case \( b = -1 \) and \( c = 1 \). We know that \( f \) has four basic intervals, \( I_1, I_2, I_3 \) and \( I_4 \), the first two in \( C_1 \) and the other two in \( C_2 \), such that \( f(I_1) = f(I_3) = C_1 \) and \( f(I_2) = f(I_4) = C_2 \). We have four possibilities for these intervals. Let \( a_0 \in I_1 \cap I_2 \) and \( b_0 \in I_3 \cap I_4 \) (see for instance Figure 13). We suppose first that \( I_2 = [a_0, p] \). If \( I_3 = [p, b_0] \) choose \( a_1 = \inf\{(f|I_3)^{-1}(b_0)\} \) and set \( I_{3_1} = [a_0, a_1] \). Then \( f \) has the subgraph \( \ominus I_1 \to I_2 \to I_3_1 \to I_1 \) with \( I_3 \cap I_2_1 = \emptyset \) and by Proposition 2, \( \text{Per}(f) = N \). If \( I_3 = [b_0, p] \) denote \( b_1 = \inf\{(f|I_3)^{-1}(a_0)\} \) and \( I_{3_1} = [b_1, b_0] \). Then \( f \) has the subgraph \( \ominus I_4 \to I_3_1 \to I_2 \to I_4 \) with \( I_2 \cap I_3_1 = \emptyset \) and by Proposition 2, \( \text{Per}(f) = N \).

We consider now case \( I_2 = [p, a_0] \). If \( I_3 = [p, b_0] \) set \( b_1 = \sup\{(f|I_3)^{-1}(a_0)\} \) and \( I_{3_2} = [b_1, b_0] \). Then \( f \) has the subgraph \( \ominus I_1 \to I_2 \to I_3_2 \to I_1 \) with \( I_2 \cap I_3_2 = \emptyset \) and by Proposition 2, \( \text{Per}(f) = N \). If \( I_3 = [b_0, p] \) write \( a_1 = \sup\{(f|I_3)^{-1}(b_0)\} \) and \( I_{3_2} = [a_1, a_0] \). Then \( f \) has the subgraph \( \ominus I_1 \to I_2 \to I_3_2 \to I_1 \) with \( I_3 \cap I_2_2 = \emptyset \) and by Proposition 2, \( \text{Per}(f) = N \). Therefore, if \( b = -1 \) and \( c = 1 \), then \( \text{MPer}(f) = N \).

We consider now case \( b = 1 \) and \( c = -1 \). We know that \( f \) has four basic intervals, \( I_1, I_2, I_3 \) and \( I_4 \), the first two in \( C_1 \) and the other two in \( C_2 \), such that \( f(I_1) = f(I_3) = C_1 \) and \( f(I_2) = f(I_4) = C_2 \). We have again four possibilities for these intervals. Let \( a_0 \in I_1 \cap I_2 \) and \( b_0 \in I_3 \cap I_4 \) (see for instance Figure 14). We take the interval \( I_2 \) to be \([a_0, p] \). If \( I_3 = [p, b_0] \) define \( b_1 = \sup\{(f|I_3)^{-1}(a_0)\} \) and \( I_{3_2} = [b_1, b_0] \). It follows that
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Fig. 13. Examples of maps with \( \{a, d\} \subset \{-1, 0, 1\} \), \(|a| + |d| = 2\), \(b = -1\) and \(c = 1\).

Fig. 14. Examples of maps with \( \{a, d\} \subset \{-1, 0, 1\} \), \(|a| + |d| = 2\), \(b = 1\) and \(c = -1\).

\(f\) has the subgraph \( \bigcirc I_4 \to I_3 \rightleftharpoons I_2 \to I_4 \) with \( I_2 \cap I_3 = \emptyset \) and by Proposition 2, \( \text{Per}(f) = \mathbb{N} \). If \( I_3 = [b_0, p] \) consider \( a_1 = \inf \{(f|I_2)^{-1}(b_0)\} \) and \( I_{22} = [a_0, a_1] \). Then \( f \) has the subgraph \( \bigcirc I_1 \to I_{22} \rightleftharpoons I_3 \to I_1 \) with \( I_3 \cap I_{22} = \emptyset \) and we get by Proposition 2, \( \text{Per}(f) = \mathbb{N} \).

