This is a preprint of: "Periodic orbits for a generalized Hénon-Heiles Hamiltonian system with an additional singular gravitational term", Jaume Llibre, Clàudia Valls, *Europhys. Lett. EPL*, vol. 134(60005), 2021.
DOI: [10.1209/0295-5075/134/60005]

PERIODIC ORBITS FOR A GENERALIZED HÉNON-HEILES HAMILTONIAN SYSTEM WITH AN ADDITIONAL SINGULAR GRAVITATIONAL TERM

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Abstract. Using the averaging theory of first order we study analytically the existence of two families of periodic orbits of a generalized Hénon-Heiles Hamiltonian system. Moreover we characterize when this generalized Hénon-Heiles Hamiltonian system has or has not a second C^1 first integral independent with the Hamiltonian.

1. Introduction and statement of results

The classical Hénon-Heiles Hamiltonian

$$H = \frac{1}{2}(p_x^2 + p_y^2 + x^2 + y^2) + x^2 y - \frac{y^3}{3}.$$

was introduced in 1964 as a model for studying the existence of a third integral of motion of a star in an rotating meridian plane of a galaxy in the neighborhood of a circular orbit [14] and it becomes a paradigm for nonlinear dynamics of Hamiltonian systems.

In this paper we study the generalized Hénon-Heiles Hamiltonian system with an additional singular gravitational term of the form

$$H_\varepsilon = \frac{1}{2}(p_x^2 + p_y^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{y^3}{3} - \varepsilon^5 \frac{1}{x^2 + y^2}, \tag{1}$$

where $\varepsilon \geq 0$ is a small parameter (note that when $\varepsilon = 0$ the Hamiltonian H_0 is the classical Hénon-Heiles Hamiltonian).

We study the periodic dynamics of the Hénon-Heiles Hamiltonian system with the additional singular gravitational term $1/(x^2 + y^2)$. The Hénon-Heiles modelizes how stars move around a galactic center. The addition of this singular gravitational term allows to modelize the motion of the stars in a pseudo or post-Newtonian dynamics. Thus this model allows to predict phenomena which cannot be detected by the classical Newtonian mechanics.

Other generalizations of the Hénon-Heiles Hamiltonian system (1) with different additional singular gravitational terms was introduced in [20] where the authors classify numerically sets of starting conditions for the trajectories. The additional singular gravitational term in our system provides more accurate and realistic dynamics of a test particle moving in the central region of a galaxy and creates a

1991 *Mathematics Subject Classification.* 34C35, 34D30.
Key words and phrases. Generalized Hénon-Heiles potential, periodic orbits, averaging theory.

singularity that cannot be modeled by the classical Hénon-Heiles Hamiltonian. This problem has attracted the activity of an extensive number of works (see for instance the works of [2, 3, 4, 5, 7, 8, 9, 10, 11, 12, 13, 14, 16, 18, 19].

The Hamiltonian system associated to the Hamiltonian (1) is

$$
\begin{aligned}
\dot{x} &= p_x, \\
\dot{y} &= p_y, \\
\dot{p}_x &= -x - 2xy - \varepsilon^5 \frac{2x}{(x^2+y^2)^2}, \\
\dot{p}_y &= -y - x^2 + y^2 - \varepsilon^5 \frac{2x}{(x^2+y^2)^2}.
\end{aligned} \tag{2}
$$

Here the dot denotes derivative with respect to the time t. After the equilibrium points the periodic orbits are the most simple interesting orbits of a differential system as they provide information on the motion in their neighborhoods (after studying their type of stability) and, if there are isolated periodic orbits having some multiplier distinct from 1 in the energy levels of the Hamiltonian system, this orbit prevents the existence of a second $\mathcal{C}^1$ first integral independent with the Hamiltonian, see details in section 2.

In section 2 we present a brief introduction to the averaging theory of first order, the notion of Liouville-Arnol'd integrability and a result on the existence of a second $\mathcal{C}^1$ first integral independent with the Hamiltonian for a Hamiltonian system.

Using the averaging theory we shall compute two families of periodic orbits of the generalized Hénon–Heiles Hamiltonian system (2), and we obtain the following result.

