This is a preprint of: "Limit cycles of planar piecewise differential systems with linear Hamiltonian saddles", Jaume Llibre, Clàudia Valls, *Symmetry (Basel)*, vol. 13, 1128:1–10, 2021.
DOI: [10.3390/sym13071128]

PIECEWISE DIFFERENTIAL SYSTEMS WITH ONLY LINEAR HAMILTONIAN SADDLES CAN CREATE LIMIT CYCLES?

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Abstract. We study the continuous and discontinuous planar piecewise differential systems formed only by linear Hamiltonian saddles and separated by one or two parallel straight lines. When these piecewise linear differential systems are either continuous or discontinuous and are separated by one straight-line, or are continuous and are separated by two parallel straight lines, we show that they have no limit cycles. On the other hand, when these piecewise linear differential systems are discontinuous and are separated by two parallel straight lines, we show that they can have at most one limit cycle. Moreover we show that this upper bound is reached by providing an example of such a system with one limit cycle.

1. Introduction and statement of the main results

The study of limit cycles (i.e., periodic orbits of a differential system in $\mathbb{R}^2$ isolated in the set of all periodic orbits of that system) goes back essentially to Poincaré [22] at the end of the nineteenth century and their existence became important in the applications because many phenomena are related with their existence, see for instancethe Van der Pol oscillator [25, 26], or the Belousov–Zhabotinskii chemical reaction [3, 27]. The study of continuous piecewise linear differential systems separated by one or two parallel straight lines appears in a natural way in control theory (see for instance the books [2, 9, 11, 12, 17, 21]). The easiest continuous piecewise linear differential systems are the ones formed by two linear differential systems and separated by a straight line and for such systems it is known that they can have at most one limit cycle (see for instance [7, 14, 18, 19] and the references therein).

In the present paper we first show that if both linear differential systems are Hamiltonian saddles, then the continuous piecewise linear differential system has not limit cycles.

Theorem 1. *A continuous piecewise linear differential system separated by one straight line formed by two Hamiltonian linear saddles has no limit cycles.*

The proof of Theorem 1 is given in section 3. Theorem 1 can be extended to continuous piecewise linear differential systems separated by two parallel straight lines and formed by three linear Hamiltonian saddles.

2000 *Mathematics Subject Classification.* 34C05, 34C23, 34C25, 34C29.
Key words and phrases. Limit cycles, linear Hamiltonian saddles, continuous piecewise linear differential systems, discontinuous piecewise differential systems, first integrals.

Theorem 2. *A continuous piecewise linear differential system separated by two parallel straight lines and formed by three Hamiltonian linear saddles has no limit cycles.*

Theorem 2 is proved in section 4.

The study of discontinuous piecewise linear differential systems separated by straight lines goes back to Andronov et al. [1] and nowadays they had attracted the attention of many authors mainly because these systems appear in mechanics, electrical circuits, economy, etc, (see for instance the books [6, 23], the surveys [20, 24] and the references therein).

In planar discontinuous piecewise linear differential systems we can have two kinds of limit cycles: the *slidding limit cycle* and the *crossing limit cycle*. A slidding limit cycle contains some segment of the lines of discontinuity, and a crossing limit cycle does not contain any of such segments. In the present paper, we only focus our study on the crossing limit cycles and in all the paper when we talk about limit cycles, we are referring to crossing limit cycles.

As for the continuous case, the easiest discontinuous piecewise linear differential systems are the ones formed by two linear differential systems and separated by a straight line. It is known that such systems can have three limit cycles but it is not known if three is the maximum number of limit cycles that such systems can exhibit (see [4, 5, 8, 10, 13, 15]).

We first show, as in the continuous time, that if both linear differential systems are linear Hamiltonian saddles, then the discontinuous piecewise linear differential system has no limit cycles.

Theorem 3. *A discontinuous piecewise linear differential system separated by one straight line and formed by two linear Hamiltonian saddles has no limit cycles.*

Theorem 3 is proved in section 5. It can be extended to the case in which the discontinuous piecewise linear differential system are separated by two parallel straight lines and in this case the upper bound on the number of limit cycles is one.

Theorem 4. *A discontinuous piecewise linear differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles can have at most one limit cycle. Moreover there are systems in this class having one limit cycle, see Figure 1.*

Theorem 4 is proved in section 6. We remark that it is clear from the proof of Theorem 4 that there are discontinuous piecewise linear differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles that do not have limit cycles.

