POLYNOMIAL HAMILTONIAN SYSTEMS OF DEGREE 3 WITH NILPOTENT SADDLES

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ABSTRACT. We provide normal forms and the global phase portraits in the Poincaré disk for all Hamiltonian planar polynomial vector fields of degree 3 symmetric with respect to the $x-$axis having a nilpotent saddle at the origin.

1. INTRODUCTION AND STATEMENT OF THE RESULTS

Let (P, Q) be an analytic map from $\mathbb{R}^2$ into itself. The qualitative theory of ordinary differential equations in the plane provide a qualitative description of the behavior of each orbit (that is, a curve represented by a solution of a differential equation $x' = P$, $y' = Q$) instead of giving explicitly the solutions. More exactly, if $(x(t), y(t))$ is an orbit of that system with maximal interval of definition (α, ω), one of the objectives is to describe its behavior when $t \to \alpha$ or $t \to \omega$, i.e. the α and ω-limit sets of this orbit. To this end, it suffices:

(i) to describe the local phase portraits of singular points;
(ii) to determine the number and the location of limit cycles;
(iii) to determine the α and ω-limit sets of all separatrices of the differential system.

In this paper we will focus on the first one (i) for Hamiltonian systems. We recall that Hamiltonian systems are relevant for many physical studies. Let $H(x, y)$ be a real polynomial in the variables x and y. Then a system of the form

$$(1) \qquad\qquad x' = H_y \quad y' = -H_x$$

is called a *polynomial Hamiltonian system*. Here the prime denotes derivative with respect to the independent variable t. A system of the form (1) is called a *a polynomial Hamiltonian system of degree d*, if the maximum of the degrees of H_y and H_x is d.

Quadratic systems have been widely studied in the last 100 years, and more than 1.000 papers have been published about them. The classification of centers for real planar polynomial differential systems started with the classification of centers for quadratic polynomial differential systems, and these results go back mainly to Dulac [10], Kapteyn [12, 13] and Bautin [3]. In [15] Vulpe provides all the global phase portraits of quadratic polynomial differential systems having a center. Schlomiuk, Guckenheimer and Rand in [14, pages 3 and 4 and 13] describes a brief history of the problem of the center in general, and it includes a list of 300 papers covering this topic. There are many partial results for the centers of planar polynomial differential systems of degree larger than two. Recently Colak, Llibre and Valls [4, 5, 6, 7] provided the global phase portraits on the Poincaré disk of all Hamiltonian planar polynomial vector fields having only linear and cubic homogeneous terms which have a linear type center or a nilpotent center at the origin, together with their bifurcation diagrams.

Dulak [10] was the first to detect that centers can pass to saddles through a complex change of variables, see for more details [11]. Despite the fact that the classification of centers for real planar polynomial differential systems have been widely studied very few results exist in the case of saddles. For the case of quadratic systems having an integrable saddle, its phase portraits were provided in [2]. For the case in which there exists a nilpotent saddle and the degree of the system is greater than two no result exists on the classification of the phase portraits. This is the content of this paper. These systems have too many parameters and so we will restrict our study on cubic polynomial Hamiltonian systems with a nilpotent saddle at the origin such that are $\mathbb{Z}_2$ symmetric.

Vector fields with symmetry appear very often in applications so the study of symmetric vector fields is not old fashioned and nowadays it has been an increasing interest in systems that are $\mathbb{Z}_2$ symmetric. We recall that a vector field

$$X = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$$

has a $\mathbb{Z}_2$-symmetry if is invariant by the change of variables $(x, y, t) \mapsto (R(x,y), -t)$ (called *reversible*), or by the change of variables $(x, y, t) \mapsto (R(x,y), t)$ (called *equivariant*) in which

$$R(x,y) = (x, -y), \text{ or } R(x,y) = (-x, y), \text{ or } R(x,y) = (-x, -y).$$

Since this class of systems is still too wide, we will study it in separated papers. In this paper we will focus on the reversible and equivariant systems that are symmetric with respect to the x-axis, i.e. in which $R(x,y) = (x, -y)$. Note that they must satisfy

$$(2) \qquad MX(x,y) = \mp X(x, -y) \quad \text{where } M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In short, in this paper we classify the global phase portraits of all Hamiltonian planar polynomial vector fields of degree three reversible and equivariant with respect to the $x-$axis having a nilpotent saddle at the origin. The classification of the global phase portraits of Hamiltonian planar polynomial vector fields of degree three reversible and equivariant with respect to the $x-$axis having a nilpotent center at the origin was given in [8]. To do such a classification we will use the Poincaré compactification of polynomial vector fields. The Poincaré compactification that we shall use for describing the global phase portraits of our Hamiltonian systems is standard. For all the definitions and results on the Poincaré compactification see Chapter 5 of [9]. We say that two vector fields on the Poincaré disk are topologically equivalent if there exists a homeomorphism from one into the other which sends orbits to orbits preserving or reversing the direction of the flow. To state our main result we introduce some notation. Set

(3)
$$b_1 = -\frac{2}{27}(8 - 15a + 6a^2 + a^3 - \sqrt{(4 - 5a + a^2)^2}),$$
$$\ell_0 = \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b = b_1, a \in (0, 1)\},$$
$$R_0 = \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : 0 < b < b_1, a \in (0, 1)\},$$
$$R_1 = \mathbb{R} \times \mathbb{R}^+ \setminus (\ell_0 \cup R_0),$$

and

$$
\begin{aligned}
b_2 &= -\frac{2}{27}(-8 + 15a - 6a^2 - a^3 - \sqrt{(4 - 5a + a^2)^3}), \\
b_3 &= -\frac{2}{27}(-8 + 15a - 6a^2 - a^3 + \sqrt{(4 - 5a + a^2)^3}), \\
p_0 &= (2(2 + \sqrt{3}), 4(7 + 4\sqrt{3})), \quad p_1 = (4, 8), \\
\ell_1 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b = a^2, a \in (0, 2(2 + \sqrt{3}))\}, \\
\ell_2 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b = b_2, a \in (4, 2(2 + \sqrt{3}))\}, \\
\ell_3 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b = b_2, a \in (2(2 + \sqrt{3}), \infty)\}, \\
\ell_4 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b = a^2, a \in (2(2 + \sqrt{3}), \infty)\}, \\
\ell_5 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b = b_3, a \in (4, \infty)\}, \\
R_2 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b > 0, a \in (-\infty, 0]\} \\
&\quad \cup \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b > a^2, a \in (0, 2(2 + \sqrt{3}))\} \\
&\quad \cup \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b > b_2, a \in (2(2 + \sqrt{3}), \infty)\}, \\
R_3 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : 0 < b < a^2, a \in (0, 4)\} \\
&\quad \cup \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b_2 < b < a^2, a \in [4, 2(2 + \sqrt{3}))\} \\
&\quad \cup \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : 0 < b < b_3, a \in [4, \infty)\}, \\
R_4 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b_3 < b < b_2, a \in (4, 2(2 + \sqrt{3})]\} \\
&\quad \cup \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : b_3 < b < a^2, a \in (2(2 + \sqrt{3}), \infty)\}, \\
R_5 &= \{(a, b) \in \mathbb{R} \times \mathbb{R}^+ : a^2 < b < b_2, a \in (2(2 + \sqrt{3}), \infty)\}.
\end{aligned}
$$
(4)

Theorem 1. *A Hamiltonian planar polynomial vector field of degree three with a nilpotent saddle at the origin and $\mathbb{Z}_2$-symmetric with $R(x, y) = (x, -y)$, after a linear change of variables and a rescaling of its independent variable can be written as one of the following six classes:*

- (I) $x' = y$, $y' = x^3$;
- (II) $x' = y + y^3$, $y' = x^3$;
- (III) $x' = y - y^3$, $y' = x^3$;
- (IV) $x' = y + x^2 y + a y^3$, $y' = x^3 - x y^2$ *with* $a \in \mathbb{R}$;
- (V) $x' = y - x^2 y + a y^3$, $y' = x^3 + x y^2$ *with* $a \in \mathbb{R}$;
- (VI) $x' = y + 2xy + a x^2 y$, $y' = x^3 - y^2 - a x y^2$ *with* $a \in \mathbb{R}$;
- (VII) $x' = y + 2xy + a x^2 y + y^3$, $y' = b x^3 - y^2 - a x y^2$, *with* $b > 0$, $a \in \mathbb{R}$;
- (VIII) $x' = y + 2xy + a x^2 y - y^3$, $y' = b x^3 - y^2 - a x y^2$, *with* $b > 0$, $a \in \mathbb{R}$.