Suppose that \( I_2 = [p, a_0] \). If \( I_3 = [p, b_0] \) set \( a_1 = \sup \{(f|I_2)^{-1}(b_0)\} \) and \( I_{21} = [a_0, a_1] \). Then \( f \) has the subgraph \( \bigcirc I_1 \to I_{21} \rightleftharpoons I_3 \to I_1 \) with \( I_3 \cap I_{21} = \emptyset \) and by Proposition 2, \( \text{Per}(f) = \mathbb{N} \). Therefore, if \( b = 1 \) and \( c = -1 \), then \( \text{MPer}(f) = \mathbb{N} \). Hence, if \(|a| + |d| = 2\) and \(bc = -1\) then \( \text{MPer}(f) = \mathbb{N} \).

This completes the proof of Statement (c1). □

Proof. [Proof of Statement (c21) of Theorem B] We assume now that \( a = 1 \) and \( d = 0 \). If \( bc = 0 \) then \( \text{MPer}(f) = \{1\} \) as it can be deduced from the examples of Figure 15. We suppose that \( b \) and \( c \) are such that \(|bc| > 1\) and \((b, c) \notin \{(2, 1), (2, -1), (-2, 1), (-2, -1)\}\). From the graph of \( f \) (see for instance Figure 16) it follows that there are three basic intervals \( I_1, I_2 \) and \( I_3, I_1, I_2 \subset C_1, I_3 \subset C_2 \), such that either \( p \notin I_1 \cap I_2 \)
or \( p \not\in I_1 \cap I_3 \) and \( f \) has the subgraph of Proposition 2, so \( \text{Per}(f) \supset \mathbb{N} \setminus \{2\} \).

![Fig. 15. Examples of maps with \( a = 1, d = 0 \) and \( bc = 0 \).](image1)

![Fig. 16. Examples of maps with \( a = 1, d = 0, (b, c) \not\in \{(2, 1), (2, -1), (-2, 1), (-2, -1)\} \) and \(|bc| > 1\).](image2)

If \( (b, c) \not\subset \{-2, -1, 1, 2\} \) then we can choose \( I_2 \) and \( I_3 \) such that \( I_2 \cap I_3 = \emptyset \) and by Proposition 2, \( 2 \in \text{Per}(f) \). If \( (b, c) \subset \{-2, -1, 1, 2\} \) in general there do not exist two basic intervals \( I_i \) and \( I_j \), \( I_i \neq I_j \), such that \( p \not\in I_i \cap I_j \) and \( I_i \preceq I_j \). If they exist then by Lemma 1 considering the non–repetitive loop \( I_i \rightarrow I_j \rightarrow I_i \) there is a periodic point \( z \) of \( f \) with period 2. If they do not exist (see for instance (c) and (d) of Figure 16) and \( (b, c) \in \{(1, -2), (-1, -2)\} \), \( 2 \in \text{Per}(f) \) as we can see from the examples of Figure 17.

Now we will prove that if \( (b, c) \in \{(1, -2), (-1, 2)\} \) or \( |b| = |c| = 2 \) then \( 2 \in \text{Per}(f) \).

![Fig. 17. Examples of maps with \( a = 1, d = 0, (b, c) \in \{(1, 2), (-1, 2)\} \) and \( 2 \not\in \text{Per}(f) \).](image3)

We suppose first that \( (b, c) \in \{(1, -2), (-1, 2)\} \). We know that \( f \) has four basic intervals, \( I_1, I_2, I_3 \) and \( I_4 \), the first three in \( C_1 \) and \( I_4 = C_2 \), such that \( f(I_1) = f(I_4) = C_1 \) and \( f(I_2) = f(I_3) = C_2 \). Let \( p \) and \( a_0 \) be the endpoints of \( I_2 \), \( a_0 \) and \( a_1 \) the endpoints of \( I_1 \), \( a_1 \) and \( p \) the endpoints of \( I_3 \) (see for instance Figure 18).

