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Projective structures on a hyperbolic 3–orbifold

Joan Porti and Stephan Tillmann

Abstract We compute and analyse the moduli space of those real projective structures on a hyperbolic 3–orbifold that are modelled on a single ideal tetrahedron in projective space. Parameterisations are given in terms of classical invariants, *traces*, and geometric invariants, *cross ratios.*

AMS Classification 57M25, 57N10

Keywords 3–orbifold, projective geometry

1 Introduction

This note studies certain real projective structures on a 3-dimensional orbifold, O, which is obtained by taking the one tetrahedron triangulation with two vertices of S^3, deleting the vertices and modelling the edge neighbourhoods on $\mathbb{R}^3/\langle r\rangle$, where r is a rotation by $120°$ (see Figure 1). This orbifold supports a unique complete hyperbolic structure; this has two Euclidean $(3,3,3)$–turnover cusps and is of finite volume.

In [6], the philosophy was put forward that strictly convex projective manifolds behave like hyperbolic manifolds sans Mostow rigidity. This paper computes a moduli space of projective structures on O that are modelled on an ideal tetrahedron. This moduli space, denoted $\mathrm{Mod}(O)$, turns out to be a smooth, open 2–dimensional disc. We obtain two parameterisations of $\mathrm{Mod}(O)$: one in terms of algebraic invariants, *traces*, and one in terms of geometric invariants, *cross-ratios.*

The complete hyperbolic structure on O is singled out as the only structure on $\mathrm{Mod}(O)$ having *standard cusps*, whilst the remaining structures on $\mathrm{Mod}(O)$ all have *generalised cusps.* It is also characterised as the unique fixed point of a natural involution on $\mathrm{Mod}(O)$. The problem to decide which of the structures on $\mathrm{Mod}(O)$ are *properly convex* appears to be difficult by elementary means, but is completely solved by the theoretical results of Cooper, Long and Tillmann [6, 5], and of Ballas, Danciger and Lee [2]; the answer is *all of them.* This problem has also motivated some of Choi's work [4].

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2 Projective structures modelled on triangulations

In n–dimensional real projective space, n–simplices are overly congruent. Given any two n–simplices, there is a projective transformation taking one to the other. Given an n–simplex, there is a n–dimensional family of projective transformations taking it to itself whilst fixing each of its vertices. The following notions can be defined in all dimensions, but we restrict to the case $n = 3$.

The space $\mathbb{RP}^3$ will be viewed as the set of 1–dimensional vector subspaces of $\mathbb{R}^4$ with the induced topology. The set of projective transformations, $\mathrm{PGL}(4, \mathbb{R})$, then corresponds to the quotient of $\mathrm{GL}(4, \mathbb{R})$ by its centre, the group of all non-zero multiples of the identity matrix. If Δ is the 3–simplex with vertices corresponding to the standard unit vectors $e_1, \ldots, e_4$ in $\mathbb{R}^4$ and containing $\sum e_i$ in its interior, then the family of projective transformations stabilising Δ and fixing its vertices corresponds to the set of diagonal matrices in $\mathrm{GL}(4, \mathbb{R})$ having all entries positive or all entries negative. This gives a 3–dimensional family of projective transformations.

Let M be an arbitrary, ideally triangulated 3–orbifold of finite topological type. We recall that an ideal 3–simplex is a 3–simplex with its four vertices removed, and an ideal triangulation of M is an expression of M as a *finite* collection of ideal 3–simplices with their faces glued in pairs, so that the branching locus of the orbifold is contained in the 2–skeleton. We assume that the interior of each ideal i–simplex is embedded and that M has finitely many ends, each homeomorphic to the product of a 2-orbifold with a half line $[0 + \infty)$. The end-compactification $\overline{M}$ of M is the result of adding the vertices of the ideal 3–simplices, topologically it is the one-point compactification of each end of M.

A real projective structure on M is (an equivalence class of) a pair (dev, ρ), where $\mathrm{dev}\colon \widetilde{M} \to \mathbb{RP}^3$ is a locally injective map and $\rho\colon \pi_1(M) \to \mathrm{PGL}(4, \mathbb{R})$ is a representation of the orbifold fundamental group which makes dev ρ-equivariant. Since the orbifold universal cover $\widetilde{M}$ is non-compact, some of its ends may be homeomorphic to $\mathbb{R}^2 \times (0, 1)$. We therefore make some additional assumptions.