Moreover, the global phase portraits of these six families are topologically equivalent to the following of Figure 1:

- *for systems* (I)*:* 1.1;
- *for systems* (II)*:* 1.2;
- *for systems* (III)*:* 1.3;
- *for systems* (IV)*:* 1.2 *if* $a > 0$; 1.4 *if* $a = 0$; 1.5 *if* $-1 < a < 0$; 1.6 *if* $a = -1$ *and* 1.7 *if* $a < -1$;
- *for systems* (V)*:* 1.1 *if* $a = 0$; 1.2 *if* $a > 0$ *and* 1.3 *if* $a < 0$;
- *for systems* (VI)*:* 1.1 *if* $a \leq 0$; 1.4 *if* $a > 1$; 1.8 *if* $a = 1$ *and* 1.9 *if* $a \in (0, 1)$;
- *for systems* (VII)*:* 1.2 *if* $(a, b) \in R_1$; 1.10 *if* $(a, b) \in R_0$ *and* 1.11 *if* $(a, b) \in \ell_0$;
- *for systems* $(VIII)$*:* 1.3 *if* $(a, b) \in R_2$; 1.5 *if* $(a, b) \in R_3$; 1.7 *if* $(a, b) \in R_5$; 1.12 *if* $(a, b) \in R_3$; 1.13 *if* $(a, b) \in R_3$; 1.14, 1.15, 1.16, *or* 1.17 *if* $(a, b) \in R_4$; 1.18 *if* $(a, b) \in \ell_1$; 1.19, 1.20, *or* 1.21 *if* $(a, b) \in \ell_2$; 1.21 *if* $(a, b) \in \ell_5$; 1.22 *if* $(a, b) \in \ell_3$; 1.23 *if* $(a, b) \in \ell_4$; 1.24 *if* $(a, b) = p_0$ *and* 1.25 *if* $(a, b) = p_1$.

The proof of Theorem 1 is given in section 3.

2. Preliminary results

A vector field is said to have the *finite sectorial decomposition property* at a singular point q if either q is a center, a focus or a node, or it has a neighborhood consisting of a finite union of parabolic, hyperbolic or elliptic sectors. We note that all the isolated singular points of a polynomial differential system satisfy the finite vectorial decomposition property.

Theorem 2 (Poincaré Formula)**.** *Let q be an isolated singular point having the finite sectorial decomposition property. Let e, h an p denote the number of elliptic, hyperbolic and parabolic sectors of q, respectively. Then the index of q is $(e - h)/2 + 1$.*
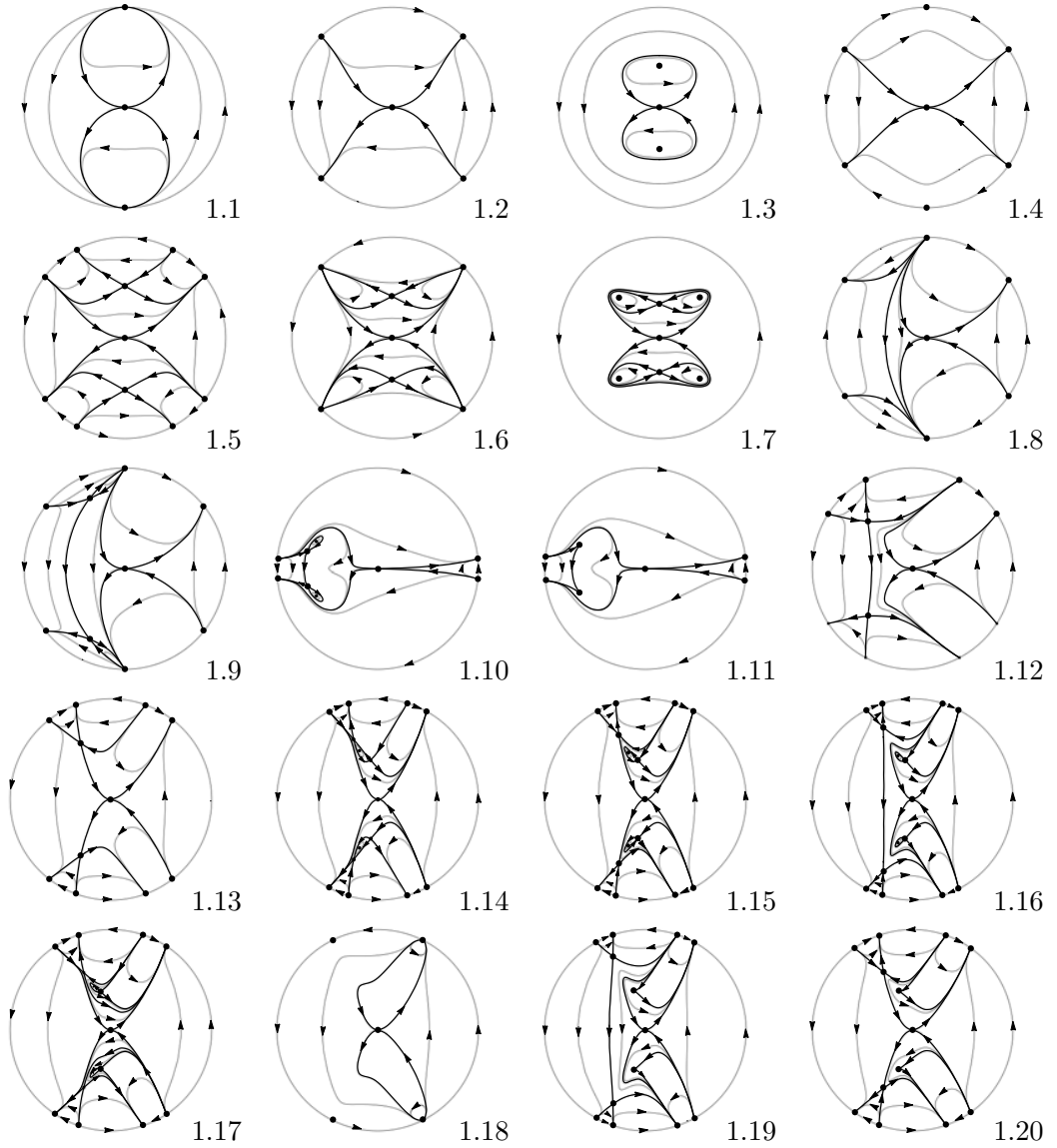
FIGURE 1. Global phase portraits of Hamiltonian planar polynomial vector field of degree three with a nilpotent saddle at the origin and $\mathbb{Z}_2$-symmetric with $R(x,y) = (x, -y)$. The separatrices are in bold.

The indices of a saddle, a center and a cusp are $-1, 1$ and 0, respectively.

Theorem 3 (Poincaré–Hopf Theorem). *For every vector field on the sphere $\mathbb{S}^2$ with a finite number of singular points, the sum of the indices of these singular points is 2.*
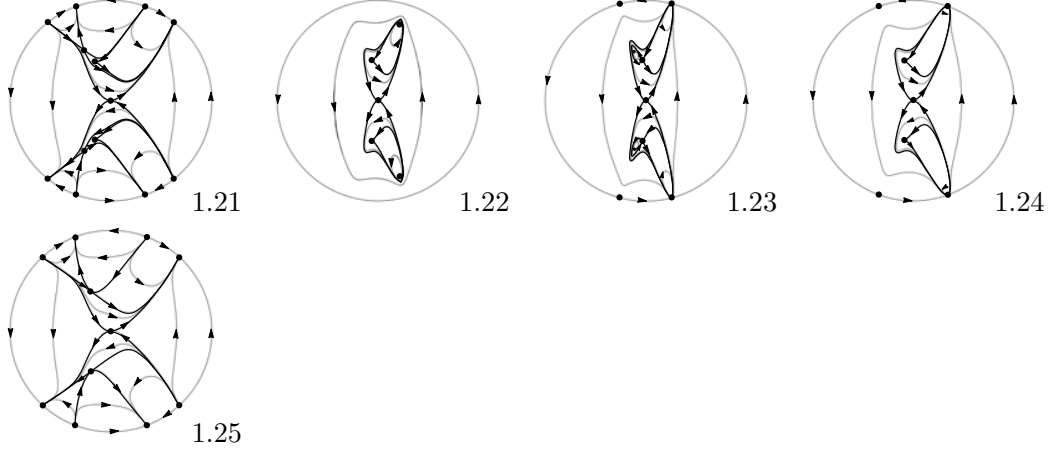
FIGURE 1. Global phase portraits of Hamiltonian planar polynomial vector field of degree three with a nilpotent saddle at the origin and $\mathbb{Z}_2$-symmetric with $R(x, y) = (x, -y)$. The separatrices are in bold.

Nilpotent singular points of Hamiltonian planar polynomial vector fields are either saddles, centers, or cusps (for more details see Theorem 3.5 of [9] and take into account that Hamiltonian systems cannot have foci).

The proof of the following proposition is basically the same as the one of Lemma 12 in [5] and so it will be omitted.