The unique linear differential systems which are Hamiltonian are the linear centers and the linear saddles. In [16] the authors studied the limit cycles of the continuous and discontinuous piecewise linear differential systems formed only by centers and separated by one or two parallel straight lines. In the present paper we do a similar study for

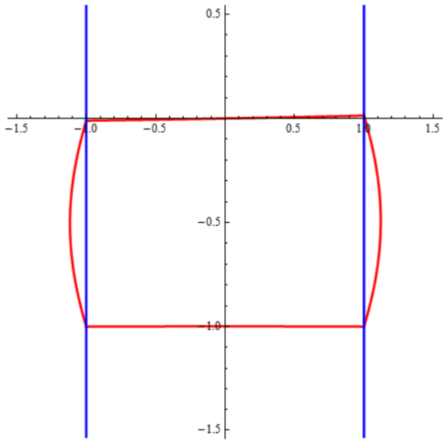

FIGURE 1. The limit cycle of the discontinuous piecewise differential system formed by the three linear Hamiltonian saddles (16), (17) and (18). This limit cycles is travelled in counterclockwise sense.

the continuous and discontinuous piecewise linear differential systems formed only by Hamiltonian linear saddles and separated by either one or two parallel straight lines.

The paper is organized such that in section 2 before the proof of the main theorems of the paper we present a normal form of a linear differential system having a linear weak saddle and we characterize the continuous and discontinuous piecewise linear differential system separated by any number κ of parallel straight lines for $\kappa \geq 1$ and formed by $\kappa + 1$ linear Hamiltonian saddles. We prove it for $k = 1$ and then we state it in the general case.

2. Preliminaries

The following lemma, provides a normal form for an arbitrary linear differential system having a linear Hamiltonian saddle.

Proposition 5. *A differential system having a linear Hamiltonian saddle can be written as*

$$\dot{x} = -bx - \frac{4b^2 - \omega^2}{4a} y + d, \quad \dot{y} = ax + by + c, \quad \text{when } a \neq 0,$$

or

$$\dot{x} = -bx + By + d, \quad \dot{y} = by + c, \quad \text{with } b \neq 0 \text{ when } a = 0.$$

Proof. Consider a general linear differential system

$$\dot{x} = Ax + By + d, \quad \dot{y} = ax + by + c,$$

in $\mathbb{R}^2$ and assume that it has a Hamiltonian saddle. The eigenvalues of this system are

$$\frac{A + b \pm \sqrt{4aB + (A-b)^2}}{2}.$$

Since this system has a Hamiltonian saddle $A+b=0$ and $4aB+(A-b)^2=\omega^2$ for some $\omega\neq 0$. Hence, $A=-b$ and $4aB+4b^2=\omega^2$. So if $a\neq 0$ we have $B=-(4b^2-\omega^2)/(4a)$ and if $a=0$ then $b\neq 0$. This completes the proof of the proposition. □

Corollary 6. *A differential system having a linear Hamiltonian saddle can be written as*

$$\dot{x}=-bx-\delta y+d,\quad \dot{y}=\alpha x+by+c, \tag{1}$$

with $\alpha\in\{0,1\}$.

Proof. Note that in view of Proposition 5 any linear differential system having a Hamiltonian saddle can be written as

$$\dot{x}=-bx-\delta y+d,\quad \dot{y}=ax+by+c, \tag{2}$$

where $\delta\in\mathbb{R}$ if $a=0$, and if $a\neq 0$ then $\delta=(4b^2-\omega^2)/(4a)$ with $\omega\neq 0$. It is possible to do a rescaling of the independent variable since it does not change the orbits and so it will not change the number of crossing limit cycles. After a rescaling of the independent variable of the form $\tau=at$ if $a\neq 0$ we can assume that equation (2) can be written as in (1) where $\delta\in\mathbb{R}$ and $\alpha=0$ if $a=0$, and if $a\neq 0$ then $\delta=(4b^2-\omega^2)/(4a)$ with $\omega\neq 0$ and $\alpha=1$. So, $\alpha\in\{0,1\}$. □

The first integral of system (1) is

$$H(x,y)=-\frac{\alpha}{2}x^2-bxy-\frac{\delta}{2}y^2-cx+dy. \tag{3}$$

3. Proof of Theorem 1

Assume that we have a continuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles. Without loss of generality we can assume that the straight line of continuity is $x=0$. It follows from Corollary 6 that we can assume that the systems in $x<0$ and $x>0$ are written in the form (1).

