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# Minimizers for the Thin One-Phase Free Boundary Problem

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**Abstract**

We consider the “thin one-phase" free boundary problem, associated to minimizing a weighted Dirichlet energy of the function in $\mathbb{R}^{n+1}_+$ plus the area of the positivity set of that function in $\mathbb{R}^n$. We establish full regularity of the free boundary for dimensions $n \leq 2$, prove almost everywhere regularity of the free boundary in arbitrary dimension, and provide content and structure estimates on the singular set of the free boundary when it exists. All of these results hold for the full range of the relevant weight.

While our results are typical for the calculus of variations, our approach does not follow the standard one first introduced by Alt and Caffarelli in 1981. Instead, the nonlocal nature of the distributional measure associated to a minimizer necessitates arguments that are less reliant on the underlying PDE. 

## 1 Introduction

This article is devoted to the study of the regularity properties of a weighted version of the thin one-phase problem. More precisely, we investigate even, nonnegative minimizers of the following functionals: denote $x \in \mathbb{R}^{n+1}$ by $x = (x', y) \in \mathbb{R}^n \times \mathbb{R}$, and for $\beta \in (-1, 1)$ we define

$$\mathcal{J}(v, \Omega) := \int_\Omega |y|^\beta |\nabla v|^2 \, dx + m(\{v > 0\} \cap \mathbb{R}^n \cap \Omega), \tag{1.1}$$

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where $m$ stands for the $n$-dimensional Lebesgue measure. Here and throughout the paper the integration is done with respect to the $(n+1)$-dimensional Lebesgue measure unless stated otherwise. This functional is finite for open sets, $\Omega$, and functions in the weighted Hilbert space,

$$H^1(\beta, \Omega) := \{v \in L^2(\Omega; |y|^\beta) : \nabla v \in L^2(\Omega; |y|^\beta)\},$$

equipped with the usual weighted norm.

Our main concern is to investigate fine regularity properties of the free boundary of minimizers $v$ of (1.1), that is, the set

$$F(v) := \partial_{\mathbb{R}^n}\{v(x,0) > 0\} \cap \Omega.$$

Since the free boundary lies on a codimension 1 subspace of the ambient space $\mathbb{R}^{n+1}$, such a problem is called a *thin one-phase free boundary problem*. This type of free boundary problem has been investigated for the first time by Caffarelli, Roquejoffre, and the last author in [7] in relation to the theory of semipermeable membranes (see, e.g., [21]). As we will describe later,. this is an analogue of the classical one-phase problem (also called the Bernoulli problem) but for the fractional Laplacian.

The Bernoulli problem was first treated in a rigorous mathematical way by Alt and Caffarelli in the seminal paper [2]: in the Bernoulli problem we consider minimizers of (1.1) where $\beta = 0$, and the second term is replaced by $\mathcal{L}^{n+1}(\{v > 0\} \cap \Omega)$ (where $\mathcal{L}^{n+1}$ stands for the Lebesgue measure in $\mathbb{R}^{n+1}$). In particular, for the Bernoulli problem, the free boundary fully sits in the ambient space, $\mathbb{R}^{n+1}$. In [2], the authors provided a general strategy to attack this type of problem. Out of necessity we needed to modify this blueprint in several substantial ways (see below for a more detailed comparison). For more information on the one-phase problem (and some of its variants) we refer to the book of Caffarelli and Salsa (and references therein) [8] and to the more recent survey of De Silva, Ferrari, and Salsa [14].

As noticed in [7], problem (1.1) is related in a tight way to the *standard* one-phase free boundary problem but with the Dirichlet energy replaced by the Gagliardo seminorm $[u]_{\dot{H}^\alpha}$ for $\alpha = \frac{1-\beta}{2} \in (0,1)$. This connection suggests that the thin one-phase problem is actually intrinsically a *nonlocal* problem, though the energy in (1.1) is clearly local.

**Connection with the Fractional One-Phase Problem**

As previously mentioned, the functional $\mathcal{J}$ introduced by Caffarelli, Roquejoffre, and the last author in [7] is a local version of the following nonlocal free boundary problem: given a function $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with suitable decay at infinity, we can define its fractional Laplacian at $x \in \mathbb{R}^n$ by

$$(-\Delta)^\alpha f(x) = c_{n,\alpha}\,\text{p. v.} \int_{\mathbb{R}^n} \frac{f(x) - f(\xi)}{|x-\xi|^{n+2\alpha}}\, d\xi.$$

At the formal level, we are interested in solutions of the free boundary problem

$$\begin{cases}(-\Delta)^\alpha f = 0 & \text{in } \Omega \cap \{f > 0\},\\ \partial_\nu^\alpha f = A & \text{on } \Omega \cap F(f),\end{cases} \tag{1.2}$$

where $\partial_\nu^\alpha f(x) := \lim_{\Omega\cap\{f>0\}\ni\xi\to x} \frac{f(\xi)-f(x)}{((\xi-x)\cdot\nu(x))^\alpha}$ and where $f$ satisfies a given "Dirichlet boundary condition" on the complement of $\Omega$.

As in the case of the classical Laplacian (see [2]), we are interested in obtaining equation (1.2) as the Euler-Lagrange equation of a certain functional. Given a locally integrable function $f$, consider its fractional Sobolev energy

$$[f]_{\dot{H}^\alpha(\mathbb{R}^n)} := \iint_{\mathbb{R}^{2n}} \frac{|f(x)-f(\xi)|^2}{|x-\xi|^{n+2\alpha}}\, d\xi\, dx.$$

Since we want to study competitors that vary only in a certain domain $\Omega$, it is natural to consider only the integration region that may suffer variations when changing candidates. Thus, we define the energy

$$J(f,\Omega) := c_{n,\alpha} \iint_{\mathbb{R}^{2n}\setminus(\Omega^c)^2} \frac{|f(x)-f(\xi)|^2}{|x-\xi|^{n+2\alpha}}\, d\xi\, dx + m(\{f>0\}\cap\Omega). \tag{1.3}$$

We say that $f \in L^1_{\text{loc}}$ is a minimizer of $J$ in $\Omega$ if $J(f,\Omega)$ is finite and $J(f,\Omega) \le J(g,\Omega)$ for every $g$ satisfying that $f - g \in \dot{H}^\alpha(\mathbb{R}^n)$ and such that $f(x) = g(x)$ for almost every $x \in \Omega^c$. We say that $f$ is a global minimizer if it is a minimizer for every open set $\Omega \subset \mathbb{R}^n$. Note that both terms in (1.3) are in competition, since a minimizer of the fractional Sobolev energy in $\Omega$ is $\alpha$-harmonic and, thus, if it is nonnegative outside of $\Omega$, it is strictly positive inside of $\Omega$, maximizing the second term.

Consider now the Poisson kernel for fixed $n \in \mathbb{N}$ and $0 < \alpha < 1$,

$$P_y(\xi) := P_{n,\alpha}(\xi, y) = c_{n,\alpha} \frac{|y|^{2\alpha}}{|(\xi,y)|^{n+2\alpha}} \quad \text{for every } (\xi,y) \in \mathbb{R}^n \times \mathbb{R}. \tag{1.4}$$

The Poisson extension of $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ is given by

$$u(x',y) := f * P_y(x') = \int_{\mathbb{R}^n} P_{n,\alpha}(\xi,y) f(x'-\xi)\, d\xi \quad \text{for every } (x',y) \in \mathbb{R}^n \times \mathbb{R}. \tag{1.5}$$

By [9], with a convenient choice of the constant, one gets

$$\lim_{y \searrow 0} y^{1-2\alpha} u_y(x',y) = -(-\Delta)^\alpha f(x')$$

in every point where $f$ is regular enough. Moreover, the extension satisfies the localized equation $\nabla\cdot(|y|^\beta \nabla u) = 0$ weakly, away from $\mathbb{R}^n \times \{0\}$. The whole point is that local minimizers of (1.3) can be extended via the previous Poisson kernel $P_y$ to (even) minimizers of (1.1) (see the Appendix for a precise statement). Therefore, the thin one-phase problem appears as a "localization" of the one-phase problem for the fractional Laplacian. Notice that—and this is of major importance for us—this localization technique does not carry over to other types of nonlocal operators besides pure powers of second-order elliptic operators. This is a major drawback of the theory, in the sense that, at the moment, it seems to be impossible to tackle one-phase problems involving more general operators than the fractional Laplacian. The main point is we do not know how to prove any kind of monotonicity for general integral operators.

This connection between the nonlocal analogue of the Bernoulli problem and our thin one-phase problem allows us to simplify several arguments by working in the purely nonlocal setting. However, this underlying nonlocality is also the reason why several results, which came more easily in the setting of [2], are nontrivial or substantially harder for us. For example, perturbations of solutions need to take into account long-range effects that make classical, local perturbation arguments much more difficult.

In the paper [7], the authors proved basic properties of the minimizers for the functional $\mathcal{J}$ such as optimal regularity, nondegeneracy near the free boundary, and positive densities of phases. Also they provided an argument for $n = 2$ showing that Lipschitz free boundaries are $C^1$. A feature of the functional $\mathcal{J}$ is that the weight $|y|^\beta$ is either degenerate or singular at $\{y = 0\}$ (except in the case $\beta = 0$). Such weights belong to the Muckenhoupt class $A_2$ and the seminal paper of Fabes, Kenig, and Serapioni [26] investigated regularity issues for elliptic PDEs involving such weights (among other things). After that, [19] proved an $\varepsilon$-regularity result and [1] showed the existence of a monotonicity formula for this setting.

In the case $\beta = 0$, the problem is still degenerate in the sense that derivatives near the free boundary blow up. The case $\beta = 0$ has been thoroughly investigated in the series of papers by De Silva, Savin, and Roquejoffre [16–18].

The main goal of our paper is to provide a full picture of the regularity of the free boundary for any power $\beta \in (-1, 1)$, both in terms of measure-theoretic statements and partial (or full) regularity results. From this point of view our contribution is a complement to the paper by De Silva and Savin [18] for $\beta = 0$. It has to be noticed that the standard approach to regularity of Lipschitz free boundaries as developed by Caffarelli (see the monograph [8]) does not seem to work in our setting.

**Our Approach to Regularity**

In [2] (and many subsequent works), the minimizing property of the solution is used to prove that the distributional Laplacian of that solution is an Ahlfors-regular measure supported on the free boundary. This implies (among other things) that the free boundary is a set of (locally) finite perimeter, and thus almost every point on the free boundary has a measure-theoretic tangent. One can then work purely with the weak formula (i.e., the analogue of (1.2)) to prove a “flat implies smooth” result which, together with the existence almost everywhere of a measure theoretic tangent, has as a consequence that the free boundary is almost everywhere a smooth graph and the free boundary condition in (1.2) holds in a classical sense at the smooth points.

A similar "flat implies smooth" result exists in our context (this is essentially due to De Silva, Savin, and the last author [19]; see Theorem 2.4 below). However, showing that the free boundary is the boundary of a set of finite perimeter proves to be much more difficult. Due to the nonlocal nature of the problem, $-\mathrm{div}(|y|^{\beta}\nabla u)$ (considered as a distribution) is not supported on the free boundary. Furthermore, the scaling of this measure does not allow us to conclude that the free boundary has the correct dimension (much less that it is Ahlfors regular).

To prove finite perimeter, we take the following approach inspired by the work of de Silva and Savin: after establishing some preliminaries we prove crucial compactness results. This, along with a monotonicity formula originally due to Allen [1] allows us to run a dimension reduction argument in the vein of Federer or (in the context of free boundary problems) Weiss [38]. With this tool in hand, we show that the set of points at which no blowup is flat is a set of lower dimension. Locally finite perimeter and regularity for the reduced boundary then follow from a covering argument and some standard techniques.

Here and throughout the paper, we will denote the ball of radius $r$ in $\mathbb{R}^{n+1}$ centered at the origin by $B_r$, and $B'_r := B_r \cap \mathbb{R}^n \times \{0\}$. Moreover, for the definition of $\mathbf{H}^{\beta}$, see Section 2. We may then summarize our regularity results in the following theorem.

THEOREM 1.1. *[Main regularity theorem] Let* $u \in \mathbf{H}^{\beta}(B_1)$ *be a (nonnegative, even) local minimizer of* $\mathcal{J}$ *in* $B_1 \subset \mathbb{R}^{n+1}$*. Let* $B'_{1,+}(u) := \{x = (x', 0) \in B_1 : u(x) > 0\}$*, let* $F(u)$ *be the boundary of* $B'_{1,+}(u)$ *inside of* $B'_1$*, and assume that* $0 \in F(u)$*. Then*:

(1) $B'_{1,+}(u)$ (*as a subset of* $\mathbb{R}^n \times \{0\}$) *is a set of locally finite perimeter in* $B'_1$.
(2) *We can write the free boundary as a disjoint union* $F(u) = \mathcal{R}(u) \cup \Sigma(u)$*, where* $\mathcal{R}(u)$ *is open inside* $F(u)$*, and for* $x \in \mathcal{R}(u)$ *there exists an* $r_x > 0$ *such that* $B(x, r_x) \cap F(u)$ *can be written as the graph of a* $C^{1,s}$*-continuous function.*
(3) *Furthermore, the set* $\Sigma(u)$ *is of Hausdorff dimension* $\leq n-3$ (*and, therefore, of* $\mathcal{H}^{n-1}$*-measure zero*). *In particular, for* $n \leq 2$*,* $\Sigma(u)$ *is empty, and moreover, if* $n = 3$ *then* $\Sigma(u)$ *is discrete.*

*The constants* (*implicit in the set of finite perimeter, and the Hölder continuity of the functions whose graph gives the free boundary*) *depend on* $n$ *and* $\beta$ *but not on* $\|u\|_{\mathbf{H}^{\beta}(B_1)}$.

As usual $\Sigma(u) \subset F(u)$ is called the *singular set* of the free boundary: the set of points around which $F(u)$ cannot be parametrized as a smooth graph and all the blowups will be nontrivial minimal cones; see Theorem 2.4.

Our second contribution concerns the structure and size of the singular set. It builds on recent major works on quantitative stratification [32] extended to free boundary problems (in particular, the one-phase problem) by Edelen and the first author [22].

THEOREM 1.2. *Let $u \in \mathbf{H}^\beta(B_1)$ be a (nonnegative, even) local minimizer of $\mathcal{J}$ in $B_1$ and $0 \in F(u)$. Let $B'_{1,+}(u) := \{x = (x', 0) \in B_1 : u(x) > 0\}$ and $F(u)$ be the boundary of $B'_{1,+}(u)$ inside $B'_1$. Then, there exists a $k^*_\alpha \geq 3$ such that $\Sigma(u)$ is $(n - k^*_\alpha)$-rectifiable and*

$$\mathcal{H}^{n-k^*_\alpha}(\Sigma(u) \cap D) \leq C_{n,\alpha,\mathrm{dist}(D,\partial B_1)} \quad \text{for every } D \Subset B_1.$$

In [15], De Silva and Jerison constructed a singular minimizer for the Alt-Caffarelli one-phase problem in dimension 7, giving the dimension bound $k^* \leq 8$ in the previous theorem in this case (see [22]). This result is not known for the thin one-phase problem. The reason is that the one-phase problem, seen from the nonlocal point of view involving the fractional Laplacian, is related to the so-called nonlocal minimal surfaces introduced by Caffarelli, Roquejoffre, and Savin [6]. Indeed, in [33], the authors proved that a fractional version of Allen-Cahn equation converges variationally to the standard perimeter functional for $\alpha \geq 1/2$ and to the so-called nonlocal minimal surfaces for $\alpha < 1/2$. We can then conjecture the bound $k^*_\alpha \leq 8$ for $\alpha \geq 1/2$ by analogy with the result for the standard one-phase problem, but the bound for $\alpha < 1/2$ is not clear at all. However, one knows that there is no singular cone in dimension 2 for nonlocal minimal surfaces [34] and that the Bernstein problem is known for those in dimensions 2 and 3 [28].

We would like also to make a last remark about a result that is of purely nonlocal nature. In the case of the one-phase problem, one can show that the distributional Laplacian is a Radon measure along the free boundary. In the case of the thin one-phase free boundary problem, due to the nonlocality of the problem, such a behavior does not happen in the sense that we will show that the fractional Laplacian is an absolutely continuous measure with respect to $n$-dimensional Lebesgue measure with a precise behavior. This phenomenon is of purely nonlocal nature and similar to the fact that the fractional harmonic measure is of trivial nature. More precisely, every minimizer $u$ satisfies $\nabla \cdot (|y|^\beta \nabla u) = 0$ weakly, away from $\mathbb{R}^n \cap \{u \leq 0\}$. Thus, equation (1.2) above can be understood as an Euler-Lagrange equation for the functional $\mathcal{J}$ in the sense that the restriction to $\mathbb{R}^n$ of a given minimizer $u$ in $\Omega \subset \mathbb{R}^{n+1}$, harmonic away from $\mathbb{R}^n \times \{0\}$ and with asymptotic behavior $u(x, y) = \mathcal{O}(|(x, y)|^\alpha)$, is always a solution to (1.2) for $A = A(\alpha)$ at "nice" points of the free boundary.

A brief summary of this paper follows. In Sections 3 and 4 we discuss compactness of minimizers and we recall Allen's monotonicity formula to derive some immediate consequences. In Section 5 we show that the positive phase is a set of locally finite perimeter, establishing the first part of Theorem 1.1 (modulo energy bounds), and we show that the singular set can be identified using the Allen-Weiss density. Section 6 is devoted to deducing full regularity of minimizers in $\mathbb{R}^{2+1}$ concluding the proof of Theorem 1.1.

Once we have established the finite perimeter, in Section 7 we remove the dependence of the estimates on the energy of the minimizer in the previous theorems, using a rather subtle argument that combines results from all the previous sections. A crucial step is to analyze some basic properties of the distributional fractional Laplacian of our minimizer. As stated above this analysis will not be enough to establish that the positivity set of the minimizer is a set of locally finite perimeter. We believe that many of these results may be of independent interest. For example, corresponding results for the classical Bernoulli problem have been used to understand the free boundary problems for harmonic measure (see [31]).

Finally, Section 8 is devoted to the proof of Theorem 1.2.

**Notation**

We denote the constants that depend on the dimension $n$, $\alpha$, and perhaps some other fixed parameters that are clear from the context by $C$. Their value may change from one occurrence to another. On the other hand, constants with subscripts such as $C_0$ retain their values along the text. For $a, b \geq 0$, we write $a \lesssim b$ if there is $C > 0$ such that $a \leq Cb$. We write $a \approx b$ to mean $a \lesssim b \lesssim a$.

Let $u$ be a continuous function in $\mathbb{R}^{n+1}$. Then we write $\Omega_+(u) := \Omega \cap \{u > 0\}$, and we denote the zero phase, the positive phase, and the free boundary by

$$\begin{aligned}
\Omega_0(u) &:= \{x \in \mathbb{R}^n \times \{0\} : u(x) = 0\}^\circ, \\
\Omega'_+(u) &:= \Omega_+ \cap (\mathbb{R}^n \times \{0\}) = \{x \in \mathbb{R}^n \times \{0\} : u(x) > 0\}, \\
F(u) &:= F_\Omega(u) = \partial(\Omega_+(u) \cap \mathbb{R}^n \times \{0\}) \cap \Omega,
\end{aligned}$$

respectively. Here both the boundary and the interior are taken with respect to the standard topology in $\mathbb{R}^n$. Note that $\mathbb{R}^n \times \{0\}$ is the disjoint union of $\Omega_0(u)$, $\Omega'_+(u)$, and $F(u)$ whenever $u$ is nonnegative. We also call $F_{\text{red}}(u) = F_{\text{red},\Omega}(u)$ the points of $F_\Omega(u)$ where the free boundary is expressed locally as a $C^1$ surface. Finally, let $\Sigma(u) = \Sigma_\Omega(u) = F_\Omega(u) \setminus F_{\text{red},\Omega}(u)$. In general, we will write $\Omega' := \Omega \cap (\mathbb{R}^n \times \{0\})$.