1. Structure is modelled on projective simplices:

Let $\widetilde{M}$ be the orbifold universal cover of M. Assume that M is a good orbifold, so that $\widetilde{M}$ is a manifold. The ideal simplices of M lift to $\widetilde{M}$. Let $\widehat{M}$ be the result of adding the vertices to the lifted simplices, with the corresponding identifications induced by side parings. For instance, for an ideally triangulated hyperbolic 3-orbifold of finite volume, $\widetilde{M}$ is the hyperbolic 3-space $\mathbb{H}^3$ and $\widehat{M}$ consists in adding the set of points in the ideal boundary $\partial_\infty \mathbb{H}^3 \cong S^2$ fixed by parabolic isometries, which is a dense countable subset of $\partial_\infty \mathbb{H}^3$. In general, the set $\widehat{M} \setminus \widetilde{M}$ is a countable set of points in which the triangulation is not locally finite. Furthermore, the end-compactification $\overline{M}$ of M is the quotient of $\widehat{M}$ by the action of the orbifold fundamental group of M.

We assume that $\mathrm{dev}\colon \widetilde{M} \to \mathbb{RP}^3$ *extends to a continuous, equivariant map* $\widehat{\mathrm{dev}}\colon \widehat{M} \to \mathbb{RP}^3$.

In this case, $\widehat{\mathrm{dev}}$ is ρ_0–equivariantly homotopic to a map $\widehat{\mathrm{dev}}_0$ with the property that $\widehat{\mathrm{dev}}_0(\Delta)$ is a projective simplex (of dimension 0, 1, 2 or 3) for every simplex Δ in $\widehat{M}$. In particular, the map is possibly not locally injective.

2. Structure is non-collapsed: Let Δ a 3–simplex in $\widehat{M}$.

We assume that the images of the vertices of Δ *under* $\widehat{\text{dev}}$ *are in general position.*

In this case, the above homotopy between $\widehat{\text{dev}_0}$ and $\widehat{\text{dev}}$ can be assumed to preserve $\widetilde{M}$, and $\widehat{\text{dev}_0}$ maps any simplex to a simplex of the same dimension. Moreover, it may be assumed to do so by a linear map and in particular, it is locally injective at interior points of 3–simplices.

Definition 1 Let M be an ideally triangulated 3–orbifold with the property that the ideal triangulation restricts to an ideal triangulation of the singular locus. A *real projective structure modelled on the triangulation* is a real projective structure (dev, ρ) on M which is modelled on projective simplices and non-collapsed. If the triangulation consists of a single 3–simplex, we will also say that the structure is modelled on a 3–simplex.

3 The moduli space

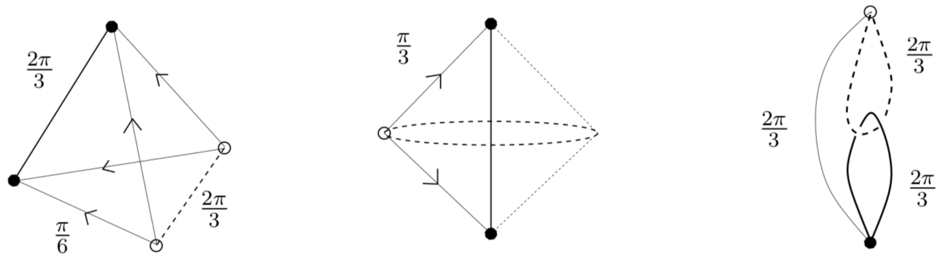

Figure 1: To obtain O, first glue the faces meeting along one of the edges with cone angle $2\pi/3$ to obtain a spindle, and then identify the boundary discs of the spindle. The result is S^3 minus two points, with the labelled graph (minus its vertices) as the singular locus. The hyperbolic structure can be obtained by identifying the ideal 3–simplex with an ideal hyperbolic 3–simplex with shape parameter $\frac{1}{2}+\frac{\sqrt{-3}}{6}$. The fundamental group of O admits, up to conjugation, exactly two irreducible representations into $\text{SL}(2,\mathbb{C})$. They are complex conjugates and correspond to holonomies for the hyperbolic structure.

