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Limit Cycles of Planar Piecewise Differential Systems with Linear Hamiltonian Saddles

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Abstract: We provide the maximum number of limit cycles for continuous and discontinuous planar piecewise differential systems formed by linear Hamiltonian saddles and separated either by one or two parallel straight lines. We show that when these piecewise differential systems are either continuous or discontinuous and are separated by one straight line, or are continuous and are separated by two parallel straight lines, they do not have limit cycles. On the other hand, when these systems are discontinuous and separated by two parallel straight lines, we prove that the maximum number of limit cycles that they can have is one and that this maximum is reached by providing an example of such a system with one limit cycle. When the line of discontinuity of the piecewise differential system is formed by one straight line, the symmetry of the problem allows to take this straight line without loss of generality as the line $x = 0$. Similarly, when the line of discontinuity of the piecewise differential system is formed by two parallel straight lines due to the symmetry of the problem, we can assume without loss of generality that these two straight lines are $x = \pm 1$.

Keywords: crossing limit cycles; linear Hamiltonian saddles; continuous piecewise linear differential systems; discontinuous piecewise differential systems

MSC: 34C07; 34C25

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1. Introduction

The study of limit cycles of differential systems in $\mathbb{R}^2$ (i.e., periodic orbits of a differential system in $\mathbb{R}^2$ isolated in the set of all periodic orbits of that system) goes back essentially to Poincaré [1] at the end of the nineteenth century and their existence became important in application due to their relation with real world phenomena, see for instance the limit cycle of van der Pol equation [2,3], or the one of the Belousov–Zhavotinskii model [4,5].

Continuous piecewise linear differential systems separated by straight lines appear naturally in control theory (see, for instance, Refs. [6–11]). The easiest continuous piecewise linear differential systems are the ones formed by two linear differential systems separated by a straight line and for such systems it is well known that one is the upper bound on the number of limit cycles that they can have and that this upper bound is reached (see, for instance, Refs. [12–15] and the references therein).

The unique linear differential systems which are Hamiltonian are linear centers and linear saddles. In [16] the authors obtained the maximum number of limit cycles of continuous and discontinuous piecewise differential systems formed by linear centers and separated by either one or two parallel straight lines. In the present paper we do a symmetric study for continuous and discontinuous piecewise differential systems formed by linear Hamiltonian saddles and separated by either one or two parallel straight lines.

In the following two theorems we prove that if the continuous differential systems are formed by linear Hamiltonian saddles, then if they are separated by either one straight line (Theorem 1) or by two parallel straight lines (Theorem 2), they do not have limit cycles.

Theorem 1. *A continuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles does not have limit cycles.*

Theorem 2. *A continuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles does not have limit cycles.*

Theorems 1 and 2 are proved in Sections 3 and 4, respectively.

The study of discontinuous piecewise linear differential systems separated by straight lines goes back to Andronov et al. [17]. Nowadays, they have attracted the attention of many authors mainly because these systems appear in mechanics, electrical circuits, economy, etc. (see, for instance, the books [18,19], the surveys [20,21] and the references therein). To provide an explicit example, consider a planar Coulomb friction damping vibration system of the form

$$m\ddot{x} + kx + \mu mg\,\mathrm{sgn}(\dot{x}) = 0$$

where x is the displacement of the oscillator mass m, k is the stiffness coefficient of spring, μ is the coefficient Coulomb friction, g is the gravitational acceleration, and $\mathrm{sgn}(\dot{x})$ is the sign function of relative sliding speed $\dot{x} = y$ for $\dot{x} > 0$ and $\dot{x} < 0$. Note that this model is equivalent to

$$\begin{aligned} \dot{x} &= y, \quad \dot{y} = -\frac{k}{m}x - \mu g \quad \text{if } y > 0 \\ \dot{x} &= y, \quad \dot{y} = -\frac{k}{m}x + \mu g \quad \text{if } y < 0, \end{aligned}$$

which is a discontinuous planar piecewise linear differential system separated by the straight line $y = 0$.

In planar discontinuous piecewise linear differential systems, we can have two kinds of limit cycles: The sliding limit cycle and the crossing limit cycle. A sliding limit cycle contains a segment of the discontinuity lines, and a crossing limit cycle only contains isolated points of the discontinuity lines. In the present paper we only study crossing limit cycles, but for simplicity we shall call them limit cycles instead of crossing limit cycles.