Throughout the paper we will often fix $\beta \in (-1, 1)$ but then refer to $\alpha \in (0, 1)$ or vice versa. These two numbers are always connected by the relationship $\alpha = \frac{1-\beta}{2}$.

## 2 Preliminaries

In this section, we provide the known results concerning the problem under consideration. We say that a function $u$ is *even* if it is symmetric with respect to the hyperplane $\mathbb{R}^n \times \{0\}$, that is, $u(x', y) = u(x', -y)$. The function spaces that we will consider are the following

$$\mathbf{H}^\beta(\Omega) := \{u \in H^1(\beta, \Omega) : u \text{ is even and nonnegative}\}$$

and

$$\mathbf{H}^\beta_{\text{loc}}(\Omega) := \left\{u \in L^2_{\text{loc}}(\Omega) : u \in \mathbf{H}^\beta(B) \text{ for every ball } B \Subset \Omega\right\}.$$

We will omit $\Omega$ in the notation when it is clear from the context.

DEFINITION 2.1. We say that a function $u \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\Omega)$ is a (local) minimizer of $\mathcal{J}$ in a domain $\Omega$ if for every ball $B \Subset \Omega$ and for every function $v \in \mathbf{H}^{\beta}(B)$ such that the traces $v|_{\partial B} \equiv u|_{\partial B}$, the inequality

$$\mathcal{J}(u, B) \leq \mathcal{J}(v, B)$$

holds.

As usual for several free boundary problems, it is a natural question to exhibit a particular (global) solution so that one gets an idea of the qualitative properties of general solutions. Let us consider the following function: for every $x \in \mathbb{R}^n$ let

$$f_{n,\alpha}(x) := c_{n,\alpha}(x_n)^{\alpha}_{+},$$

where $a_+ = \max\{0, a\}$. If $n = 1$, $f_{1,\alpha}$ is a solution to (1.2) for a convenient choice of $c_{1,\alpha}$ (see [4, theorem 3.1.4]). In fact, one can see that the same is true for $n \geq 1$ using Fubini's theorem conveniently, with

$$-(-\Delta)^{\alpha} f_{n,\alpha}(x) = c_{n,\alpha}(x_n)^{-\alpha}_{-}, \tag{2.1}$$

where $a_- = \max\{0, -a\}$.

As a toy question we wonder whether the trivial solutions are minimizers. Indeed, this is the case, as we will see later in Section 4.1.

PROPOSITION 2.2. *Let $n \in \mathbb{N}$ and $0 < \alpha < 1$. Then the trivial solution $u_{n,\alpha} := f_{n,\alpha} * P_y$ is a minimizer of $\mathcal{J}$ in every ball $B \subset \mathbb{R}^{n+1}$.*

Next we collect the main properties of minimizers in the unit ball proven in [7, theorems 1.1–1.4, prop. 3.3, and cor. 3.4].

THEOREM 2.3. *If $u \in \mathbf{H}^{\beta}(B_1)$ is a minimizer of $\mathcal{J}$ in $\Omega = B_1$ with $\|u\|_{\dot{\mathbf{H}}^{\beta}(B_1)} := \|\nabla u\|_{L^2(B_1,|y|^{\beta})} \leq E_0$ and $x_0 \in F(u) \cap B_{\frac{1}{2}}$, then it satisfies*

- (P1) *Optimal regularity* (see [7, theorem 1.1]): $\|u\|_{\dot{C}^{\alpha}(B_{1/2})} \leq C(1 + E_0)$.
- (P2) *Nondegeneracy* (see [7, theorem 1.2]:) $u(x) \geq C \operatorname{dist}(x, F(u))^{\alpha}$ *for* $x \in B'_{1/2,+}$.
- (P3) *Interior corkscrew condition* (see [7, prop. 3.3]:) *there exists* $x_+ \in B'_r(x_0)$ *so that* $B'(x_+, C_0 r) \subset \Omega'_+(u)$.
- (P4) *Positive density* (see [7, theorem 1.3]:) $|\Omega_0 \cap B'_r(x_0)| \gtrsim r^n$.
- (P5) *Blowups are minimizers* (see [7, cor. 3.4]): *The limit of a blowup sequence* $u_k(x) := \frac{u(x_0+\rho_k x)}{\rho_k^{\alpha}}$ *converging weakly in* $H^1(\beta, B_1)$ *and uniformly is a global minimizer.*
- (P6) *Normal behavior at the free boundary* (see [7, theorem 1.4]): *the boundary condition in* (1.2) *is satisfied at every point on the free boundary with a measure-theoretic normal* (see [24]) *for a prescribed value of A.*

*All the constants depend on n and α, and also on $E_0$ except for the ones in* P1 *and* P2.

A major tool in the present paper is an $\epsilon$-regularity result, i.e., in the language of free boundaries a statement of the type "flatness implies smoothness." In [19], the authors proved such an $\epsilon$-regularity result for viscosity solutions to the overdetermined system associated to minimizers of $\mathcal{J}$. Here we establish that all local minimizers are in fact viscosity solutions. While this verification may be standard for experts in the field, we include it here for the sake of completeness.

THEOREM 2.4 ($\epsilon$-regularity). *There exists $\epsilon > 0$ depending only on $n$, $\alpha$, and $E_0$ such that for every nonnegative, even minimizer $u$ of the energy* (1.1) *on a ball $B \subset \mathbb{R}^{n+1}$ with $\|u\|_{\mathbf{H}^\beta(B)} \le E_0 r(B)^{\frac{n}{2}}$ and*

$$\{(x,0) \in B : x_n \le -\epsilon\} \subset B_0(u) \subset \{(x,0) \in B : x_n \le \epsilon\}, \tag{2.2}$$

*we have that $F(u) \in C^{1,\gamma}_{\mathrm{loc}}(\frac{1}{2}B)$, with $0 < \gamma < 1$.*

Note that the dependence on $E_0$ will be removed in Section 7.

PROOF. We say that $u$ is a *viscosity solution* to

$$\begin{cases} \nabla \cdot (|y|^\beta \nabla u) = 0 & \text{in } B_1^+(u), \\ \lim_{t\to 0+} \frac{u(x_0+t\nu(x_0),0)}{t^\alpha} = 1, & \text{for } (x_0,0) \in F(u) \end{cases} \tag{2.3}$$

if

(i) $u \in C(B_1)$, $u \ge 0$,
(ii) $u \in C^{1,1}_{\mathrm{loc}}(B_{1,+}(u))$, $u$ is even, and it solves $\nabla \cdot (|y|^\beta \nabla u) = 0$ in the viscosity sense, and
(iii) any strict *comparison subsolution* (resp., supersolution) cannot touch from below (resp., from above) at a point $(x_0, 0) \in F(u)$.

We claim that

$$\text{every nonnegative even minimizer is a viscosity solution.} \tag{2.4}$$

Conditions (i) and (ii) have been verified in [19, 36]. To verify our claim it suffices to prove condition (iii) above: that any strict comparison subsolution cannot touch $u$ from below at a point $(x_0, 0) \in F(u)$. The analogous claim for strict comparison supersolutions will follow in the same way.

Let us recall (see, e.g., definition 2.2 in [19]) that $w \in C(B_1)$ is a strict comparison subsolution (resp., supersolution) to (2.3) if

(a) $w \ge 0$,
(b) $w$ is even with respect to $\{y = 0\}$,
(c) $w \in C^2(\{w > 0\})$,
(d) $\mathrm{div}(|z|^\beta \nabla w) \ge 0$ in $B_1 \setminus \{y = 0\}$,
(e) $F(w)$ is locally given by the graph of a $C^2$ function and for any $x_0 \in F(w)$ we may write

$$w(x,y) = aU((x-x_0)\cdot\nu(x_0), y) + o(\|(x-x_0,y)\|^\alpha), \quad (x,y) \to (x_0,0). \tag{2.5}$$

Here $U$ is the extension of the trivial solution (see [19]), and $\nu(x_0)$ is the unit normal to $F(w)$ considered as a subset of $\mathbb{R}^n$ pointing into $\{w > 0\}$ and $a \geq 1$.

(f) Furthermore, either the inequality is strict in (d) or $a > 1$ in (e).

So assume that $w \geq u$ where $w$ is a strict comparison subsolution and $u$ is some minimizer and that $w = u$ at $(x_0, 0) \in F(u)$. Since $u(x_0, 0) = 0$ it follows that $(x_0, 0) \in F(w)$ and with a harmless rotation we can guarantee that $\nu((x_0, 0)) = e_n$. We want to show that $e_n$ is also the measure-theoretic unit normal to $F(u)$. Indeed, since $F(w)$ is $C^2$ there must exist a ball $B \subset \{w > 0\}$ that is tangent to $F(w)$ at $(x_0, 0)$. It must then be the case that $B \subset \{u > 0\}$ as well. Thus $(x_0, 0) \in F(u)$ has a tangent ball from the inside, which, by proposition 4.5 in [7] implies that $u$ has the asymptotic expansion

$$u(x, y) = U((x - x_0) \cdot \nu(x_0), y) + o(\|(x - x_0, y)\|^\alpha), \quad (x, y) \to (x_0, 0).$$

If $u \geq w$, this implies that $w$ must satisfy the expansion in (2.5) with $a = 1$ at the point $x_0$. This, in turn, implies that $\operatorname{div}(|z|^\beta \nabla w) > 0$ in $B_1 \backslash \{y = 0\}$ (by the definition of a strict subsolution). Furthermore, since $w \in C^2$ where $\{w > 0\}$, we can guarantee that $\operatorname{div}(|z|^\beta \nabla w) \geq 0$ in all of $B_1 \cap \{w > 0\}$.

Let us return to the ball $B$ that is a subset of $\{u > 0\}$ and $\{w > 0\}$ and for which $(x_0, 0) \in \bar{B}$. We know that $w - u \neq 0$ in $B \setminus \{y = 0\}$ (this is because $w$ strictly satisfies the differential inequality in $B$ away from $\{y = 0\}$), and we know that $w - u$ is a subsolution in $B$. Furthermore, $(x_0, 0) \in B$ is a strict maximum, so by the Hopf lemma in [5, prop. 4.11] it must be that

$$\lim_{t \downarrow 0^+} \frac{(w - u)(x_0 + t\nu(x_0), 0)}{t^\alpha} > 0.$$

This contradicts the fact that $u$ and $w$ both satisfy (2.5) at $(x_0, 0)$ with $a = 1$. Therefore, $(x_0, 0)$ must not have been a touching point and $u$ is indeed a viscosity solution.

Since, $u$ is a viscosity solution, [19, theorem 1.1] applies and we get the desired $\varepsilon$-regularity. ☐

## 3 Compactness of Minimizers

In this section we prove important results on the compactness of minimizers. As we mentioned above, our contribution is that convergent sequences of minimizers also converge in the relevant weighted Sobolev spaces strongly rather than just weakly. This will prove essential to the compactness arguments used later in this paper.

### 3.1 Caccioppoli Inequality

First we want to show that the distribution $\lambda := \nabla \cdot (|y|^\beta \nabla u)$ is in fact a Radon measure with support in the complement of the positive phase as long as $u$ is a minimizer. In Section 7 we will come back to this measure to understand its behavior around the free boundary.

LEMMA 3.1. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega, |y|^\beta)$ be such that $\nabla \cdot (|y|^\beta \nabla u) = 0$ weakly in $\Omega_+(u)$, i.e., for every $\eta \in C_c^\infty(\Omega_+(u))$,*

$$\langle \nabla \cdot (|y|^\beta \nabla u), \eta \rangle := -\int (|y|^\beta \nabla u) \nabla \eta = 0. \tag{3.1}$$

*Then $\lambda := \nabla \cdot (|y|^\beta \nabla u)$ is a positive Radon measure supported on $\{u = 0\}$ and for every $v \in W^{1,2}(\Omega, |y|^\beta) \cap C_c(\Omega)$*

$$\int v \, d\lambda = -\int |y|^\beta \nabla u \cdot \nabla v. \tag{3.2}$$

PROOF. Indeed, by (3.1) the quantity

$$\begin{aligned} -\int |y|^\beta \nabla u \cdot \nabla \zeta &= -\int |y|^\beta \nabla u \cdot \nabla \Big( \zeta \max\Big\{ \min\Big\{ 2 - \frac{u}{\varepsilon}, 1 \Big\}, 0 \Big\} \Big) \\ &\geq -\int_{\Omega \cap \{0 < u < 2\varepsilon\}} |y|^\beta |\nabla u| |\nabla \zeta| \xrightarrow{\varepsilon \to 0} 0 \end{aligned}$$

defines a positive functional on positive $\zeta \in C_c^{0,1}(\Omega)$. Moreover, for compact $K \subset \Omega$, consider a Lipschitz function $f_K$ such that $\chi_K \leq f_K \leq \chi_\Omega$. If $\zeta \in C_c^{0,1}(K)$, by the positivity shown above we obtain

$$-\int |y|^\beta \nabla u \cdot \nabla \zeta \leq -\|\zeta\|_{L^\infty} \int |y|^\beta \nabla u \cdot \nabla f_K \leq C_{K,u} \|\zeta\|_{L^\infty}$$

and, by Hahn-Banach's theorem, we can extend the functional to a positive functional in $C_c(\Omega)$, that is given by integration against a positive Radon measure by the Riesz representation theorem.

The fact that (3.2) holds for all functions in $W^{1,2}(\Omega, |y|^\beta) \cap C_c(\Omega)$ follows by a standard density argument. □

The Caccioppoli inequality is the first step to proving convergence in a Sobolev sense. It will also be useful when we remove the a priori dependence of our results on the Sobolev norm of the minimizer.

LEMMA 3.2 (Caccioppoli inequality). *Let $B \subset \mathbb{R}^{n+1}$ be a ball of radius $r$ centered on $\mathbb{R}^n \times \{0\}$, and let $u \in W^{1,2}(B, |y|^\beta)$ be such that $\nabla \cdot (|y|^\beta \nabla u) = 0$ weakly in $B \cap \{u > 0\}$. Then*

$$\int_{\frac{1}{2}B} |y|^\beta |\nabla u|^2 \leq \frac{4}{r^2} \int_{B \setminus \frac{1}{2}B} |y|^\beta u^2.$$

PROOF. Let $\eta$ be a Lipschitz function such that $\chi_{\frac{1}{2}B} \leq \eta \leq \chi_B$ and with $|\nabla\eta| \leq \frac{1}{r}$. By Lemma 3.1

$$0 = \int_B u\eta^2 \, d\lambda = \int_B |y|^\beta \nabla u \cdot \nabla(u\eta^2).$$

By the Leibniz rule

$$\int_B |y|^\beta \eta^2 |\nabla u|^2 = -\int_B |y|^\beta 2u\eta \nabla u \cdot \nabla\eta,$$

and using Hölder's inequality we get

$$\int_{\frac{1}{2}B} |y|^\beta |\nabla u|^2 \leq \int_B |y|^\beta \eta^2 |\nabla u|^2 \leq \int_B |y|^\beta 4u^2 |\nabla\eta|^2 \leq \frac{4}{r^2} \int_{B\setminus\frac{1}{2}B} |y|^\beta u^2.$$

□

LEMMA 3.3. *Let $u \in \mathbf{H}^\beta(B_r)$ be a minimizer of $\mathcal{J}$ in $B_{2r}$ and $0 \in F(u)$. Then*

$$\begin{aligned} r^{-n/2}\|\nabla u\|_{L^2(\frac{1}{2}B_r;|y|^\beta)} &\leq r^{-\alpha}\|u\|_{L^\infty(B_r)} \\ &\leq \|u\|_{\dot{C}^\alpha(B_r)} \leq C\big(1 + r^{-n/2}\|\nabla u\|_{L^2(B_{2r};|y|^\beta)}\big). \end{aligned}$$

PROOF. The first inequality is an immediate consequence of Caccioppoli, the middle estimate is trivial, and the last follows from P1 in Theorem 2.3. □

### 3.2 Compactness

In the following lemma we prove the compactness of minimizers in the relevant Sobolev spaces. For convenience, we also detail several compactness results that were either already proven in [7] or are standard consequences of the nondegeneracy estimates in Theorem 2.3. Nevertheless, we include full proofs here for the sake of completeness. We note here (as we did above and will do again below) that while we currently need to assume the uniform bound on the Hölder norm of the functions $u_k$, we can get rid of this assumption in the light of the results of Section 7.

LEMMA 3.4 (Compactness results). *Let $\{u_k\}_{k=1}^\infty \subset \mathbf{H}^\beta_{\text{loc}}(\Omega)$ be a sequence of minimizers in a domain $\Omega \subset \mathbb{R}^{n+1}$ with $\|u_k\|_{\dot{C}^\alpha(\Omega)} \leq E_0$ with nonempty free boundary. Then there exists a subsequence converging to some $u_0 \in \mathbf{H}^\beta_{\text{loc}}(\Omega)$ such that for every bounded open set $G \Subset \Omega$ we have*

(1) *$u_k \to u_0$ in $C^\beta(G)$ for every $\beta < \alpha$,*
(2) *$u_k \to u_0$ in $L^p(G)$ for every $p \leq \infty$,*
(3) *$\partial\{u_k > 0\} \cap \overline{G} \to \partial\{u_0 > 0\} \cap \overline{G}$ in the Hausdorff distance,*
(4) *$\chi_{\{u_k>0\}} \to \chi_{\{u_0>0\}}$ in $L^1(G')$, and*
(5) *$\nabla u_k \to \nabla u_0$ in $L^p(G;|y|^\beta)$ for every $p \leq 2$.*

PROOF. The first claim follows from uniform Hölder continuity and compact embeddings of Hölder spaces. The claim (2) follows from (1) easily.

We now prove the third claim. Let $\epsilon > 0$. We will first show that for $x \in \mathbb{R}^n \times \{0\}$ we have

$$d(x, F(u_0)) > \epsilon \Rightarrow d(x, F(u_k)) > \frac{\epsilon}{2} \tag{3.3}$$

for large $k$. This implies that $F(u_k) \subset \{x\colon d(F(u_0), x) < 2\epsilon\}$ for $k$ large enough.

Let $B(x,\epsilon) \subset F(u_0)^c$. If $u_0$ is positive in $B(x,\epsilon)$, then it is bounded from below by a positive number in $B(x,\epsilon/2)$. In this case $u_k$ are also positive in $B(x,\epsilon/2)$ for large $k$ due to uniform convergence in $G$. Thus $B(x,\epsilon/2) \subset F(u_k)^c$ for large $k$. If $u \equiv 0$ in $B'(x,\epsilon)$, then due to the uniform convergence we know that for $k$ large enough $u_k < C\epsilon^\alpha$ in $B'(x,\epsilon)$, where $C$ is a constant given by P2 in Theorem 2.3 so that $u_k$ has no free boundary points in $B(x,\epsilon/2)$ for all large $k$. This proves (3.3).

Next we will show that for all large $k$

$$F(u_0) \subset \{x\colon d(F(u_k), x) < \epsilon\}. \tag{3.4}$$

If this was not true we could find a point $x \in F(u_0)$ and a subsequence of $u_k$ such that $B'(x,\epsilon) \subset F(u_k)^c$ for every $k$ in the subsequence. If the subsequence contains infinitely many $u_k$ such that $u_k \equiv 0$ in $B(x,\epsilon)$, then also $u_0 \equiv 0$ due to uniform convergence. Otherwise the sequence contains infinitely many $u_k$ for which $B(x,\epsilon)$ is contained in the positive phase. In this case the nondegeneracy implies that in $B(x,\epsilon/2)$ we have $u_k > C\epsilon^\alpha$, with $C$ independent of $k$. Again uniform convergence implies the same lower bound for $u_0$, which contradicts our choice $x \in F(u_0)$.