Proposition 4. *Let X_ε be a real Hamiltonian planar polynomial vector field of degree three. Then X_ε can be written as*

$$\begin{aligned}
x' &= a_{10}x + a_{01}y + a_{20}x^2 + a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y \\
&\quad + a_{12}xy^2 + a_{03}y^3, \\
y' &= b_{10}x - a_{10}y + b_{20}x^2 - 2a_{20}xy - \frac{a_{11}y^2}{2} + b_{30}x^3 - 3a_{30}x^2y \\
&\quad - a_{21}xy^2 - \frac{a_{12}y^3}{3} + \varepsilon x.
\end{aligned}$$

Suppose that p is an isolated singular point of X_ε different from the origin. If $a_{10}^2 + a_{01}b_{10} = 0$ but $a_{01} \neq 0$, then the following statements hold:

 a) *If p is non–elementary, then it is nilpotent.*
 b) *If p is a non–elementary singular point of X_0, then it is an elementary singular point of X_ε with $\varepsilon \neq 0$.*
 c) *If p is a cusp of X_0, then for $\varepsilon \neq 0$ small enough such that $\varepsilon a_{01} < 0$, the origin of X_ε is a linear type center and the local phase portrait of X_ε at p is a center–loop (i.e. a hyperbolic saddle with a loop and a center inside the loop).*

3. Proof of Theorem 1

3.1. **Normal form.** Without loss of generality we can assume that a cubic planar Hamiltonian system with a nilpotent saddle at the origin is given by
(5)
$$x' = y + a_5 x^2 + 2a_6 xy + 3a_7 y^2 + a_9 x^3 + 2a_{10} x^2 y + 3a_{11} xy^2 + 4a_{12} y^3,$$
$$y' = -(3a_4 x^2 + 2a_5 xy + a_6 y^2 + 4a_8 x^3 + 3a_9 x^2 y + 2a_{10} xy^2 + a_{11} y^3).$$

which corresponds to equation (1) where,

$$H(x,y) = y^2/2 + a_4 x^3 + a_5 x^2 y + a_6 xy^2 + a_7 y^3 + a_8 x^4 + a_9 x^3 y$$
$$+ a_{10} x^2 y^2 + a_{11} xy^3 + a_{12} y^4.$$

By hypothesis, systems (5) cannot be invariant under $(x, y, t) \to (x, -y, t)$ (they should satisfy condition (2) which is not possible) and so there are no equivariant systems (5). For the case of reversible ones, we impose condition (2) and so we have that $a_5 = a_7 = a_9 = a_{11} = 0$. Hence systems (5) become

(6)
$$x' = y + 2a_6 xy + 2a_{10} x^2 y + 4a_{12} y^3,$$
$$y' = -(3a_4 x^2 + a_6 y^2 + 4a_8 x^3 + 2a_{10} xy^2).$$

Since systems (6) must have a saddle at the origin, by Theorem 3.5 of [9] we must have $a_4 = 0$ and $a_8 < 0$. Therefore we obtain

(7)
$$x' = y + 2a_6 xy + 2a_{10} x^2 y + 4a_{12} y^3,$$
$$y' = -(a_6 y^2 - 4a_8 x^3 + 2a_{10} xy^2), \quad a_8 > 0.$$

Now we provide the normal form to system (7) to transform it into other systems with less parameters.

Assume $a_6 \neq 0$ and $a_{12} \neq 0$. By the change of coordinates and reparametrization of time of the form

(8)
$$x \to \alpha X, \quad y \to \beta Y, \quad t \to \gamma \tau,$$

with $\alpha = 1/a_6$, $\beta = 1/(2\sqrt{|a_{12}|})$ and $\gamma = 2\sqrt{|a_{12}|}/a_6$, systems (7) can be written as in (VII) when $a_{12} > 0$ and as in $(VIII)$ when $a_{12} < 0$.

Now assume $a_6 \neq 0$ and $a_{12} = 0$. By the change of coordinates and reparametrization of time as in (8) with $\alpha = 1/a_6$, $\beta = 2\sqrt{a_8}/a_6^2$, and $\gamma = a_6/(2\sqrt{a_8})$ we get system (VI).

Let now $a_6 = 0$ and $a_{10} \neq 0$. By the change of coordinates and reparametrization of time as in (8) with $\alpha = 1/\sqrt{2|a_{10}|}$, $\beta = \sqrt{a_8}/|a_{10}|$ and $\gamma = \sqrt{|a_{10}|}/\sqrt{2a_8}$ we get systems (IV) when $a_{10} > 0$ and (V) when $a_{10} < 0$.

Assume now $a_6 = a_{10} = 0$ and $a_{12} \neq 0$. By the change of coordinates and reparametrization of time as in (8) with $\alpha = 1/(2(a_8|a_{12}|)^{1/4})$, $\beta = 1/(2\sqrt{|a_{12}|})$ and $\gamma = |a_{12}|^{1/4}/a_8^{1/4}$ we get systems (II) when $a_{12} > 0$ and (III) when $a_{12} < 0$.

Finally, let $a_6 = a_{10} = a_{12} = 0$. The change of coordinates and reparametrization of time as in (8) with $\alpha = \beta = 1/(2\sqrt{a_8})$ and $\gamma = 1$ transforms system (7) into system (I).

In short, we have proved the first part of Theorem 1. Now we study the global phase portraits of each of systems (I)–(VI) separately.

3.2. **Global phase portrait of systems (I).** Consider systems (I)

$$x' = y, \quad y' = x^3.$$

The origin is the only finite singular point of the system, which is a saddle. In the local chart U_1 systems (I) become

$$u' = 1 - u^2 v^2, \quad v' = -uv^3.$$

When $v = 0$ there are no infinite singular points on the local chart U_1. In the local chart U_2 we get

$$u' = v^2 - u^4, \quad v' = -u^3 v.$$

The origin is an infinite singular point of the system, whose linear part is zero. Applying the blow-up technique (see [1] for more details) we obtain that the origin of U_2 is the union of two elliptic and four parabolic sectors. So, the global phase portraits of systems (I) are topologically equivalent to the phase portrait 1.1 of Figure 1.

3.3. **Global phase portrait of systems (II).** Consider systems (II)

$$x' = y + y^3, \quad y' = x^3.$$

It is easy to see that the origin is the unique finite singular point. We will now investigate the infinite singular points of systems (II). In the local chart U_1 systems (II) become

$$u' = 1 - u^2 v^2 - u^4, \quad v' = -uv(v^2 + u^2).$$

When $v = 0$ the infinite singular points are $P_\pm = (\pm 1, 0)$. The eigenvalues of the linear part of systems (II) at $(1, 0)$ are $-4, -1$ and so it is a stable node. The eigenvalues of the linear part at $(-1, 0)$ are $4, 1$ and it is an unstable node.

In U_2 systems (II) become

$$u' = -1 + v^2 - u^4, \quad v' = -vu^3.$$

The origin is not a singular point on the local chart U_2. Therefore the global phase portraits of systems (II) are topologically equivalent to phase portrait 1.2 of Figure 1.

3.4. **Global phase portrait of systems (III).** Consider systems (III)

$$x' = y - y^3, \quad y' = x^3.$$

The finite singular points are $E_0 = (0,0)$ and $E_\pm = (0, \pm 1)$. Since the singular point E_+ is nilpotent, using Theorem 3.5 of [9] we obtain that $(0,1)$ is a center. Doing the same for $(0,-1)$ we also get that it is a center.

In the local chart U_1 systems (III) are

$$u' = 1 - u^2 v^2 + u^4, \quad v' = uv(u^2 - v^2).$$

When $v = 0$ there are no singular points on the local chart U_1. Now we should check the origin of U_2. In U_2 systems (III) become

$$u' = -1 + v^2 - u^4, \quad v' = -vu^3,$$

and so the origin is not singular.

According to this local information the global phase portraits of systems (III) are topologically equivalent to the phase portrait 1.3 of Figure 1.

3.5. **Global phase portrait of systems (IV).** Consider systems (IV)

$$x' = y + x^2 y + a y^3, \quad y' = x^3 - x y^2$$

with $a \in \mathbb{R}$. When $a \geq 0$ systems (IV) have only the origin as its finite singular point. When $-1 \leq a < 0$, among the origin, systems (IV) have the two singular points $F_\pm = \left(0, \pm 1/\sqrt{-a} \right)$. The eigenvalues of the linear part at the singular points $F_\pm$ are $\pm \sqrt{2}/\sqrt{-a}$ and so they are both saddles. Finally, when $a < -1$, among the origin, systems (IV) have the two singular points $F_\pm$ (which are saddles) and the four singular points

$$E_{\pm,\pm} = \left(\pm \frac{1}{\sqrt{-1-a}}, \pm \frac{1}{\sqrt{-1-a}} \right).$$

The eigenvalues of the linear part at these singular points are $\pm 2/\sqrt{1+a}$ and so all four are centers.