We have system

$$\dot{x}=-b_1x-\delta_1y+d_1,\quad \dot{y}=\alpha_1x+b_1y+c_1, \tag{4}$$

in $x<0$ with the first integral

$$H_1=-\frac{\alpha_1}{2}x^2-b_1xy-\frac{\delta_1}{2}y^2-c_1x+d_1y, \tag{5}$$

and system

$$\dot{x}=-b_2x-\delta_2y+d_2,\quad \dot{y}=\alpha_2x+b_2y+c_2, \tag{6}$$

in $x>0$ with the first integral

$$H_2=-\frac{\alpha_2}{2}x^2-b_2xy-\frac{\delta_2}{2}y^2-c_2x+d_2y. \tag{7}$$

Since we must have a continuous piecewise differential system, both systems must coincide on $x=0$ and so $\delta_1=\delta_2$, $d_1=d_2$, $b_1=b_2$ and $c_1=c_2$.

Note that if the continuous piecewise differential system has a periodic orbit candidate to be a limit cycle, because the two differential systems are linear Hamiltonian saddles such a periodic orbit must intersect the line $x = 0$ in exactly two points, namely $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Since H_1 and H_2 are two first integrals, we have that

$$H_1(0, y_1) = H_1(0, y_2) \quad \text{and} \quad H_2(0, y_1) = H_2(0, y_2), \tag{8}$$

that is

$$(2d_2 - \delta_2(y_1 + y_2))(y_1 - y_2) = 0.$$

So the periodic orbits of these continuous piecewise differential systems are in a continuum of periodic orbits and consequently this differential system has no limit cycles. This completes the proof of the theorem.

4. Proof of Theorem 2

Assume that we have a continuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles. Without loss of generality we can assume that the straight lines of discontinuity are $x = -1$ and $x = 1$. It follows from Corollary 6 that we can assume that the systems in $x < -1$, $-1 < x < 1$ and $x > 1$ are written as in (1).

We have system (4) with first integral (5) in $x < -1$, system (6) with first integral (7) in $-1 < x < 1$, and system

$$\dot{x} = -b_3 x - \delta_3 y + d_3, \quad \dot{y} = \alpha_3 x + b_3 y + c_3, \tag{9}$$

with first integral

$$H_3 = -\frac{\alpha_3}{2}x^2 - b_3 xy - \frac{\delta_3}{2}y^2 - c_3 x + d_3 y, \tag{10}$$

in $x > 1$.

Since we must have a continuous piecewise differential system, systems (4) and (6) must coincide in $x = -1$, and systems (6) and (9) must coincide in $x = 1$. Doing so we obtain

$$b_1 = b_2 = b_3, \quad d_1 = d_2 = d_3, \quad \delta_1 = \delta_2 = \delta_3, \quad c_1 = c_3 + \alpha_1 - 2\alpha_2 + \alpha_3, \quad c_2 = c_3 - \alpha_2 + \alpha_3.$$

Note that if the continuous piecewise differential systems has a periodic orbit candidate to be a limit cycle, because its three differential systems are linear Hamiltonian saddles such a periodic orbit must intersect each line $x = \pm 1$ in exactly two points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since H_1, H_2 and H_3 are three first integrals, we have that

$$\begin{aligned} H_1(-1, y_1) - H_1(-1, y_2) = 0, \quad H_2(-1, y_2) - H_2(1, y_3) = 0, \\ H_3(1, y_3) - H_3(1, y_4) = 0, \quad H_2(1, y_4) - H_2(-1, y_1) = 0, \end{aligned} \tag{11}$$

Doing so we get

$$\begin{aligned}
&(2b_3 + 2d_3 - \delta_3(y_1 + y_2))(y_1 - y_2) = 0,\\
&4c_3 - 4\alpha_2 + 4\alpha_3 + 2(b_3 + d_3)y_2 + 2(b_3 - d_3)y_3 - \delta_3(y_2^2 - y_3^2) = 0,\\
&(2b_3 - 2d_3 + \delta_3(y_3 + y_4))(y_3 - y_4) = 0,\\
&4c_3 - 4\alpha_2 + 4\alpha_3 + 2(b_3 + d_3)y_1 + 2(b_3 - d_3)y_4 - \delta_3(y_1^2 - y_4^2) = 0.
\end{aligned}$$