Theorem 2 *The set of real projective structures on* O *modelled on a 3–simplex is parameterised by a connected component,* X*, of the set of all* $(w,x,y,z) \in \mathbb{R}^4$ *subject to the following two equations:*

$$w + x + y + z = 3 + wy, \tag{3.1}$$

$$wy = zx. \tag{3.2}$$

The structures corresponding to any two distinct points of X *are neither isotopic nor projectively equivalent. Moreover,* X *is diffeomorphic to an open disc.*

The involution $(w,x,y,z) \to (y,z,w,x)$ *on* X *has exactly one fixed point,* $(3,3,3,3) \in X$*, that corresponds to the complete hyperbolic structure on* O*.*

Before proving the theorem, we describe the zero set of (3.1) and (3.2).

Lemma 3 *The zero set of* (3.1) *and* (3.2) *in* $\mathbb{R}^4$ *is a smooth surface with two components, each diffeomorphic to a disc. One of the components,* X, *contains the point with coordinates* $w = x = y = z = 3$, *and it is distinguished from the other component by the inequalities* $x > 1$ *and* $z > 1$ *(equivalently* $y > 1$ *and* $w > 1$*).*

Proof of the lemma From (3.1) we replace $z = w\,y - w - x - y + 3$ in (3.2), so the set we want to describe is diffeomorphic to

$$\{(x, y, w) \in \mathbb{R}^3 \mid -wxy + wx + wy + x^2 + xy - 3x = 0\}.$$

We change variables to simplify computations: set $w = a + 1$, $x = b + 1$, $y = c + 1$. Then the subvariety becomes

$$\{(a, b, c) \in \mathbb{R}^3 \mid b^2 - abc + a + c = 0\}.$$

We view the defining equation as a quadratic equation on b, whose discriminant is

$$\mathrm{Disc} = a^2c^2 - 4a - 4c.$$

It can be checked that the vanishing locus of Disc consist of precisely two smooth curves, each homeomorphic to $\mathbb{R}$ (for instance, by computing c for each value of a, as Disc is quadratic on any of the variables). Also by elementary methods, one can show that the set $\{(a, c) \in \mathbb{R}^2 \mid \mathrm{Disc} \geq 0\}$ is homeomorphic to two half planes. Then the set we want to describe is the 2–to–1 branched covering of these half planes, branched along their boundaries, and therefore the subvariety is diffeomorphic to two smooth discs. Furthermore the open quadrant $\{(a, b) \in \mathbb{R}^2 \mid a > 0,\ c > 0\}$ contains one of the components of $\mathrm{Disc} \geq 0$ and is disjoint from the other one. Finally, the points $(a, c) = (0, 0)$ and $(a, c) = (2, 2)$ lie in $\mathrm{Disc} = 0$, but in different components. □

Proof of Theorem 2 Denote the ideal 3–simplex by $[v_1, v_2, v_3, v_4]$. The face pairings are $\alpha[v_1, v_2, v_3] = [v_1, v_2, v_4]$ and $\beta[v_2, v_3, v_4] = [v_1, v_3, v_4]$. The orbifold fundamental group is generated by the face pairings, and we have the following presentation:

$$\pi_1^{orb}(O) = \langle \alpha, \beta : \alpha^3 = \beta^3 = (\alpha\beta\alpha^{-1}\beta^{-1})^3 = 1\rangle. \tag{3.3}$$

To determine all projective structures of O modelled on a 3–simplex up to projective equivalence, it suffices to fix a projective 3–simplex, Δ, and to determine all representations ρ of $\pi_1(O)$ with $\rho(\alpha)$ and $\rho(\beta)$ as the corresponding face pairings. Here we have used that all 3–simplices are projectively equivalent. Choosing $\Delta = [e_1, e_2, e_3, e_4]$, the most general form of lifts of the face pairings is:

$$A = \begin{pmatrix} s_1 & 0 & 0 & a_1 \\ 0 & s_2 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & s_4 & a_4 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} b_1 & t_1 & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ b_3 & 0 & t_3 & 0 \\ b_4 & 0 & 0 & t_4 \end{pmatrix}, \tag{3.4}$$

subject to $s_1s_2s_4a_3 \neq 0$ and $t_1t_3t_4b_2 \neq 0$.