To show the fourth claim we notice that $F(u_0)$ has zero $n$-dimensional Lebesgue measure by the Lebesgue differentiation theorem and the positive density of the zero phase. Take an open set $V \supset F(u_0)$ with $m(V \cap G') < \epsilon$. For large $k$ we have $F(u_k) \cup F(u_0) \subset V \cap G'$, so $\|\chi_{\{u_k>0\}} - \chi_{\{u_0>0\}}\|_{L^1(G')} < \epsilon$.

Also the sequence is uniformly bounded in $H^{1,p}(G; |y|^\beta)$ by the Caccioppoli inequality. This implies by compactness [29, theorem 1.31] the weak convergence of $\nabla u_k$ in $L^p(G; |y|^\beta)$. To obtain strong convergence, use Lemma 3.5 below. □

It remains to show that weak convergence implies strong convergence.

LEMMA 3.5. *Any sequence of minimizers* $\{u_k\}_{k=0}^\infty$ *in* $\Omega \subset \mathbb{R}^{n+1}$ *with* $u_k \to u_0$ *uniformly and* $\nabla u_k \rightharpoonup \nabla u_0$ *weakly in* $L^2_{\text{loc}}(\Omega, |y|^\beta)$ *satisfies that* $\nabla u_k \to \nabla u_0$ *strongly in* $L^2_{\text{loc}}(\Omega, |y|^\beta)$.

PROOF. Let $\eta \in C_c^{0,1}(\Omega)$ be a nonnegative function. We claim that for every $\varepsilon > 0$ there exists $j_0$ so that

$$\int |y|^\beta \eta |\nabla u - \nabla u_j|^2 \le \varepsilon \quad \text{for } j \ge j_0.$$

First we isolate the main difficulty

$$\int |y|^\beta \eta |\nabla u_0 - \nabla u_j|^2 = \int |y|^\beta \eta (\nabla u_0 - \nabla u_j)\cdot \nabla u_0 - \int |y|^\beta \eta(\nabla u_0 - \nabla u_j)\cdot \nabla u_j.$$

By weak convergence,

$$\left| \int |y|^\beta \eta (\nabla u_0 - \nabla u_j)\cdot \nabla u_0 \right| \le \varepsilon/4$$

for $j$ big enough. Note that this is true even if the $u_j$ are not minimizers. The bound on the second term, however, needs the minimization property.

We observe that

$$\begin{aligned} &\int |y|^\beta \eta (\nabla u_0 - \nabla u_j)\cdot \nabla u_j \\ &= \underbrace{\int |y|^\beta (\nabla u_0 - \nabla u_j)\cdot \nabla(\eta u_j)}_{=:\mathrm{I}} - \underbrace{\int |y|^\beta u_j (\nabla u_0 - \nabla u_j)\cdot \nabla \eta}_{=:\mathrm{II}}. \end{aligned} \tag{3.5}$$

To estimate I in (3.5), let $\lambda_j$ be the measures corresponding to $u_j$ from Lemma 3.1. By (3.2) we get that

$$\int |y|^\beta (\nabla u_0 - \nabla u_j)\cdot \nabla(\eta u_j) = \int \eta u_j \, d\lambda_0 - \int \eta u_j \, d\lambda_j.$$

Since $\lambda_j$ is supported on $\{u_j = 0\}$, we have that

$$\int \eta u_j \, d\lambda_j = 0$$

for every $j$ (including $j = 0$ as $u_0$ is also a minimizer to $\mathcal{J}$, see corollary 3.4 in [7]).

To finish the estimate on I in (3.5), we observe that

$$\int \eta u_j \, d\lambda_0 = \int \eta(u_j - u_0) d\lambda_0 \le \sup_{\mathrm{supp}\ \eta} |u_j - u_0| \int \eta \, d\lambda_0.$$

By uniform convergence on compact subsets, $\sup_{\mathrm{supp}\ \eta} |u_j - u_0| \le \frac{\epsilon}{4\|\eta\|_{L^1(\lambda_0)}}$ for $j$ big enough.

We turn towards estimating II in (3.5):

$$\begin{aligned} |\mathrm{II}| &= \left| \int |y|^\beta u_j (\nabla u_0 - \nabla u_j)\cdot \nabla \eta \right| \\ &\le \left| \int |y|^\beta (\nabla u_0 - \nabla u_j)\cdot (u_0 \nabla \eta) \right| \\ &\quad + \sup_{\mathrm{supp}\ \eta} |u_j - u_0| \|\nabla u_0 - \nabla u_j\|_{L^2(\Omega, |y|^\beta)} \|\nabla \eta\|_{L^2(\Omega,|y|^\beta)}. \end{aligned} \tag{3.6}$$

The first term goes to zero by weak convergence of $\nabla u_j$ to $\nabla u_0$. The second term satisfies

$$\sup_{\operatorname{supp}\eta} |u_j - u_0| \|\nabla u_0 - \nabla u_j\|_{L^2(\operatorname{supp}\eta, |y|^\beta)} \|\nabla \eta\|_{L^2(\Omega, |y|^\beta)} \leq \varepsilon/4$$

for $j$ big enough, by uniform convergence and the uniform bound of the norm $\|\nabla u_j\|_{L^2(\operatorname{supp}\ \eta, |y|^\beta)}$ derived from the Caccioppoli inequality in Lemma 3.2 together with uniform convergence. □

Lemma 3.4 implies that minimizers converge to minimizers (which was observed in Corollary 3.4 in [7]), but also implies the stronger fact that the energy is continuous under this convergence:

COROLLARY 3.6. *Let $u_k$ be a sequence of minimizers in $\Omega \subset \mathbb{R}^{n+1}$ with $u_k \to u_0$ locally uniformly and $\sup_k \|u_k\|_{\mathbf{H}^\beta} < \infty$. Then $u_0$ is also a minimizer to $\mathcal{J}$ in $\Omega$ and for any $B \Subset \Omega$ we have $\mathcal{J}(u_k, B) \to \mathcal{J}(u_0, B)$ after passing to a subsequence.*

## 4 Monotonicity Formula and Some Immediate Consequences

From [1] we have the following monotonicity formula:

THEOREM 4.1 (Monotonicity formula, see [1, theorem 4.3]). *Let $u$ be a minimizer in $B_\delta(x_0)$ for the functional $\mathcal{J}$ with $x_0 \in F(u)$. Then the function*

$$r \mapsto \Psi_r^u(x) := \Psi(r) = \frac{\mathcal{J}(u, B_r(x_0))}{r^n} - \frac{\alpha}{r^{n+1}} \int_{\partial B_r(x_0)} |y|^\beta u^2 \, d\mathcal{H}^n$$

*is defined and nondecreasing in $(0, \delta)$, and for $0 < \rho < \sigma < \delta$, it satisfies*

$$\Psi(\sigma) - \Psi(\rho) = \int_{B_\sigma(x_0) \setminus B_\rho(x_0)} |y|^\beta \, \frac{2|\alpha u(x) - (x - x_0) \cdot \nabla u(x)|^2}{|x_0 - x|^{n+2}} \, dx \geq 0.$$

As a consequence, the blowup limits are cones, in the sense of the following corollary.

COROLLARY 4.2. *Let $u$ be a minimizer in $B_\delta(x_0)$ with $x_0 = (x_0', 0)$. Consider a decreasing sequence $0 < \rho_k \xrightarrow{k \to \infty} 0$ and the associated rescalings $u_k(x) := \frac{u(x_0 + \rho_k x)}{r^\alpha}$. Then the* Allen-Weiss density

$$\Psi_0^u(x_0) := \lim_{r \searrow 0} \Psi_r^u(x_0)$$

*is well-defined. Furthermore, for every bounded open set $D \subset \mathbb{R}^{n+1}$ and $k \geq k(D)$ this subsequence $u_k$ is bounded in $H^{1,2}(D; |y|^\beta)$ and, passing to a subsequence $u_{k_j}$, converges* (*in the sense of Lemma* 3.4) *to $u_0$, which is a globally defined minimizer of $\mathcal{J}$ that is homogeneous of degree $\alpha$.*

The proof is the same as in [38, theorem 2.8].

*Remark* 4.3 (Nonuniqueness of blowups). We call the function $u_0$ appearing in Corollary 4.2 a *blowup* of $u$ at $x_0$. A priori, the function $u_0$ may depend on the subsequence $u_{k_j}$. However, a simple scaling argument shows that for all radii $r \geq 0$ and all blowups $u_0$ to $u$ at $x_0$ we have

$$\Psi_r^{u_0}(0) \equiv \Psi_0^u(x_0).$$

## 4.1 Dimension Reduction

We use the homogeneity of the blowups to obtain dimension estimates on the points in the free boundary for which there exists a nonflat blowup. This process is known as "dimension reduction" and has been applied to a variety of situations (see [38] for its application to the Bernoulli problem).

The first lemma shows that blowup limits have additional symmetry:

LEMMA 4.4. *Let $u \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ be an $\alpha$-homogeneous minimizer of $\mathcal{J}$ and let $x_0 \in F(u) \setminus \{0\}$. Then any blowup limit $u_0$ at $x_0$ is invariant in the direction of $x_0$; i.e., for every $x \in \mathbb{R}^{n+1}$ and every $\lambda \in \mathbb{R}$,*

$$u_0(x + \lambda x_0) = u_0(x).$$

PROOF. Let $x \in \mathbb{R}^{n+1}$, and consider its decomposition $x = \widetilde{x} + \lambda x_0$ with $\widetilde{x} \in \langle x_0 \rangle^{\perp}$. We only need to check that

$$u_0(x) = u_0(\widetilde{x}). \tag{4.1}$$

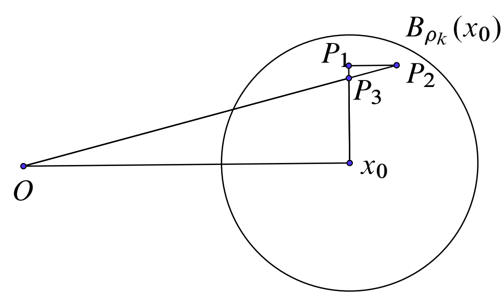


FIGURE 4.1. The distance $\mathrm{dist}(P_1, P_3) = \mathcal{O}(\rho_k^2)$.

Consider a ball $B = B(0, r) \subset \mathbb{R}^{n+1}$ so that $\widetilde{x}, x \in B$. Let $\{\rho_k\}$ be a sequence of radii converging to 0 and such that $u_k(x) := \frac{u(x_0 + \rho_k x)}{\rho_k^{\alpha}}$ converges to $u_0$ uniformly on $B_r$. For $k$ big enough, $\|u_k - u_0\|_{L^\infty(B_r)} < \varepsilon$. Then,

$$|u_0(x) - u_0(\widetilde{x})| \leq 2\varepsilon + |u_k(x) - u_k(\widetilde{x})|. \tag{4.2}$$

To control the last term above, we use the homogeneity of $u$. Writing $P_1 := x_0 + \rho_k \widetilde{x}$ and $P_2 := x_0 + \rho_k x$, we have $\rho_k^{\alpha} u_k(\widetilde{x}) = u(P_1)$ and $\rho_k^{\alpha} u_k(x) = u(P_2)$.

Let $P_3$ be the intersection between the line through $P_1$ and $x_0$ and the line through the origin and $P_2$ (see Figure 4.1). By homogeneity of $u$

$$u(P_2) = u(P_3)\left(\frac{|P_2|}{|P_3|}\right)^{\alpha} = u(P_3)\left(1 \pm \frac{|P_2 - P_3|}{|P_3|}\right)^{\alpha}.$$

Thus,

$$\begin{aligned}\rho_k^{\alpha}|u_k(x) - u_k(\widetilde{x})| &\le \left|u(P_1) - u(P_3)(1 + \mathcal{O}(\rho_k))^{\alpha}\right| \\ &\le |u(P_1) - u(P_3)| + |u(P_3)|\mathcal{O}(\rho_k).\end{aligned}$$

By Thales' theorem, $|P_1 - P_3| = \frac{|P_1 - P_2||P_3 - x_0|}{|x_0|} = \mathcal{O}(\rho_k^2)$, and using the $\dot{C}^{\alpha}$ character of $u$ and the fact that $u(x_0) = 0$, we get

$$\rho_k^{\alpha}|u_k(x) - u_k(\widetilde{x})| \le \|u\|_{\dot{C}^{\alpha}}\big(|P_1 - P_3|^{\alpha} + |P_3|^{\alpha}\mathcal{O}(\rho_k)\big) = \mathcal{O}(\rho_k^{2\alpha}) + \mathcal{O}(\rho_k),$$

and (4.1) follows by (4.2) since $\rho_k \to 0$. □

We then recall that a minimizer with a translational symmetry is actually a minimizer without that symmetry in one dimension less. This is known as "cone splitting":

LEMMA 4.5. *Let $u \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ be an $\alpha$-homogeneous minimizer of $\mathcal{J}$ in $\mathbb{R}^{n+1}$ that is invariant in the direction $e_n$. Then $\widetilde{u}(x', y) := u(x', 0, y)$ is a minimizer of $\mathcal{J}$ in one dimension less.*

PROOF. The proof is a slight variation of [38, proof of lemma 3.2]. □

Next we provide a nonstandard proof of Proposition 2.2, to show that the trivial solution is a minimizer. We use (P5) in a sequence of conveniently chosen blowups and a dimension reduction argument. Note that the proposition could also be proven via a classical dimension reduction argument.

PROOF OF PROPOSITION 2.2. Consider a nonzero minimizer $u$ with nonempty free boundary (see [7, prop. 3.2] for its existence), choose a free boundary point $x_0 \in F(u)$ and consider $u_0$ to be a blowup weak limit at this point, which exists and is $\alpha$-homogeneous by Lemma 4.2. Then $u_0$ is also a global minimizer by (P5) and not null by the nondegeneracy condition.

Next we argue by induction: given $0 \le j \le n-2$ let $u_j$ be an $\alpha$-homogeneous global minimizer different from 0 such that it is invariant in a $j$-dimensional linear subspace $H_j \subset \mathbb{R}^n$, i.e., for every $v \in H_j$ and every $x' \in \mathbb{R}^n$,

$$u_j(x', y) = u(x' + v, y).$$

Consider a point $x_j \in F(u_j) \setminus (H_j \times \{0\})$ that exists as long as $j < n-1$ by the interior corkscrew condition and positive density, and let $u_{j+1}$ be a blowup limit at this point, which is again an $\alpha$-homogeneous global minimizer. We claim that $u_{j+1}$ is invariant in fact in the $(j+1)$-dimensional subspace $H_j + \langle x_j' \rangle$.

Indeed, $u_{j+1}$ is invariant in $\langle x_j' \rangle$ by Lemma 4.4. On the other hand, since $u_j$ is invariant in $H_j$, so are the functions in the blowup sequence, and thus, $u_{j+1}$ is invariant in $H_j$. Thus, for $v \in H_j$, $v_0 \in \langle x_j' \rangle$, and $x \in \mathbb{R}^{n+1}$ we get

$$u(x + v + x_j') = u(x + v) = u(x),$$

and the claim follows.

Thus, after $n - 1$ steps, we obtain $u_{n-1}$, which is an $\alpha$-homogeneous global minimizer invariant in an $(n-1)$-dimensional space $H_{n-1}$, with nonempty free boundary. Thus,

$$u_{n-1}(x', 0) = C_{n,\alpha}(x_n')_+^{\alpha},$$

where the constant is given by (P6). The proposition follows by Proposition A.1. □

## 4.2 Upper Semicontinuity

Next we show that Allen-Weiss energy at a fixed radius is continuous both with respect to the minimizer and with respect to the point:

LEMMA 4.6. *Let $u_j \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\Omega)$ be minimizers of $\mathcal{J}$ in $\Omega$ and $u_j \to u_0$ in the sense of Lemma* 3.4. *Then, for $x_j \to x_0$ and $r < \operatorname{dist}(x_0, \partial\Omega)$,*

$$\Psi_r^{u_j}(x_j) \xrightarrow{j \to \infty} \Psi_r^{u_0}(x_0).$$

PROOF. Let $\varepsilon > 0$. We want to check that for $j$ big enough,

$$\left| \Psi_r^{u_j}(x_j) - \Psi_r^{u_0}(x_0) \right| \le \varepsilon.$$

We will consider the three terms of the energy separately. For the first term,

$$\int_{B_r(x_j)} |y|^{\beta} |\nabla u_j|^2 - \int_{B_r(x_0)} |y|^{\beta} |\nabla u_0|^2 \le r^n \varepsilon/3$$

follows from the $L^2$ convergence of the gradients. Indeed, if $\delta_j := |x_j - x_0| \le \delta$ for $j$ big enough and $B_{r+\delta} \subset \Omega$, then

$$\begin{aligned}
&\int_{B_r(x_j)} |y|^{\beta} |\nabla u_j|^2 - \int_{B_r(x_0)} |y|^{\beta} |\nabla u_0|^2 \\
&\quad \le \int_{B_r(x_j)} |y|^{\beta} \left( |\nabla u_j|^2 - |\nabla u_0|^2 \right) + \int_{B_r(x_j) \Delta B_r(x_0)} |y|^{\beta} |\nabla u_0|^2 \\
&\quad \le \int_{B_{r+\delta}(x_j)} |y|^{\beta} \left( |\nabla u_j|^2 - |\nabla u_0|^2 \right) + \int_{(B_{r+\delta_j} \setminus B_{r-\delta_j})(x_0)} |y|^{\beta} |\nabla u_0|^2 \\
&\quad \le r^n \varepsilon/3.
\end{aligned}$$

For the measure, we estimate

$$\left| \int_{B_r(x_j)'} \chi_{\Omega_+(u_j)} dm - \int_{B_r(x_0)'} \chi_{\Omega_+(u_0)} dm \right| \le r^n \varepsilon/3$$

for $j$ big enough as a consequence of $\chi_{\Omega_+(u_j)} \to \chi_{\Omega_+(u_0)}$ in $L^1_{\mathrm{loc}}$ as before. The fact that

$$\alpha \left| \int_{\partial B_r(x_j)} u_j^2 - \int_{\partial B_r(x_0)} u_0^2 \right| \le r^{n+1} \varepsilon/3$$

for $j$ big enough is a straight consequence of the uniform convergence and the continuity of $u_0$. □

It is well-known that the limit of a decreasing sequence of continuous functions is upper semicontinuous (see [11, theorem 1.8]). The monotonicity formula also implies the following result.

LEMMA 4.7. *Let $u_j \in \mathbf{H}^\beta_{\mathrm{loc}}(\Omega)$ be minimizers of $\mathcal{J}$ in $\Omega$ and $u_j \xrightarrow{j\to\infty} u_0$ in the sense of Lemma 3.4, with $x_j \in F(u_j)$ for $j \in \mathbb{N}$. Then, if $x_j \to x_0$ and $r_j \to 0$,*

$$\limsup_j \Psi_0^{u_j}(x_j) \le \limsup_j \Psi_{r_j}^{u_j}(x_j) \le \Psi_0^u(x_0).$$

PROOF. The first inequality comes from monotonicity.

To see that

$$\limsup_j \Psi_{r_j}^{u_j}(x_j) \le \Psi_0^{u_0}(x_0),$$

it is enough to check that for every $r > 0$

$$\limsup_j \Psi_{r_j}^{u_j}(x_j) \le \Psi_r^{u_0}(x_0),$$

or by using monotonicity, it suffices to show that for every $\varepsilon > 0$ and $j$ big enough,

$$\Psi_r^{u_j}(x_j) - \Psi_r^{u_0}(x_0) \le \varepsilon.$$

But this is true for $j$ big enough because the left-hand side converges to 0 by the continuity of the energy from Lemma 4.6. □

# 5 Measure-Theoretic Properties

## 5.1 Finite Perimeter

We will show that $\Omega'_+(u)$ is a set of locally finite perimeter. Then $F_{\mathrm{red}}(u)$ will coincide with the measure-theoretic reduced boundary by the $\epsilon$-regularity theorem, see [2, secs. 4.6 and 4.7].