We consider an ordering in the intervals \( I_1, I_2 \) and \( I_3 \) in such a way that \( p \) is the smallest element of \( I_2 \) and the greatest of \( I_3 \). Under these assumptions set \( I_2 = [p, a_0], I_1 = [a_0, a_1] \) and \( I_3 = [a_1, p] \). Define

![Fig. 18. Examples of maps with \( a = 1, d = 0 \) and \( (b, c) \in \{(1, -2), (-1, 2)\} \).](image4)
$b_0 = \sup \{(f|I_4)^{-1}(a_0)\}$, $I_{4_1} = [p,b_0]$ and $I_{4_2} = [b_0,p]$. Set $a_2 = \inf \{(f|I_3)^{-1}(b_0)\}$ and $I_{3_1} = [a_1,a_2]$. If $(b,c) = (1,-2)$ we have $I_{4_2} \rightleftharpoons I_{3_1}$ and $I_{4_2} \cap I_{3_1} = \emptyset$. If $(b,c) = (-1,2)$ we get $I_{4_1} \rightleftharpoons I_{3_1}$ and $I_{4_1} \cap I_{3_1} = \emptyset$. So, by Lemma 1, $2 \in \Per(f)$.

Suppose now that $|b| = |c| = 2$. We know that $f$ has five basic intervals, $I_1$, $I_2$, $I_3$, $I_4$ and $I_5$, the first three in $C_1$ and the other two in $C_2$, such that $f(I_2) = f(I_3) = C_2$ and $f(I_1) = f(I_4) = f(I_5) = C_1$. Taking an ordering similar to the previous case define the intervals $I_2 = [p,a_0]$, $I_1 = [a_0,a_1]$, $I_3 = [a_1,p]$, $I_4 = [p,b_0]$ and $I_5 = [b_0,p]$ (see for instance Figure 19). Set $a_2 = \sup \{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_2,a_0]$. If $c = 2$ we have $I_{2_2} \rightleftharpoons I_5$ and $I_{2_2} \cap I_5 = \emptyset$. If $c = -2$ we have $I_{2_2} \rightleftharpoons I_4$ and $I_{2_2} \cap I_4 = \emptyset$. So, by Lemma 1, $2 \in \Per(f)$. Therefore, if $|bc| > 1$ and $(b,c) \notin \{(2,1),(-2,1),(-2,1),(1,2)\}$ we have $\MPer(f) = \mathbb{N} \setminus \{2\}$ if $(b,c) \in \{(1,2),(-1,-2)\}$ and $\MPer(f) = \mathbb{N}$ otherwise.

![Figure 19](image1.png)

Fig. 19. Examples of maps with $a = 1$, $d = 0$ and $|b| = |c| = 2$.

![Figure 20](image2.png)

Fig. 20. Examples of maps with $a = 1$, $d = 0$ and $(b,c) \in \{(2,1),(-2,1)\}$.
We assume that $|bc| > 1$ and $(b, c) \in \{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$. We know that $f$ has four basic intervals, $I_1, I_2, I_3$ and $I_4$, the first two in $C_1$ and the others in $C_2$, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = C_2$. Let $p$ and $a_0$ be the endpoints of $I_1$ and $I_2$, and $b_0$ and $p$ the endpoints of $I_3$ and $I_4$ (see for instance Figures 20 and 21). For each pair $(b, c)$ we have two possibilities for the intervals $I_1$ and $I_2$. If $(b, c) \in \{(2, 1), (-2, 1)\}$ and $I_2 = [a_0, p]$ write $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_1} = [a_0, a_1]$. Then $f$ has the subgraph $C I_1 \rightarrow I_{2_1} \rightleftharpoons I_3 \rightarrow I_1$ with $I_3 \cap I_{2_1} = \emptyset$ and by Proposition 2, $\text{Per}(f) = \mathbb{N}$. If $I_2 = [p, a_0]$ consider $a_1 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_1, a_0]$. Then $f$ has the subgraph $C I_1 \rightarrow I_{2_2} \rightleftharpoons I_3 \rightarrow I_1$ with $I_3 \cap I_{2_2} = \emptyset$ and by Proposition 2, $\text{Per}(f) = \mathbb{N}$.