The easiest discontinuous piecewise differential systems are the ones formed by two linear differential systems and separated by a straight line. There are examples of such systems with three limit cycles, but it is not known if three is the maximum number of limit cycles that such systems can exhibit (see [22–27]).

We first show, as for the continuous systems, that discontinuous piecewise differential systems formed by linear Hamiltonian saddles and separated by one straight line do not have limit cycles.

Theorem 3. *A discontinuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles does not have limit cycles.*

Theorem 3 is proven in Section 5. It can be extended to discontinuous systems separated by two parallel straight lines but in this case the upper bound on the number of limit cycles is one and this upper bound is reached. Thus, we have proved the extension of the 16th Hilbert problem on the maximum number of limit cycles for the polynomial differential systems of a given degree to the class of discontinuous piecewise differential systems formed by three linear Hamiltonian saddles separated by two parallel straight lines.

Theorem 4. *A discontinuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles can have at most one limit cycle. Moreover, there are systems in this class with one limit cycle, see Figure 1.*

Theorem 4 is proven in Section 6. We remark that it is clear from the proof of Theorem 4 that there are systems in the statement of Theorem 4 that do not have limit cycles.

The paper is organized in such a way that in Section 2, before the proof of the main theorems, we provide a normal form for an arbitrary differential system with linear Hamiltonian saddles.

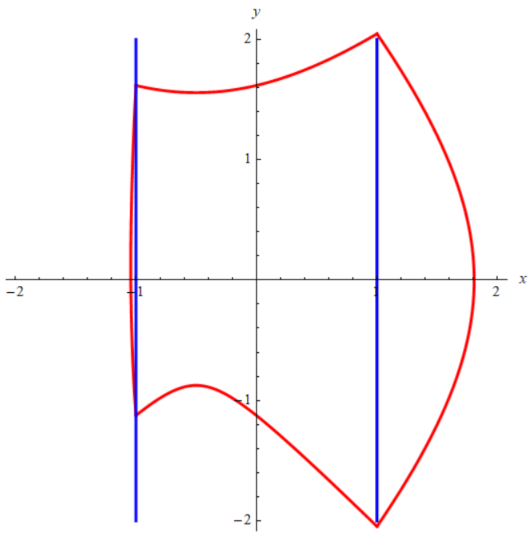

Figure 1. The limit cycle of the discontinuous piecewise differential system formed by the three linear Hamiltonian saddles (19), (20) and (21). This limit cycle has travelled in a counterclockwise sense.

2. Preliminaries

2.1. Piecewise Differential Systems

A piecewise differential system on $\mathbb{R}^2$ is a pair of C^r (with $r \geq 1$) differential systems in $\mathbb{R}^2$ separated by a smooth codimension one manifold Σ. The line of separation Σ of the piecewise differential system is defined by $\Sigma = h^{-1}(0)$, where $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a differentiable function with 0 as a regular value. Note that Σ is the separating boundary of the regions $\Sigma^+ = \{(x,y) \in \mathbb{R}^2 \,|\, h(x,y) > 0\}$ and $\Sigma^- = \{(x,y) \in \mathbb{R}^2 \,|\, h(x,y) < 0\}$. So the piecewise C^r vector field associated to a piecewise differential system with line of discontinuity Σ is

$$Z(x,y) = \begin{cases} X(x,y), & \text{if } h(x,y) \geq 0, \\ Y(x,y), & \text{if } h(x,y) \leq 0. \end{cases} \tag{1}$$

As usual, system (1) is denoted by $Z = (X, Y, \Sigma)$ or simply by $Z = (X, Y)$ when the separation line Σ is well understood.

When the piecewise differential system, with the vector field $Z = (X, Y)$ given in (1) satisfies $X(x,y) = Y(x,y)$ at all the points (x,y) such that $h(x,y) = 0$, we say that we have a continuous piecewise differential system. Otherwise, we say that we have a discontinuous piecewise differential system.