The solutions (y_1, y_2, y_3, y_4) of these last systems satisfying the necessary condition $y_1 < y_2$ and $y_3 < y_4$ are

$$y_2 = \frac{2(b_3 + d_3)}{\delta_3} - y_1, \quad y_3 = \frac{d_3 - b_3}{\delta_3} \pm \frac{\sqrt{\Delta}}{\delta_3}, \quad y_4 = \frac{d_3 - b_3}{\delta_3} \mp \frac{\sqrt{\Delta}}{\delta_3},$$

where $\Delta = (b_3 - d_3)^2 - 4(c_3 - \alpha_2 + \alpha_3)\delta_3 - 2(b_3 + d_3)\delta_3 y_1 + \delta_3^2 y_1^2$. Note that we only have two solutions taking the upper signs of y_3, y_4 or the lower signs of y_3, y_4. Hence all the periodic orbits of the continuous piecewise differential system are in a continuum of periodic orbits and consequently this differential system has no limit cycles. This completes the proof of the theorem.

5. Proof of Theorem 3

Assume that we have a discontinuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles. Without loss of generality we can assume that the straight line of continuity is $x = 0$. It follows from Corollary 6 that we can assume that the systems in $x < 0$ and $x > 0$ are written in the form (1).

We have system (4) with first integral (5) in $x < 0$, and system (6) with first integral (7) in $x > 0$.

Note that if the discontinuous piecewise differential system has a periodic orbit candidate to be a limit cycle, because its two differential systems are linear Hamiltonian saddles such a periodic orbit must intersect the line $x = 0$ in exactly two points, namely $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Since H_1 and H_2 are two first integrals we have that (8) must be satisfied, that is

$$(2d_1 - \delta_1(y_1 + y_2))(y_1 - y_2) = 0, \quad (2d_2 - \delta_2(y_1 + y_2))(y_1 - y_2) = 0$$

The solutions (y_1, y_2) of this last system satisfying the necessary condition $y_1 < y_2$ either do not exist if $d_1/\delta_1 \neq d_2/\delta_2$, or there is a continuum of solutions. So the periodic orbits of the discontinuous piecewise linear differential systems are in a continuum of periodic orbits, and consequently this differential system has no limit cycles. This completes the proof of the theorem.

6. Proof of Theorem 4

Assume that we have a discontinuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles. Without loss of generality we can assume that the straight lines of discontinuity are $x = -1$ and $x = 1$.

It follows from Corollary 6 that we can assume that the systems in $x < -1$, $-1 < x < 1$ and $x > 1$ are written as in (1).

We have system (4) with first integral (5) in $x < -1$, system (6) with first integral (7) in $-1 < x < 1$, and system (9) with Hamiltonian (10) in $x > 1$. Note that if the discontinuous piecewise differential systems has a periodic orbit candidate to be a limit cycle, because its three differential systems are linear Hamiltonian saddles such a periodic orbit must intersect each line $x = \pm 1$ in exactly two points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since H_1, H_2 and H_3 are three first integrals, we have that system (11) must be satisfied. Doing so we get

$$
\begin{aligned}
&(2(b_1 + d_1) - \delta_1(y_1 + y_2))(y_1 - y_2) = 0,\\
&4c_2 + 2(b_2 + d_2)y_2 + 2(b_2 - d_2)y_3 - \delta_2(y_2^2 - y_3^2) = 0,\\
&(2(b_3 - d_3) + \delta_3(y_3 + y_4))(y_3 - y_4) = 0,\\
&4c_2 + 2(b_2 + d_2)y_1 + 2(b_2 - d_2)y_4 - \delta_2(y_1^2 - y_4^2) = 0.
\end{aligned} \tag{12}
$$

Assume first that $\delta_1 = \delta_3 = 0$ the solutions of equations (12) are $d_1 = c_1 - b_1$, $d_3 = b_c + c_3$ and $y_3 = f_1(y_2)$, $y_4 = f_2(y_1)$ being f_1, f_2 functions in the variables y_2 and y_1, respectively. In this case the periodic orbits of the discontinuous piecewise linear differential systems are in a continuum of periodic orbits and consequently this differential system has no limit cycles.