Since the above does not take division by the centre into account, the equation $A^3 = cI_4$ for $c \neq 0$ is projectively equivalent to $A^3 = I_4$, since the equation $c^3 = 1$ always has a non-zero real root. Similarly for B. This gives:

$$A = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & a_3 \\ 0 & 0 & -a_3^{-1} & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & -b_2^{-1} & 0 & 0 \\ b_2 & 0 & 0 & 0 \\ b_3 & 0 & 1 & 0 \\ b_4 & 0 & 0 & 1 \end{pmatrix}, \tag{3.5}$$

subject to $a_3 b_2 \neq 0$. Note that both matrices are elements of $\mathrm{SL}(4, \mathbb{R})$. Since we are interested in representations up to conjugacy, one may conjugate the above to give:

$$A = \begin{pmatrix} 1 & 0 & 0 & a_1 \\ 0 & 1 & 0 & a_2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ b_3 & 0 & 1 & 0 \\ b_4 & 0 & 0 & 1 \end{pmatrix}. \tag{3.6}$$

It now remains to analyse $(ABA^{-1}B^{-1})^3 = cI_4$. With (3.6), one first notes that $c = 1$. In particular, any representation of $\pi_1(O)$ into $\mathrm{PGL}(4, \mathbb{R})$ lifts to a representation into $\mathrm{SL}(4, \mathbb{R})$. By a case-by-case analysis, one obtains the following cases:

Case 1: $a_1 = a_2 = b_3 = b_4 = 0$. In this case there is a single representation which in fact satisfies $A^3 = B^3 = ABA^{-1}B^{-1} = I_4$. The corresponding developing map is not locally injective, and hence that there is no corresponding real projective structure.

Case 2: One obtains a single equation:

$$(a_1 + a_2)(b_3 + b_4) = 3 + a_1 a_2 b_3 b_4. \tag{3.7}$$

Analysis of which representations are conjugate yields that pairs A, B and A', B' give conjugate representations in $\mathrm{PGL}(4, \mathbb{R})$ if and only if they are conjugate by

$$M = \mathrm{diag}(m, m, m^{-1}, m^{-1})$$

for some $m \neq 0$. The effect on the quadruples is:

$$(a'_1, a'_2, b'_3, b'_4) = (ma_1, ma_2, m^{-1}b_3, m^{-1}b_4). \tag{3.8}$$

Note that M corresponds to a projective transformation stabilising any subsimplex of Δ. Since no face pairing is a reflection, we need to check local injectivity at the edges of the triangulation. It follows from inspection that local injectivity at the axis of A is equivalent to not both a_1 and a_2 to be contained in $(-\infty, 0]$, and local injectivity at the axis of B is equivalent to not both b_1 and b_2 to be contained in $(-\infty, 0]$. Thus, the corresponding pair of equivariant map and representation, (dev, ρ), can be replaced by a projective structure given by $(M \circ \mathrm{dev}, M \circ \rho \circ M^{-1})$ with $M = \mathrm{diag}(m, m, m^{-1}, m^{-1})$ that is locally injective at both the axes of A and B unless ($a_1, a_2 \leq 0$ and $b_1, b_2 \geq 0$) or ($a_1, a_2 \geq 0$ and $b_1, b_2 \leq 0$). But equation (3.7) has no solutions of this form.

It remains to check local injectivity around the axis of the commutator $C = ABA^{-1}B^{-1}$. The description of dev around the axis of C is obtained by gluing translates of Δ following a cyclic order. Knowing that dev is locally injective on the edges of A and B, we have that this process of gluing simplices around the axis of C turns either once or several times: namely either dev is locally injective or it is locally a cyclic branched covering, branched on the axis of C. Moreover, since we require that $C^3 = I_4$, by continuity the degree of this branched covering is locally constant along the parameter space. This degree is the criterion for choosing one of the components of the parameter space and discarding the other. Before that, we describe the parameter space, then we will check whether it is locally injective or a branched covering in a given point on each component.