We consider the case $|bc| = 1$. First assume that $bc = 1$. As we can see from the examples of Figure 22, $2 \notin \text{MPer}(f)$. Now we will prove that $\text{Per}(f) = \mathbb{N} \setminus \{2\}$.

If $(b, c) \in \{(-2, -1), (2, -1)\}$ and $I_2 = [a_0, p]$ set $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_1} = [a_0, a_1]$. Then $f$ has the subgraph $C I_1 \rightarrow I_{2_1} \rightleftharpoons I_3 \rightarrow I_1$ with $I_3 \cap I_{2_1} = \emptyset$ and by Proposition 2, $\text{Per}(f) = \mathbb{N}$. If $I_2 = [p, a_0]$ consider $a_1 = \sup\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_2} = [a_1, a_0]$. Then $f$ has the subgraph $C I_1 \rightarrow I_{2_2} \rightleftharpoons I_3 \rightarrow I_1$ with $I_3 \cap I_{2_2} = \emptyset$ and by Proposition 2, $\text{Per}(f) = \mathbb{N}$. Therefore, if $|bc| > 1$ and $(b, c) \in \{(2, 1), (2, -1), (-2, 1), (-2, -1)\}$, $\text{MPer}(f) = \mathbb{N}$.

We consider the case $|bc| = 1$. First assume that $bc = 1$. As we can see from the examples of Figure 22, $2 \notin \text{MPer}(f)$. Now we will prove that $\text{Per}(f) = \mathbb{N} \setminus \{2\}$.

We know that $f$ has three basic intervals, $I_1, I_2$ and $I_3$, the first two in $C_1$ and $I_3 = C_2$, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = C_2$. We have two possibilities for the intervals $I_1$ and $I_2$: either $p$ is the smallest element of $I_1$ and the greatest of $I_2$ or $p$ is the smallest element of $I_2$ and the greatest of $I_1$ (see for instance Figure 23). In the assumption that $b = c = 1$, if $I_1 = [p, a_0]$, write $b_0 = \inf\{(f|I_3)^{-1}(a_0)\}$, $I_{3_1} = [p, b_0]$, $a_1 = \inf\{(f|I_2)^{-1}(b_0)\}$ and $I_{2_1} = [a_0, a_1]$. Then $f$ has the subgraph $C I_3 \rightarrow I_{3_1} \rightarrow I_{3_2} \rightarrow I_3$ and by Proposition 2, $\text{Per}(f) \supset \mathbb{N} \setminus \{2\}$. If $I_1 = [a_0, p]$, define $b_0 = \sup\{(f|I_3)^{-1}(a_0)\}$, $I_{3_2} = [b_0, p]$,
Fig. 23. Examples of maps with $a = 1$, $d = 0$ and $bc = 1$.

Assume now that $bc = -1$. We know that $f$ has three basic intervals, $I_1$, $I_2$ and $I_3$, the first two in $C_1$ and $I_3 = C_2$, such that $f(I_1) = f(I_3) = C_1$ and $f(I_2) = C_2$. We have two possibilities for the intervals $I_1$ and $I_2$: either $p$ is the smallest element of $I_1$ and the greatest of $I_2$ or $p$ is the smallest element of $I_2$ and the greatest of $I_1$ (see for instance Figure 24). Define $b_0 = \inf \{(f|I_3)^{-1}(a_0)\}$, $I_{3_1} = [p, b_0]$, $I_{3_2} = [b_0, p]$ and
\[ a_1 = \inf \{(f|I_2)^{-1}(b_0)\}. \]