DEFINITION 5.1. For every $0 < \alpha < 1$ we can define $k^*_\alpha$ as the infimum of

$$\left\{k \in \mathbb{N} : \exists \text{ an } \alpha\text{-homogeneous minimizer } u \in \mathbf{H}^\beta_{\mathrm{loc}}(\mathbb{R}^{k+1}) \text{ s.t. } \Sigma(u) = \{0\}\right\}.$$

Note that, to the best of our knowledge, there is no result showing that $k^*_\alpha$ needs to be finite.

LEMMA 5.2. *Let $u$ be an $\alpha$-homogeneous minimizer of $\mathcal{J}$ in $\mathbb{R}^{n+1}$ with $n < k^*_\alpha$. Then $u$ is a rotation of the trivial solution.*

See [38, sec. 3] for the proof.

From the positive density properties, we know that $k_\alpha^* \geq 2$. From the homogeneity of the blowups we find out that the free boundary in $\mathbb{R}^{1+1}$ is in fact a collection of isolated points. Later in Theorem 6.1 we will show that in fact $k_\alpha^* \geq 3$.

LEMMA 5.3 (Isolated singularities). *Let $u \in \mathbf{H}^\beta_{\mathrm{loc}}(\Omega)$ for $\Omega \subset \mathbb{R}^{1+1}$ be a minimizer of $\mathcal{J}$ in $\Omega$. Then $F(u)$ has no accumulation points in $\Omega$.*

PROOF. Arguing by contradiction, we assume that $F(u)$ has an interior accumulation point which, without loss of generality, we assume to be the origin.

Let $(x_k, 0)$ be a sequence of singular points converging to 0 with $x_k > 0$. Consider the blowup rescaling $u_k(x) := \frac{u(x_k x)}{x_k^\alpha}$. Note that $u_k(0,0) = u_k(1,0) = 0$. Moreover, by the interior corkscrew condition, there exist $z_k \in (1/2, 3/2)$ such that $u_k|_{B'_c(z_k,0)} > 0$, so $u(z_k, 0) \gtrsim C$ by the nondegeneracy condition.

Choosing a subsequence, we may assume that $z_k \to z_0 \geq 1/2$, and $u_k \to u_0$ in the sense of Lemma 3.4. In particular, $u_0$ is homogeneous by Corollary 4.2, reaching a contradiction with the fact that $u_0(1,0) = 0$ and $u_0(z_0, 0) \gtrsim C$. □

We will prove the local finiteness of the perimeter of the free boundary adapting a proof of De Silva and Savin in [18]. Our proof is essentially the same, but we repeat it for the sake of completeness.

As in [18] we say that a set $A \subset \mathbb{R}^n$ satisfies the property $(\mathrm{P}_t)$ if the following holds: for every $x \in A$ there exists an $r_x > 0$ such that for every $0 < r < r_x$, every subset $S$ of $B(x,r) \cap A$ can be covered with a finite number of balls $B(x_i, r_i)$ with $x_i \in S$ such that

$$\sum_i r_i \leq r^t/2. \tag{5.1}$$

LEMMA 5.4. *If $\mathcal{H}^t(\Sigma(U)) = 0$ for every minimal cone $U$ in $\mathbb{R}^{n+1}$, $\mathcal{H}^t(\Sigma(u)) = 0$ for every minimizer $u$ of $\mathcal{J}$ defined on $\Omega \subset \mathbb{R}^{n+1}$.*

PROOF. We first show that $\Sigma(u)$ satisfies the property $(P_t)$. If $(P_t)$ does not hold, we find a point $y \in \Sigma(u)$ for $(P_t)$ is violated for a sequence $r_k \to 0$. We consider the blowup sequence

$$u_{r_k}(x) = r_k^{-\alpha} u(y + r_k x). \tag{5.2}$$

By Corollary 3.6 we may assume, by taking a subsequence, that $u_{r_k}$ converges to a minimal cone $U$. By our assumptions we may cover $\Sigma(U) \cap B(0,1)$ with a finite collection of balls $\{B(x_i, \frac{\rho_i}{10})\}_{i=1}^k$ with

$$\sum_i \rho_i^t \leq \frac{1}{2}.$$

By Lemma 3.4 we know that free boundaries converge in the Hausdorff sense and thus the set $F(u_{r_k}) \cap B(0,1) \setminus \bigcup_i B(x_i, \rho_i/5)$ is flat for all large $k$. From

Theorem 2.4 we infer that all singularities must be covered by the same balls; that is, for all $k \geq k_0$,

$$\Sigma(u_{r_k}) \cap B(0,1) \subset \bigcup_i B(x_i, \rho_i/5). \tag{5.3}$$

After rescaling we see that $u$ satisfies the condition for property $(P_t)$ in the ball $B(y, r_k)$, which is a contradiction. Therefore the property $(P_t)$ holds as claimed.

Consider the set $D_k := \{y \in \Sigma(u) : r_y \geq 1/k\}$. Fix a point $y_0 \in D_k$. By property $(P_t)$ applied to $r_0 = 1/k$, we find a finite cover of $D_k \cap B(y_0, r_0)$ with balls $B(y_i, r_i)$, $y_i \in D_k$, satisfying

$$\sum_i r_i^t \leq r_0^t/2.$$

Similarly, for each ball $B(y_i, r_i)$ in the cover we use the property $(P_t)$ to find a finite number of balls $B(y_{ij}, r_{ij})$, $y_{ij} \in D_k$, which cover $D_k \cap B(y_i, r_i)$ and satisfy

$$\sum_j r_{ij}^t \leq r_i^t/2,$$

and thus $\sum_{i,j} r_{ij} \leq r_0^t/4$. By repeating the argument $N$ times we obtain a cover of $D_k \cap B(y_0, r_0)$ by balls $B(z_l, r_l)$ that satisfies

$$\sum_l r_l^t \leq 2^{-N} r_0^t.$$

This implies that $\mathcal{H}^t(B(y_0, r_0) \cap D_k) = 0$ and thus $\mathcal{H}^t(D_k) = 0$. By countable additivity we obtain the claim. □

LEMMA 5.5. *If $\mathcal{H}^t(\Sigma(U)) = 0$ for some $t > 0$ and for every minimal cone in $\mathbb{R}^{n+1}$, we then have that $\mathcal{H}^{t+1}(\Sigma(V)) = 0$ for every minimal cone $V$ in $\mathbb{R}^{(n+1)+1}$.*

PROOF. Without loss of generality we may assume $\Sigma(V) \neq \{0\}$. Let $x \in \Sigma(V) \setminus \{0\}$. By Corollary 3.6 the blowups at any point of $\Sigma(V) \cap \partial B$ converge to a minimal cone in dimension $(n+1)+1$ up to a subsequence. Let $V_x$ be a blowup at $x$. By Lemma 4.5, $V_x$ is a minimal cone that is invariant in at least one direction. By Lemma 4.5 , by using our assumption, this implies that $\mathcal{H}^{t+1}(\Sigma(V_x)) = 0$, and thus the singular set of every possible blowup cone of any minimizer $V$ has zero $\mathcal{H}^{t+1}$-measure.

Arguing as in Lemma 5.4 we obtain $\mathcal{H}^{t+1}(\Sigma(V)) = 0$. □

Combining Lemmas 5.3, 5.4, and 5.5 we obtain the following corollary. Notice that we will be able to replace $n-1$ by $n-2$ by Theorem 6.1.

COROLLARY 5.6. *Every minimizer satisfies*

$$\mathcal{H}^{n-1}(\Sigma(u)) = 0.$$

LEMMA 5.7. *Let $u \in \mathbf{H}^\beta(2B)$ be a minimizer of $\mathcal{J}$ in $2B$ with $\|u\|_{\dot{C}^\alpha(2B)} < E_0$. Then there exists a constant $C$ depending on $n$, $\alpha$, and $E_0$ and a finite collection of balls $\{B(X_i, r_i)\}$ s.t.*

$$\mathcal{H}^{n-1}\left((F(u) \cap B) \setminus \bigcap_{i=1}^{m} B(X_i, r_i)\right) \leq C \tag{5.4}$$

*and*

$$\sum_{i=1}^{m} r_i^{n-1} \leq \frac{1}{2}. \tag{5.5}$$

PROOF. The proof is by contradiction. For $k \in \mathbb{N}$ assume $\|u_k\|_{\dot{C}^\alpha(2B)} < E_0$ and the left-hand side of (5.4) is bounded below by $k > 0$ for every collection of balls satisfying (5.5). By Lemma 3.2 we know the sequence $u_k$ is bounded in $\mathbf{H}^\beta(B)$. Taking a subsequence we may assume that $u_k$ converges locally uniformly to a minimizer $u$ (see Corollary 3.6).

By Corollary 5.6 the set of singularities $\Sigma(u)$ has $\mathcal{H}^{n-1}$-measure zero, and thus they can be covered with finitely many balls $B_i$ satisfying (5.5).

Since $F(u) \setminus \Sigma(u)$ is a $C^{1,\gamma}$-surface by Theorem 2.4, using the Hausdorff convergence of the free boundaries we apply again Theorem 2.4 to see that $F(u_k) \cap B_1 \setminus \bigcup_{i=1}^{M} B_i$ are also $C^{1,\gamma}$-surfaces converging to $F(u) \cap B_1 \setminus \bigcup_{i=1}^{M} B_i$ uniformly in the $C^1$-norm. This is a contradiction with the assumption that the Hausdorff measure blows up as $k$ goes to $\infty$. □

The fact that the free boundary has finite perimeter follows now from the same iteration argument as [18, lemma 5.10].

LEMMA 5.8. *Let $u$ be as in Lemma 5.7. Then for some constant $C$ depending only on $E_0$,*

$$\mathcal{H}^{n-1}(F(u) \cap B) \leq C. \tag{5.6}$$

PROOF. By Lemma 5.7 we find a finite collection of balls $B_{r_i}$ such that

$$F(u) \cap B \subset \Gamma \cup \bigcup B_{r_i}, \tag{5.7}$$

with $\mathcal{H}^{n-1}(\Gamma) \leq C$ and $\sum r_i^{n-1} \leq \frac{1}{2}$.

Applying Lemma 5.7 again for each ball $B_{r_i}$, we have

$$F(u) \cap B_{r_i} \subset \Gamma_i \cup \bigcup B_{r_{ij}}, \tag{5.8}$$

with $\mathcal{H}^{n-1}(\Gamma_i) \leq C r_i^{n-1}$ and $\sum r_{ij}^{n-1} \leq \frac{1}{2} r_i^{n-1}$. Moreover, we have

$$\begin{aligned} \mathcal{H}^{n-1}\Big((F(u) \cap B_1) \cap \bigcup_{i,j} B_{r_{ij}}\Big) &\leq \mathcal{H}^{n-1}(\Gamma) + \sum \mathcal{H}^{n-1}(\Gamma_i) \\ &\leq C\Big(1 + \sum_{i,j} r_{ij}^{n-1}\Big) \leq C\left(1 + \frac{1}{2}\right). \end{aligned}$$

Continuing inductively, after $k$ steps we have that

$$F(u) \cap B_1 \subset \Gamma' \cup \bigcup_{q=1}^{N} B_{r_q}, \tag{5.9}$$

with

$$\mathcal{H}^{n-1}(\Gamma') \le C\left(\sum_{i=0}^{k} 2^{-i}\right) \le 2C,$$

and $\sum r_q^{n-1} \le 2^{-k}$. This gives the claim. □

Finally, the fact that $\{u > 0\} \cup \Omega$ has locally finite perimeter in $\Omega$ follows from the previous lemma and well-known results of Federer; see, for example, [3, prop. 3.62] or [27, 4.5.11].

### 5.2 Energy Gap

Next we will check that the Allen-Weiss density can also be used to identify singular points. First, let us state a useful identity for minimizers (which is also valid in the context of variational solutions in the sense of [37]).

LEMMA 5.9 (See [1, prop. 3.4]). *Let $u \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\Omega)$ be a minimizer to* (1.1) *in* $\Omega$. *For every $B \Subset \Omega$ we have*

$$\int_B |y|^\beta |\nabla u|^2 = \int_{\partial B} |y|^\beta u \nabla u \cdot \nu \, d\mathcal{H}^n. \tag{5.10}$$

Let $u$ be a minimizer and $x_0 \in F(u)$. If we consider a blowup $u_0$ at $x_0$, then

$$\begin{aligned}\Psi_0^u(x_0) &= \Psi_1^{u_0}(0)\\ &= \int_{B_1} |y|^\beta |\nabla u_0|^2 + m(\{u_0 > 0\} \cap \mathbb{R}^n \cap B_1) - \alpha \int_{\partial B_1} |y|^\beta u_0^2 \, d\mathcal{H}^n.\end{aligned}$$

By Lemma 5.9 we get

$$\begin{aligned}\Psi_1^{u_0}(0) = &\int_{\partial B_1(x_0)} |y|^\beta u_0 \nabla u_0 \cdot \nu \, d\mathcal{H}^n + m(\{u_0 > 0\} \cap \mathbb{R}^n \cap B_1)\\ &- \alpha \int_{\partial B_1} |y|^\beta u_0^2 \, d\mathcal{H}^n.\end{aligned}$$

Since $\nabla u_0(x) \cdot \nu(x) = \frac{\alpha}{|x|} u_0(x)$ almost everywhere on the sphere, the first and the third terms cancel out, and we obtain

$$\Psi_1^{u_0}(0) = m(\{u_0 > 0\} \cap B_1').$$

Thus, the density $\Psi_0^u$ at a free boundary point is given by the area of the positive phase of any blowup at the same point.

We write $\omega_n := m(B_1')$ for the volume of the $n$-dimensional ball.

PROPOSITION 5.10. *Every homogeneous minimizer $u \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\mathbb{R}^{n+1})$ has density*

$$\Psi^u_1(0) = m(\{u > 0\} \cap B'_1) \geq \frac{\omega_n}{2},$$

*and equality is only attained when $u$ is the trivial minimizer.*

PROOF. Let $u$ be a minimizer such that $\Psi^u_1(0) \leq \frac{\omega_n}{2}$.

Let $x_1 \in F_{\mathrm{red}}(u)$. Being a regular point, $\Psi^u_0(x_1) = \frac{\omega_n}{2}$. On the other hand, by the homogeneity and the continuity in Lemma 4.6,

$$\lim_{r\to\infty} \Psi^u_r(x_1) = \lim_{r\to\infty} \Psi^u_1(x_1/r) = \Psi^u_1(0) \leq \frac{\omega_n}{2}.$$

Combining both assertions with the monotonicity of $\Psi$ we get that $\Psi^u_r(x_1) \equiv \frac{\omega_n}{2}$. But using the second formula in Theorem 4.1, one can see that this is true only whenever $\Psi$ is $\alpha$-homogeneous with respect to $x_1$. Thus, $u$ is 1-symmetric and invariant in the direction of $\langle x_1\rangle$.

By Corollary 5.6 $F_{\mathrm{red}}(u)$ has full $\mathcal{H}^{n-1}$ measure on $F(u)$. Thus, we can find $x_1, \dots x_{n-1} \in F_{\mathrm{red}}(u)$ linearly independent. By the previous discussion $u$ is invariant on an $(n-1)$-dimensional affine manifold, and thus, it is the trivial solution. □

COROLLARY 5.11 (Energy gap). *There exists $\overline{\epsilon} > 0$ depending only on $n$ and $\alpha$ such that every minimizer $u \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\Omega)$ and every singular point $x_0 \in \Sigma(u)$ satisfy*

$$\Psi^u_1(x_0) - \frac{\omega_n}{2} \geq \overline{\epsilon}.$$

PROOF. Assume the conclusion to be false. Then there exist $u_j$ minimizers in $B_1$ with

$$\Psi^{u_j}_1(0) \leq \frac{\omega_n}{2} + 1/j.$$

Passing to a subsequence, $u_j \to u_0$ as in Lemma 3.4. Using Lemma 4.7 we get that

$$\Psi^{u_0}_1(0) = \lim_j \Psi^{u_j}_1(0) \leq \frac{\omega_n}{2}.$$

But then $u_0$ is the trivial cone by Proposition 5.10. Since $F(u_j) \to F(u)$ in the Hausdorff distance, using $\epsilon$-regularity (see Theorem 2.4) we get that $u_j$ is the trivial cone for $j$ big enough. □

The value $\overline{\epsilon}$ above depends on the constants and on $\|u\|_{\dot{C}^\alpha}$ in a neighborhood of $x_0$. In Section 7 we will show that $\overline{\epsilon}$ does not depend on $u$ at all.

## 6 Full Regularity in $\mathbb{R}^{2+1}$

In the case of $n = 2$, we prove full regularity of the free boundary for minimizers of our functional. Note that this result does not depend on the previous sections except that we use dimension reduction and blowups to deduce regularity of the free boundary.

THEOREM 6.1. *Let $n = 2$. Then there is no singular minimal cone. In particular, the free boundary $F(u)$ of every minimizer $u$ is $C^{1,\alpha}$ everywhere.*

PROOF. We follow closely the arguments in [18, theorem 5.5], building on [34]. The case $\beta = 0$ has been considered in [18]. The idea is to construct a competitor by a perturbation argument. We note at this point that the argument is two dimensional in nature and does not generalize to higher dimensions. Recall the functional under consideration:

$$\mathcal{J}(u, \Omega) = \int_\Omega |y|^\beta \, |\nabla u|^2 + m(\{u > 0\} \cap \mathbb{R}^n \cap \Omega).$$

Let $V$ be a nontrivial minimal cone. Define, as in [18], the Lipschitz continuous function

$$\psi_R(t) = \begin{cases} 1, & 0 \le t \le R, \\ 2 - \frac{\ln(t)}{\ln(R)}, & R \le t \le R^2, \\ 0, & t \ge R^2. \end{cases} \tag{6.1}$$

Define now the bi-Lipschitz change of coordinates

$$Z(x', y) = (x', y) + \psi_R(|(x', y)|) e_1$$

and set $V_R^+(Z) = V(x', y)$. Clearly, one has

$$D_{(x',y)} Z = \mathrm{Id} + A$$

where $\|A\| \le |\psi_R'(|(x', y)|)| << 1$. Defining now $V_R^-$ exactly as $V_R^+$ changing $\psi_R$ into $-\psi_R$, the very same computation as in [18] gives

$$\mathcal{J}(V_R^+, B_{R^2}) + \mathcal{J}(V_R^-, B_{R^2}) \le 2\mathcal{J}(V, B_{R^2}) + \int_{B_{R^2}} |y|^\beta \, |\nabla V|^2 \|A\|^2.$$

Now, we have

$$\int_{B_{R^2}} |y|^\beta |\nabla V|^2 \|A\|^2 = \int_R^{R^2} \int_{\partial B_r} |y|^\beta |\nabla V|^2 \|A\|^2 \, d\mathcal{H}^n \, dr.$$

Now since $V$ is homogeneous of degree $\alpha$ by assumption, the function $g(x, y) = |y|^\beta \, |\nabla V|^2$ is homogeneous of degree $\beta + 2\alpha - 2 = -1$. Therefore by a trivial change of variables on the sphere of radius $r$ and using the fact that $n = 2$, we get the very same estimate

$$\int_{B_{R^2}} |y|^\beta |\nabla U|^2 \|A\|^2 \le \frac{C}{\ln(R)} \xrightarrow{R \to \infty} 0.$$

The rest of the proof follows verbatim [18, p. 1318], since this is only based on energy considerations, and we refer the reader to it. □

# 7 Uniform Bounds Around the Free Boundary

The optimal regularity bound and the nondegeneracy described in Theorem 2.3 were obtained in [7] with bounds that depend on the seminorm $\|u\|_{\dot{\mathbf{H}}^\beta(B_1)}$. As a consequence, this dependence propagates to many of our estimates above. In this chapter we use the seminorm dependent estimates (e.g., Lemma 5.8) to prove seminorm *independent* nondegeneracy estimates. Re-running the arguments above yields the seminorm independent results presented in our main Theorem 1.1.