In order to establish a definition for the trajectories of a discontinuous piecewise differential system Z and investigate its behavior, we need a criterion for the transition of the orbits between Σ^+ and Σ^- across Σ. The contact between the vector field X (or Y) and the line of discontinuity Σ is characterized by the derivative of h in the direction of the vector field X, i.e.,

$$Xh(p) = \langle \nabla h(p), X(p) \rangle,$$

where $\langle .,. \rangle$ is the usual inner product in $\mathbb{R}^2$. The basic results of the discontinuous piecewise differential systems in this context were stated by Filippov [28]. We can divide the line of discontinuity Σ in the following sets:

(a) Crossing set: $\Sigma^c : \{p \in \Sigma : Xh(\mathbf{x}) \cdot Yh(\mathbf{x}) > 0\}$.
(b) Escaping set: $\Sigma^e : \{p \in \Sigma : Xh(\mathbf{x}) > 0 \text{ and } Yh(\mathbf{x}) < 0\}$.
(c) Sliding set: $\Sigma^s : \{p \in \Sigma : Xh(\mathbf{x}) < 0 \text{ and } Yh(\mathbf{x}) > 0\}$.

The escaping Σ^e or sliding Σ^s regions are, respectively, defined on points of Σ where both vector fields X and Y simultaneously point outwards or inwards from Σ while the interior of its complement in Σ defines the crossing region Σ^c (see Figure 2). The complementary of the union of these regions is the set formed by the tangency points between X or Y with Σ.

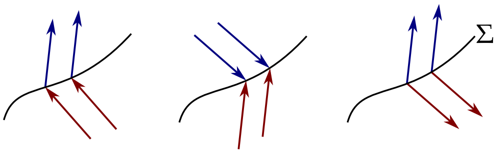

Figure 2. Crossing, sliding and escaping regions, respectively.

2.2. Linear Hamiltonian Saddles

A linear differential system which is Hamiltonian and has a saddle will be called simply a linear Hamiltonian saddle.

The following lemma provides a normal form for an arbitrary differential system with linear Hamiltonian saddles.

Proposition 1. *Differential systems with a linear Hamiltonian saddle can be written as*

$$\dot{x} = -bx - \delta y + d, \quad \dot{y} = \alpha x + by + c, \tag{2}$$

with $\alpha \in \{0,1\}$, $b, \delta, d, c \in \mathbb{R}$. Moreover, if $\alpha = 0$ then $c = 0$, and if $\alpha = 1$ then $\delta = b^2 - \omega^2$ with $\omega \neq 0$.

Proof. A general linear differential system in $\mathbb{R}^2$ can be written as

$$\dot{x} = a_0 x + b_0 y + c_0, \quad \dot{y} = a_1 x + b_1 y + c_1.$$

Consider that it has a Hamiltonian saddle. The eigenvalues of this system are

$$\frac{a_1 + b_0 \pm \sqrt{4a_0 b_1 + (a_1 - b_0)^2}}{2}.$$

This system has a Hamiltonian saddle $a_1 + b_0 = 0$ and $4a_0 b_1 + (a_1 - b_0)^2 = \omega^2$ for some $\omega \neq 0$. Hence, $a_1 = -b_0$ and $4a_0 b_1 + 4b_0^2 = \omega^2$ for some $\omega \neq 0$. So if $a_0 \neq 0$ we have $b_1 = -(4b_0^2 - \omega^2)/(4a_0)$ and if $a_0 = 0$, then $b_0 \neq 0$. Therefore any linear differential system with a Hamiltonian saddle can be written as

$$\dot{x} = -bx - \delta y + d, \quad \dot{y} = ax + by + c, \tag{3}$$

where $\delta \in \mathbb{R}$ if $a = 0$, and if $a \neq 0$ then $\delta = (4b^2 - \omega^2)/(4a)$ with $\omega \neq 0$. It is possible to do a rescaling of the independent variable because it does not change the orbits, and so it does not change the number of crossing limit cycles. After a rescaling of the independent variable of the form $\tau = at$ if $a \neq 0$, we can assume that Equation (3) can be written as in (2) where $\delta \in \mathbb{R}$ and $\alpha = 0$ if $a = 0$, and if $a \neq 0$, then $\delta = b^2 - \omega^2$ for some $\omega \neq 0$ and $\alpha = 1$. So $\alpha \in \{0,1\}$ and if $\alpha = 0$, then $b \neq 0$ and setting $Y = y - c/b$ we get

$$\dot{x} = -bx - \delta Y + d_2, \quad \dot{Y} = bY, \tag{4}$$

with $d_2 = d + \delta c/b$. This completes the proof of the proposition. □

A first integral of system (2) is

$$H(x,y) = -\frac{\alpha}{2}x^2 - bxy - \frac{\delta}{2}y^2 - cx + dy. \tag{5}$$

3. Proof of Theorem 1

Take a continuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles. Without loss of generality and taking into account the symmetry of the problem we can assume that the straight line of continuity is $x = 0$. It follows from Proposition 1 that we can assume that the systems in $x < 0$ and $x > 0$ are written in the form (2).