If \( I_1 = [a_0, p] \) let \( I_{21} = [p, a_1] \) and \( I_{22} = [a_1, a_0] \). Consider \( a_2 = \inf \{(f|I_1)^{-1}(a_1)\} \). We write \( I_{11} = [a_0, a_2] \) and \( I_{12} = [a_2, p] \). If \( b = 1 \) and \( c = -1 \) (see (a) of Figure 24) \( f \) has the subgraph

We consider the non-repetitive loops \( I_{11} \to I_{21} \to I_{32} \to I_{11} \) and \( I_{12} \to I_{11} \to I_{21} \to I_{32} \to I_{12} \to \ldots \to I_{12} \) of lengths 3 and \( n \geq 4 \), respectively. From the first loop and by Lemma 1 there is a periodic point \( z \) of \( f \) with period 3; from the second loop and by Lemma 1 there is a periodic point \( z \) of \( f \) with period \( n \geq 4 \). Moreover, \( I_{31} \supseteq I_{23} \) and \( I_{31} \cap I_{23} = \emptyset \), so, by Lemma 1, \( 2 \in \text{Per}(f) \). Hence, \( \text{Per}(f) = N \). If \( b = -1 \) and \( c = 1 \) (see (b) of Figure 24) \( f \) has the subgraph

Now from the non-repetitive loops \( I_{11} \to I_{21} \to I_{31} \to I_{11} \) and \( I_{12} \to I_{11} \to I_{21} \to I_{31} \to I_{12} \to \ldots \to I_{12} \) of lengths 3 and \( n \geq 4 \), respectively, and \( I_{32} \supseteq I_{23} \) and \( I_{32} \cap I_{23} = \emptyset \), it follows that \( \text{Per}(f) = N \).

If \( I_1 = [p, a_0] \) let \( I_{21} = [a_0, a_1] \), \( I_{22} = [a_1, p] \). Define \( a_2 = \sup \{(f|I_1)^{-1}(a_1)\} \), \( I_{11} = [a_0, a_2] \) and \( I_{12} = [a_2, p] \). If \( b = 1 \) and \( c = -1 \) (see (c) of Figure 24) \( f \) has the subgraph

Again from the non-repetitive loops \( I_{12} \to I_{22} \to I_{32} \to I_{12} \) and \( I_{11} \to I_{12} \to I_{22} \to I_{32} \to I_{11} \to \ldots \to I_{11} \) of lengths 3 and \( n \geq 4 \), respectively, \( I_{32} \supseteq I_{23} \) and \( I_{32} \cap I_{23} = \emptyset \), \( \text{Per}(f) = N \). If \( b = -1 \) and \( c = 1 \) (see (d) of Figure 24) \( f \) has the subgraph

We consider the non-repetitive loops \( I_{12} \to I_{22} \to I_{32} \to I_{12} \) and \( I_{11} \to I_{12} \to I_{22} \to I_{32} \to I_{11} \to \ldots \to I_{11} \) of lengths 3 and \( n \geq 4 \), respectively, \( I_{31} \supseteq I_{23} \) and \( I_{31} \cap I_{23} = \emptyset \). We obtain that \( \text{Per}(f) = N \). Therefore, if \( a = 1 \), \( d = 0 \) and \( bc = -1 \), \( \text{MPer}(f) = N \). This completes the proof of Statement (c21).

**Proof.** [Proof of Statement (c22) of Theorem B] If \( a = 0 \) and \( d = 1 \), by using the same kind of arguments that in the case \( a = 1 \) and \( d = 0 \), and interchanging \( b \) and \( c \), we obtain Statement (c22).

**Proof.** [Proof of Statement (c23) of Theorem B] We suppose that \( a = -1 \) and \( d = 0 \). If \( bc = 0 \) then \( \text{MPer}(f) = \{1\} \) as it can be seen from the examples of Figure 25. The cases in which \( \text{MPer}(f) \) is either \( N \setminus \{2\} \) or \( N \) can be proved following exactly the same kind of arguments that in the proof of Statement (c21).
Assume now that $bc = -1$. From the examples of Figure 26 we can see that $3 \not\in \text{MPer}(f)$.