Assume now that $\delta_1 = 0$ and $\delta_3 \neq 0$ the solutions of equations (12) are $d_1 = -b_1$,

$$
y_3 = \frac{2(d_3 - b_3)}{\delta_3} - y_4 \tag{13}
$$

and $y_2 = f_1(y_4)$, $y_1 = f_2(y_4)$ being f_1, f_2 functions in the variable y_4, respectively. In this case the periodic orbits of the discontinuous piecewise linear differential systems are in a continuum of periodic orbits and consequently this differential system has no limit cycles.

Assume now that $\delta_1 \neq 0$ and $\delta_3 = 0$ the solutions of equations (12) are $d_1 = -b_1$,

$$
y_1 = \frac{2(b_1 + d_1)}{\delta_1} - y_2, \tag{14}
$$

$d_3 = b_3$, and $y_2 = f_1(y_2)$, $y_1 = f_2(y_2)$ being f_1, f_2 functions in the variable y_2, respectively. In this case the periodic orbits of the discontinuous piecewise linear differential systems are in a continuum of periodic orbits and consequently this differential system has no limit cycles.

Finally, assume that $\delta_1\delta_3 \neq 0$. The solution of the first and third equations are (14) and (13). Introducing these solutions into the second and fourth equations in (12) we get

$$
\begin{aligned}
e_1 = {} & 4((b_3 - d_3)^2\delta_2 - (b_2 - d_2)(b_3 - d_3)\delta_3 + c_2\delta_3^2) + 2(b_2 + d_2)\delta_3^2 y_2\\
& + 2\delta_3(2(b_3 - d_3)\delta_2 - (b_2 - d_2)\delta_3)y_4 - \delta_2\delta_3^2(y_2^2 - y_4^2) = 0
\end{aligned} \tag{15}
$$

and

$$e_2 = 4(\delta_1((b_1+d_1)(b_2+d_2)+c_2\delta_1)+(b_1+d_1)^2\delta_2) - 2\delta_1((b_2+d_2)\delta_1 - 2(b_1+d_1)\delta_2)y_2 \\ + 2(b_2-d_2)\delta_1^2 y_4 - \delta_1^2\delta_2(y_2^2-y_4^2) = 0.$$

Taking

$$e_3 = \delta_1^2 e_1 - \delta_3^2 e_2 = 0$$

and solving in y_4 we get

$$y_4 = \frac{A_0}{A_1} + \frac{A_2}{A_1} y_2$$

where

$$A_0 = (b_3-d_3)^2\delta_1^2\delta_2 - (b_2-d_2)(b_3-d_3)\delta_1^2\delta_3 + (b_1+d_1)((b_1+d_1)\delta_2 - (b_2+d_2)\delta_1)\delta_3^2,$$
$$A_1 = \delta_1^2\delta_3((d_3-b_3)\delta_2 + (b_2-d_2)\delta_3),$$
$$A_2 = \delta_1((b_2+d_2)\delta_1 - (b_1+d_1)\delta_2)\delta_3^2,$$

whenever $A_1 \neq 0$. The case with $A_1 = 0$ yields $d_2 = b_2 + (d_3-b_3)\delta_2/\delta_3$. Introducing it into $e_3 = 0$ and solving in y_2 we obtain $y_2 = y_1 = (b_1 - c_1 + d_1)/\delta_1$ which is not possible. So, we an assume that $A_1 \neq 0$. Now introducing y_4 into the first equation in (15) and solving in y_2 we get

$$y_2 = y_{2\pm} = \frac{(b_1+d_1)}{\delta_1} \pm \frac{\sqrt{\Delta}}{A_4}$$

where

$$A_4 = \delta_1^2\delta_2\delta_3^2(b_3\delta_1\delta_2 - d_3\delta_1\delta_2 + 2d_2\delta_1\delta_3 - b_1\delta_2\delta_3 - d_1\delta_2\delta_3)(b_3\delta_1\delta_2 - d_3\delta_1\delta_2 \\ - 2b_2\delta_1\delta_3 + b_1\delta_2\delta_3 + d_1\delta_2\delta_3)),$$
$$\Delta = 4\delta_1^4\delta_2\delta_3^2((b_3-d_3)\delta_2 + (-b_2+d_2)\delta_3)^2((b_3-d_3)\delta_1\delta_2 + 2d_2\delta_1\delta_3 \\ - (b_1+d_1)\delta_2\delta_3)((b_3-d_3)\delta_1\delta_2 - 2b_2\delta_1\delta_3 + (b_1+d_1)\delta_2\delta_3)((b_3-d_3)^2\delta_1^2\delta_2 \\ - 2(b_2-d_2)(b_3-d_3)\delta_1^2\delta_3 + (2\delta_1((b_1+d_1)(b_2+d_2)+2c_2\delta_1) - (b_1+d_1)^2\delta_2)\delta_3^2),$$

whenever $A_4 \neq 0$ and if $A_4 = 0$ then there is at most one solution y_2.