The coordinates given in the statement of the theorem can be expressed in terms of classical invariants of the chosen lift of ρ:

$$\begin{aligned} w &= a_1 b_4 = 2 + \operatorname{tr} AB, \\ x &= a_1 b_3 = 2 + \operatorname{tr} A^{-1}B, \\ y &= a_2 b_3 = 2 + \operatorname{tr} A^{-1}B^{-1}, \\ z &= a_2 b_4 = 2 + \operatorname{tr} AB^{-1}, \end{aligned}$$

and (3.7) can be expressed in terms of these, giving the first equation given in (3.1). The second arises from $wy = a_1 a_2 b_3 b_4 = xz$. It can now be verified that the zero set of (3.1) and (3.2) corresponds to the quotient of the action of $\mathbb{R} \setminus \{0\}$ given in (3.8) on the set of all (a_1, a_2, b_3, b_4) subject to (3.7). We apply Lemma 3 to conclude that the parameter space has two components. Next we check wheter local injectivity of dev holds or not on a point on each component of this parameter space.

On the component X we consider $(w, x, y, z) = (3, 3, 3, 3)$: it is a straightforward computation that it corresponds to the holonomy of the hyperbolic structure. In particular the devoloping map is locally injective along the axis of $C = ABA^{-1}B^{-1}$.

On the other component of the parameter space we consider $(w, x, y, z) = (1, 1, 1, 1)$: we take $a_1 = a_2 = b_3 = b_4 = 1$. We look at the cycle of consecutive translates of Δ than gives a full turn around $e_1 e_4$, the axis of $C = ABA^{-1}B^{-1}$:

$$\begin{aligned}(\Delta, A(\Delta), AB(\Delta), ABA^{-1}(\Delta), C(\Delta), CA(\Delta), CAB(\Delta), CABA^{-1}(\Delta),& \\ C^2(\Delta), BAB^{-1}(\Delta), BA(\Delta), B(\Delta)).& \quad (3.9)\end{aligned}$$

This cycle has 12 elements, because C has order 3 and the edge has four representatives in Δ. The matrix C acts on this cycle by a cyclic permutation. By direct computation, when $a_1 = a_2 = b_3 = b_4 = 1$:

$$CAB = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Hence $CAB(\Delta) = \Delta$ and the cycle (3.9) turns at least twice around the edge, i.e. dev is locally a branched covering of degree ≥ 2. (We prove in the Remark 5 that the degree is precisely 2.) Hence we discard the whole component.

Let X be the component of the parameter space containing $(w, x, y, z) = (3, 3, 3, 3)$. We have shown that it parameterizes projective structures, we aim to show that these structures are properly convex. For that purpose, we look at the peripheral subgroups of $\pi_1^{orb}(O)$. There are two conjugacy classes of peripheral subgroups, corresponding to the ends of O and represented respectively by the stabilizer of v_1 and the stabilizer of v_4. The stabilizer of v_1 is the group generated by α and $\gamma = [\alpha, \beta]$, it is the fundamental group of the link of the end of O corresponding to v_1 (which is a turnover $S^2(3, 3, 3)$):

$$\pi_1^{orb}(S^2(3, 3, 3)) \cong \langle \alpha, \gamma \mid \alpha^3 = \gamma^3 = (\alpha^2\gamma)^3 = 1 \rangle.$$

Similarly, for v_4 the peripheral group is generated by β and γ. The maximal torsion-free subgroup of the peripheral group of v_1 has index 3, it is generated by $\alpha\gamma$ and $\gamma\alpha$ and it is isomorphic to $\mathbb{Z}^2$.

Lemma 4 *For $(w, x, y, z) \in X \setminus \{(3, 3, 3, 3)\}$, up to conjugation we have*

$$A = \rho(\alpha) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{and} \qquad AC = \rho(\alpha\gamma) = \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with $\lambda_i \in \mathbb{R}$, $\lambda_i > 0$ and $\lambda_1 \neq \pm 1$.