We know that $f$ has three basic intervals, $I_1$, $I_2$ and $I_3$, the first two in $C_1$ and $I_3 = C_2$. We have two possibilities for the intervals $I_1$ and $I_2$: either $p$ is the smallest element of $I_1$ and the greatest of $I_2$ or $p$ is the smallest element of $I_2$ and the greatest of $I_1$ (see for instance Figure 27). Denote $b_0 = \inf\{(f|_{I_3})^{-1}(a_0)\}$, $I_{31} = [p, b_0]$ $I_{32} = [b_0, p]$ and $a_1 = \inf\{(f|_{I_2})^{-1}(b_0)\}$.

If $I_1 = [a_0, p]$ let $I_{21} = [p, a_1]$ and $I_{22} = [a_1, a_0]$. Consider $a_2 = \inf\{(f|_{I_1})^{-1}(a_1)\}$. Write $I_{11} = [a_0, a_2]$ and $I_{12} = [a_2, p]$. If $b = 1$ and $c = -1$ (see (a) of Figure 27) $f$ has the subgraph $\circ I_{11} \rightarrow I_{12} \rightarrow I_{21} \rightarrow I_{22} \rightarrow I_{11}$. We consider the non-repetitive loop $I_{11} \rightarrow I_{12} \rightarrow I_{21} \rightarrow I_{22} \rightarrow I_{11} \rightarrow \ldots \rightarrow I_{11}$ of length $n \geq 4$. By Lemma 1 there is a periodic point $z$ of $f$ with period $n \geq 4$. Moreover, $I_{31} \leftrightarrow I_{22}$ and
$I_{31} \cap I_{22} = \emptyset$, so, by Lemma 1, $2 \in \text{Per}(f)$. Hence, $\text{Per}(f) = \mathbb{N} \setminus \{3\}$. If $b = -1$ and $c = 1$ (see (b) of Figure 27) $f$ has the subgraph $\gamma(I_{11} \rightarrow I_{12} \rightarrow I_{21} \rightarrow I_{31} \rightarrow I_{11})$. We consider the non-repetitive loop $I_{11} \rightarrow I_{12} \rightarrow I_{21} \rightarrow I_{31} \rightarrow I_{11} \rightarrow \ldots \rightarrow I_{11}$ of length $n \geq 4$. By Lemma 1 there is a periodic point $z$ of $f$ with period $n \geq 4$. Moreover, $I_{32} \nsubseteq I_{22} \text{ and } I_{32} \cap I_{22} = \emptyset$, so, by Lemma 1, $2 \in \text{Per}(f)$. Hence, $\text{Per}(f) = \mathbb{N} \setminus \{3\}$.

If $I_1 = [p,a_0]$ let $I_{21} = [a_0,a_1]$ and $I_{22} = [a_1,p]$. Consider $a_2 = \sup\{ (f|I_{1})^{-1}(a_1) \}$. Write $I_{11} = [p,a_2]$ and $I_{12} = [a_2,a_0]$. If $b = 1$ and $c = -1$ (see (c) of Figure 27) $f$ has the subgraph $\gamma(I_{12} \rightarrow I_{11} \rightarrow I_{22} \rightarrow I_{31} \rightarrow I_{11})$. From the non-repetitive loop $I_{12} \rightarrow I_{11} \rightarrow I_{22} \rightarrow I_{31} \rightarrow I_{11} \rightarrow \ldots \rightarrow I_{12}$ of length $n \geq 4$, $I_{32} \nsubseteq I_{22}$ and $I_{32} \cap I_{22} = \emptyset$, we obtain that $\text{Per}(f) = \mathbb{N} \setminus \{3\}$. If $b = -1$ and $c = 1$ (see (d) of Figure 27) $f$ has the subgraph $\gamma(I_{12} \rightarrow I_{11} \rightarrow I_{22} \rightarrow I_{32} \rightarrow I_{12})$. Using the non-repetitive loop $I_{12} \rightarrow I_{11} \rightarrow I_{22} \rightarrow I_{32} \rightarrow I_{12} \rightarrow \ldots \rightarrow I_{12}$ of length $n \geq 4$, $I_{32} \nsubseteq I_{22}$ and $I_{32} \cap I_{22} = \emptyset$, we get that $\text{Per}(f) = \mathbb{N} \setminus \{3\}$. Therefore, if $a = -1$, $d = 0$ and $bc = -1$, $\text{MPer}(f) = \mathbb{N} \setminus \{3\}$. This completes the proof of Statement (c23).