We will now investigate the infinite singular points of systems (IV) as well as we provide their global phase portraits. We distinguish between the cases $a < -1$, $-1 \leq a < 0$ and $a \geq 0$.

If $a \geq 0$, on the local chart U_1 systems (IV) become

$$(9) \qquad u' = 1 - 2u^2 - au^4 - u^2 v^2, \quad v' = -uv(1 + au^2 + v^2).$$

When $v = 0$ and $a \geq 0$ there are two singular points

$$Q_\pm = \left(\pm \frac{1}{\sqrt{1 + \sqrt{1+a}}}, 0 \right).$$

The eigenvalues of the linear part at $Q_\pm$ are

$$\mp \frac{\sqrt{1+a}}{\sqrt{1+\sqrt{1+a}}}, \quad \mp 4 \frac{\sqrt{1+a}}{\sqrt{1+\sqrt{1+a}}}$$

which are negative in the case of Q_+ and are positive in the case of Q_-. Hence Q_+ is an attracting node and Q_- is a repelling node.

On the local chart U_2 systems (IV) can be written as

(10) $u' = a + 2u^2 + v^2 - u^4, \quad v' = uv(1 - u^2).$

The origin of U_2 is a singular point if and only if $a = 0$. In this case it is linearly zero. Using the blow-up technique we get that it is the union of two hyperbolic sectors. Hence, when $a > 0$ the global phase portraits of systems (IV) are topologically equivalent to the phase portrait 1.2 of Figure 1, and when $a = 0$ it is topologically equivalent to 1.4 of Figure 1.

Assume now $-1 \leq a < 0$. On the local chart U_1 systems (IV) become (9). When $v = 0$, and $-1 \leq a < 0$ there are four singular points on the local chart U_1 which are $Q_\pm$ (previously seen that are a stable node and an unstable node, respectively), and the singular points

$$P_\pm = \left(\pm \frac{1}{\sqrt{1 - \sqrt{1+a}}}, 0 \right).$$

The eigenvalues of the linear part at $P_\pm$ are

$$\pm \frac{\sqrt{1+a}}{\sqrt{1 - \sqrt{1+a}}}, \quad \pm 4 \frac{\sqrt{1+a}}{\sqrt{1 - \sqrt{1+a}}}$$

which for $-1 < a < 0$ are positive in the case of P_+ and are negative in the case of P_-. Hence P_+ is an unstable node and P_- is a stable node. If $a = -1$, then $P_\pm$ and $Q_\pm$ coincide and we get two singular points which are linearly 0. Applying blow-up techniques we get that the points $P_\pm = Q_\pm$ are the union of two elliptic sectors and four parabolic sectors.

On the local chart U_2 systems (IV) become (10) and since $a \neq 0$ the origin is not a singular point.

Gluing all this local information together with the fact that the system is symmetric and that the value of the Hamiltonian on the two saddles is the same and different from the value of the Hamiltonian at the origin (and so the origin and the two saddles cannot be connected) we get that the global phase portraits of systems (IV) are topologically equivalent to phase portrait 1.5 of Figure 1 if $-1 < a < 0$ and to phase portrait 1.6 of Figure 1 if $a = -1$.

Finally, if $a < -1$, on the local chart U_1 systems (IV) become (9). When $v = 0$, and $a < -1$ there are no infinite singular points on the local chart

U_1. On U_2 systems (IV) become (10) and since $a \neq 0$ the origin is not a singular point.

Gluing all this local information together with the fact that the system is symmetric and that the saddles cannot be connected with the saddle at the origin, we get that the global phase portraits of system (IV) when $a < -1$ is topologically equivalent to the phase portrait 1.7 of Figure 1.

3.6. **Global phase portrait of systems (V).** Consider system (V), i.e.

$$x' = y - x^2y + ay^3, \quad y' = x^3 + xy^2$$

with $a \in \mathbb{R}$. When $a \geq 0$ the only finite singular point is the origin and when $a < 0$ among the origin, there are two new finite singular points

$$\left(0, \pm\frac{1}{\sqrt{-a}}\right).$$

It is easy to see that all of them are centers because the eigenvalues of their linear parts are $\pm i\sqrt{2}/\sqrt{-a}$ which are purely imaginary.

We will now investigate the infinite singular points of systems (V).

In the local chart U_1 systems (V) can be written as

$$u' = 1 + 2u^2 - u^2v^2 - au^4, \quad v' = -uv(-1 + au^2 + v^2).$$

The infinite singular points are the roots of the polynomial $1 + 2u^2 - au^4$. Therefore, if $a \leq 0$ there are no infinite singular points on the local chart U_1. If $a > 0$ the points

$$Q_{\pm} = \left(\pm\frac{1}{\sqrt{-1 + \sqrt{1 + a}}}, 0\right)$$

are infinite singular points on U_1. The eigenvalues at the singular points $Q_{\pm}$ are

$$\mp\frac{\sqrt{1 + a}}{\sqrt{-1 + \sqrt{1 + a}}}, \quad \mp4\frac{\sqrt{1 + a}}{\sqrt{-1 + \sqrt{1 + a}}}.$$

So, Q_+ is an unstable node and Q_- is a stable node.

Now we investigate the origin of the local chart U_2. In the local chart U_2 systems (V) can be written as

$$u' = a - 2u^2 - u^4 + v^2, \quad v' = -uv(1 + u^2).$$

The origin of U_2 is a singular point if and only if $a = 0$. In this case it is linearly zero. Using a blow-up technique we obtain that the origin of U_2 behaves as in system (I), that is, is the union of two elliptic and four parabolic sectors.

If $a > 0$, there are no finite singular points among the origin, two nodes in the local chart U_1 (one stable and one unstable) and the origin of U_2 is not

a singular point. Therefore the global phase portrait of systems (V) with $a > 0$ are topologically equivalent to the phase portrait 1.2 of Figure 1.

When $a = 0$, there are no finite singular points among the origin, there are no singular points in the local chart U_1 and the origin of U_2 which is the union of two elliptic and four parabolic sectors. Therefore the global phase portrait of systems (V) with $a = 0$ are topologically equivalent to the phase portrait 1.1 of Figure 1.

When $a < 0$ we have two centers among the saddle at the origin as finite singular points and no infinite singular points. Hence the global phase portrait of systems (V) with $a < 0$ are topologically equivalent to the phase portrait 1.3 of Figure 1.

3.7. **Global phase portrait of systems (VI).** Consider systems (VI)

$$x' = y + 2xy + ax^2y, \quad y' = x^3 - y^2 - axy^2,$$

with $a \in \mathbb{R}$.

If $0 < a < 1$, besides the origin there exist two finite singular points which are (x_-, y_-) and (x_-, y_+) where

$$x_- = \frac{-1 - \sqrt{1-a}}{a}, \quad y_\pm = \pm\sqrt{\frac{(1+\sqrt{1-a})^3}{a^3\sqrt{1-a}}}.$$

It is easy to see that all of them are saddles because the eigenvalues of their linear parts are

$$\pm 2\sqrt{1-a}\sqrt{\frac{(1+\sqrt{1-a})^3}{a^3\sqrt{1-a}}},$$

respectively.

If $a \geq 1$ or $a \leq 0$ the unique finite singular point is the origin.

In the local chart U_1 systems (VI) can be written as

$$u' = 1 - 2au^2 - 3u^2v - u^2v^2, \quad v' = -uv(a + 2v + v^2).$$

The infinite singular points are the roots of the polynomial $1 - 2au^2$. Therefore, if $a \leq 0$ there are no infinite singular points on the local chart U_1. If $a > 0$ the points $Q_\pm = \left(\pm 1/(\sqrt{2}\sqrt{a}), 0\right)$ are infinite singular points. The eigenvalues at the singular points $Q_\pm$ are $\mp 2\sqrt{2}\sqrt{a}$ and $\mp\sqrt{a}/\sqrt{2}$. Hence, Q_+ is a stable node and Q_- is an unstable node.

Now we investigate the origin of the local chart U_2. In the local chart U_2 systems (VI) can be written as

$$u' = 2au^2 - u^4 + 3uv + v^2, \quad v' = auv - u^3v + v^2.$$

The origin of U_2 is a singular point which is linearly zero. Using a blow-up technique we obtain that if $a > 1$ it is the union of two hyperbolic sectors; if

$a = 1$ is the union of two hyperbolic and four parabolic sectors, and if $a < 1$ it is the union of two elliptic and four parabolic sectors.