Theorem 1. *The generalized Hamiltonian system* (2) *for* ε *sufficiently small in each Hamiltonian level* $H = \varepsilon^2 h > 0$ *has two periodic solutions of the form*

$$
\begin{aligned}
&(x(t,\varepsilon), y(t,\varepsilon), p_x(t,\varepsilon), p_y(t,\varepsilon)) = \\
&(\varepsilon\sqrt{h}\cos t + O(\varepsilon^2), \pm\varepsilon\sqrt{h}\sin t + O(\varepsilon^2), -\varepsilon\sqrt{h}\sin t + O(\varepsilon^2), \pm\varepsilon\sqrt{h}\cos t + O(\varepsilon^2)).
\end{aligned}
$$

Theorem 1 is proved in section 3.

Using the existence of the two periodic orbits provided by Theorem 1, we can state and prove the second main result of the paper.

Theorem 2. *The generalized Hamiltonian system* (2) *for* ε *sufficiently small in each Hamiltonian level* $H = \varepsilon^2 h > 0$ *satisfies*

(a) *either it is Liouville-Arnol'd integrable and the gradients of the two constants of motion are linearly dependent on some points of the two periodic orbits found in Theorem 1,*
(b) *or it is not Liouville–Arnol'd integrable with any second* C^1 *first integral.*

Theorem 2 is proved in section 4.

2. Preliminaries

2.1. **The averaging theory of first order.** Here we summarize the averaging theory of first order for finding periodic orbits. See [17, Theorem 11.5] for the proof of the result presented in this section.

Theorem 3. *We assume that the non-autonomous differential system*

$$\dot{x}(t) = \varepsilon F_1(t, x) + \varepsilon^2 R(t, x, \varepsilon), \tag{3}$$

being $F_1 : \mathbb{R} \times D \to \mathbb{R}^n$ *and*$R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \to \mathbb{R}^n$ *functions T-periodic in the* t *variable, and* D *is an open subset of* $\mathbb{R}^n$*, satisfies that: the functions* $F_1, R, \partial F_1/\partial x, \partial^2 F_1/\partial x^2$ *and* $\partial R/\partial x$ *are defined, continuous and bounded by a constant* M *(independent of* ε*) in* $\mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f)$*. We define* $f_1 : D \to \mathbb{R}^n$ *as*

$$f_1(z) = \int_0^T F_1(s, z) ds.$$

If a satisfies $f_1(a) = 0$ *and* $|\partial f_1/\partial y||_{y=a} \neq 0$*, then for* $|\varepsilon| > 0$ *small enough, there exists a* $T-$*periodic solution* $x(\cdot, \varepsilon)$ *of system* (3) *verifying that* $x(0, \varepsilon) \to a$ *when* $\varepsilon \to 0$*.*

2.2. **Periodic orbits and the Liouville–Arnol'd integrability.** First we present some results on the Liouville–Arnol'd integrability of Hamiltonian systems with two degrees of freedom, and also on the periodic orbits of the differential equations, see more details in [1, 6] and [6, Subsection 7.1.2], respectively. We emphasize that these results work in Hamiltonian systems with an arbitrary number of degrees of freedom.

It is well known that a Hamiltonian system with Hamiltonian H of two degrees of freedom is *integrable in the sense of Liouville–Arnol'd* if it has a second first integral C independent with H (i.e. the gradient vectors of H and C are *independent* in all the points of the phase space except perhaps in a set of zero Lebesgue measure). A flow defined on a subspace of the phase space is *complete* if its solutions are defined for all time $t \in \mathbb{R}$.