The question of seminorm independence may seem purely technical; however, independence allows the compactness arguments of the next section to work without additional assumptions on the minimizers involved.

## 7.1 Uniform Nondegeneracy

We will begin by showing uniform nondegeneracy from scratch to deduce uniform Hölder character from this fact, reversing the usual arguments in the literature.

The following lemma was shown in [1, cor. 4.2] in a more general setting. Here we give a more basic approach based on [2, lemma 3.4]. The main difference is that where Alt and Caffarelli could use the energy to directly control the $H^1$ norm of the minimizer, in our case we need to find an alternative because the measure term of the functional is computed on the thin phase (as opposed to the $H^1$ norm which is computed on the whole space). To bypass this difficulty we will use Allen's monotonicity formula.

The drawback of our approach is that we need the ball to be centered on the free boundary, while in the original lemma, Alt and Caffarelli could center the ball in the zero phase, allowing for a slightly better result.

LEMMA 7.1. *Let $u$ be a minimizer in $B_r$ with $0 \in F(u)$. Then $\sup_{\partial B_r} u \geq C r^\alpha$ with $C$ depending only on $n$ and $\alpha$.*

PROOF. By rescaling we can assume that $r = 1$.

Let $\mathcal{L}u := -\nabla \cdot (|y|^\beta \nabla u)$, consider $\Gamma(x) = \frac{1}{|x|^{n-2\alpha}}$, which is a solution of $\mathcal{L}\Gamma = 0$ away from the origin (or $\Gamma(x) = \log|x|$ if $n = 1$ and $\alpha = 1/2$), and let

$$\widetilde{v}(x) := \ell \frac{\max\{1 - \Gamma(2x), 0\}}{1 - \Gamma(2)} \quad \text{where } \ell := \sup_{\partial B_1} u.$$

It follows that $u \leq \widetilde{v}$ on $\partial B_1$ and thus

$$\mathcal{J}(u, B_1) \leq \mathcal{J}(\min\{u, \widetilde{v}\}, B_1),$$

and observing that $\widetilde{v} = 0$ on $B_{1/2}$ and $\widetilde{v} > 0$ on the annulus $A := B_1 \setminus B_{1/2}$, we get

$$\begin{aligned}
&\int_{B_{1/2}} |y|^\beta |\nabla u|^2 + m\big(B'_{1/2,+}(u)\big) \\
&\quad \le \int_A |y|^\beta (|\nabla(\min\{u,\widetilde{v}\})|^2 - |\nabla u|^2) + m(A'_+(\min\{u,\widetilde{v}\})) - m(A'_+(u)) \\
&\quad \le -2\int_A |y|^\beta \nabla \max\{u-\widetilde{v},0\}\cdot\nabla\widetilde{v}.
\end{aligned}$$

By Green's theorem, writing $d\sigma = |y|^\beta \, d\mathcal{H}^n$ we get

$$\begin{aligned}
\int_{B_{1/2}} |y|^\beta |\nabla u|^2 + m\Big(B'_{1/2,+}(u)\Big) &\le -2\int_{\partial B_{1/2}} u\partial_\nu \widetilde{v}\, d\sigma \\
&= C_{n,\alpha}\ell \int_{\partial B_{1/2}} u\, d\sigma,
\end{aligned} \tag{7.1}$$

with $C_{n,\alpha} > 0$.

Using the monotonicity formula and Proposition 5.10, we get that $\psi^u(r) \ge \psi^u(0) \ge \frac{\omega(B_1)}{2}$, and therefore

$$\frac{\alpha}{r}\int_{\partial B_r} u^2\, d\sigma + \frac{\omega(B_1) r^d}{2} \le \mathcal{J}_r(u), \tag{7.2}$$

so using Hölder's inequality and the AM-GM inequality we obtain

$$\begin{aligned}
\int_{\partial B_{1/2}} u\, d\sigma \le \left(\int_{\partial B_{1/2}} u^2\, d\sigma\right)^{\frac12} C_{n,\alpha}^{\frac12} &\le \frac12 \int_{\partial B_{1/2}} u^2\, d\sigma + \frac12 C_{n,\alpha} \\
&\le C_{n,\alpha}\mathcal{J}_{1/2}(u).
\end{aligned} \tag{7.3}$$

Combining (7.1), (7.2), and (7.3) we obtain

$$0 < \mathcal{J}_{1/2}(u) \le C_{n,\alpha}\ell\mathcal{J}_{1/2}(u),$$

and therefore $\ell \ge C_{n,\alpha}^{-1}$.

□

To show averaged nondegeneracy we need a mean value principle that is well-known, but we include its proof for the sake of completeness.

LEMMA 7.2 (Mean value principle). *Let $u \in H^1(\beta,\Omega)$ be a weak solution to $\mathcal{L}u := \nabla\cdot(|y|^\beta \nabla u) = 0$ in $\Omega$, and let $x_0 \in \mathbb{R}^n \times \{0\}$ with $B_r(x_0) \subset \Omega$. Then*

$$u(x_0) = \fint_{B_r} u\, d\omega$$

*where the mean is taken with respect to the measure $d\omega := |y|^\beta \, dx$.*

PROOF. Changing variables, we have that

$$A(\rho) := \frac{1}{\rho^{\beta+n+1}} \int_{B_\rho(x_0)} |y|^\beta u(x)dx = \int_{B_1} |y|^\beta u(\rho x + x_0)dx.$$

On the other hand, set

$$\begin{aligned}\widetilde{A}(\rho) &:= \int_{B_1} |y|^\beta \nabla u(\rho x + x_0) \cdot x\, dx \\ &= \int_{B_\rho(x_0)} \left(\frac{|y|}{\rho}\right)^\beta \frac{\nabla u(x) \cdot (x - x_0)}{\rho} \frac{dx}{\rho^{n+1}} \\ &= \frac{1}{2\rho^{\beta+n+2}} \int_{B_\rho(x_0)} |y|^\beta \nabla u(x) \cdot \nabla |x - x_0|^2\, dx.\end{aligned}$$

Since $u$ is a weak solution to $\nabla\cdot(|y|^\beta \nabla u) = 0$ in $\Omega$, we can apply Green's formula twice to obtain

$$\begin{aligned}\widetilde{A}(\rho) &= \frac{1}{2\rho^{\beta+n+2}} \int_{\partial B_\rho(x_0)} |x - x_0|^2 |y|^\beta \nabla u(x) \cdot \nu\, dx \\ &= \frac{1}{2\rho^{\beta+n}} \int_{\partial B_\rho(x_0)} |y|^\beta \nabla u(x) \cdot \nu\, dx = 0.\end{aligned}$$

Because $u$ is absolutely continuous on lines (see [24, theorem 4.21]), for almost every $x$ we have $\int_\rho^r \nabla u(tx + x_0) \cdot x\, dt = u(rx + x_0) - u(\rho x + x_0)$. Applying Fubini's theorem we get

$$\begin{aligned}\int_\rho^r \widetilde{A}(t)dt &= \int_{B_1} |y|^\beta \int_\rho^r \nabla u(tx + x_0) \cdot x\, dt\, dx \\ &= \int_{B_1} |y|^\beta (u(rx + x_0) - u(\rho x + x_0))dx = A(r) - A(\rho).\end{aligned}$$

So $A(r) - A(\rho) = 0$ for all $\rho < r$.

On the other hand, taking the mean with respect to the measure $d\omega := |y|^\beta\, dx$ and using the continuity of $u$ (see [26, theorem 2.3.12]) we obtain

$$\begin{aligned}\left| u(x_0) - \frac{1}{\omega(B_1)} \lim_{\rho\to 0} A(\rho) \right| &= \lim_{\rho\to 0} \frac{1}{\omega(B_\rho(x_0))} \left| \int_{B_\rho(x_0)} (u(x_0) - u(x))d\omega(x) \right| \\ &\le \lim_{\rho\to 0} o_{\rho\to 0}(1) = 0.\end{aligned}$$ □

COROLLARY 7.3. *Let $u$ be a minimizer in $B_r$ with $0 \in F(u)$ and let $d\sigma = |y|^\beta\, d\mathcal{H}^n$. Then $⨍_{\partial B_r} u\, d\sigma \ge C r^\alpha$ with $C$ depending only on $n$ and $\alpha$.*

PROOF. Let $v$ be the $\mathcal{L}$-harmonic replacement of $u$ in $B_r$, that is, the solution to

$$\begin{cases} \mathcal{L}v = 0 & \text{in } B_r, \\ v \equiv u & \text{on } \partial B_r; \end{cases} \tag{7.4}$$

see [29, theorem 3.17]. After differentiating with respect to the radius, by the mean value principle we get that $v(0) = \fint_{\partial B_r} u\, d\sigma$. By the comparison principle and the Harnack inequality we get that

$$Cr^\alpha \le \sup_{B_{r/2}} u \le \sup_{B_{r/2}} v \le C \fint_{\partial B_r} u\, d\sigma. \tag{7.5}$$ □

### 7.2 Behavior of the Distributional Fractional Laplacian

Next we use an idea of [2] and investigate the behavior of the distributional $\alpha$-Laplacian of the minimizer introduced in Section 3. As mentioned in the introduction, in [2] this investigation immediately yields that the positivity set is a set of locally finite perimeter, and more precisely, that it is Ahlfors regular of the correct dimension. However, the nonlocal nature of this problem indicates that the distributional fractional Laplacian may not be supported on the free boundary, and thus we cannot expect to immediately gain such strong geometric information.

First we can bound the growth of the fractional Laplacian measure around a free boundary point. Note that this growth is the natural counterpart to the upper Ahlfors regularity in the case of Alt-Caffarelli minimizers.

THEOREM 7.4. *Let $u \in \mathbf{H}^\beta(B_{2r}(x_0))$ be a minimizer of $\mathcal{J}$ in $B_{2r}(x_0)$, and let $x_0 \in F(u)$. Then, we have*

$$\lambda(B_r(x_0)) \le C r^{n-\alpha}.$$

*In particular, $\lambda(F(u)) = 0$.*

A glance at (2.1) will convince the reader that these estimates are sharp, for they cannot be improved even in the case of the trivial solution.

PROOF. Without loss of generality we may assume that $x_0 = 0$. Let $\mathcal{L}u := -\nabla \cdot (|y|^\beta \nabla u)$ and let $v$ be the $\mathcal{L}$-harmonic replacement of $u$ in $B_{2r}$; see (7.4). Write $d\sigma = |y|^\beta\, d\mathcal{H}^n$ and $M := \fint_{\partial B_{2r}} u\, d\sigma$. By Harnack's inequality (see [7], for instance) and the mean value principle in Lemma 7.2,

$$\inf_{B_r} v \ge C v(0) = CM.$$

We have that

$$\lambda(B_r) = \int_{B_r} d\lambda \le \frac{1}{CM} \int_{B_r} v\, d\lambda.$$

Since $u \equiv 0$ in the support of $\lambda$ and $u$ is $\mathcal{L}$-subharmonic (see [2, lemma 2.2]) we get

$$\int_{B_r} v\, d\lambda = \int_{B_r} (v-u) d\lambda \le \int_{B_{2r}} (v-u) d\lambda.$$

By the properties of the measure $\lambda$, we obtain

$$\int_{B_{2r}} (v-u) d\lambda = -\int_{B_{2r}} |y|^\beta \nabla(v-u) \cdot \nabla u = \int_{B_{2r}} |y|^\beta (|\nabla u|^2 - |\nabla v|^2),$$

and using the definition of the functional and the fact that $u$ is a minimizer, we get

$$\int_{B_{2r}} |y|^\beta (|\nabla u|^2 - |\nabla v|^2) = \mathcal{J}(u, B_{2r}) - m\big(B_{2r}^+(u)\big) - \mathcal{J}(v, B_{2r}) + m(B'_{2r}) \le C r^n.$$

Altogether, we have that

$$\lambda(B_r) \le \frac{1}{CM} C r^n,$$

and since uniform nondegeneracy (see Corollary 7.3) implies that $M \ge C r^\alpha$, we can conclude the proof of the first statement.

To show the second one, note that since the free boundary has locally finite $(n-1)$-dimensional Hausdorff measure, given a set $E \subset F(u)$ and $k \in \mathbb{N}$ we can find a collection of balls $I_k = \{B_i^k\}_i$ such that

$$E \subset \bigcup_{B \in I_k} B, \quad \sup_{B \in I_k} r(B) \le 1/k, \quad \text{and} \quad \sum_{B \in I_k} r(B)^{n-1} \le 2\mathcal{H}^{n-1}(E).$$

Thus,

$$\lambda(E) \le \sum_{B \in I_k} \lambda(B) \lesssim \sum_{B \in I_k} r(B)^{n-\alpha} \le \sup_{B \in I_k} r(B)^{1-\alpha} \sum_{B \in I_k} r(B)^{n-1} \xrightarrow{k \to \infty} 0.$$

□

Next we study the measure away from the free boundary. We should emphasize here that even though the estimates in Lemma 7.5 and Theorem 7.6 depend on $E_0$, they will be used to remove the dependence of our other estimates on $E_0$. More precisely, Theorem 7.6 will play a role in establishing the continuity of the Green function in Lemma 7.9. This qualitative fact is used to prove the quantitative uniform Hölder character in Theorem 7.8.

After proving Theorem 7.8, we may drop the hypothesis $\|u\|_{\mathbf{H}^\beta(B_2)} \le E_0$ from both Lemma 7.5 and Theorem 7.6.

LEMMA 7.5. *If $u \in \mathbf{H}^\beta_{\mathrm{loc}}(B_2)$ is a minimizer of $\mathcal{J}$ in the ball $B_2$ with $\|u\|_{\mathbf{H}^\beta(B_2)} \le E_0$ and $0 \in F(u)$, then for every $x_0 = (x', 0) \in B_{1,0}(u)$ we get*

$$\lim_{y \to 0} |y|^\beta \big|u_y(x', y)\big| \approx C \operatorname{dist}(x_0, F(u))^{-\alpha}.$$

*Moreover, for every ball $B$ centered at $\mathbb{R}^n \times \{0\}$ with $B' \Subset B_{1,0}(u)$, we have that*

$$|y|^\beta \big|u_y(x', y)\big| \le C \operatorname{dist}(x_0, F(u))^{-\alpha}$$

*for $|y| < C_B \operatorname{dist}(x, F(u))$, where the constant $C_B$ may depend on $B$.*

PROOF. Let $u$ be a minimizer, and let $B := B_r(x_0)$ with $B' \subset B_{1,0}(u)$. By [35, lemma 2.2], we can write $u(x', y) = |y|^{1-\beta} g(x') + \mathcal{O}(y^2)$, where $g$ is a $C^{1+\beta}(\frac{1}{2}B')$ function, with a uniform control on the error term in terms of $\|u\|_{L^2(B, |y|^\beta)}$. In particular, $\lim_{y \to 0} |y|^{\beta-1} u(x', y) = g(x')$.

Let us define

$$\widetilde{u}(x', y) := \begin{cases} u(x', y) & \text{if } y \geq 0, \\ -u(x', -y) & \text{if } y < 0. \end{cases} \tag{7.6}$$

It is clear that $\mathcal{L}\widetilde{u} \equiv 0$ in $B$. According to [36, lemma 3.26, cor. 3.29] $v(x', y) = |y|^\beta y^{-1}\widetilde{u}(x', y)$ is an even $C^\infty(\frac{1}{2}B)$ function in $\mathbf{H}^{2-\beta}(B)$ (note that $1 < 2-\beta < 3$ is out of the usual range of $\beta$) and satisfying $\nabla \cdot (|y|^{2-\beta}\nabla v) = 0$. The mean value principle (see Lemma 7.2) applies also to this case, so

$$g(x_0') = v(x_0) = \frac{1}{\int_{\frac{1}{2}B} |y|^{2-\beta}} \int_{\frac{1}{2}B} |y|^{2-\beta} v(x) = C \frac{1}{r^{2-\beta+n+1}} \int_{\frac{1}{2}B} |y| u(x),$$

and using P1–P3 if $r = \operatorname{dist}(x_0, F(u))$, we get

$$g(x_0') = v(x_0) \approx C r^{\beta-2+1+\alpha} = C r^{-\alpha}.$$

On the other hand, on the upper half-plane we have

$$u_y = (y^{1-\beta} v)_y = (1-\beta) y^{-\beta} v + y^{1-\beta} v_y,$$

so

$$y^\beta u_y(x', y) = (1-\beta) v(x', y) + y v_y(x', y)$$

and

$$\lim_{y \to 0^+} y^\beta u_y(x', y) = (1-\beta) g(x') \approx r^{-\alpha},$$

the limit being uniform on compact subsets of $B$. □

THEOREM 7.6. *If $u \in \mathbf{H}^\beta_{\text{loc}}(B_2)$ is a minimizer of $\mathcal{J}$ in $B_2$ with $\|u\|_{\mathbf{H}^\beta(B_2)} \leq E_0$, then the measure $\lambda$ is absolutely continuous with respect to the Lebesgue measure, and for $m$-almost every $x \in B_1'(u)$ we have that*

$$\frac{d\lambda}{dm}(x) = 2 \lim_{y \to 0} |y|^\beta u_y(x', y) \approx \chi_{B_{1,0}(u)}(x) \operatorname{dist}(x, F(u))^{-\alpha},$$

*with constants depending on $n$, $\alpha$, and $E_0$.*

PROOF. By Theorem 7.4 we only need to show absolute continuity in $B_{1,0}(u) \cup B'_{1,+}(u)$. For $x = (x', 0) \in B'_{1,+}(u)$ by [9, lemma 4.2] we have that

$$\lim_{y \to 0} |y|^\beta u_y(x', y) = 0,$$

and, for $x \in B_{1,0}(u)$ we have seen in Lemma 7.5 that

$$\lim_{y \to 0} |y|^\beta u_y(x', y) \approx \operatorname{dist}(x, F(u))^{-\alpha},$$

showing the second part of the statement.

Consider a ball $B_r(x_0)$ with $x_0 \in \mathbb{R}^n \times \{0\}$ and a collection of even smooth functions $\chi_{B_r} \le \psi_k \le \chi_{B_{r+\frac{1}{k}}}$. Then

$$\lambda(B_r) \le -\int |y|^\beta \nabla u \cdot \nabla \psi_k \le \lambda(B_{r+(1/k)}), \tag{7.7}$$

and for every $\varepsilon > 0$ we use the Green's theorem to get

$$-\int |y|^\beta \nabla u \cdot \nabla \psi_k = -\int_{|y|\le\varepsilon} |y|^\beta \nabla u \cdot \nabla \psi_k - \int_{|y|=\varepsilon} |y|^\beta \psi_k \nabla u \cdot \nu \, dm.$$

Using the symmetry properties and taking limits,

$$-\int |y|^\beta \nabla u \cdot \nabla \psi_k = 2 \lim_{\varepsilon\to 0} \int \varepsilon^\beta \psi_k(x',\varepsilon) u_y(x',\varepsilon) dm(x'). \tag{7.8}$$

Next we want to apply the dominated convergence theorem. Let us begin by considering a ball $B_r(x_0) \subset B_1$ centered in the zero phase, with $\mathrm{dist}(B'_r(x_0), F(u)) \ge 2r$. In this case, by Lemma 7.5 we have

$$\varepsilon^\beta u_y(x',\varepsilon) \lesssim r^{-\alpha}, \tag{7.9}$$

with constants depending perhaps on $u$ and $B_r$ as well.