If $a > 1$, there are no finite singular points among the origin, two nodes in the local chart U_1 (one stable and another unstable) and the origin of U_2 is the union of two hyperbolic sectors. Therefore the global phase portraits of systems (VI) with $a > 1$ are topologically equivalent to the phase portrait 1.4 of Figure 1.

If $a = 1$, there are no finite singular points among the origin, two nodes in the local chart U_1 (one stable and one unstable) and the origin of U_2 is the union of two hyperbolic sectors and four parabolic sectors. Therefore the global phase portrait of systems (VI) with $a = 1$ is topologically equivalent to the phase portrait 1.8 of Figure 1.

If $a \in (0, 1)$, there are two finite singular among the origin which are saddles, two nodes in the local chart U_1 (one stable and one unstable) and the origin of U_2 is union of two elliptic and four parabolic sectors, separated by the infinity. Taking into account the symmetry of the system and that both saddles cannot be connected with the saddle at the origin because the value of the Hamiltonian on the two saddles is different from the value of the Hamiltonian at the origin, we get that the global phase portraits of systems (VI) with $a \in (0, 1)$ are topologically equivalent to the phase portrait 1.9 of Figure 1.

If $a \leq 0$, there are no finite singular points besides the origin, there are no singular points in the local chart U_1 and the origin of U_2 is the union of two elliptic and four parabolic sectors, separated by the infinity. Hence, the global phase portrait of systems (VI) with $a \leq 0$ are topologically equivalent to the phase portrait 1.1 of Figure 1.

3.8. **Global phase portrait of systems (VII).** Consider systems (VII)

$$x' = y + 2xy + ax^2y + y^3, \quad y' = bx^3 - y^2 - axy^2,$$

with $b > 0$, $a \in \mathbb{R}$. We first compute the infinite singular points because the finite ones are more elaborated in this case. In the local chart U_1 systems (VII) become

$$u' = b - 2au^2 - u^4 - 3u^2v - u^2v^2, \quad v' = -auv - u^3v - 2uv^2 - uv^3.$$

The infinite singular points are the roots of the polynomial $b - 2au^2 - u^4$. Taking into account that $b > 0$ there are only two infinite singular points which are

$$Q_\pm = \left(\pm \sqrt{-a + \sqrt{a^2 + b}}, 0 \right).$$

The eigenvalues at these singular points are

$$\mp \sqrt{-a + \sqrt{a^2 + b}} \sqrt{a^2 + b}, \quad \mp 4\sqrt{-a + \sqrt{a^2 + b}} \sqrt{a^2 + b}.$$

Hence, Q_+ is a stable node and Q_- is an unstable node.

Now we investigate the origin of the local chart U_2. In the local chart U_2 systems (VII) can be written as

$$u' = 1 + 2au^2 - bu^4 + 3uv + v^2, \quad v' = auv - bu^3v + v^2.$$

and so the origin of U_2 is not a singular point.

Now we compute the finite singular points of system (VII). From equation $x' = 0$ we get that either $y = 0$ or $y = \pm\sqrt{-ax^2 - 2x - 1}$. The unique finite singular point with $y = 0$ is the origin. Substituting $y = \pm\sqrt{-ax^2 - 2x - 1}$ into equation $y' = 0$ we get the equation

(11) $$\mathcal{C} = \left(a^2 + b\right)x^3 + 3ax^2 + (a + 2)x + 1 = 0.$$

Since $b > 0$, $\mathcal{C} = 0$ is always a cubic equation because the coefficient of x^3 is always positive. We compute the discriminant of the cubic $\mathcal{C}$ in (11) and we get that it is equal to

$$\mathcal{D} = 4a^2 - 12a^3 + 12a^4 - 4a^5 - 32b + 60ab - 24a^2b - 4a^3b - 27b^2.$$

If $\mathcal{D} > 0$, $\mathcal{C}$ has three distinct real roots; if $\mathcal{D} < 0$, it has one real root and two distinct complex roots and if $\mathcal{D} = 0$, it has at least two equal real roots. The solution of $\mathcal{D} = 0$ gives the two curves in the parameter space $(a, b) \in \mathbb{R} \times \mathbb{R}^+$ that we denote by b_0 and b_1:

$$b_0 = -\frac{2}{27}(8 - 15a + 6a^2 + a^3 + \sqrt{(4 - 5a + a^2)^3}),$$

$$b_1 = -\frac{2}{27}(8 - 15a + 6a^2 + a^3 - \sqrt{(4 - 5a + a^2)^3})$$

(b_1 was previously defined in (3)). We can see that b_0 is always less or equal than zero in its domain of definition, so we do not consider it because $b > 0$. On the other hand, b_1 is positive for $a \in (-\infty, 0) \cup (0, 1)$, zero for $a = 0, 1$, it does not exist for $a \in (1, 4)$ and is negative for $a \geq 4$.

We have the following lemma.

Lemma 5. *Systems (VII) have, among the origin, at most two nilpotent singular points and they occur when the parameters a and b satisfy $\mathcal{D} = 0$.*

Proof. We need to show that x', y' in systems (VII) and the determinant of the Jacobian matrix on systems (VII) cannot vanish simultaneously at more than two points besides the origin. The determinant of de Jacobian matrix is

$$\mathcal{E} = -3a^2x^2y^2 - 3abx^4 - 6axy^2 + 3ay^4 + ay^2 - 6bx^3 - 9bx^2y^2 - 3bx^2 - 4y^2.$$

Substituting the solution of $x' = 0$ with $y = \pm\sqrt{-ax^2 - 2x - 1}$, into $\mathcal{E}$ we get

$$\mathcal{E}' = 2\left(ax^2 + 2x + 1\right)\left(3a^2x^2 + 6ax + a + 3bx^2 + 2\right).$$

So we are interested in the solutions of system $\mathcal{C} = \mathcal{E}' = 0$. Note that if $ax^2 + 2x + 1 = 0$, then $y = 0$ and the unique singular point with $y = 0$ is the origin. By computing the Groëbner basis of the polynomials $\mathcal{C}$ and $3a^2x^2 + 6ax + a + 3bx^2 + 2$ we get a set of six polynomials in the variables a, b, x, the first two being $-\mathcal{D}$ and also

$$(9b^2 + 72b)x + 4a^4 + 4a^3 - 2a^2b - 20a^2 + 28ab + 12a + 46b.$$

Since $b > 0$ the coefficient in x of this polynomial is different from zero. Therefore there exist a unique solution x of $\mathcal{C} = \mathcal{E}' = 0$ and it occurs when $\mathcal{D} = 0$. This solution can provide at most two possible solutions for y of system $x' = y' = 0$.

We recall that in view of Proposition 4 the non–elementary singular points are nilpotent. So the lemma is proved. □

Now taking into account that we are interested only on the regions in the half plane $(a, b) \in \mathbb{R} \times \mathbb{R}^+$ where there is real roots of the cubic equation $\mathcal{C} = 0$, we divide the half plane $(a, b) \in \mathbb{R} \times \mathbb{R}^+$ in the three regions R_0, R_1, ℓ_0 given in (3).

On the region R_1 the unique finite singular points of system (VII) is the origin and there are only two infinite singular points on the local chart U_1: a stable and an unstable node. Therefore, the global phase portrait of systems (VII) on the region R_1 is topologically equivalent to the phase portrait 1.2 of Figure 1.

On the region R_0 there are five finite singular points of system (VII): the origin which is a nilpotent saddle and four more finite singular points which are hyperbolic (see Lemma 5), and there are only two infinite singular points on the local chart U_1: a stable and an unstable node. Therefore the known singular points have total index 2 on the Poincaré sphere. By Theorem 3, the four remaining finite singular points must have total index 0. Since they are hyperbolic, by the symmetry of the system, they must be two centers and two saddles. Taking again into account the symmetry of the system and the fact that the two saddles have the same value of the Hamiltonian but different from the value of the Hamiltonian at the origin and so the saddles cannot be connected with the saddle at the origin, we obtain that the global phase portraits of systems (VII) on the region R_0 are topologically equivalent to the phase portrait 1.10 of Figure 1.

On the curve ℓ_0 there are three finite singular points of system (VII): the origin which is a nilpotent saddle and two more finite singular points which are either hyperbolic or nilpotent (see Lemma 5), and there are only two infinite singular points on the local chart U_1: a stable and an unstable node. Therefore the known singular points have total index 2 on the Poincaré sphere. By Theorem 3, the two remaining finite singular points must have total index 0. Since they are either hyperbolic or nilpotent, by

symmetry of the system, they must be two cusps. Hence, the global phase portrait of systems (VII) on the region ℓ_0 is topologically equivalent to the phase portrait 1.11 of Figure 1 (note that they correspond to the fact that the saddles and the centers forming the center–loop in the region R_0 have coalesced into the cusps).