We have system

$$\dot{x} = -b_1 x - \delta_1 y + d_1, \quad \dot{y} = \alpha_1 x + b_1 y + c_1, \tag{6}$$

in $x < 0$ with the first integral

$$H_1 = -\frac{\alpha_1}{2}x^2 - b_1 xy - \frac{\delta_1}{2}y^2 - c_1 x + d_1 y, \tag{7}$$

and system

$$\dot{x} = -b_2 x - \delta_2 y + d_2, \quad \dot{y} = \alpha_2 x + b_2 y + c_2, \tag{8}$$

in $x > 0$ with the first integral

$$H_2 = -\frac{\alpha_2}{2}x^2 - b_2 xy - \frac{\delta_2}{2}y^2 - c_2 x + d_2 y. \tag{9}$$

Since the piecewise differential system is continuous, both systems must coincide on $x = 0$ and so $\delta_1 = \delta_2$, $d_1 = d_2$, $b_1 = b_2$ and $c_1 = c_2$.

Note that if the continuous piecewise differential system has a limit cycle taking into account that the two differential systems are linear Hamiltonian saddles, this system must have a periodic orbit intersecting the discontinuity line $x = 0$ in exactly two points, namely $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Since H_1 and H_2 are two first integrals, we have that

$$H_1(0,y_1) = H_1(0,y_2) \quad \text{and} \quad H_2(0,y_1) = H_2(0,y_2), \tag{10}$$

that is

$$(2d_2 - \delta_2(y_1 + y_2))(y_1 - y_2) = 0.$$

So all periodic orbits of these systems are in a continuum of periodic orbits yielding the non-existence of limit cycles. This completes the proof of the theorem.

4. Proof of Theorem 2

Assume that we have a continuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles. Without loss of generality and due to the symmetry of the problem we can assume that the straight lines of discontinuity are $x = -1$ and $x = 1$. It follows from Proposition 1 that we can assume that the systems in $x < -1$, $-1 < x < 1$ and $x > 1$ are written as in (2).

We have system (6) with first integral (7) in $x < -1$, system (8) with first integral (9) in $-1 < x < 1$, and system

$$\dot{x} = -b_3 x - \delta_3 y + d_3, \quad \dot{y} = \alpha_3 x + b_3 y + c_3, \tag{11}$$

with first integral

$$H_3 = -\frac{\alpha_3}{2}x^2 - b_3 xy - \frac{\delta_3}{2}y^2 - c_3 x + d_3 y, \tag{12}$$

in $x > 1$.

Since the piecewise differential system is continuous, systems (6) and (8) must coincide in $x = -1$, and systems (8) and (11) must coincide in $x = 1$. Doing so we obtain

$$b_1 = b_2 = b_3, \quad d_1 = d_2 = d_3, \quad \delta_1 = \delta_2 = \delta_3, \quad c_1 = c_3 + \alpha_1 - 2\alpha_2 + \alpha_3, \quad c_2 = c_3 - \alpha_2 + \alpha_3.$$

Note that if the continuous piecewise differential system has a limit cycle taking into account that the two differential systems are linear Hamiltonian saddles, this system must have a periodic orbit intersecting each discontinuity line $x = \pm 1$ in exactly two points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since H_1, H_2 and H_3 are three first integrals, we have that

$$\begin{aligned} H_1(-1,y_1) - H_1(-1,y_2) = 0, \quad H_2(-1,y_2) - H_2(1,y_3) = 0, \\ H_3(1,y_3) - H_3(1,y_4) = 0, \quad H_2(1,y_4) - H_2(-1,y_1) = 0, \end{aligned} \tag{13}$$

Doing so we get

$$\begin{aligned} &(2b_3 + 2d_3 - \delta_3(y_1 + y_2))(y_1 - y_2) = 0, \\ &4c_3 - 4\alpha_2 + 4\alpha_3 + 2(b_3 + d_3)y_2 + 2(b_3 - d_3)y_3 - \delta_3(y_2^2 - y_3^2) = 0, \\ &(2b_3 - 2d_3 + \delta_3(y_3 + y_4))(y_3 - y_4) = 0, \\ &4c_3 - 4\alpha_2 + 4\alpha_3 + 2(b_3 + d_3)y_1 + 2(b_3 - d_3)y_4 - \delta_3(y_1^2 - y_4^2) = 0. \end{aligned}$$