Theorem 4. *Consider a Hamiltonian system with two degrees of freedom defined on the phase space* M *with Hamiltonian* H *and having a second first integral* C *independent with* H*. Let* $I_{hc} = \{p \in M : H(p) = h \text{ and } C(p) = c\} \neq \emptyset$ *be. If* (h, c) *is a regular value of the map* (H, C)*, then the following statements hold.*

- (a) I_{hc} *is a two dimensional submanifold of* M *invariant under the flow of the Hamiltonian system.*
- (b) *If the flow on a connected component* I^*_{hc} *of* I_{hc} *is complete, then* I^*_{hc} *is diffeomorphic either to the torus* $\mathbb{S}^1 \times \mathbb{S}^1$*, or to the cylinder* $\mathbb{S}^1 \times \mathbb{R}$*, or to the plane* $\mathbb{R}^2$*. If* I^*_{hc} *is compact, then the flow on it is always complete and* $I^*_{hc} \approx \mathbb{S}^1 \times \mathbb{S}^1$*.*
- (c) *Under the assumptions of statement* (b) *the flow on* I^*_{hc} *is conjugated to a linear flow on either* $\mathbb{S}^1 \times \mathbb{S}^1$*, or on* $\mathbb{S}^1 \times \mathbb{R}$*, or on* $\mathbb{R}^2$*.*

Consider the autonomous differential system

$$\dot{x} = f(x),$$

where $f\colon U \to \mathbb{R}^n$ is C^2, and U is an open subset of $\mathbb{R}^n$. We write its general solution as $\phi(t, x_0)$ with $\phi(0, x_0) = x_0 \in U$ and t belonging to its maximal interval of definition.

We say that $\phi(t, x_0)$ is *T-periodic* with $T > 0$ if and only if $\phi(T, x_0) = x_0$ and $\phi(t, x_0) \neq x_0$ for $t \in (0, T)$. The periodic orbit associated to the periodic solution $\phi(t, x_0)$ is $\gamma = \{\phi(t, x_0), t \in [0, T]\}$. The variational equation associated to the T-periodic solution $\phi(t, x_0)$ is

$$\dot{M} = \left(\frac{\partial f(x)}{\partial x} \Big|_{x=\phi(t,x_0)} \right) M, \tag{4}$$

where M is an $n \times n$ matrix. The *monodromy matrix* associated to the T-periodic solution $\phi(t, x_0)$ is the solution $M(T, x_0)$ of (4) satisfying that $M(0, x_0)$ is the identity matrix. The eigenvalues λ of the monodromy matrix associated to the periodic solution $\phi(t, x_0)$ are called the *multipliers* of the periodic orbit.

For an autonomous differential system, one of the multipliers is always 1, and its corresponding eigenvector is tangent to the periodic orbit.

A periodic solution of an autonomous Hamiltonian system always has two multipliers equal to one. One multiplier is 1 because the Hamiltonian system is autonomous, and the other 1 is due to the existence of the first integral given by the Hamiltonian.

Theorem 5. *If a Hamiltonian system with two degrees of freedom and Hamiltonian H is Liouville–Arnol'd integrable, and C is a second first integral such that the gradients of H and C are linearly independent at each point of a periodic orbit of the system, then all the multipliers of this periodic orbit are equal to* 1.

Theorem 5 is due to Poincaré [16] (see section 36), and see also [15]. It provides a tool for studying the non Liouville–Arnol'd integrability, independently of the class of differentiability of the second first integral. The main problem for applying this theorem is to find periodic orbits having multipliers different from 1.

3. Proof of Theorem 1

It is well known that for Hamiltonian system with more than one degree of freedom their periodic orbits generically live on cylinders filled out of periodic orbits, see [1]. Therefore it is not possible to apply directly Theorem 3 to a Hamiltonian system, because then the determinant of the Jacobian matrix of the function f_1 at some of its zeros will be always zero. So Theorem 3 must be applied to every fixed Hamiltonian level where generically the periodic orbits appear isolated.

From the statement of Theorem 3 we see that the differential system associated to the Hamiltonian system where we want to apply such a theorem needs to have a small parameter, so we will do the rescaling $(x, y, p_x, p_y) = \varepsilon(X, Y, P_X, P_Y)$ using the parameter ε. In these new variables system (2) becomes

$$
\begin{aligned}
\dot{X} =& \; p_X, \\
\dot{Y} =& \; p_Y, \\
\dot{P}_X \; =& -X - \varepsilon 2X\Big(\frac{1}{(X^2+Y^2)^2} + Y\Big), \\
\dot{P}_Y \; & -Y - \varepsilon\Big(\frac{2Y}{(X^2+Y^2)^2} + X^2 - Y^2\Big),
\end{aligned} \tag{5}
$$

and the Hamiltonian in these new variables becomes

$$
H = \frac{1}{2}(P_X^2 + P_Y^2 + X^2 + Y^2) - \varepsilon\frac{3 - 3X^4Y - 2X^2Y^3 + Y^5}{3(X^2+Y^2)}.
$$