3.9. **Global phase portrait of systems (VIII).** Consider system (VIII)

$$x' = y + 2xy + ax^2y - y^3, \quad y' = bx^3 - y^2 - axy^2,$$

with $b > 0$, $a \in \mathbb{R}$. We first compute the infinite singular points because the finite ones are more elaborated in this case. In the local chart U_1 system (VIII) becomes

$$u' = b - 2au^2 + u^4 - 3u^2v - u^2v^2, \quad v' = -auv + u^3v - 2uv^2 - uv^3.$$

The infinite singular points are the roots of the polynomial $b - 2au^2 + u^4$. If $b > a^2$ there are no infinite singular points on the local chart U_1.

If $b = a^2$ and $a < 0$, there are also no infinite singular points on the local chart U_1.

If $b = a^2$ and $a > 0$ there are two infinite singular points on the local chart U_1 which are $(\pm\sqrt{a}, 0)$. They are nilpotent. Using Theorem 3.5 in [9] and applying blow-up techniques we obtain that they both consist in the union of one hyperbolic and one elliptic sector separated by the infinity.

If $0 < b < a^2$ with $a < 0$ there are also no infinite singular points on the local chart U_1.

If $0 < b < a^2$ with $a > 0$ there are four singular points on the local chart U_1: $u_\pm^0 = (\pm\sqrt{a + \sqrt{a^2 - b}}, 0)$ and $u_\pm^1 = (\pm\sqrt{a - \sqrt{a^2 - b}}, 0)$. Computing the eigenvalues of the Jacobian matrix at the singular points $u_\pm^0$ we get that they are $\pm\sqrt{a + \sqrt{a^2 - b}}\sqrt{a^2 - b}$ and $\pm 4\sqrt{a + \sqrt{a^2 - b}}\sqrt{a^2 - b}$, so u_+^0 is an unstable node and u_-^0 is a stable node. Moreover, the eigenvalues of the Jacobian matrix at the singular points $u_\pm^1$ we get that they are $\mp\sqrt{a - \sqrt{a^2 - b}}\sqrt{a^2 - b}$ and $\mp 4\sqrt{a - \sqrt{a^2 - b}}\sqrt{a^2 - b}$, so u_+^1 is a stable node and u_-^1 is an unstable node.

Now we investigate the origin of the local chart U_2. In the local chart U_2 systems (VIII) can be written as

$$u' = -1 + 2au^2 - bu^4 + 3uv + v^2, \quad v' = auv - bu^3v + v^2$$

and so the origin of U_2 is not a singular point.

Now we compute the finite singular points of system (VIII) in a similar way that in system (VII). From equation $x' = 0$ we get that either $y = 0$ or $y = \pm\sqrt{ax^2 + 2x + 1}$. The unique finite singular point with $y = 0$ is the
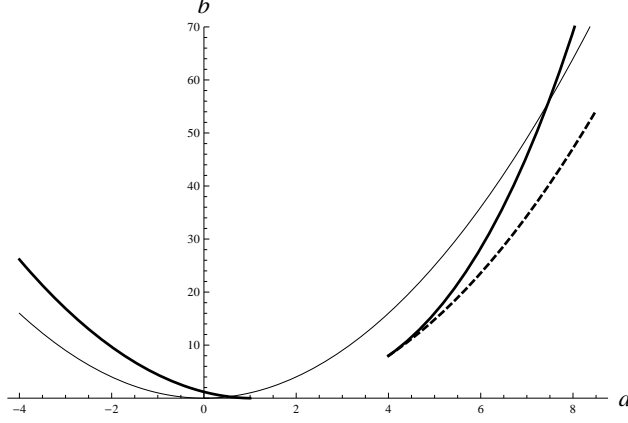
FIGURE 2. The curves where the number of finite singular points can change. The plot of the curves $b = b_2$ (continous thick line), $b = b_3$ (dashed line), and $b = a^2$ (continuous thin line) for $b > 0$.

origin. Substituting $y = \pm\sqrt{ax^2 + 2x + 1}$ into equation $y' = 0$ we get the equation

(12) $$\mathcal{C} = \left(b - a^2\right)x^3 - 3ax^2 + (-a - 2)x - 1 = 0.$$

If $b - a^2 \neq 0$, then $\mathcal{C} = 0$ is a cubic equation whereas if $b - a^2 = 0$ it is quadratic. We compute the discriminant of the cubic $\mathcal{C}$ in (12) and we get that it is equal to

$$\mathcal{D} = 4a^2 - 12a^3 + 12a^4 - 4a^5 + 32b - 60ab + 24a^2b + 4a^3b - 27b^2.$$

The solution of $\mathcal{D} = 0$ gives the two curves b_2 and b_3 introduced in (4). So the number of finite singular points can change only at the curves $b = b_2$, $b = b_3$ and $b = a^2$.

Note that

- b_3 is less than or equal to zero for $a \in (-\infty, 1]$, it does not exist for $a \in (1, 4)$ and it is positive for $a \geq 4$;
- b_2 is positive for $a \in (-\infty, 0) \cup (0, 1) \cup [4, \infty)$, zero for $a = 1$ and it does not exist for $a \in (1, 4)$;
- $b = a^2$ is always positive for $a \neq 0$ and zero for $a = 0$;
- $b_2 > a^2$ for $a \in (-\infty, 2(2 - \sqrt{3})) \cup (2(2 + \sqrt{3}), \infty)$, $b_2 = a^2$ for $a = 2(2 \pm \sqrt{3})$ and $0 < b_2 < a^2$ for $a \in (2(2 - \sqrt{3}), 1) \cup [4, 2(2 + \sqrt{3}))$;
- $b_3 < a^2$ for $a \in [4, \infty)$ and $b_3 = b_2$ for $a = 4$.

The plot of the curves $b = b_2$, $b = b_3$ and $b = a^2$ for $b > 0$ is given in Figure 2.

Proceeding in a similar way than in the proof of Lemma 5 we have the following lemma.

Lemma 6. *Systems (VIII) have, among the origin, at most two nilpotent singular points and they occur when the parameters a and b belong to the curve $\mathcal{D}$.*

Proof. As in system (VII) we need to show that x', y' in systems (VIII) and the determinant of the Jacobian matrix on systems (VIII) cannot vanish simultaneously at more than two points besides the origin. The determinant of de Jacobian matrix evaluated at the solution of $x' = 0$ with $y = \pm\sqrt{ax^2 + 2x + 1}$ is

$$\mathcal{E}' = 2(ax^2 + 2x + 1)t(3bx^2 - 3a^2x^2 - 6ax - a - 2).$$

As above, if $ax^2 + 2x + 1 = 0$, then $y = 0$ and the unique singular point with $y = 0$ is the origin. By computing the Groëbner basis of the polynomials $\mathcal{C}$ and $3bx^2 - 3a^2x^2 - 6ax - a - 2$ we get a set of six polynomials in the variables a, b, x, which are the polynomial $-\mathcal{D}$ and also

$$\mathcal{F}_1 = \left(9b^2 - 72b\right)x + 4a^4 + 4a^3 + 2a^2b - 20a^2 - 28ab + 12a - 46b,$$

$$\mathcal{F}_2 = (3ab - 12b)x + 2a^3 - 4a^2 + 2a - 9b,$$

$$\mathcal{F}_3 = \left(6a^2 - 6a - 9b\right)x - 2a^2 + 10a - 8,$$

$$\mathcal{F}_4 = 9bx^2 + 9bx + 2a^2 - 4a + 2,$$

$$\mathcal{F}_5 = 3ax^2 + (2a + 4)x + 3.$$

The coefficient in x of the polynomial $\mathcal{F}_1$ is zero when $b = 8$, so if $b \neq 8$ there is a unique solution of system $\mathcal{C} = \mathcal{E}' = 0$ for x. When $b = 8$, the coefficient in x of the polynomial $\mathcal{F}_2$ is zero when $a = 4$, so if $b = 8$ and $a \neq 4$ there is also a unique solution for x. Finally if $b = 8$ and $a = 4$, then $\mathcal{F}_i = 0$ for $i = 1, 2, 3$, $F_4 = 18(2x + 1)^2$ and $\mathcal{F}_5 = 3(2x + 1)^2$. So in this case there is also a unique solution for x. In short, there exist a unique solution x of $\mathcal{C} = \mathcal{E}' = 0$ and it occurs when $\mathcal{D} = 0$. This solution can provide at most two possible solutions for y of system $x' = y' = 0$.

We recall that in view of Proposition 4 the non–elementary singular points are nilpotent. So the lemma is proved. $\qquad\square$

In some regions of the parameter space (a, b) there could be finite singular points of system (VIII) that are saddles. Now we investigate the possible connections among these saddles. Note that the origin is always a nilpotent saddle of systems (VIII) and that the remaining singular points of system (VIII) always go in pairs: if $(\overline{x}, \overline{y})$ is a singular point, then so is $(\overline{x}, -\overline{y})$. On the other hand due to the symmetry of the system, the phase portraits of systems (VIII) are symmetric with respect to the x–axis.