Proof. [Proof of Statement (c24) of Theorem B] If $a = 0$ and $d = -1$, by using the same kind of arguments that in the case $a = -1$ and $d = 0$, and interchanging $b$ and $c$, we obtain Statement (c24).

Proof. [Proof of Statement (c3) of Theorem B] We suppose that $a = d = 0$. If $bc = 0$ or $bc = 1$ we can deduce from the examples of Figure 28 that $\text{MPer}(f) = \{1\}$.

If $bc = -1$ then $\text{MPer}(f) = \{1,2\}$ (see for instance Figure 29).

We assume now that $|bc| = 2$. Since $a = d = 0$ we may assume without loss of generality that $|b| = 1$ and $|c| = 2$. We consider first case $bc = -2$. Clearly, $\{1,2\} \subseteq \text{Per}(f)$, no other odd number belongs to $\text{MPer}(f)$ and $4 \notin \text{MPer}(f)$ as it can be deduced from Figure 30. Now we will prove that $n \in \text{Per}(f)$ for any $n$ even larger than 4.

We know that $f$ has three basic intervals, $I_1$, $I_2$ and $I_3$, the first two in $C_1$ and $I_3 = C_2$, such that $f(I_1) = f(I_2) = C_2$ and $f(I_3) = C_1$ (see for instance Figure 31). Consider $b_0 = \inf\{ (f|I_3)^{-1}(a_0) \}$, $a_1 = \inf\{ (f|I_1)^{-1}(b_0) \}$, $b_1 = \inf\{ (f|I_3)^{-1}(a_1) \}$. Set $I_{11} = [p,a_1]$, $I_{12} = [a_1,a_0]$, $I_{21}$ the interval with endpoints $b_1$ and $p$, $I_{32}$ the interval with endpoints $b_1$ and $b_0$, and $I_{33}$ the interval with endpoints $b_0$ and $p$. Then $f$ has the subgraph $I_{32} \rightarrow I_{21} \rightarrow I_{33} \rightarrow I_{32}$.
Minimal set of periods for continuous self–maps of a bouquet of circles

Fig. 30. Examples of maps with $a = d = 0$, $bc = -2$ and $4 \notin \text{Per}(f)$.

We consider the non–repetitive loops $I_{3_1} \to I_{1_2} \to I_{3_2}$ and $I_2 \to I_{3_1} \to I_{1_2} \to I_{3_3} \to I_2 \to \ldots \to I_{3_3} \to I_2$ of lengths 2 and $n$ even, $n \geq 6$, respectively. We have $I_{3_1} \cap I_{1_2} = \emptyset$, so, from the first loop and by Lemma 1 there is a periodic point $z$ of $f$ with period 2; from the second loop and by Lemma 1 there is a periodic point $z$ of $f$ with period $n$ even $n \geq 6$. Therefore, if $bc = -2$ then $\text{MPer}(f) = \{1\} \cup (2\mathbb{N} \setminus \{4\})$.

We suppose that $bc = 2$. No odd number other than 1 belongs to $\text{MPer}(f)$, as it can be seen from the examples of Figure 32. Also from Figure 32 we can deduce that $2 \notin \text{MPer}(f)$. Now we will prove that $n \in \text{Per}(f)$ for any $n$ even larger than 2.

We consider now case $|bc| > 2$. We must separate case $|b| = |c| = 2$ from the others. If $|b| > 2$ or $|c| > 2$ then there are three basic intervals $I_1$, $I_2$ and $I_3$ such that $I_2 \cap I_3 = \emptyset$ and $I_1 \rightleftharpoons I_3 \rightleftharpoons I_2$ (see for instance Figure 34). By Lemma 1 there is a periodic point $z$ of $f$ with period $n$ even $n \geq 4$. Therefore, if $bc = 2$ then $\text{MPer}(f) = \{1\} \cup (2\mathbb{N} \setminus \{2\})$.