When $A_4 \neq 0$, since

$$y_1 = y_{1\pm} = \frac{2(d_1+b_1)}{\delta_1} - y_{2\pm} = \frac{d_1+b_1}{\delta_1} \mp \frac{\sqrt{\Delta}}{A_4} = y_{2\mp},$$

there is at most one solution with $y_1 > y_2$ and $y_3 < y_4$. In summary, we have proved that at most we can have one limit cycle. Now we shall prove that the discontinuous piecewise linear differential system having a Hamiltonian saddle in each of these three pieces has also one limit cycle. This will complete the proof of Theorem 4.

The Hamiltonians of the three linear Hamiltonian systems with a saddle are

$$H_1(x,y) = 1606 + 5\sqrt{20954161} + 32256x + 5(-6130 + \sqrt{20954161})y + 7680x^2 - 7680y^2,$$
$$H_2(x,y) = 4615x - \sqrt{20954161}x + (\sqrt{20954161} - 7645)y + (-4573 + \sqrt{20954161})x^2 - \\ (\sqrt{20954161} - 4615)xy - 3072y^2,$$
$$H_3(x,y) = 519 + \sqrt{1317937} - (1024 + \sqrt{1317937})x - 505y + 512x^2 - 512y^2,$$

where the Hamiltonian system in the half-plane $x < -1$ is

$$\begin{aligned} \dot{x} &= -30650 + 5\sqrt{20954161} - 15360y, \\ \dot{y} &= -32256 - 15360x; \end{aligned} \tag{16}$$

the Hamiltonian system in the strip $-1 < x < 1$ is

$$\begin{aligned} \dot{x} &= -7645 + \sqrt{20954161} + (4615 - \sqrt{20954161})x - 6144y, \\ \dot{y} &= -4615 + \sqrt{20954161} + (9146 - 2\sqrt{20954161})x - (4615 + \sqrt{20954161})y; \end{aligned} \tag{17}$$

and the Hamiltonian system in the half-plane $x > 1$ is

$$\begin{aligned} \dot{x} &= -505 - 1024y, \\ \dot{y} &= 1024 + \sqrt{1317937} - 1024x. \end{aligned} \tag{18}$$

These three linear differential systems are saddles because the determinant of their linear part are -235929600, $13940638 - 3058\sqrt{20954161} < 0$ and -1048576, respectively.

The discontinuous piecewise differential system formed by the three linear differential systems (16), (17) and (18) in order to have one limit cycle intersecting the two discontinuous straight lines $x = \pm 1$ at the points $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, these points must satisfy system (11), and this system has a unique solution satisfying that $y_1 > y_2$ and $y_3 < y_4$, namely

$$(y_1, y_2, y_3, y_4) = \left(\frac{1}{3} \left(\frac{\sqrt{20954161}}{512} - \frac{2297}{256} \right), -1, -1, \frac{7}{512} \right).$$

Drawing the corresponding limit cycle associated to this solution we obtain the limit cycle of Figure 1.

7. Conclusions

We have studied the continuous and discontinuous planar piecewise differential systems formed only by linear Hamiltonian saddles separated by one or two parallel straight lines.

Such continuous piecewise differential systems appear in control theory, while the discontinuous ones appear in mechanics, electrical circuits, economy, etc. When these piecewise differential systems are continuous separated by either one or two parallel straight lines, we prove that they have no limit cycles. But when the piecewise differential systems are discontinuous separated two parallel straight lines, we show that they can have at most one limit cycle, and that there exist systems with either zero or one limit cycle. In the case in which the piecewise differential systems are discontinuous and separated only by one straight line, they cannot have limit cycles.

Acknowledgements

The first author is supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grant PID2019-104658GB-I00, the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2019.

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