Lemma 7. *If $b = (a - 1)^2$ and $a > 1$, the saddles $(-1, \pm\sqrt{a - 1})$ can be connected to the saddle at the origin.*

Proof. The Hamiltonian of system (VIII) is

$$H = \frac{1}{2}ax^2y^2 - \frac{bx^4}{4} + xy^2 - \frac{y^4}{4} + \frac{y^2}{2}.$$

Since the Hamiltonian H at the origin is zero, a saddle $(\overline{x}, \overline{y})$ can be connected with the saddle at the origin if the value of the Hamiltonian evaluated at $(\overline{x}, \overline{y})$ is equal to zero. We compute the Groëbner basis of x', y', H and we obtain a set of eight polynomial equations in the variables x, y, a, b whose solutions with $(x, y) \neq (0, 0)$ are

$$x = -1, \qquad y = \pm\sqrt{a-1}, \qquad b = (a-1)^2.$$

The singular points $(x, y) = (-1, \pm\sqrt{a-1})$ are defined for $a > 1$ (recall that $b \neq 0$) and the eigenvalues of the Jacobian matrix at these critical points are $\pm\sqrt{2}(a-1)$, so they are saddles. Therefore the saddles $(x, y) = (-1, \pm\sqrt{a-1})$ on the curve $b = (a-1)^2$ can be connected with the saddle at the origin. $\qquad\square$

Lemma 8. *Let $(\overline{x}, \overline{y})$ and $(\widetilde{x}, \widetilde{y})$ with $\widetilde{x} \neq \overline{x}$ be two saddles different from the origin. They can be connected when (a, b) satisfy one of the following relations*

$$(13) \quad b = a^2, \qquad b = a^2 - 2a, \qquad b = \frac{a^3 - a^2}{a+2}, \qquad b = b_2, \qquad b = b_3.$$

Proof. It is not difficult to prove that the solutions of $x' = y' = 0$ are $y = \pm\sqrt{ax^2 + 2x + 1}$ with

$$x = x_0 = \frac{1}{b-a^2}\left(a - \frac{2^{1/3}A}{3\,C} + \frac{C}{3\,2^{1/3}}\right),$$

$$x = x_1 = \frac{1}{b-a^2}\left(a + \frac{(1+i\sqrt{3})A}{3\,2^{2/3}C} - \frac{(1-i\sqrt{3})C}{6\,2^{1/3}}\right),$$

$$x = x_2 = \frac{1}{b-a^2}\left(a + \frac{(1-i\sqrt{3})A}{3\,2^{2/3}C} - \frac{(1+i\sqrt{3})C}{6\,2^{1/3}}\right),$$

where $A = 3a^3 - 3a^2 - 3ab - 6b$, $C = (B+K)^{1/3}$, $B = -27a^2b + 54ab + 27b^2$ and $K = \sqrt{4A^3 + B^2}$.

Let H_0, H_1 and H_2 be the value of the Hamiltonian evaluated at the solutions with x_0, x_1 and x_2 respectively. We factorize $H_0 - H_1$ and we drop the denominators obtaining a polynomial equation $g_0 = 0$. We consider two additional equations

$$g_1 = C^3 - (B+K) = 0, \quad g_2 = K^2 - (4A^3 + B^2) = 0.$$

We treat C and K as variables and we eliminate them in equation $g_0 = 0$ by means of resultants in the following way. First we eliminate the variable C by doing the resultant $R_1 = \text{Res}[g_0, g_1, C]$ and then we eliminate the variable

K by doing the resultant $R_2 = \text{Res}[R_1, g_2, K]$. Hence we obtain $R_2 = k\overline{R}_2$ where

$$\overline{R}_2 = b^2 \left(a^2 - b\right)^6 \left(a^2 - 2a - b\right)^2 \left(a^3 - a^2 - ab - 2b\right)^{12}$$
$$\left(4a^5 - 12a^4 - 4a^3b + 12a^3 - 24a^2b - 4a^2 + 60ab + 27b^2 - 32b\right)^3,$$

and k is a large number. By properties of the resultants, the set of solutions of equation $\overline{R}_2 = 0$ contains all solutions of $g_0 = 0$ and probably new ones. If we do the same for equations $H_1 - H_3 = 0$ and $H_2 - H_3 = 0$ we arrive to the same equation $\overline{R}_2 = 0$. The solutions of this equation are

$$b = a^2, \qquad b = a^2 - 2a, \qquad b = \frac{a^3 - a^2}{a + 2}, \qquad b = b_2, \qquad b = b_3.$$

So these are the curves where the connection of saddles can occur. Note that the solutions x_0, x_1 and x_2 are not defined when $b = a^2$ and that on the curves $b = b_2$ and $b = b_3$ the number of finite singular points can change. $\square$

Later on we will see that on the region R_2 there are no saddles different form the origin and that on the region R_4 there are at most four saddles different from the origin. Clearly $(a-1)^2 < a^2$ when $a > 1$. Moreover, it is not difficult to see that

$$\begin{cases} b_2 < (a-1)^2 & \text{for } a < 5, \\ b_2 = (a-1)^2 & \text{for } a = 5, \quad \text{and} \\ b_2 > (a-1)^2 & \text{for } a > 5 \end{cases} \quad \begin{cases} (a^3 - a^2)/(a+2) = b_3 & \text{for } a = 4, \\ (a^3 - a^2)/(a+2) < b_3 & \text{for } a > 4. \end{cases}$$

The possible curves with connection of saddles are plotted in Figure 3. We have plotted them only in the interval $[0, 6]$ for clarity.

As for system (VII) we only need to distinguish between regions with different number of real solutions of $\mathcal{C} = 0$. Recall that the number of positive real solutions of $\mathcal{C} = 0$ can change at the curves $b = b_2$, $b = b_3$, $b = a^2$. Moreover, for the infinite singular points we need to distinguish between the regions $b = a^2$ with $a > 0$, $b < a^2$ with $a > 0$ and the rest. Doing so, we end up with $p_0, p_1, \ell_1, \ell_2, \ell_3, \ell_4, \ell_5, R_2, R_3, R_4$ and R_5 given in (4).

The number of real solutions of $\mathcal{C} = 0$ are: one double solution on p_0; one triple solution on p_1; no solutions on ℓ_1; two solutions (one being double) on ℓ_2, ℓ_3 and ℓ_5; two simple solutions on ℓ_4; three simple solutions on R_4 and R_5, and one simple solution on R_2 and R_3.

On the region R_2 there are no infinite singular points and besides the origin there are two finite singular points which are hyperbolic (see Lemma 6). Therefore the known singular points have total index - 2 on the Poincaré sphere. By Theorem 3, the two remaining finite singular points must have total index 4. Since they are hyperbolic, taking into account the symmetry of the system they must be two centers. Hence, the global phase portrait
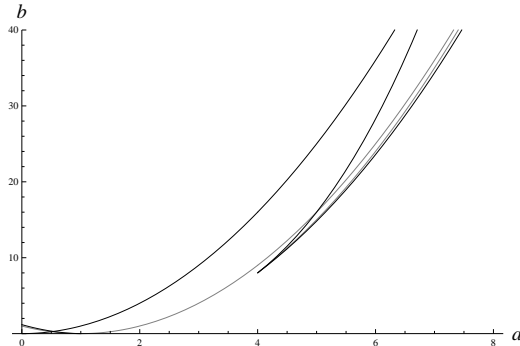
FIGURE 3. The black lines correspond to the curves where the number of finite singular points can change (see Figure 2 for more details), and the gray lines correspond to the curves where the connection of saddles can occur. The connection with the saddle at the origin can occur on the upper gray line, and the connection between two saddles different from the origin can occur on the lower gray line.

of systems (VIII) on the region R_2 is topologically equivalent to the phase portrait 1.3 of Figure 1.

On the region R_3 besides the origin there are two finite singular points which are hyperbolic (see Lemma 6). On the local chart U_1 there are four nodes (two stable and two instable) and the origin of U_2 is not a singular point. Therefore the known singular points have total index 6 on the Poincaré sphere. By Theorem 3, the two remaining finite singular points must have total index -4. Since they are hyperbolic, taking into account the symmetry of the system they must be two saddles. By Lemma 7 they can be connected with the saddle at the origin on the curve $b = (a-1)^2$. The global phase portraits of systems (VIII) on the region R_3 are topologically equivalent to the phase portraits 1.5, 1.12 and 1.13 of Figure 1. We recall that all these phase portraits are realized. The case 1.5 is realized in the region $R_3 \cap \{b < (a-1)^2\}$ (in this case there is no saddle–connections and we can take for example $a = 8$ and $b = 10$). The case 1.12 is realized in the region $R_3 \cap \{b > (a-1)^2\}$ (in this case there is a connection between the two saddles that are not at the origin and we can take for instance $a = 2$ and $b = 3/2$). Finally, the case 1.13 is realized in the region $R_3 \cap \{b = (a-1)^2\}$ (in this case there is a saddle connection between the two saddles and the saddle at the origin and we can take for instance $a = 9/2$, $b = 49/4$).