From the statement of Theorem 3 we see that the differential system where we want to apply such a theorem needs to be periodic in the independent variable. Therefore in the Hamiltonian system (5) we change the variables (X, Y, P_X, P_Y) to $(r, \theta, \rho, \alpha)$ given by $(X, Y, P_X, P_Y) = (r\cos\theta, \rho\cos(\theta+\alpha), r\sin\theta, \rho\sin(\theta+\alpha))$, and later on we will take as the new independent variable the θ. In these new variables system (5) becomes

(6)

$$
\begin{aligned}
\dot{r} =& \; \varepsilon r\sin\theta\cos\theta\left(-2\rho\cos(\alpha+\theta) - \frac{2}{\left(\rho^2\cos^2(\alpha+\theta) + r^2\cos^2\theta\right)^2}\right), \\
\dot{\theta} =& \; -1 + \varepsilon\cos^2\theta\left(-2\rho\cos(\alpha+\theta) - \frac{2}{\left(\rho^2\cos^2(\alpha+\theta) + r^2\cos^2\theta\right)^2}\right), \\
\dot{\rho} =& \; \varepsilon\sin(\alpha+\theta)\left(-r^2\cos^2\theta + \rho^2\cos^2(\alpha+\theta) - \frac{2\rho\cos(\alpha+\theta)}{\left(\rho^2\cos^2(\alpha+\theta) + r^2\cos^2\theta\right)^2}\right), \\
\dot{\alpha} =& \; \varepsilon\left(\cos^2\theta\left(\frac{2}{\left(\rho^2\cos^2(\alpha+\theta) + r^2\cos^2\theta\right)^2} + \left(2\rho - \frac{r^2}{\rho}\right)\cos(\alpha+\theta)\right)\right. \\
& \left. + \cos^2(\alpha+\theta)\left(\rho\cos(\alpha+\theta) - \frac{2}{\left(\rho^2\cos^2(\alpha+\theta) + r^2\cos^2\theta\right)^2}\right)\right).
\end{aligned}
$$

This system is not Hamiltonian but it has the first integral (that we also name as H)

$$
\begin{aligned}
H = \frac{1}{2}(\rho^2 + r^2) + & \frac{\epsilon}{3\left(\rho^2\cos^2(\alpha+\theta) + r^2\cos^2\theta\right)}\left(-\rho^5\cos^5(\alpha+\theta)\right. \\
& \left. + 2\rho^3 r^2\cos^2\theta\cos^3(\alpha+\theta) + 3\rho r^4\cos^4\theta\cos(\alpha+\theta) - 3\right).
\end{aligned}
$$

In order to apply Theorem 3 we take θ as the new independent variable. Hence the differential system (6) becomes

(7)
$$
\begin{aligned}
r' = & \ \varepsilon\left(\frac{2r\sin\theta\cos\theta}{(\rho^2\cos^2(\alpha+\theta)+r^2\cos^2\theta)^2}+2\rho r\sin\theta\cos\theta\cos(\alpha+\theta)\right)+O(\varepsilon^2),\\
\rho' = & \ \epsilon\Big(-\rho^2\sin(\alpha+\theta)\cos^2(\alpha+\theta)+\frac{2\rho\sin(\alpha+\theta)\cos(\alpha+\theta)}{(\rho^2\cos^2(\alpha+\theta)+r^2\cos^2\theta)^2}\\
& +r^2\cos^2\theta\sin(\alpha+\theta)\Big)+O(\varepsilon^2),\\
\alpha' = & \ \varepsilon\Big(-\rho\cos^3(\alpha+\theta)-2\rho\cos^2\theta\cos(\alpha+\theta)+\frac{2\cos^2(\alpha+\theta)}{(\rho^2\cos^2(\alpha+\theta)+r^2\cos^2\theta)^2}\\
& -\frac{2\cos^2\theta}{(\rho^2\cos^2(\alpha+\theta)+r^2\cos^2\theta)^2}+\frac{r^2\cos^2\theta\cos(\alpha+\theta)}{\rho}\Big)+O(\varepsilon^2),
\end{aligned}
$$

here the prime denotes derivative with respect to the variable θ. Note that system (7) is 2π-periodic in the variable θ.