The solutions (y_1, y_2, y_3, y_4) of these last systems satisfying the necessary condition $y_1 < y_2$ and $y_3 < y_4$ are

$$y_2 = \frac{2(b_3 + d_3)}{\delta_3} - y_1, \quad y_3 = \frac{d_3 - b_3}{\delta_3} \pm \frac{\sqrt{\Delta}}{\delta_3}, \quad y_4 = \frac{d_3 - b_3}{\delta_3} \mp \frac{\sqrt{\Delta}}{\delta_3},$$

where $\Delta = (b_3 - d_3)^2 - 4(c_3 - \alpha_2 + \alpha_3)\delta_3 - 2(b_3 + d_3)\delta_3 y_1 + \delta_3^2 y_1^2$. Note that we only have two solutions taking the upper signs of y_3, y_4 or the lower signs of y_3, y_4. Hence all periodic orbits of these systems are in a continuum of periodic orbits yielding the non existence of limit cycles. This completes the proof of the theorem.

5. Proof of Theorem 3

Assume that we have a discontinuous piecewise differential system separated by one straight line and formed by two linear Hamiltonian saddles. Without loss of generality and due to the symmetry of the problem we can assume that the straight line of continuity is $x = 0$. It follows from Proposition 1 that we can assume that the systems in $x < 0$ and $x > 0$ are written in the form (2).

We have system (6) with first integral (7) in $x < 0$, and system (8) with first integral (9) in $x > 0$.

Note that, if the discontinuous piecewise differential system has a limit cycle taking into account that the two differential systems are linear Hamiltonian saddles, this system must have a periodic orbit intersecting the discontinuity line $x = 0$ in exactly two points, namely $(0, y_1)$ and $(0, y_2)$ with $y_1 < y_2$. Since H_1 and H_2 are two first integrals we have that (10) must be satisfied, that is

$$(2d_1 - \delta_1(y_1 + y_2))(y_1 - y_2) = 0, \quad (2d_2 - \delta_2(y_1 + y_2))(y_1 - y_2) = 0 \tag{14}$$

The solutions (y_1, y_2) of (14) satisfying the condition $y_1 < y_2$ either do not exist if $d_1/\delta_1 \neq d_2/\delta_2$, or there is a continuum of solutions. So the periodic orbits of the discontinuous piecewise linear differential systems are in a continuum of periodic orbits, and consequently this differential system has no limit cycles. This completes the proof of the theorem.

6. Proof of Theorem 4

Take a discontinuous piecewise differential system separated by two parallel straight lines and formed by three linear Hamiltonian saddles. Without loss of generality and due to the symmetry of the problem we can assume without loss of generality that the straight lines of discontinuity are $x = -1$ and $x = 1$. It follows from Proposition 1 that we can assume that the systems in $x < -1$, $-1 < x < 1$ and $x > 1$ are written as in (2).

We have system (6) with first integral (7) in $x < -1$, system (8) with first integral (9) in $-1 < x < 1$, and system (11) with Hamiltonian (12) in $x > 1$. Note that if the discontinuous piecewise differential system has a limit cycle taking into account that the two differential systems are linear Hamiltonian saddles, this system must have a periodic orbit intersecting each discontinuity line $x = \pm 1$ in exactly two points, namely $(-1, y_1)$, $(-1, y_2)$, $(1, y_3)$ and $(1, y_4)$, with $y_1 > y_2$ and $y_3 < y_4$. Since H_1, H_2 and H_3 are three first integrals, we have that system (13) must be satisfied. Doing so we get

$$
\begin{aligned}
&(2(b_1 + d_1) - \delta_1(y_1 + y_2))(y_1 - y_2) = 0,\\
&4c_2 + 2(b_2 + d_2)y_2 + 2(b_2 - d_2)y_3 - \delta_2(y_2^2 - y_3^2) = 0,\\
&(2(b_3 - d_3) + \delta_3(y_3 + y_4))(y_3 - y_4) = 0,\\
&4c_2 + 2(b_2 + d_2)y_1 + 2(b_2 - d_2)y_4 - \delta_2(y_1^2 - y_4^2) = 0.
\end{aligned}
\tag{15}
$$

Assume first that $\delta_1 = \delta_3 = 0$ the solutions of Equation (15) are $d_1 = c_1 - b_1$, $d_3 = b_c + c_3$ and $y_3 = f_1(y_2)$, $y_4 = f_2(y_1)$ being f_1, f_2 functions in the variables y_2 and y_1, respectively. In this case all periodic orbits of the these systems are in a continuum of periodic orbits yielding the non-existence of limit cycles.