If instead $B'_r(x_0) \Subset B'_{1,+}(u)$, by [36, theorem 3.28] $u$ is an even $C^\infty$ function on $B'_r(x_0)$, so $|y|^\beta u_y = \mathcal{O}(|y|^{1+\beta})$. Thus

$$\varepsilon^\beta u_y(x',\varepsilon) \lesssim r^{2-2\alpha}. \tag{7.10}$$

In both cases, the dominated convergence theorem applies, and

$$\lim_{\varepsilon\to 0} \int_{B_{r+(1/k)}\cap\{y=\varepsilon\}} \varepsilon^\beta \psi_k u_y \, dm = \int_{B'_{r+(1/k)}} \psi_k \lim_{\varepsilon\to 0} (\varepsilon^\beta u_y(\cdot,\varepsilon)) dm,$$

and by (7.7) and (7.8), we obtain

$$\lambda(B_r) \le 2 \int_{B'_{r+(1/k)}} \psi_k \lim_{\varepsilon\to 0} (\varepsilon^\beta u_y(\cdot,\varepsilon)) dm \le \lambda(B_{r+(1/k)}).$$

In particular, $\lim_{\varepsilon\to 0}(\varepsilon^\beta u_y(\cdot,\varepsilon)) \in L^1_{\mathrm{loc}}(B_{1,0}(u) \cup B'_{1,+}(u))$, and taking limits in $k$ we get

$$\lambda(B_r) = 2 \int_{B'_r} \lim_{\varepsilon\to 0} (\varepsilon^\beta u_y(\cdot,\varepsilon)) dm. \qquad \square$$

A consequence of our control of the behavior of $\lambda$ is that we can establish the existence of exterior corkscrews. We should note that exterior corkscrews can also be obtained by a purely geometric argument given the nondegeneracy and positive density of Theorem 2.3 (see, e.g., the proof of proposition 10.3 in [13]).

COROLLARY 7.7. *If $u \in$ is a minimizer in $B_2$ with $\|u\|_{\mathbf{H}^\beta(B_2)} \le E_0$, then $B'_{1,+}(u)$ satisfies the exterior corkscrew condition, i.e. there exists a constant $C_1$ such that* *for every $x \in F(u)$ and every $0 < r < \operatorname{dist}(x, \partial B_1)$ one can find $x_0 \in B_r(x)$ so that*

$$B(x_0, C_1 r) \cap B'_{1,+}(u) = \varnothing.$$

PROOF. This is a consequence of Theorems 7.4 and 7.6, and the positive density condition for the zero phase. Indeed, given a ball $B_r \subset \mathbb{R}^{n+1}$, combining both theorems we get

$$\begin{aligned} r^{n-\alpha} &\gtrsim \lambda(B_{1,0}(u) \cap B_r) \geq C_{E_0} \int_{B_{1,0}(u)\cap B_r} \operatorname{dist}(x, \partial B_1)^{-\alpha} \\ &\geq C\Big( \sup_{B_{1,0}(u)\cap B_r} \operatorname{dist}(x, \partial B_1)\Big)^{-\alpha} |B_{1,0}(u) \cap B_r|, \end{aligned}$$

and the positive density condition implies that

$$|B_{1,0}(u) \cap B_r| \geq C_{E_0} r^n.$$

Thus,

$$\sup_{B_{1,0}(u)\cap B_r} \operatorname{dist}(x, \partial B_1) \geq C_{E_0} r,$$

which is equivalent to the exterior corkscrew condition. □

### 7.3 Uniform Hölder Character

The uniform nondegeneracy of Section 7.1 lets us conclude uniform control on the Hölder norm of $u$.

THEOREM 7.8. *Let $u$ be a minimizer of $\mathcal{J}$ in $B_r$ with $0 \in F(u)$. Then $|u(x)| \leq C|x|^\alpha$ for every $x \in \partial B_{r/2}$ with $C$ depending only on $n$ and $\alpha$.*

PROOF. Again we set $v$ to be the $\mathcal{L}$-harmonic replacement of $u$ inside of $B_r$ as in (7.4). Let $\widetilde{u} := v - u$ so that

$$\mathcal{L}\widetilde{u} = \mathcal{L}v - \mathcal{L}u = -\lambda = -\nabla \cdot (|y|^\beta \nabla u)$$

and $\widetilde{u} \in H_0^{1,2}(B_r; |y|^\beta)$.

Consider the Green function $G : B_r \times B_r \to \mathbb{R}$ such that $\mathcal{L}G(\cdot, z) = \delta_z$ and $G(\cdot, z) \in H^{1,2}_{\text{loc}}(\overline{B_r} \setminus \{z\})$ with null trace on $\partial B_r$ (see [25, prop. 2.4]). By [25, prop. 2.1, lemma 2.7] there exists $p_0 > 1$ so that $\widetilde{u}$ is the unique function in $H_0^{1,p_0}(B_r; |y|^\beta)$ such that $\mathcal{L}\widetilde{u} = \lambda$, and moreover

$$\widetilde{u}(z) = \int_{B_r} G(z, x)\, d\lambda(x) \tag{7.11}$$

for almost every $z \in B_r$.

Below, in Lemma 7.9, we will see that the equality (7.11) is in fact valid for every $z \in B_{r/4}$, that is, $\widetilde{u} = \int_{B_r} G(\cdot, x)\, d\lambda(x)$. In particular,

$$v(0) = \widetilde{u}(0) = \int_{B_r} G(0, x)\, d\lambda(x).$$

Next we use the following estimate (see [25, theorem 3.3]): let $z, x \in B_{r/4}$. Then

$$G(z,x) \approx \int_{|x-z|}^{r} \frac{s\,ds}{w(B(x,s))},$$

where $w$ is the $A_2$ weight $w(x) = |y|^\beta$. Computing, for $x = (x', y)$ we obtain

$$w(B(x,s)) \approx s^n \int_{y-s}^{y+s} |t|^\beta \,dt \approx s^n \max\{|y|, s\}^{\beta+1}.$$

First we assume that $n - 2\alpha > 0$. Thus, if $x \in B'_{r/4}$, then

$$G(z,x) \approx \int_{|x-z|}^{r} s^{-n-\beta}\,ds \approx |x-z|^{-n-\beta+1} = |x-z|^{2\alpha-n}. \tag{7.12}$$

Note that $\lambda(B_r) \le C r^{n-\alpha}$ by Theorem 7.4. Thus, writing $A_{t,s} := B_s \setminus B_t$, we have that

$$\begin{aligned} v(0) &= \int_{B_r} G(0,x)d\lambda(x) \\ &\le \int_{cr^{2\alpha-n}}^{\infty} \lambda(\{x \in B_{r/4} : G(0,x) > t\})dt + \int_{A_{r/4,r}} G(0,x)d\lambda(x). \end{aligned}$$

By the strong maximum principle, the Green function in the annulus is bounded by $Cr^{n-2\alpha}$. This fact, together with Theorem 7.4, implies that

$$v(0) \le \int_{cr^{2\alpha-n}}^{\infty} \lambda\Big(B_{Ct^{\frac{-1}{(n-2\alpha)}}}\Big)dt + Cr^\alpha \le C\int_{cr^{2\alpha-n}}^{\infty} t^{-\frac{n-\alpha}{n-2\alpha}}\,dt + Cr^\alpha = Cr^\alpha.$$

By the mean value theorem we conclude that

$$⨍_{\partial B_r} v\,d\sigma \le Cr^\alpha,$$

where $d\sigma = |y|^\beta \,d\mathcal{H}^n$. The theorem follows by observing that, as in (7.5), the mean of $v$ dominates $u$ by $\sup_{\partial B_{r/2}} u \le \sup_{\partial B_{r/2}} v \le C ⨍_{\partial B_r} v\,d\sigma$.

In case $n - 2\alpha = 0$, which could only happen for $n = 1$ and $\alpha = 1/2$, estimate (7.12) reads as

$$G(z,x) \approx \log\Big(\frac{r}{|x-z|}\Big),$$

and the proof follows the same steps.

In case $n - 2\alpha < 0$, then estimate (7.12) reads as

$$G(z,x) \approx r^{n-2\alpha},$$

and the estimate is even better compared to the above. □

LEMMA 7.9. *$\int_{B_r} G(z,x)d\lambda(x)$ is continuous in $z \in B_{r/4}$.*

PROOF. Let $\varepsilon < r/2$ and let $z_1, z_2 \in B_{r/4}$, with $|z_1 - z_2| \leq \varepsilon/2$. Then

$$\begin{aligned}\int_{B_r} &|G(z_1,x) - G(z_2,x)| d\lambda(x) \\ &\leq \int_{B_r \setminus B_\varepsilon(z_1)} |G(z_1,x) - G(z_2,x)| d\lambda(x) \\ &\quad + \int_{B_\varepsilon(z_1)} G(z_1,x) d\lambda(x) + \int_{B_\varepsilon(z_1)} G(z_2,x) d\lambda(x).\end{aligned} \tag{7.13}$$

Next we use (7.12) and Theorems 7.4 and 7.6. By decomposing the domain on dyadic annuli, in case $n - 2\alpha > 0$ we get

$$\begin{aligned}\int_{B_\varepsilon(z_1)} &G(z_1,x) d\lambda(x) \\ &\leq \sum_{j \leq 0} \int_{A_{2^{j-1}\varepsilon, 2^j\varepsilon}(z_1)} G(z_1,x) d\lambda(x) \\ &\lesssim \sum_{j \leq 0} \lambda(B_{2^j\varepsilon}(z_1))(2^{j-1}\varepsilon)^{2\alpha - n} \lesssim \varepsilon^\alpha \sum_{j \leq 0} 2^{j\alpha}.\end{aligned} \tag{7.14}$$

In case $n - 2\alpha = 0$ we obtain

$$\varepsilon^\alpha \sum_{j \leq 0} 2^{j\alpha} \log\left(\frac{r}{2^j\varepsilon}\right)$$

on the right-hand side instead, and in case $n - 2\alpha < 0$ we obtain

$$\varepsilon^{n-\alpha} r^{2\alpha - n} \sum_{j \leq 0} 2^{j(n-\alpha)}.$$

In every case, by fixing $\varepsilon$ small enough, this term can be as small as wanted. The same will happen with the last term on the right-hand side of (7.13).

On the other hand, by [26, theorem 2.3.12] Green's function is uniformly continuous on the set $\{(z,x) \in B_r \times B_r : |z - x| > \varepsilon\}$, so

$$|G(z_1,x_1) - G(z_2,x_2)| \leq \delta_\varepsilon(|z_1 - z_2| + |x_1 - x_2|) \quad \text{with } \delta_\varepsilon(t) \xrightarrow{t \to 0} 0.$$

Thus,

$$\int_{B_r \setminus B_\varepsilon(z_1)} |G(z_1,x) - G(z_2,x)| d\lambda(x) \leq \delta_\varepsilon(|z_1 - z_2|)\lambda(B_r) \to 0.$$

Assuming that $|z_1 - z_2|$ is small enough, we obtain that

$$\int_{B_r} |G(z_1,x) - G(z_2,x)| d\lambda(x)$$

is as small as wanted and the claim follows. □

*Remark* 7.10. In light of Theorem 7.8 and the Caccioppoli inequality (see Section 3.1), arguing as in [7, theorem 1.1] we obtain that every minimizer $u$ in a ball $B_r$ with $0 \in F(u)$ has uniform $C^\alpha$ character in $B_{r/2}$ and the same for the $\mathbf{H}^\beta$ norm. Moreover, using [7, theorem 1.2] we can find interior corkscrew points with constants not depending on these norms. This allows us to remove the a priori dependence on $\|u\|_{\mathbf{H}^\beta}$ from all of our results above.

### 7.4 Lower Estimates for the Distributional Fractional Laplacian

Next we bound the growth of the measure around a free boundary point from below. None of these results will be used in the present paper, but we include them to give a complete picture of the tools under consideration.

THEOREM 7.11. *Let $u \in \mathbf{H}^\beta(B_{2r})$ be a minimizer of $\mathcal{J}$ in $B_{2r}$ such that $0 \in F(u)$. Then we have*

$$\lambda(B_r) \geq C r^{n-\alpha}.$$

PROOF. Let $\mathcal{L}u := -\nabla\cdot(|y|^\beta \nabla u)$ and let $v$ be the $\mathcal{L}$-harmonic replacement of $u$ in $B_r$ (see (7.4)). Let $\widetilde{u} := v - u$, and consider the Green function $G : B_r \times B_r \to \mathbb{R}$ as in the proof of Theorem 7.8.

Let $0 < \kappa < 1$ to be fixed later. By P1–P3 in Theorem 2.3, there exists a point $z_0 \in B_{\kappa r}$ with

$$u(z_0) \approx (\kappa r)^\alpha, \tag{7.15}$$

with constants depending only on $n$ and $\alpha$ by Remark 7.10. By *P1* there is a constant $c$ such that for every $z \in B(z_0, c\kappa r)$ we have that $u(z) \approx (\kappa r)^\alpha$. Since $\lambda$ is supported on the zero phase of $u$, the ball $B(z_0, c\kappa r)$ is away from its support, and

$$\widetilde{u}(z) = \int_{B_r \setminus B(z_0, c\kappa r)} G(z,x)\, d\lambda(x).$$

Using the strong maximum principle (see [29, theorem 6.5]) and (7.12), for almost every $z \in B(z_0, c\kappa r/2)$ we get

$$\begin{aligned} \widetilde{u}(z) &\leq \lambda(B_r) \sup_{x \in B_r \setminus B(z_0, c\kappa r)} G(z,x) = \lambda(B_r) \sup_{x \in B_{r/4} \setminus B(z_0, c\kappa r)} G(z,x) \\ &\approx \lambda(B_r) \sup_{x \notin B(z_0, c\kappa r)} |x - z|^{2\alpha - n} = \lambda(B_r)(c\kappa r)^{2\alpha - n}. \end{aligned}$$

That is,

$$\widetilde{u}(z) \lesssim \lambda(B_r)(c\kappa r)^{2\alpha - n}. \tag{7.16}$$

On the other hand, note that $u$ is continuous. By the Riesz representation theorem, there exists a probability measure $\omega_{\mathcal{L}}^z$ such that

$$v(z) = \int_{\partial B_r} u(x)\, d\omega_{\mathcal{L}}^z(x).$$

We can choose $r$ so that $\partial B_r$ intersects a big part of a corkscrew ball; i.e., assume that there exists a point $\xi_0 \in \partial B_r'$ that is the center of a ball $B'(\xi_0, cr)$ where $u$

has positive values. This can be done by the interior corkscrew condition, with all the constants involved depending only on $n$ and $\alpha$. Then, changing the constant if necessary, all points $\xi \in B(\xi_0, cr)$ satisfy that $u(\xi) \geq Cr^\alpha$ by the nondegeneracy condition and the optimal regularity. Call $U := \partial B_r \cap B(\xi_0, cr)$. Then

$$v(z) \gtrsim r^\alpha \omega_{\mathcal{L}}^z(U).$$

But $\omega_{\mathcal{L}}^z(U)$ is bounded below by a constant by [29, lemma 11.21] and the Harnack inequality (use a convenient Harnack chain). All in all, we have that

$$v(z) \gtrsim r^\alpha. \tag{7.17}$$

Combining (7.16), (7.15), and (7.17) and choosing $\kappa$ small enough, depending on $n$ and $\alpha$, we get

$$\lambda(B_r) \gtrsim \frac{\widetilde{u}(z_0)}{(c\kappa r)^{2\alpha-n}} \geq \frac{Cr^\alpha - C'(\kappa r)^\alpha}{(c\kappa r)^{2\alpha-n}} \geq C_{n,\alpha} r^{n-\alpha}$$

for $\kappa$ small enough.

In case $n - 2\alpha = 0$, that is, for $n = 1$ and $\alpha = 1/2$, using similar changes as in the proof of Theorem 7.8 we get

$$\widetilde{u}(z) \lesssim \lambda(B_r) \sup_{x \notin B(z_0, c\kappa r)} \log\left(\frac{r}{|x-z|}\right) \approx \lambda(B_r)|\log \kappa|$$

instead of (7.16). In case $n - 2\alpha < 0$, the proof is even easier than before. □

*Remark* 7.12. Theorem 7.11 implies that the $(n-\alpha)$-Hausdorff measure of the free boundary is locally finite. This does not suffice to show finite perimeter of the positive phase; therefore we had to use the approach in Section 5.

The following theorem summarizes the information that we have gathered so far about the measure $\lambda$.

THEOREM 7.13. *If $u \in \mathbf{H}^\beta_{\text{loc}}(\Omega)$ is a minimizer of $\mathcal{J}$ in $\Omega$, then the measure $\lambda$ is absolutely continuous with respect to the Lebesgue measure in $\Omega'(u)$. Moreover, given $x_0 \in F(u)$ and $r > 0$ such that $B_{2r}(x_0) \subset \Omega$, then*

$$\lambda(B_r(x_0)) \approx r^{n-\alpha}, \tag{7.18}$$

*and for almost every $x \in B'_r(x_0)$ we have that*

$$\frac{d\lambda}{dm}(x) = 2 \lim_{y \to 0} |y|^\beta u_y(x', y) \approx \chi_{\Omega_0(u)}(x) \operatorname{dist}(x, F(u))^{-\alpha},$$

*with constants depending only on n and $\alpha$.*

## 8 Rectifiability of the Singular Set

In this section we use the Rectifiable-Reifenberg and quantitative stratification framework of Naber-Valtorta [32] to prove Hausdorff measure and structure results for the singular set. Recall that $k^*_\alpha$ is the first dimension in which there exists nontrivial $\alpha$-homogeneous global minimizers to (1.1) defined in Section 5.

THEOREM 8.1. *Let* $u \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\Omega)$ *be a minimizer of* (1.1) *in a domain* $\Omega$. *Then* $\Sigma(u)$ *is* $(n-k^*_\alpha)$*-rectifiable, and for every* $D \Subset \Omega$ *we have*

$$\mathcal{H}^{n-k^*_\alpha}(\Sigma(u) \cap D) \leq C_{n,\alpha,\mathrm{dist}(D,\partial\Omega)}.$$

Part of the power of this framework is that it is very general. One needs certain compactness properties on the minimizers and a connection between the drop in the monotonicity formula and the local flatness of the singular set (see Theorem 8.14 below). To avoid redundancy and highlight the original contributions of this article, we omit many details here and try to focus on the estimates needed to apply this framework to minimizers of (1.1). Whenever we omit details, we will refer the interested reader to the relevant parts of [22].

The key first step is to introduce the appropriate formulation of quantitative stratification. First introduced by Cheeger and Naber [10] in the context of manifolds with Ricci curvature bounded from below, this is a way to quantify the intuitive fact that $F(u)$ should "look" $(n-k^*_\alpha)$-dimensional near a point $x_0 \in F(u)$ at which the blowups have $(n-k^*_\alpha)$-linearly independent translational symmetries.

### 8.1 Quantitative Stratification for Minimizers to $\mathcal{J}$

We have seen in Section 4.1 that homogeneous functions have linear spaces of translational symmetry. Here we want to quantify (both in terms of size and stability) how far a function is from having no more than $k$ directions of translational symmetry.