System (7) in the variables (r,ρ,α) (with $\rho=\rho_0+\varepsilon\rho_1+O(\varepsilon^2)$) has the first integral

(8)
$$
\begin{aligned}
H = & -h+\tfrac{1}{2}\left(\rho_0^2+r^2\right)+\varepsilon\rho_0\rho_1+\varepsilon\frac{1}{3\left(\rho_0^2\cos^2(\alpha+\theta)+r^2\cos^2\theta\right)}(-3\\
& +3\rho_0 r^4\cos^4\theta\cos(\alpha+\theta)+2\rho_0^3r^2\cos^2\theta\cos^3(\alpha+\theta)-\rho_0^5\cos^5(\alpha+\theta))+O(\varepsilon^2).
\end{aligned}
$$

We fix the value of the first integral H at $\varepsilon^2 h>0$ in order that the averaging theory can provide information about the periodic orbits of system (7). Computing ρ from equation (8) we obtain

$$
\begin{aligned}
\rho = & \ \sqrt{2h-r^2}+\frac{\varepsilon\big(3+\sqrt{2h-r^2}\cos(\alpha+\theta)\big)}{3\sqrt{2h-r^2}(2h\cos^2(\alpha+\theta)+r^2\sin\alpha\sin(\alpha+2\theta))}\\
& \cdot\big(-3r^4\cos^4\theta+2r^2(r^2-2h)\cos^2\theta\cos^2(\alpha+\theta)\\
& +(r^2-2h)^2\cos^4(\alpha+\theta)\big)+O(\varepsilon^2).
\end{aligned}
$$

Now substituting ρ in system (7), this differential system reduces to

(9)
$$
\begin{aligned}
r' = & \ \varepsilon\left(\frac{2r\sin\theta\cos\theta}{((2h-r^2)\cos^2(\alpha+\theta)+r^2\cos^2\theta)^2}+2r\sqrt{2h-r^2}\sin\theta\cos\theta\cos(\alpha+\theta)\right)\\
& +O(\varepsilon^2)=F_{11}(\theta,r,\alpha)+O(\varepsilon^2),\\
\alpha' = & \ \varepsilon\Big(-\sqrt{2h-r^2}\cos^3(\alpha+\theta)+\frac{2\cos^2(\alpha+\theta)}{((2h-r^2)\cos^2(\alpha+\theta)+r^2\cos^2\theta)^2}\\
& -2\sqrt{2h-r^2}\cos^2\theta\cos(\alpha+\theta)+\frac{r^2\cos^2\theta\cos(\alpha+\theta)}{\sqrt{2h-r^2}}\\
& -\frac{2\cos^2\theta}{(2h-r^2)\cos^2(\alpha+\theta)+r^2\cos^2\theta)^2}\Big)+O(\varepsilon^2)=F_{12}(\theta,r,\alpha)+O(\varepsilon^2).
\end{aligned}
$$

Now system (9) satisfies all the assumptions for applying Theorem 3, i.e. it has the form (3) with $x = (r, \alpha)$, $t = \theta$, $T = 2\pi$ and $F_1 = (F_{11}, F_{12})$. The averaged functions of F_{11} and F_{12} in the period 2π are

$$\begin{aligned} f_{11}(r,\alpha) = & \frac{2\pi \cot\alpha}{r^2\sqrt{\sin^2\alpha(2h-r^2)}}, \\ f_{12}(r,\alpha) = & \frac{4\pi \csc^2\alpha(r^2-h)\sqrt{\sin^2\alpha(2h-r^2)}}{r(r^3-2hr)^2}. \end{aligned}$$