Assume now that $\delta_1 = 0$ and $\delta_3 \neq 0$ the solutions of Equation (15) are $d_1 = -b_1$,

$$y_3 = \frac{2(d_3 - b_3)}{\delta_3} - y_4 \tag{16}$$

and $y_2 = f_1(y_4)$, $y_1 = f_2(y_4)$ being f_1, f_2 functions in the variable y_4, respectively. In this case all periodic orbits of these systems are in a continuum of periodic orbits yielding the non-existence of limit cycles.

Assume now that $\delta_1 \neq 0$ and $\delta_3 = 0$, the solutions of Equation (15) are $d_1 = -b_1$,

$$y_1 = \frac{2(b_1 + d_1)}{\delta_1} - y_2, \tag{17}$$

$d_3 = b_3$, and $y_2 = f_1(y_2)$, $y_1 = f_2(y_2)$ being f_1, f_2 functions in the variable y_2, respectively. In this case, all periodic orbits of the these systems are in a continuum of periodic orbits yielding the non-existence of limit cycles.

Finally, assume that $\delta_1\delta_3 \neq 0$. The solution of the first and third equations are (17) and (16). Introducing these solutions into the second and fourth equations in (15) we get

$$
\begin{aligned}
e_1 = {}& 4((b_3 - d_3)^2\delta_2 - (b_2 - d_2)(b_3 - d_3)\delta_3 + c_2\delta_3^2) + 2(b_2 + d_2)\delta_3^2 y_2\\
&+ 2\delta_3(2(b_3 - d_3)\delta_2 - (b_2 - d_2)\delta_3)y_4 - \delta_2\delta_3^2(y_2^2 - y_4^2) = 0
\end{aligned}
\tag{18}
$$

and

$$
\begin{aligned}
e_2 = {}& 4(\delta_1((b_1 + d_1)(b_2 + d_2) + c_2\delta_1) + (b_1 + d_1)^2\delta_2) - 2\delta_1((b_2 + d_2)\delta_1 - 2(b_1 + d_1)\delta_2)y_2\\
&+ 2(b_2 - d_2)\delta_1^2 y_4 - \delta_1^2\delta_2(y_2^2 - y_4^2) = 0.
\end{aligned}
$$

Taking

$$e_3 = \delta_1^2 e_1 - \delta_3^2 e_2 = 0$$

and solving in y_4 we get

$$y_4 = \frac{A_0}{A_1} + \frac{A_2}{A_1} y_2$$

where

$$
\begin{aligned}
A_0 &= (b_3 - d_3)^2\delta_1^2\delta_2 - (b_2 - d_2)(b_3 - d_3)\delta_1^2\delta_3 + (b_1 + d_1)((b_1 + d_1)\delta_2 - (b_2 + d_2)\delta_1)\delta_3^2,\\
A_1 &= \delta_1^2\delta_3((d_3 - b_3)\delta_2 + (b_2 - d_2)\delta_3),\\
A_2 &= \delta_1((b_2 + d_2)\delta_1 - (b_1 + d_1)\delta_2)\delta_3^2,
\end{aligned}
$$

whenever $A_1 \neq 0$. The case with $A_1 = 0$ yields $d_2 = b_2 + (d_3 - b_3)\delta_2/\delta_3$. Introducing this into $e_3 = 0$ and solving in y_2, we obtain $y_2 = y_1 = (b_1 - c_1 + d_1)/\delta_1$, which is not possible. So, we assume that $A_1 \neq 0$. Now introducing y_4 into the first equation in (18) and solving in y_2 we get