DEFINITION 8.2. We write $V^k$ for the collection of linear $k$-dimensional subspaces of $\mathbb{R}^n$. A function $u$ is said to be *$k$-symmetric* if it is $\alpha$-homogeneous with respect to some point, and there exists a $L \in V^k$ so that

$$u(x+v) = u(x) \quad \text{for every } v \in L.$$

A function $u$ is said to be $(k,\epsilon)$-symmetric in a ball $B$ if for some $k$-symmetric $\tilde{u}$ we have

$$r^{-2-n} \int_B |y|^\beta |u-\tilde{u}|^2 \, dy < \epsilon.$$

Next we define the $k$-stratum $S^k(u)$, the $(k,\epsilon)$-stratum $S^k_\epsilon(u)$, and the $(k,\epsilon,r)$-stratum $S^k_{\epsilon,r}(u)$. A key insight here is to define these strata by the blowups having $k$ or fewer symmetries as opposed to exactly $k$ symmetries.

DEFINITION 8.3. Let $0 \leq k \leq n$, $0 < \varepsilon < \infty$, and $0 < r < d_\Omega(x) := \mathrm{dist}(x, \partial\Omega)$, let $u$ be a continuous function in $\Omega$, and let $x \in F(u)$. We say that:

- $x \in S^k(u)$ if $u$ has no $(k+1)$-symmetric blowups at $x$;
- $x \in S^k_\epsilon(u)$ if $u$ is not $(k+1,\epsilon)$-symmetric in $B_s(x)$ for $0 < s \leq \min\{1, d_\Omega(x)\}$;
- $x \in S^k_{\epsilon,r}(u)$ if $u$ is not $(k+1,\epsilon)$-symmetric in $B_s(x)$ for $r \leq s \leq \min\{1, d_\Omega(x)\}$.

If it is clear from the context we will omit $u$ from the notation.

We now detail some standard properties of the strata defined above and how they interact with the free boundary $F(u)$. While the proofs are mostly standard, we give the details as the scaling associated to the problem (1.1) adds some technical difficulties. This proof also provides a blueprint for fleshing out the details in Sections 8.3 and 8.4.

LEMMA 8.4. *Let* $0 \leq j \leq k \leq n$, $0 < \varepsilon \leq \tau < \infty$, $0 < r \leq s < \operatorname{dist}(x, \partial\Omega)$, *and let* $u \in \mathbf{H}^{\beta}_{\mathrm{loc}}(\Omega)$ *be a minimizer in* $\Omega$. *Then*:

(1) $S^0 \subset S^1 \subset \cdots \subset S^{n-1} = S^n = F(u)$. *Moreover, for the reduced boundary, we have that* $F_{\mathrm{red}}(u) \subset S^{n-1} \setminus S^{n-2}$ *and* $\Sigma(u) \subset S^{n-k^*_\alpha}$.

(2) *We have* $S^j_\tau \subset S^k_\epsilon \subset S^k$, *and moreover,* $S^k = \bigcup_{\epsilon>0} S^k_\epsilon$.

(3) *Also* $S^j_\tau \subset S^j_{\tau,r} \subset S^k_{\epsilon,s}$ *and moreover,* $S^k_\epsilon = \bigcap_{r>0} S^k_{\epsilon,r}$.

(4) *The sets* $S^k_\epsilon$ *are closed, in both* $x$ *and* $u$: *if* $u_i \xrightarrow{L^2_{\mathrm{loc}}(\Omega;|y|^\beta)} u$ *and* $x_i \to x$ *with* $x_i \in S^k_\epsilon(u_i)$, *then* $x \in S^k_\epsilon(u)$.

(5) *If* $u_i \xrightarrow{L^2_{\mathrm{loc}}(\Omega;|y|^\beta)} u$, $\epsilon_i \to 0$, *and* $u_i$ *are* $(k, \epsilon_i)$*-symmetric in* $B_1$, *then* $u$ *is* $k$*-symmetric in* $B_1$.

PROOF.

(1) The inclusions $S^k \subset S^{k+1}$ of the first property are trivial. The last equalities are consequences of the nondegeneracy. The fact that $F_{\mathrm{red}}(u) \cap S^{n-2} = \varnothing$ can be deduced from the Hausdorff convergence of the free boundaries described in Lemma 3.4 and Theorem 2.4. Finally, $\Sigma(u) \subset S^{n-k^*_\alpha}$ is a consequence of Lemmas 4.5 and 5.2.

(2) The inclusions $S^j_\tau \subset S^k_\epsilon$ of the second property come from the definitions: if $x \notin S^k_\epsilon$ then there exist a ball $B \subset \Omega$ centered at $x$ and a $(k+1)$-symmetric $\tilde{u}$ so that $r(B)^{-2-n} \int_B |y|^\beta |u - \tilde{u}|^2 dy < \epsilon \leq \tau$. But $\tilde{u}$ is also $(j+1)$-symmetric. Thus, $x \notin S^j_\tau$.

The fact that $S^k_\epsilon \subset S^k$ is a consequence of the uniform convergence on Lemma 3.4: if $x \notin S^k$, then $u$ has a $(k+1)$-symmetric blowup sequence $u_i \to u_0$ at $x$ converging uniformly. Thus,

$$\begin{aligned}
&\int_{B_{\rho_i}(x_0)} |y|^\beta \left| u(x) - \rho_i^\alpha u_0\left(\frac{x - x_0}{\rho_i}\right)\right|^2 dx \\
&\quad = \rho_i^{\beta+2\alpha+n+1} \int_{B_1} |y|^\beta \left| \frac{u(x_0 + \rho_i x)}{\rho_i^\alpha} - u_0(x)\right|^2 dx \\
&\quad \leq \rho_i^{n+2} \omega(B_1) \|u_i - u_0\|_{L^\infty}.
\end{aligned}$$

That is,

$$\rho_i^{-n-2} \int_{B_{\rho_i}(x)} |y|^\beta \left| u(x) - r_i^\alpha u_0\left(\frac{x - x_0}{\rho_i}\right)\right|^2 dx \xrightarrow{i\to\infty} 0.$$

Therefore, for every $\varepsilon$ there exists a ball small enough so that $u$ is $(k+1,\varepsilon)$-symmetric in it. In particular, $S^k \supset \bigcup_{\epsilon>0} S^k_\epsilon$.

To see the converse, assume that $x \notin \bigcup_\epsilon S^k_\epsilon$. Then for every $i \in \mathbb{N}$ there exist a $(k+1)$-symmetric function $\widetilde{u}_i$, invariant with respect to $L_i \in V^{k+1}$, and $r_i < \min\{1, \operatorname{dist}(x, \partial\Omega)\}$ such that

$$\frac{1}{r_i^{n+2}} \int_{B_{r_i}} |y|^\beta |u(x) - \widetilde{u}_i(x)|^2 \, dx < \frac{1}{i}.$$

In the case when $r_i$ stays away from 0, since $r_i < 1$, we can take a subsequence converging to $r_0 \in (0,1)$, and one can see that $u$ is $(k+1)$-symmetric in the ball $B_{r_0}(x_0)$. Otherwise, consider

$$u_i := \frac{u(x_0 + r_i x)}{r_i^\alpha} \quad \text{and} \quad \widetilde{u}_{i,i} = \frac{\widetilde{u}_i(x_0 + r_i x)}{r_i^\alpha}.$$

By taking subsequences, we can assume that $L_i \to L_0$ locally in the Hausdorff distance, and that $u_i \to u_0$ locally uniformly. One can check also using the Hölder character of $u$ that $\{\widetilde{u}_{i,i}\}$ is uniformly bounded in $L^2(B; |y|^\beta)$, so taking subsequences again, we can assume the existence of $\widetilde{u}_0$ so that $\widetilde{u}_{i,i} \to \widetilde{u}_0$ in $L^2(B; |y|^\beta)$. This function will be $(k+1)$-symmetric, being invariant in the directions of $L_0$. By the triangle inequality we get

$$\begin{aligned}\int_{B_1} |y|^\beta |u_0 - \widetilde{u}_0|^2 \, dx \lesssim \int_{B_1} |y|^\beta |u_0 - u_i|^2 \, dx + \int_{B_1} |y|^\beta |u_i - \widetilde{u}_{i,i}|^2 \, dx \\ + \int_{B_1} |y|^\beta |\widetilde{u}_{i,i} - \widetilde{u}_0|^2 \, dx.\end{aligned}$$

The first and the last integrals converge to 0 by our choice of the subsequence. For the middle term, just change variables as before:

$$\int_{B_1} |y|^\beta |u_i - \widetilde{u}_{i,i}|^2 \, dx = \frac{1}{r_i^{n+2}} \int_{B_{r_i}} |y|^\beta |u(x) - \widetilde{u}_i(x)|^2 \, dx \to 0.$$

Thus we have that $u_0 = \widetilde{u_0}$ and therefore, $x \notin S_k$.

(3) The inclusions $S^j_\tau \subset S^j_{\tau,r} \subset S^k_{\epsilon,s}$ of the third property come from the definitions and thus $S^k_\epsilon \subset \bigcap_{r>0} S^k_{\epsilon,r}$. The converse implication is also trivial.

(4) The closedness is obtained by a contradiction argument again. It is straightforward but we write it here for the sake of completeness.

Assume by contradiction that $x \notin S^k_\epsilon(u)$. Then there exist a $(k+1)$-symmetric function $\widetilde{u}$ and a radius $r$ such that

$$\epsilon_0 := \frac{1}{r^{n+2}} \int_{B_r(x)} |y|^\beta |u(x) - \widetilde{u}(x)|^2 \, dx < \epsilon.$$

Let $\tau < 1$ be fixed and consider $i_0 \in \mathbb{N}$ so that $B_{\tau r}(x_i) \subset B_r(x)$ for every $i \geq i_0$. By the triangle inequality

$$\frac{1}{(\tau r)^{n+2}} \int_{B_{\tau r}(x_i)} |y|^\beta |u_i(x) - \widetilde{u}(x)|^2 \, dx \\ \leq \frac{1}{(\tau r)^{n+2}} \|u_i - u\|^2_{L^2(B_{\tau r}(x_i);|y|^\beta)} + \frac{\epsilon_0}{\tau^{n+2}}.$$

We define $\tau$ so that $\frac{\epsilon_0}{\tau^{n+2}} = \frac{\epsilon+\epsilon_0}{2}$. Choose $i_0$ big enough so that every $i \geq i_0$ satisfies that $\|u_i - u\|^2_{L^2(B_{\tau r}(x_i);|y|^\beta)} < (\tau r)^{n+2} \frac{\epsilon - \epsilon_0}{2}$. Then $x_i \notin S^k_\epsilon(u_i)$, contradicting the hypothesis.

(5) Assume that $\widetilde{u}_i$ is invariant with respect to $L_i \in V^{k+1}$ and

$$\int |y|^\beta |u_i - \widetilde{u}_i|^2 \leq \epsilon_i.$$

Consider a subsequence $\{u_i\}$ so that the varieties $L_i \to L$ locally in the Hausdorff distance. Using the triangle inequality as in (4), it follows that $u$ is $(k, \delta_i)$-symmetric with $\delta_i \to 0$. □

PROPOSITION 8.5. *There exists $\epsilon(n, \alpha) > 0$ such that if $u \in \mathbf{H}^\beta_{\mathrm{loc}}(\Omega)$ is a minimizer of $\mathcal{J}$ in a domain $\Omega \subset \mathbb{R}^{n+1}$, then $\Sigma(u) \subset S^{n-k^*_\alpha}_\epsilon(u)$.*

PROOF. It is enough to show that if $u$ is a minimizer of $\mathcal{J}$ in $B_2(0)$, then $\Sigma(u) \cap B_1(0) \subset S^{n-k^*_\alpha}_\epsilon(u)$.

By contradiction, let us assume that there is a sequence of positive numbers $\epsilon_i \xrightarrow{i\to\infty} 0$, functions $u_i$ minimizing $\mathcal{J}$ in $B_2(0)$ and $x_i \in \Sigma(u_i) \cap B_1(0)$, $r_i \in (0,1]$, with $u_i$ being $(n - k^*_\alpha + 1, \epsilon_i)$-symmetric in $B_{r_i}(x_i)$, and let $L_i$ be an $(n - k^*_\alpha + 1)$-dimensional subspace that leaves invariant one of the admissible $(n - k^*_\alpha + 1)$-symmetric approximants. By rescaling we can assume that $r_i = 1$.

Passing to a subsequence we can assume that $L_i \to L_0 \in V^{n-k^*_\alpha+1}$ locally in the Hausdorff distance and $x_i \to x_0$. By the compactness results in Lemma 3.4 we have a uniform limit $u_0$ that is a minimizer as well, and it is $(n - k^*_\alpha + 1)$-symmetric with invariant manifold $L_0$. By Lemma 4.4 any blowup $u_{0,0}$ at $x_0$ will be $(n - k^*_\alpha + 1)$-symmetric as well. Applying Lemma 4.5 $(n - k^*_\alpha + 1)$ times, we find that the restriction of $u_{0,0}$ to the orthogonal manifold $L_0^\perp$ is a $(k^*_\alpha - 1)$-dimensional minimal cone, which, by Lemma 5.2, is the trivial solution, and so is $u_{0,0}$. Thus, $x_0$ is a regular point for $u_0$.

On the other hand, the Hausdorff convergence of Lemma 3.4 together with the improvement of flatness of Theorem 2.4 imply that for $i$ big enough $x_i \in F_{\mathrm{red}}(u_i)$, reaching a contradiction. □

## 8.2 The Refined Covering Theorem

Our estimates on the size and structure of the singular set $\Sigma(u)$ come from similar results concerning the $S^k_\epsilon(u)$. In particular, we prove the following covering result:

THEOREM 8.6. *Let $u \in \mathbf{H}^\beta(B_5)$ be a minimizer to* (1.1) *in $B_5$ with $0 \in F(u)$. For given real numbers $\epsilon > 0$, $0 < r \leq 1$, and every natural number $1 \leq k \leq n-1$, we can find a collection of balls $\{B_r(x_i)\}_{i=1}^N$ with $N \leq C_{n,\alpha,\epsilon} r^{-k}$ such that*

$$S^k_{\epsilon,r}(u) \cap B_1 \subset \bigcup_i B_r(x_i).$$

*In particular, $|B'_r(S^k_{\epsilon,r} \cap B_1)| \leq C_{n,\alpha,\epsilon} r^{n-k}$ for every $0 < r \leq 1$ and*

$$\mathcal{H}^k(S^k_\epsilon(u) \cap B_1) \leq C_{n,\alpha,\epsilon}.$$

From Proposition 8.5 and Theorem 8.6, we can conclude the following corollary, which comprises the second part of Theorem 8.1 above.

COROLLARY 8.7. *If $u \in \mathbf{H}^\beta(B_5)$ is a minimizer to* (1.1) *in $B_5$ with $0 \in F(u)$, then $\Sigma(u)$ is $(n-k^*_\alpha)$-rectifiable and for every $0 < r \leq 1$ we have*

$$|B_r(\Sigma(u) \cap B_1)| \leq C_{n,\alpha} r^{k^*_\alpha}.$$

*In particular,*

$$\mathcal{H}^{n-k^*_\alpha}(\Sigma(u) \cap B_1) \leq C_{n,\alpha}.$$

Rectifiability is encoded in the following result. We omit the details of proof here but it is a consequence of the packing result above, the Rectifiable-Reifenberg theorem of [32], and Theorem 8.14 below. For more details see sections 2 and 8 of [22] (particularly theorem 2.2 in the former and the proof of theorem 1.12 in the latter).

THEOREM 8.8. *Let $u$ be a nonnegative, even minimizer to* (1.1) *in a domain $\Omega$. Then $S^k_\epsilon(u)$ is $k$-rectifiable for every $\epsilon$, and hence each stratum $S^k(u)$ is $k$-rectifiable as well.*

The proof of Theorem 8.6 follows from inductively applying the following, slightly more technical packing result (for details see section 4 of [22]).

THEOREM 8.9. *Let $\epsilon > 0$. There exists $\eta(n,\alpha,\epsilon)$ such that, for every minimizer $u \in \mathbf{H}^\beta(B_5)$ of $\mathcal{J}$ in $B_5$ with $0 \in F(u)$ and $0 < R < 1/10$, there is a finite collection $\mathcal{U}$ of balls $B$ with center $x_B \in S^k_{\epsilon,\eta R}$ and radius $R \leq r_B \leq 1/10$ that satisfy the following properties*:

(A) *Covering control*:

$$S^k_{\epsilon,\eta R} \cap B_1 \subset \bigcup_{B \in \mathcal{U}} B.$$

(B) *Energy drop*: *For every* $B \in \mathcal{U}$,

$$\text{either} \quad r_B = R \quad \text{or} \quad \sup_{2B} \Psi^u_{2r_B} \le \sup_{B_2} \Psi^u_2 - \eta.$$

(C) *Packing*:

$$\sum_{B \in \mathcal{U}} r_B^k \le c(n, \alpha, \epsilon).$$

We construct the balls of Theorem 8.9 using a "stopping time" or "good ball/bad ball" argument. Much of this argument uses harmonic analysis and geometric measure theory and is completely independent of the original problem (1.1). However, there are a few places in which we need to connect the behavior of minimizers to the geometric structure of the singular set. Here we will sketch the good ball/bad ball argument, taking for granted the estimates needed to apply this argument to our functional. In the next few subsections we will provide these estimates. For more details on the construction itself, we refer the reader to section 7 in [22].

**Outline of the Construction in Theorem 8.9**

To find this covering we define good and bad balls as follows: imagine our ball, $B$, has radius 1. We say that $B$ is a *good ball* if at every point in $x \in S^k_\varepsilon(u) \cap B$ the monotone quantity centered at that point at some small scale, $\rho$, is not much smaller than the monotone quantity on ball $B$ (we say these points have "small density drop"). A ball $B$ is a *bad ball* if all the points in $S^k_\varepsilon(u) \cap B$ with small density drop are contained in a small neighborhood of a $(k-1)$-plane. This dichotomy follows from Theorem 8.10 in Section 8.3.

In a good ball of radius $r$ we cover $S^k_\varepsilon(u)$ with balls of radius $\rho r$, iterating the construction until we find a bad ball or until the radius of the ball becomes very small. In a bad ball, we cover $S^k_\varepsilon(u)$ away from the $(k-1)$-plane without much care. Close to the $(k-1)$-plane we cover $S^k_\varepsilon(u)$ with balls of radius $\rho r$, iterating the construction until we reach a good ball or until the radius of the ball becomes very small.

Inside long strings of good balls, the packing estimates follow from powerful tools in geometric measure theory (see Theorem 8.13 below) and the connection between the drop in monotonicity and the local flatness of the singular strata (see Theorem 8.14 below). We give more details in Section 8.4.

Inside long strings of bad balls, each of which is near the $(k-1)$-plane of the previous bad ball, we have even better packing estimates than expected (as we are effectively well approximated by planes that are lower dimensional). This leaves only points that are in many bad balls, and in most of those balls they are far away from the $(k-1)$-plane. However, at these points the monotone quantity drops a definite amount many times, which contradicts either finiteness or monotonicity. This implies that the points and scales inside the bad balls that are not close to the $(k-1)$-plane form a negligible set (the technical term is a Carleson set). We give more information about the bad balls in Section 8.3.

### 8.3 Tools for Bad Balls: Key Dichotomy

THEOREM 8.10 (Key dichotomy). *Let $\epsilon, \rho, \gamma, \eta' > 0$ be fixed numbers with $\rho\gamma < 2$. There exists an $\eta_0(n, \alpha, \epsilon, \rho, \gamma, \eta') < \rho/100$ such that for every $\eta \leq \eta_0$, every $r > 0$, every $E > 0$, and every minimizer $u \in \mathbf{H}^\beta(B_{4r})$ of $\mathcal{J}$ in $B_{4r}$ with $0 \in F(u)$ and $\sup_{B_r} \Psi^u_{2r} \leq E$, then either*

- *$\Psi^u_{\gamma\rho r} \geq E - \eta'$ on $S^k_{\epsilon,\eta r} \cap B_r$, or*
- *there exists $\ell \in L^{k-1}$ so that $\{x \in B_r : \Psi^u_{2\eta r}(x) \geq E - \eta\} \subset B_{\rho r}(\ell)$.*

The key dichotomy is a direct consequence of Lemma 8.11 below. The core idea is to make effective the following assertion: if $u$ is $k$-symmetric, then along the invariant manifold the Allen-Weiss density is constant, and every point away from the manifold will have $(k+1)$-symmetric blowups by Lemma 4.4.