On the region R_4 besides the origin there are six finite singular points which are hyperbolic (see Lemma 6). On the local chart U_1 there are four nodes (two stable and two instable) and the origin of U_2 is not a singular point. Therefore the known singular points have total index 6 on the
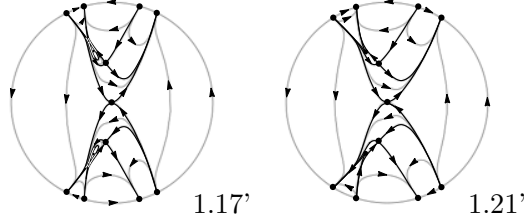
FIGURE 4. Global phase portraits of systems (VII) for $a = 7$ and $b = 34.7$ (1.17') and $b = b_2$ and $a = 4.5$ (1.21'). The separatrices are in bold.

Poincaré sphere. By Theorem 3, the six remaining finite singular points must have total index -4. Since they are hyperbolic, taking into account the symmetry of the system they must be four saddles and two centers. From Lemmas 7 and 8 the connection of saddles can occur on the curves $b = (a-1)^2$ and $b = a^2 - 2a$ (this last curve is the only curve in (13) intersecting R_4). The global phase portraits of systems (VIII) on the region R_4 are topologically equivalent to the phase portraits 1.14, 1.15, 1.16, and 1.17 of Figure 1. We recall that all phase portraits are realized. The case 1.14 is realized in the region $R_4 \cap \{b = a^2 - 2a\}$ (we can take for instance $a = 7$, $b = 35$) and it corresponds to the case where the two saddles with $y > 0$ (respectively, the two saddles with $y < 0$) are connected to each other. The case of 1.15 is realized in the region $R_4 \cap \{b = (a-1)^2\}$ (we can take for instance $a = 7$, $b = 36$) and it corresponds to the case where two saddles are connected with the saddle at the origin. The case 1.16 is realized in

$$(R_4 \cap \{(a-1)^2 < b < b_2, a < 2(2+\sqrt{3})\}) \cup (R_4 \cap \{(a-1)^2 < b < a^2, a \geq 2(2+\sqrt{3})\})$$

(we can take for instance $a = 7$, $b = 38.5$) and it corresponds to a connection between a saddle and its symmetric with respect to the x–axis. The case of 1.17 is realized in the region $R_4 \cap \{b_3 < b < (a-1)^2\} \setminus \{b = a^2 - 2a\}$. In fact in the region $R_4 \cap \{a^2 - 2a < b < (a-1)^2\}$ we can take, for instance, $a = 7$ and $b = 35.5$ and the phase portrait is 1.17 of Figure 1 whereas in the region $R_4 \cap \{b_3 < b < a^2 - 2a\}$ we can take, for instance, $a = 7$ and $b = 34.7$ and the phase portrait is 1.17' of Figure 4. Note that the phase portraits 1.17 of Figure 1 and 1.17' of Figure 4 are topologically equivalent. Indeed, in both in the half disc $y > 0$ (respectively, $y < 0$) we have a saddle connected with the four nodes at infinity and a saddle connected with two of the nodes at infinity and forming a center–loop inside the region between the separatrices of the other saddle connecting these two nodes.

On the region R_5 there are no infinite singular points and besides the origin there are six finite singular points which are hyperbolic (see Lemma 6). Therefore the known singular points have total index -2 on the Poincaré sphere. By Theorem 3, the six remaining finite singular points must have

total index 4. Since they are hyperbolic, taking into account the symmetry of the system they must be four centers and two saddles. Note that in view of Lemma 7 in this region the saddles cannot be connected with the saddle at the origin. Hence, the global phase portrait of systems (VIII) on the region R_5 is topologically equivalent to the phase portrait 1.7 of Figure 1.

On the curve ℓ_1 there are no finite singular points and there are two points on the local chart U_1 that consist in one hyperbolic and one elliptic sectors. Hence, the global phase portrait of systems (VIII) on the curve ℓ_1 is topologically equivalent to the phase portrait 1.18 of Figure 1.

On the curves ℓ_2 or ℓ_5 there are four finite singular points among the origin and there are four infinite singular points which are nodes (two stable and two unstable) and all of them are on the local chart U_1. Moreover, the finite singular points can be either hyperbolic or nilpotent and there are at most two cusps (see Lemma 6). Therefore the known singular points have total index 6 on the Poincaré sphere. By Theorem 3, the four remaining finite singular points must have total index -4. Since they are hyperbolic on nilpotent and taking into account the symmetry of the system they must be two saddles and two cusps. Note that in view of Lemma 7 there can be a connection of a pair of saddles with the saddle at the origin on the intersection between the curve $b = b_2$ and $b = (a - 1)^2$, that is, at the point $a = 5$ and $b = 16$. The global phase portrait of systems (VIII) on the curve ℓ_2 is topologically equivalent to the phase portraits 1.19, 1.20 and 1.21 of Figure 1. We recall that all phase portraits are realized. The case of 1.19 is realized on the curve ℓ_2 with $a > 5$ (we can take for instance $b = b_2$ and $a = 6$ and in this case there is a connection between the two saddles with $y \neq 0$). The case 1.20 is realized for $b = b_2$ and $a = 5$ and it corresponds to the connection of the saddles with the saddle at the origin. The case 1.21 is realized on the curve ℓ_2 with $a < 5$ (we can take for instance $b = b_2$ and $a = 4.5$, in this case all saddles are connected with the nodes at infinity). The global phase portrait of systems (VIII) on the curve ℓ_5 is topologically equivalent to the phase portrait 1.21' of Figure 4 which is topologically equivalent to the phase portrait 1.21 of Figure 1. We observe that in the phase portrait 1.19 (respectively, 1.20, 1.21 and 1.21') the saddles and the centers forming the center–loop in the phase portrait 1.16 (respectively, 1.15, 1.17 and 1.17') have coalesced.

On the curve ℓ_3 there are no infinite singular points and besides the origin there are four finite singular points that can be either hyperbolic or nilpotent and there are at most two cusps (see Lemma 6). Therefore the known singular points have total index -2 on the Poincaré sphere. By Theorem 3, the four remaining finite singular points must have total index 4. Since they are hyperbolic or nilpotent, taking into account the symmetry of the system they must be two centers and two cusps. Hence, the global phase portrait of systems (VIII) on the curve ℓ_3 is topologically equivalent

to the phase portrait 1.22 of Figure 1. Note that this corresponds to the fact that one center coalesces with the saddle (both for y positive and y negative) in phase portrait 1.7.

On the curve ℓ_4 there are two points in the local chart U_1 that consists in one hyperbolic and one elliptic sectors, and besides the origin there are four singular points which are hyperbolic (see Lemma 6). Therefore the known singular points have total index 2 on the Poincaré sphere. By Theorem 3, the four remaining finite singular points must have total index 0. Since they are hyperbolic, taking into account the symmetry of the system they must be two centers and two saddles. Taking also into account the fact that the saddles cannot be connected with the saddle at the origin, we get that the global phase portrait of systems (VIII) on the curve ℓ_4 is topologically equivalent to the phase portrait 1.23 of Figure 1.

On the point p_0 the are two points in the local chart U_1 that consists in one hyperbolic and one elliptic sector separated by the infinity and besides the origin there are two finite singular points that are hyperbolic (see Lemma 6). Therefore the known singular points have total index 2 on the Poincaré sphere. By Theorem 3, the four remaining finite singular points must have total index 0. Since they are either hyperbolic or nilpotent and there are at most two cusps (see Lemma 6), we conclude that they must be two cusps. Hence, the global phase portrait of systems (VIII) at p_0 is topologically equivalent to the phase portrait 1.24 of Figure 1. Note that in this case the saddles and the centers forming the center–loop in phase portrait 1.24 have coalesced.

On the point p_1 the are four nodes on the local chart U_1 (two stable and two unstable), the origin of U_2 is not a singular point and besides the origin there are two finite singular points which are either hyperbolic or nilpotent (see Lemma 6). Therefore the known singular points have total index 6 on the Poincaré sphere. By Theorem 3, the four remaining finite singular points must have total index -4. Taking into account the symmetry of the system they must be two saddles. Hence, the global phase portrait of systems (VIII) at p_1 is topologically equivalent to the phase portrait 1.25 of Figure 1.

Acknowledgements

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