$$y_2 = y_{2\pm} = \frac{(b_1 + d_1)}{\delta_1} \pm \frac{\sqrt{\Delta}}{A_4}$$

where

$$A_4 = \delta_1^2\delta_2\delta_3^2(b_3\delta_1\delta_2 - d_3\delta_1\delta_2 + 2d_2\delta_1\delta_3 - b_1\delta_2\delta_3 - d_1\delta_2\delta_3)(b_3\delta_1\delta_2 - d_3\delta_1\delta_2 - 2b_2\delta_1\delta_3 + b_1\delta_2\delta_3 + d_1\delta_2\delta_3)),$$
$$\Delta = 4\delta_1^4\delta_2\delta_3^2((b_3 - d_3)\delta_2 + (-b_2 + d_2)\delta_3)^2((b_3 - d_3)\delta_1\delta_2 + 2d_2\delta_1\delta_3 - (b_1 + d_1)\delta_2\delta_3)((b_3 - d_3)\delta_1\delta_2 - 2b_2\delta_1\delta_3 + (b_1 + d_1)\delta_2\delta_3)((b_3 - d_3)^2\delta_1^2\delta_2 - 2(b_2 - d_2)(b_3 - d_3)\delta_1^2\delta_3 + (2\delta_1((b_1 + d_1)(b_2 + d_2) + 2c_2\delta_1) - (b_1 + d_1)^2\delta_2)\delta_3^2),$$

whenever $A_4 \neq 0$ and if $A_4 = 0$ then there is at most one solution y_2.

When $A_4 \neq 0$, since

$$y_1 = y_{1\pm} = \frac{2(d_1 + b_1)}{\delta_1} - y_{2\pm} = \frac{d_1 + b_1}{\delta_1} \mp \frac{\sqrt{\Delta}}{A_4} = y_{2\mp},$$

there is at most one solution with $y_1 > y_2$ and $y_3 < y_4$. In summary, an upper bound for the number of limit cycles is one.

To complete the proof of Theorem 4 we provide an example of a system in this class with one limit cycle. This will complete the proof of Theorem 4.

The Hamiltonians of the three linear Hamiltonian systems with a saddle are

$$H_1(x, y) = 16x + 2y + x^2 - \frac{309 - \sqrt{157881}}{48}xy + \frac{65\left(\sqrt{157881} - 405\right)}{1536}y^2,$$
$$H_2(x, y) = x - y + x^2 - y^2,$$
$$H_3(x, y) = 8x - x^2 + y^2,$$

where the Hamiltonian system in the half-plane $x < -1$ is

$$\dot{x} = 2 - \frac{1}{48}(309 - \sqrt{157881})x - \frac{65}{768}(405 - \sqrt{157881})y,$$
$$\dot{y} = -16 - 2x + \frac{1}{48}(309 - \sqrt{157881})y; \tag{19}$$

the Hamiltonian system in the strip $-1 < x < 1$ is

$$\dot{x} = -1 - 2y, \qquad \dot{y} = -1 - 2x; \tag{20}$$

and the Hamiltonian system in the half-plane $x > 1$ is

$$\dot{x} = 2y, \qquad \dot{y} = -8 + 2x. \tag{21}$$

These three linear differential systems are saddles because the determinants of their linear part are $-8569/48 + 7\sqrt{157881}/16 < 0$, -4 and -4, respectively.

The discontinuous piecewise differential system formed by the three linear differential systems (19)–(21) in order to have one limit cycle intersecting the two discontinuous straight lines $x = \pm 1$ at the points; these points must satisfy system (13), and this system has a unique solution satisfying that $y_1 > y_2$ and $y_3 < y_4$, namely

$$(y_1, y_2, y_3, y_4) = \left(\frac{16}{65} + \frac{\sqrt{4873}}{36\sqrt{2}}, \frac{16}{65} - \frac{\sqrt{4873}}{36\sqrt{2}}, -\frac{97\sqrt{4873}}{2340\sqrt{2}}, \frac{97\sqrt{4873}}{2340\sqrt{2}}\right).$$

Drawing the corresponding limit cycle associated to this solution we obtain the limit cycle of Figure 1.

7. Discussion

As far as we know this is the first paper which studies the piecewise differential systems formed by only linear Hamiltonian saddles.

8. Conclusions

We have studied continuous and discontinuous planar piecewise differential systems formed only by linear Hamiltonian saddles separated either by one or two parallel straight lines.

When these systems are continuous and are separated by either one or two parallel straight lines, we prove that they have no limit cycles. However, when the piecewise differential systems are discontinuous separated two parallel straight lines, we show that they can have at most one limit cycle, and that there exist systems with either zero or one limit cycle. In the case in which these systems are discontinuous and are separated only by one straight line, they cannot have limit cycles.

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