We consider now case $|bc| > 2$. We must separate case $|b| = |c| = 2$ from the others. If $|b| > 2$ or $|c| > 2$ then there are three basic intervals $I_1$, $I_2$ and $I_3$ such that $I_2 \cap I_3 = \emptyset$ and $I_1 \rightleftharpoons I_3 \rightleftharpoons I_2$ (see for instance Figure 34). By Lemma 1 there is a periodic point $z$ of $f$ with
period 2, and the non-repetitive loop $I_1 \to I_3 \to I_2 \to I_3 \to \ldots \to I_2 \to I_1$ of length $n$ even larger than 2 gives a periodic point $z$ of $f$ with period $n$ even. No odd number other than 1 belongs to $\text{MPer}(f)$. Therefore, if $|b| > 2$ or $|c| > 2$, then $\text{MPer}(f) = \{1\} \cup 2\mathbb{N}$.

We suppose that $|b| = |c| = 2$. Clearly, no odd number other than 1 belongs to $\text{MPer}(f)$. Now we will prove that $n \in \text{Per}(f)$ for any $n$ even.

We know that $f$ has four basic intervals, $I_1$, $I_2$, $I_3$ and $I_4$, the first two in $C_1$ and the others in $C_2$, such that $f(I_1) = f(I_2) = C_2$ and $f(I_3) = f(I_4) = C_1$ (see for instance Figure 35). Consider $b_1 = \inf\{(f|I_3)^{-1}(a_0)\}$ and $a_1 = \inf\{(f|I_1)^{-1}(b_1)\}$. Denote $I_{11} = [p, a_1]$, $I_{12} = [a_1, a_0]$, $I_2 = [a_0, p]$, $I_{31} = [p, b_1]$, $I_{32} = [b_1, b_0]$ and $I_4 = [b_0, p]$. If $(b, c) \in \{(2, 2), (-2, 2)\}$ then $f$ has the subgraph $I_2 \simeq I_1 \simeq I_{12}$. We take the non-repetitive loops $I_4 \to I_{12} \to I_4$ and $I_2 \to I_4 \to I_{12} \to I_4 \to \ldots \to I_{12} \to I_4 \to I_2$, of lengths 2 and $n$ even larger than 2, respectively. By Lemma 1 the first loop gives a periodic point $z$ of $f$ with period 2, and the second loop gives a periodic point $z$ of $f$ with period $n$ even larger than 2. Hence, if $(b, c) \in \{(2, 2), (-2, 2)\}$, $\text{Per}(f) = \{1\} \cup 2\mathbb{N}$.

If $(b, c) = (-2, -2)$ then $f$ has the subgraph $I_4 \simeq I_{11} \simeq I_{32}$. We consider the non-repetitive loops $I_{32} \to I_{11} \to I_{32}$ and $I_4 \to I_{11} \to I_{32} \to I_{11} \to \ldots \to I_{32} \to I_{11} \to I_4$, of lengths 2 and $n$ even larger than 2, respectively. By Lemma 1 the first loop gives a periodic point $z$ of $f$ with period 2, and the second loop gives a periodic point $z$ of $f$ with period $n$ even larger than 2. Hence, if $(b, c) = (-2, -2)$, $\text{Per}(f) = \{1\} \cup 2\mathbb{N}$.
Fig. 35. Examples of maps with $a = d = 0$ and $|b| = |c| = 2$. gives a periodic point $z$ of $f$ with period $n$ even larger than 2. Hence, if $(b, c) = (-2, -2)$, $\text{Per}(f) = \{1\} \cup 2\mathbb{N}$. Therefore, if $|b| = |c| = 2$ then $M\text{Per}(f) = \{1\} \cup 2\mathbb{N}$. This completes the proof of Statement (c3).

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