Assuming the lemma, we prove that the set of properly convex projective structures on X is open and closed. For openess, the holonomy of $\langle \alpha\gamma, \gamma\alpha \rangle \cong \mathbb{Z}^2$ preserves a flag in $\mathbb{R}^4$, for any structure on X. We can argue similarly for v_4 and the peripheral subgroup generated by β and γ. Hence, we may apply Theorem 0.2 of [5] to say that set of properly convex structures is open in X. For closedness, we apply Theorem 1.5 of [2]. Namely, the compact orbifolds obtained by truncating the ends have totally geodesic boundary (two turnovers). Ballas, Danciger and Lee show in [2] that these structures yield a convex structure on the double of the truncated orbifolds, and apply a theorem of Benoist for convex structures on closed orbifolds [3]. Alternatively, one can directly refer to Theorem 0.1 of the forthcoming work [7] to obtain closedness. Thus every structure on X is properly convex. $\square$

Proof of Lemma 4 First we claim that all eigenvalues of $AC = \rho(\alpha\gamma)$ are real, for each value of the parameters $(w, x, y, z) \in X$. For that purpose, we compute the characteristic polynomial of $\rho(\alpha\gamma) = AC$, by a long but explicit computation:

$$(\lambda - 1)(\lambda^3 + (-yx - w + 2\,x + 2\,y - z)\,\lambda^2 + (zw - 2\,w + x + y - 2\,z)\,\lambda - 1)$$

and we check that the discriminant of its degree three factor is nonnegative. To check that, we write w and z in terms of x and y from (3.1) and (3.2); this only parameterizes an open dense subset of X, but it is sufficient to determine the non-negativity of the discriminant. With this parameterization of a subset X, the discriminant is

$$\frac{\left(y^2 - 3\,y + 3\right)^2 \left(x^2 - 3\,x + 3\right)^2 (-y + x)^2 \left(x^2y^2 - 3\,x^2y - 3\,xy^2 + 3\,x^2 + 3\,xy + 3\,y^2\right)^2}{(xy - x - y)^6},$$

which is always nonnegative.

From the formula of the characteristic polynomial of $AC = \rho(\alpha\gamma)$, an elementary computation shows that for $(w, x, y, z) \in X \setminus \{(3, 3, 3, 3)\}$, AC has a (real) eigenvalue $\lambda_1 \neq 1$. Up to replacing it by $1/\lambda_1^2$, we assume that λ_1 has multiplicity 1. Let $p_1 \in \mathbb{RP}^3$ be the projective point fixed by AC corresponding to $E_{\lambda_1}(AC)$, the eigenspace of eigenvalue λ_1. Set $p_2 = A(p_1)$ and $p_3 = A(p_2)$, we have $p_1 = A(p_3)$. By construction,

$$p_1 = E_{\lambda_1}(AC), \quad p_2 = E_{\lambda_1}(A^2CA^2), \quad p_3 = E_{\lambda_1}(CA), \tag{3.10}$$

where E_λ denotes the eigenspace with eigenvalue λ. Next we aim to show that p_1, p_2 and p_3 span a 3-dimensional linear subspace. We first show that $p_1 \neq p_2$. Seeking a contradiction, assume $p_1 = p_2$, then also $p_3 = p_2 = p_1$. As the product of the matrices CA, A^2CA^2, AC is 1, (3.10) would imply that $\lambda_1^3 = 1$, which contradicts $\lambda_1 \neq 1$ and $\lambda_1 \in \mathbb{R}$. Let $V \subset \mathbb{R}^4$ be the span of the three affine lines p_1, p_2 and p_3. Commutativity between AC and CA implies that $AC(p_2) = p_2$ and $AC(p_3) = p_3$. Thus V contains three different lines that are invariant by AC. If V had dimension 2, then this would imply that AC acts as a multiple of the identity on V, contradicting that λ_1 has multiplicity 1.

Hence $\dim V = 3$, as claimed. As $\dim(\mathbb{R}^4/V) = 1$ and A and C have order three, both A and C act trivially on $\mathbb{R}^4/V$. Thus every matrix in $\langle A, C\rangle$ has an eigenvector equal to 1. By looking at the action of $\langle A, C\rangle$ on the vector subspace V, one can find a matrix in $\langle AC, CA\rangle \cong \mathbb{Z}^2$ whose restriction to V has all eigenvalues different from 1. This gives the fourth vector for a basis of $\mathbb{R}^4$ so that the matrices with respect to these basis are as claimed.