We must compute the zeros (r^*, α^*) of $f_1(r,\alpha) = (f_{11}(r,\alpha), f_{12}(r,\alpha))$, and to verify that the Jacobian determinant

$$|D_{r,\alpha} f_1(r^*, \alpha^*)| \neq 0. \tag{10}$$

Solving the system $f_1(r,\alpha) = 0$ of two equations and two unknowns r and α we get two solutions (r^*, α^*) with $r^* \geq 0$, namely

$$(\sqrt{h}, \pi/2), \quad (\sqrt{h}, -\pi/2). \tag{11}$$

For these two solutions the Jacobian (11) is $196h^2\pi^2/9 \neq 0$. So, by Theorem 3 these two solutions provide two periodic solutions $(r_\pm(\theta,\varepsilon), \alpha_\pm(\theta,\varepsilon))$ of the differential system (9) with ε sufficiently small such that $(r_\pm(0,\varepsilon), \alpha_\pm(0,\varepsilon)) \to (\sqrt{h}, \pm\pi/2)$ when $\varepsilon \to 0$.

Going back to the differential system (7) we get for this system with ε sufficiently small two periodic solutions $(r_\pm(\theta,\varepsilon), \rho_\pm(\theta,\varepsilon), \alpha_\pm(\theta,\varepsilon))$ such that $(r_\pm(0,\varepsilon), \rho_\pm(0,\varepsilon),$ $\alpha_\pm(0,\varepsilon)) \to (\sqrt{h}, \sqrt{h}, \pm\pi/2)$ when $\varepsilon \to 0$.

Again going back to the differential system (6) we obtain for this system with ε sufficiently small two periodic solutions $(r_\pm(t,\varepsilon), \theta(t,\varepsilon), \rho_\pm(t,\varepsilon), \alpha_\pm(t,\varepsilon))$ such that $(r_\pm(0,\varepsilon), \theta(0,\varepsilon), \rho_\pm(0,\varepsilon), \alpha_\pm(0,\varepsilon)) \to (\sqrt{h}, -t, \sqrt{h}, \pm\pi/2)$ when $\varepsilon \to 0$.

Finally going back to the initial Hamiltonian system (2) we have for this system with ε sufficiently small two periodic solutions

$$\begin{aligned} &(x(t,\varepsilon), y(t,\varepsilon), p_x(t,\varepsilon), p_y(t,\varepsilon)) = \\ &(\varepsilon\sqrt{h}\cos t + O(\varepsilon^2), \pm\varepsilon\sqrt{h}\sin t + O(\varepsilon^2), -\varepsilon\sqrt{h}\sin t + O(\varepsilon^2), \pm\varepsilon\sqrt{h}\cos t + O(\varepsilon^2)), \end{aligned}$$

in each positive Hamiltonian level $H = \varepsilon^2 h$. This completes the proof of Theorem 1.

4. Proof of Theorem 2

Consider the two periodic solutions stated in Theorem 1. Their corresponding Jacobian $196h^2\pi^2/9 \neq 1$ playing with the energy level h. Since this Jacobian is the product of the four multipliers of these periodic solutions with two of them always equal to 1, the remainder two multipliers cannot be equal to 1. Hence, by Theorem 5, either the generalized Hénon–Heiles systems cannot be Liouville–Arnol'd integrable with any second first integral C, or it is Liouville-Arnol'd integrable and the differentials of H and C are linearly dependent on some points of these periodic orbits. Therefore the theorem is proved.

5. Conclusions

We study the periodic dynamics of the Hénon-Heiles Hamiltonian system with the additional singular gravitational term $1/(x^2+y^2)$. The Hénon-Heiles Hamiltonian modelizes how stars move around a galactic center. The addition of this singular gravitational term allows to modelize the motion of the stars in a pseudo or post-Newtonian dynamics. Thus this model allows to predict phenomena which cannot be detected by the classical Newtonian mechanics.

Using the averaging theory of first order we study analytically the existence of two families of periodic orbits of this generalized Hénon-Heiles Hamiltonian system, see Theorem 1. Moreover we characterize when this generalized Hénon-Heiles Hamiltonian system has or has not a second C^1 first integral independent with the Hamiltonian using the obtained periodic orbits, see Theorem 2.

Acknowledgments

The first author is supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación grant PID2019-104658GB-I00 (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911. The second author is partially supported by FCT/Portugal through UID/MAT/04459/2019.

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