LEMMA 8.11. *Let $\epsilon, \rho, \gamma, \eta' > 0$ be fixed numbers with $\gamma\rho < 2$. There exist $\eta_0, \theta > 0$ such that for every $\eta < \eta_0$, every $E > 0$, and every minimizer $u$ of $\mathcal{J}$ in $B_4$ with $0 \in F(u)$ and $\sup_{B_1} \Psi^u_2 \leq E$, if there exist $w_0, \ldots, w_k \in B_1$ and affine manifolds $L^i := \langle w_0, \ldots, w_i \rangle \in V^i$ with*

$$w_i \notin B_\rho(L^{i-1}) \quad \text{and} \quad \Psi^u_{2\eta}(w_i) \geq E - \eta \quad \text{for every } i \in \{0, \ldots, k\},$$

*then*

$$\Psi^u_{\gamma\rho}(x) \geq E - \eta' \quad \text{on } B_\theta(L^k) \cap B_1 \tag{8.1}$$

*and*

$$S^k_{\epsilon,\eta} \cap B_1 \subset B_\theta(L^k). \tag{8.2}$$

The proof follows (with only minor modifications) the proof in [22, lemma 3.3]. We end this subsection by formally defining the good/bad balls alluded to above:

DEFINITION 8.12. Let $x \in B_2$, $0 < R < r < 2$, and $u$ be a minimizer to $\mathcal{J}$ in $B_5$. We say that the ball $B_r(x)$ is *good* if

$$\Psi^u_{\gamma\rho r} \geq E - \eta' \quad \text{on } S^k_{\epsilon,\eta R} \cap B_r(x),$$

and otherwise we say that $B_r(x)$ is *bad*.

By Theorem 8.10 in any bad ball $B$ there exists an affine $(k-1)$-manifold $\ell_B$ with

$$\left\{w \in B : \Psi^u_{2\eta r}(w) \geq E - \eta\right\} \subset B_{\rho r}(\ell^{k-1}_B). \tag{8.3}$$

### 8.4 Tools for Good Balls: Packing Estimates and GMT

In this section we control the local flatness of the singular strata by the drop in monotonicity. To do this we introduce a key tool from geometric measure theory that estimates the flatness of a set. Given a Borel measure $\mu$, a point $x$, and a radius $r$, the beta coefficient is defined as follows:

$$\beta^k_{\mu,2}(B_r(x))^2 := \beta^k_{\mu,2}(x,r)^2 = \inf_{L \in V^k_a} \frac{1}{r^k} \int_{B_r(x)} \frac{\operatorname{dist}(z,L)^2}{r^2}\, d\mu(z) \tag{8.4}$$

where $V_a^k$ stands for the collection of $k$-dimensional affine sets of $\mathbb{R}^n$. The beta coefficients are meant to measure in a scale invariant way how far a measure is from being flat, in this case in the $L^2$ distance, although other $L^p$ versions have been used in the literature for $1 \leq p \leq \infty$ quite often, dating back to [30] (for the $L^\infty$ version) and David-Semmes [12] (for the $L^p$ version).

If we control the size of the $\beta^k$'s we can conclude size and structure estimates on the measure $\mu$. The following theorem says exactly this and represents a major technical achievement. It differs (importantly) from prior work in this area by the lack of a priori assumptions on the upper or lower densities of the measure involved.

THEOREM 8.13 (Discrete-Reifenberg Theorem; see [32, theorem 3.4]). *Let $\{B_{r_q}(q)\}_q$ be a collection of disjoint balls, with $q \in B_1(0)$ and $0 < r_q \leq 1$, and let $\mu$ be the packing measure $\mu := \sum_q r_q^k \delta_q$, where $\delta_q$ stands for the Dirac delta at $q$. There exist constants $\tau_{DR}, C_{DR} > 0$ depending only on the dimension such that if*

$$\int_0^{2r}\int_{B_r(x)} \beta_{\mu,2}^k(z,s)^2\, d\mu(z)\frac{ds}{s} \leq \tau_{DR} r^k \quad \textit{for every } x \in B_1(0),\ 0 < r \leq 1,$$

*then*

$$\mu(B_1(0)) = \sum_q r_q^k \leq C_{DR}.$$

To obtain the packing estimates required for the Discrete-Reifenberg theorem, we need to control the beta coefficients. The key estimate of this entire framework lies in the following theorem, which shows the drop in monotonicity at a given point and that a given scale controls the beta coefficient at a comparable scale.

THEOREM 8.14. *Let $\epsilon > 0$ be given. There exist $\delta(n,\alpha,\epsilon)$ and $c(n,\alpha,\epsilon)$ such that for every $u \in \mathbf{H}^\beta(B_{5r})$ minimizing $\mathcal{J}$ in $B_{5r}(x)$ with $x \in F(u)$ and*

$$\begin{cases} u \text{ is } (0,\delta)\text{-symmetric in } B_{4r}(x), \\ u \text{ is not } (k+1,\epsilon)\text{-symmetric in } B_{4r}(x), \end{cases} \tag{8.5}$$

*and every Borel measure $\mu$, we have that*

$$\beta_{\mu,2}^k(B_r(x))^2 \leq \frac{c(n,\alpha,\epsilon)}{r^k}\int_{B_r(x)} \big(\Psi_{4r}^u(w) - \Psi_r^u(w)\big)d\mu(w). \tag{8.6}$$

We follow the proof of [22, theorem 5.1] closely. First, the authors give an explicit formula for the beta coefficients.

LEMMA 8.15. *Let $X$ be the center of mass of a Borel measure $\mu$ on $B = B_r(x)$. Let $\{\lambda_i\}_{i=1}^n$ be the decreasing sequence of eigenvalues of the nonnegative bilinear form*

$$Q(v,w) := \fint_B (v\cdot(z-X))(w\cdot(z-X))d\mu(z),$$

*and let* $\{v_i\}_{i=1}^n$ *be a corresponding orthonormal sequence of eigenvectors, that is,* $v_i \cdot v_j = \delta^{ij}$ *and* $Q(v_i, v) = \lambda_i v_i \cdot v$*. Then*

$$\beta^k_{\mu,2}(B)^2 = \frac{1}{r^k}\int_B \frac{\operatorname{dist}(z, L^k)^2}{r^2}\, d\mu(z) = \frac{\mu(B)}{r^k}\frac{(\lambda_{k+1} + \cdots + \lambda_n)}{r^2},$$

*where* $L^k := X + \operatorname{span}\langle v_1, \dots, v_k\rangle$*.*

Next we find a relation between the eigenvalues of $Q$ and Allen-Weiss' energy.

LEMMA 8.16. *Under the hypothesis of Lemma* 8.15*, for every* $u \in \mathbf{H}^\beta(B_{5r})$ *minimizing* $\mathcal{J}$ *in* $B_{5r}(x)$ *and every* $i \le n$*, we have that*

$$\begin{aligned}\lambda_i \frac{2}{r^{n+2}} &\int_{A_{2r,3r}(x)} |y|^\beta (v_i \cdot Du(z))^2\, dz \\ &\le C \fint_{B_r(x)} \big(\Psi^u_{4r}(w) - \Psi^u_r(w)\big) d\mu(w).\end{aligned} \tag{8.7}$$

PROOF. The argument follows as in [22, (18) and below]. In formula (18) one needs to change $u(z)$ by $\alpha u(z)$, which can be done with exactly the same argument. □

Finally, using compactness, we bound the left-hand side of (8.16) from below.

LEMMA 8.17. *Let* $\epsilon > 0$ *be given. There exists a* $\delta(n, \alpha, \epsilon)$ *and* $c(n, \alpha, \epsilon)$ *such that, for every orthonormal basis* $\{v_i\}_{i=1}^n$ *and every* $u \in \mathbf{H}^\beta(B_{5r})$ *minimizing* $\mathcal{J}$ *in* $B_{5r}(x)$ *with* $x \in F(u)$ *and satisfying* (8.5)*, we have that*

$$\frac{1}{c(n,\alpha,\epsilon)} \le r^{-n} \int_{A_{2r,3r}(x)} |y|^\beta \sum_{i=1}^{k+1} (v_i \cdot Du(z))^2\, dz. \tag{8.8}$$

PROOF. The proof follows that of [22, (19)] and we omit it. □

PROOF OF THEOREM 8.14. By Lemmas 8.15, 8.17, and 8.16 we get that

$$\begin{aligned}\beta^k_{\mu,2}(B)^2 &\le \frac{\mu(B)}{r^{k+2}}(n-k)\lambda_{k+1} \\ &\le \frac{\mu(B)}{r^k}(n-k)c(n,\alpha,\epsilon)\sum_{i=1}^{k+1}\frac{\lambda_i}{r^{n+2}}\int_{A_{2r,3r}(x)} |y|^\beta (v_i \cdot Du(z))^2\, dz \\ &\le \frac{c(n,\alpha,\epsilon)}{r^k}\int_{B_r(x)} \big(\Psi^u_{4r}(w) - \Psi^u_r(w)\big) d\mu(w).\end{aligned}$$ □

## Appendix A Relation with the Nonlocal Bernoulli Problem

As in [20, lemma 2.1], we see that the study of minimizers of $\mathcal{J}$ includes the study of minimizers of $J$.

PROPOSITION A.1. *If $f$ is a minimizer of $J$ in the unit ball of $\mathbb{R}^n$, then $f * P_y$ is a minimizer of $\mathcal{J}$ in every ball $B$ such that $B' \Subset B'_1$.*

*If $u = f * P_y$ is a minimizer of $\mathcal{J}$, then $f$ is a minimizer for $J$. In particular, if $u$ is a minimizer of $\mathcal{J}$ in every ball, positive outside the hyperplane $\{y = 0\}$, and $u(x, y) = \mathcal{O}(|(x, y)|^\alpha)$, then $u|_{\mathbb{R}^n \times \{0\}}$ is a minimizer for $J$ in every ball.*

We follow [20, lemma 2.1]; that is, we use the following result from [6, sec. 7].

LEMMA A.2. *Let $f, g$ satisfy that $J_0(f, B_1)$ and $J_0(g, B_1) < \infty$, and suppose that $f - g$ is compactly supported in $B_1 \subset \mathbb{R}^n$. Then we have that*

$$J_0(g, B_1) - J_0(f, B_1) = c_{n,\alpha} \inf \int_\Omega |y|^\beta (|\nabla v(x, u)|^2 - |\nabla (f * P_y)(x)|^2),$$

*where the infimum is taken among all the symmetric bounded Lipschitz domains $\Omega$ with the property that $\Omega \cap (\mathbb{R}^n \times \{0\}) \subset B_1$ and among all symmetric functions $v$ with trace $g$ satisfying that $v - f * P_y$ is compactly supported on $\Omega$.*

PROOF OF PROPOSITION A.1. Let $f$ be a minimizer of $J$ in the unit ball of $\mathbb{R}^n$ and let $B_r$ be a ball such that $B'_r \Subset B'_1$. We want to show that $u := f * P_y$ is a minimizer of $\mathcal{J}$ in $B_r$.

Let $v : \mathbb{R}^{n+1} \to \mathbb{R}$ so that $v \equiv u$ in $\mathbb{R}^{n+1} \setminus B_r$ and $v \in H^1(\beta, B_r)$. Let $g$ be the trace of $v$ in $\mathbb{R}^n \times \{0\}$. By Lemma A.2 we have that

$$J_0(g, B_1) - J_0(f, B_1) \le c_{n,\alpha} \int_{B_{r+\varepsilon}} |y|^\beta (|\nabla v|^2 - |\nabla u|^2) \tag{A.1}$$

for every $\varepsilon > 0$.

Since $g|_{(B')^c} \equiv 0$, $g$ is an admissible competitor for $f$ and $J(f, B_1) \le J(g, B_1)$, i.e.,

$$\begin{aligned} J_0(g, B_1) - J_0(f, B_1) &\ge -m(\{g > 0\} \cap B_1) + m(\{f > 0\} \cap B_1) \\ &= m(\{u > 0\} \cap B'_r) - m(\{v > 0\} \cap B'_r). \end{aligned} \tag{A.2}$$

The proposition follows combining (A.1) and (A.2) and letting $\varepsilon \to 0$.

The converse follows the same sketch: every global minimizer can be expressed as the Poisson extension of its restriction to the hyperplane by Proposition B.1. □

As a consequence of the previous proposition, all the results that we have proven for minimizers of $\mathcal{J}$ also apply to minimizers of $J$:

COROLLARY A.3. *If $u : \mathbb{R}^n \to \mathbb{R}$ is a minimizer to $J$ in $B_2 \subset \mathbb{R}^n$ and $0 \in F(u)$, then $\|u\|_{C^\alpha(B_1)} \le C$, it satisfies the nondegeneracy condition $u(x) \ge C \operatorname{dist}(x, F(u))^\alpha$ for $x \in B_1$, the positive phase satisfies the corkscrew condition, every blowup limit is $\alpha$-homogeneous, and the boundary condition in* (1.2) *is satisfied at $F_{\mathrm{red}}(u)$.*

*Moreover, the positive phase $\{u > 0\} \cap B_1$ is a set of finite perimeter, the singular set is an $(n - 3)$-rectifiable set, it is discrete whenever $n = 3$, and it is empty if $n \le 2$.*

*All the constants depend only on $n$ and $\alpha$.*

## Appendix B Uniqueness of Extensions

In Proposition A.1 we have used the following result, included in [7, prop. 3.1]. Here we provide a proof that is different than the one appearing in [7].

PROPOSITION B.1. *Let* $\alpha \in (0,1)$ *and* $\beta = 1 - 2\alpha$, *and set* $\mathcal{L}u = -\operatorname{div}(|y|^\beta \nabla u)$ *in* $\mathbb{R}^{n+1}$. *Suppose that* $v : \overline{\mathbb{R}^{n+1}_+} \to \mathbb{R}$ *is nonnegative outside* $\mathbb{R}^n$, *and it is a solution to* $\mathcal{L}v = 0$ *in* $\mathbb{R}^{n+1}_+$ *with* $v(x',0) = 0$ *for all* $x' \in \mathbb{R}^n$ *and* $|v(x)| \le C|x|^\alpha$. *Then* $v \equiv 0$.

PROOF. First, since $|y|^\beta$ is $C^\infty$ away from the hyperplane $\mathbb{R}^n$, $v \in C^\infty_{\text{loc}}(\mathbb{R}^{n+1}_+)$. Let now $i \in \{1, \dots n\}$, and set

$$f_m(x) = \frac{v\big(x + \frac{1}{m}e_i\big) - v(x)}{1/m}.$$

Let $B_r = B_r(x',0)$ be a ball centered at $(x',0) \in \mathbb{R}^n \times \{0\}$ with radius $r$, and let $B_{2r}$ be its double ball. Set also $w(x) = w(x',y) = y^\beta$ for $y > 0$. Since $f_m$ is a solution of $\mathcal{L} f_m = 0$ in $B_r^+ = B \cap \mathbb{R}^{n+1}_+$, [26, theorem 2.4.3] shows that

$$\max_{B_r^+} |f_m(x)| \le C \left( \frac{1}{w(B_{2r}^+)} \int_{B_{2r}^+} |f_m|^2 w \right)^{1/2}.$$

From convergence of difference quotients (similarly to [23, theorem 3, p. 277]), if $v \in H^1(\beta, B_{2r}^+)$, the last estimate will imply that $f_m$ is uniformly bounded in $B_r^+$ by a constant $C_r$. Therefore, from the boundary Caccioppoli estimate [26, (2.4.2)] we have that

$$\int_{B_{r/2}^+} |\nabla f_m|^2 w \le \frac{C}{r^2} \int_{B_r^+} |f_m|^2 w \le \frac{C}{r^2} \int_{B_r^+} C_r^2 w \le C_{n,r,w} < \infty;$$

hence $\{f_m\}$ is bounded in $H^1(\beta, B_{r/2}^+)$. From weak compactness, a subsequence of $\{f_m\}$ converges to a solution of $\mathcal{L}u = 0$ in $B_{r/2}^+$, and since $f_m \to \partial_i v$ pointwise, we obtain that $\partial_i v$ is an $H^1(\beta, B_{r/2}^+)$ solution in $B_{r/2}^+$. Hence $\partial_i v$ is a solution to $\mathcal{L}u = 0$ in $\mathbb{R}^{n+1}_+$.

Now, for $x = (x',y) \in \mathbb{R}^{n+1}_+$, let $R = |x|$. We distinguish between two cases: $y > R/16$ and $y < R/16$.

In the first case, set $B_R$ to be the ball of radius $R$, centered at $x$. Note then that $B_{R/16} \subseteq \mathbb{R}^{n+1}_+$. Then, from [26, theorem 2.3.1], Caccioppoli's estimate, and the assumption $|v(x)| \le C|x|^\alpha$,

$$\begin{aligned} |\partial_i v(x)|^2 &\le \frac{C}{w(B_{R/32})} \int_{B_{R/32}} |\partial_i v|^2 w \le \frac{C}{w(B_{R/32})} \frac{C}{R^2} \int_{B_{R/16}} |v|^2 w \\ &\le \frac{C}{R^2} \frac{w(B_{R/16})}{w(B_{R/32})} \sup_{B_{R/16}} |v| \le C R^{2\alpha - 2}. \end{aligned}$$

In the second case, let $B_R$ be the ball centered at $(x', 0)$ with radius $R$, and denote $B_R^+ = B_R \cap \mathbb{R}_+^{N+1}$. Then $x \in B_{R/8}^+$; therefore from [26, theorem 2.4.3] and the boundary Caccioppoli estimate,

$$|\partial_i v(x)|^2 \le \frac{C}{w(B_{R/8}^+)} \int_{B_{R/8}^+} |\partial_i v|^2 w \le \frac{C}{w(B_{R/8}^+)} \frac{C}{R^2} \int_{B_{R/4}^+} |v|^2 w$$
$$\le \frac{C}{R^2} \frac{w(B_{R/4}^+)}{w(B_{R/8}^+)} \sup_{B_{R/4}^+} |v| \le C R^{2\alpha - 2}.$$

So, in all cases, $|\partial_i v(x)| \le C|x|^{\alpha-1}$. Letting $R \to \infty$ and using the maximum principle, we find that $\partial_i v = 0$ for any $i = 1, \dots, n$. Therefore $v$ does not depend on the first $n$ variables, so $v(x', y) = v(y)$. Hence, in $\mathbb{R}_+^{n+1}$,

$$0 = -\operatorname{div}(y^\beta \nabla v(y)) = -\partial_y(y^\beta v'(y)) \;\Rightarrow\; y^\beta v'(y) = \tilde{c},$$

for some constant $\tilde{c}$. From [26, theorem 2.4.6], $v$ is Hölder continuous up to the boundary; therefore, for any $y > 0$,

$$v(y) = v(y) - v(0) = \int_0^y v' = \int_0^y \tilde{c} s^{-\beta}\, ds = \frac{\tilde{c}}{1-\beta} y^{1-\beta},$$

which implies that

$$|\tilde{c}| = (1-\beta) y^{\beta-1} |v(y)| = (1-\beta) y^{\beta-1} |v(0, y)| \le (1-\beta) y^{\beta-1} y^\alpha = (1-\beta) y^{-\alpha}$$

for any $y > 0$. Letting $y \to \infty$ we obtain that $\tilde{c} = 0$; hence $v'(y) = 0$ as well, which implies that $v$ is a constant. Since $v$ vanishes on $\mathbb{R}^n$, this implies that $v \equiv 0$. □

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