Finally, the assertion on the sign of the eigenvalues follows from continuity by looking at the representation at $(w, x, y, z) = (3, 3, 3, 3)$. □

Remark 5 When $(w, x, y, z) = (1, 1, 1, 1)$, $\langle A, B\rangle$ acts as $\mathrm{Alt}(5)$ by permuting five points in $\mathbb{RP}^3$. More precisely, we take $a_1 = a_2 = b_3 = b_4 = 1$, and both A and B preserve the set of five projective points $\{\langle e_1\rangle, \langle e_2\rangle, \langle e_3\rangle, \langle e_4\rangle, \langle e_1 + e_2 - e_3 - e_4\rangle\}$ in $\mathbb{RP}^3$. The action on this set determines the group action, as a projective transformation of $\mathbb{RP}^3$ is determined by the action on five points in general position. As both A and B act as cycles of order three on this set, and their fixed points are disjoint, this is the action of the alternating group $\mathrm{Alt}(5)$. It can be shown that these five points are the vertices of a tessellation of $\mathbb{RP}^3$ defined by the translates of Δ. The stabilizer of Δ has 4 elements and the tessellation consists of 15 tetrahedra (each tetrahedron shares its vertices with two other tetrahedra of the tessellation). Furthermore, one can check that the edges invariant by A or B are adjacent to three simplices, and the edge invariant by C is adjacent to six simplices (and therefore the developing map has degree $2 = 12/6$ around the edge of C). The quotient of $\mathbb{RP}^3$ by this action of A_5 is the spherical orbifold $J \times^*_J J$ in [8].

Remark 6 By the discussion in the proof of Theorem 2 and Remark 5, the component of the parameter space described by (3.1) and (3.2) that contains $(w, x, y, z) = (1, 1, 1, 1)$ corresponds to projective structures *branched* along a singular edge, as described by Ballas and Casella in [1].

Remark 7 The proof of Theorem 2 gives an interpretation of the coordinates in terms of traces. We wish to point out that there is an alternative interpretation in terms of *cross ratios* as follows. We look for cross ratios of points in the projective line

$$l = \langle e_3, e_4\rangle$$

that contains the edge e_3e_4, namely the fixed point set of B. We consider the projective plane

$$\Pi = \langle e_1, e_2, e_3\rangle,$$

so that both Π and $A(\Pi)$ contain a face of the 3-simplex Δ. The first three points of l that we consider are the respective intersection of l with Π, $A(\Pi)$ and $A^2(\Pi)$. Their homogeneous coordinates are

$$p_3 = \Pi \cap l = [0 : 0 : 1 : 0], \quad p_4 = A(\Pi) \cap l = [0 : 0 : 0 : 1], \text{ and } p_5 = A^2(\Pi) \cap l = [0 : 0 : 1 : 1].$$

Each coordinate x, y, z and w appears as a cross ratio of p_3, p_4 and p_5 with one of the following:

$$\begin{aligned} p_x &= AB^{-1}A(\Pi) \cap l = [0 : 0 : 1 : 1 - x] \\ p_y &= ABA(\Pi) \cap l = [0 : 0 : 1 : 1 - y], \\ p_z &= A^{-1}B(\Pi) \cap l = [0 : 0 : 1 - z : 1], \\ p_w &= A^{-1}B^{-1}(\Pi) \cap l = [0 : 0 : 1 - w : 1]. \end{aligned}$$

Namely:

$$\begin{aligned} x &= (p_x, p_3; p_5, p_4), \\ y &= (p_y, p_3; p_5, p_4), \\ z &= (p_z, p_4; p_5, p_3), \\ w &= (p_w, p_4; p_5, p_3). \end{aligned}$$

Remark 8 Ballas and Casella [1] study the same example but, instead of properly convex projective structures, they consider structures equipped with peripheral flags. Their deformation space is just a point, in contrast with this setting, because for the generic structures the peripheral subgroups do not have invariant flags.

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Departament de Matemàtiques, Universitat Autònoma de Barcelona and CRM, 08193 Bellaterra, Spain

School of Mathematics and Statistics, The University of Sydney, NSW 2006, Australia

Email: porti@mat.uab.cat stephan.tillmann@sydney.edu.au