This is a preprint of: “Limit cycles of piecewise polynomial differential systems with the discontinuity line xy=0”, Tao Li, Jaume Llibre, *Commun. Pure Appl. Anal.*, vol. 20(11), 3871–3893, 2021.
DOI: [10.3934/cpaa.2021136]

LIMIT CYCLES OF PIECEWISE POLYNOMIAL DIFFERENTIAL SYSTEMS WITH THE DISCONTINUITY LINE $xy = 0$

TAO LI[1] AND JAUME LLIBRE[2]

Abstract. In this paper we study the maximum number of limit cycles bifurcating from the periodic solutions of the period annulus of the center $\dot{x} = -y((x^2+y^2)/2)^m, \dot{y} = x((x^2+y^2)/2)^m$ with $m \geq 0$ under discontinuous piecewise polynomial perturbations of degree n with four zones separated by the discontinuity set $\Sigma = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$. For such perturbations of degree n we provide an upper bound for the maximum number of the bifurcated limit cycles using the averaging theory up to an arbitrary order N, and this upper bound is reached at least for orders one and two. In particular, if $m = 0$, namely the center is linear, we improve the result of [Nonlinear Analysis: Real World Applications **41** (2018), 384–400] providing more limit cycles that bifurcate from the unperturbed periodic orbits of the period annulus of the center.

1. Introduction and statement of main result

Hilbert's 16th problem, an important subject in the qualitative theory of differential systems, asks for the maximum number of limit cycles that planar polynomial differential systems with a fixed degree can have. Since David Hilbert [15] proposed it in 1900, a large number of works were devoted to the study of this problem, see the survey paper [19]. But it is still an open problem up to now, even for quadratic differential systems. As Hilbert's 16th problem turns out to be extremely difficult, some researchers have particularized it to identify the maximum number of limit cycles bifurcating from a periodic annulus, when we perturb it inside the class of all planar polynomial differential systems with a fixed degree $n \geq 1$. In essence, this is the weak Hilbert's 16th problem, see [2, 16, 19]. If this periodic annulus is formed by the linear center $\dot{x} = -y, \dot{y} = -x$, Iliev [16] proved that $[3(n-1)/2]$ is a lower bound for the maximum number, where $[\cdot]$ denotes the integer part function. They also gave that $[N(n-1)/2]$ is an upper bound for the maximum number using the Melnikov method of order N. However the exact maximum number is unclear so far except for some special perturbations, e.g. Liénard family [14, 21]. In 2010, Buică, Giné and Llibre [6] extended the work of Iliev [16] considering the polynomial perturbations of the center

$$\dot{x} = -y\left(\frac{x^2+y^2}{2}\right)^m, \qquad \dot{y} = x\left(\frac{x^2+y^2}{2}\right)^m \tag{1}$$

with $m \geq 0$. Note that $H(x,y) = (x^2+y^2)/2$ is a first integral of system (1).

Discontinuous events are widespread in the real world, such as stick-slip motion in oscillators with dry friction [12, 20], switching in electronic circuits [4, 5], and impact in mechanical

2010 *Mathematics Subject Classification.* 34C29, 34C25, 34C05.
Key words and phrases. Averaging method, Hilbert's 16th problem, limit cycle bifurcation, piecewise polynomial system.

devices [4, 27]. To mathematically depict them, many models were established by discontinuous piecewise smooth differential systems, which consist of multiple smooth differential systems defined on different regions separated by smooth lines or curves, usually called *discontinuity set* [4]. The solution of such systems can be defined by the Filippov convention [13].

For discontinuous piecewise smooth differential systems, an isolated periodic orbit is said to be a *crossing limit cycle* if it intersects the discontinuity set only at the so-called *crossing region*, a subset of the discontinuity set where both vector fields are transverse to it and their normal components have the same sign, see [18] for more details. Analogous to polynomial differential systems, we can extend the weak Hilbert's 16th problem to discontinuous piecewise polynomial differential systems, asking for the maximum number of crossing limit cycles bifurcating from a periodic annulus under discontinuous piecewise polynomial perturbations of a fixed degree. The simplest case is the piecewise polynomial perturbations of the linear center $\dot{x} = -y, \dot{y} = x$ in two zones separated by a straight line. In this case for any given degree $n \geq 1$ Buzzi, Lima and Torregrosa [7] proved that $Nn - 1$ is an upper bound for the maximum number using Melnikov method up to order N. Moreover, they showed that this upper bound is reached for orders one and two, which means that $2n - 1$ is a lower bound. According to the works [8] (resp. [24]), this lower bound can be improved to be 3 (resp. 8 and 13) for $n = 1$ (resp. $n = 2$ and $n = 3$) up to a study of order seven (resp. five).

Since models in applications with the discontinuity that is located on multiple straight lines or curves are ubiquitous, see e.g. [1, 12, 26], this encourages us to research the crossing limit cycles of discontinuous piecewise smooth differential systems having multiple straight lines or curves as the discontinuity set, e.g. [10, 11, 25, 29, 31, 32] for some contributions.

In this paper, motivated by real applications and the works [7, 16], we bring the weak Hilbert's 16th problem to discontinuous piecewise polynomial perturbations of the center (1) in four zones $\mathcal{R}_k$ $(k = 1, 2, 3, 4)$ separated by the discontinuity set $\Sigma = \{(x, y) \in \mathbb{R}^2 : xy = 0\}$, where

$$\mathcal{R}_1 = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0\}, \qquad \mathcal{R}_2 = \{(x, y) \in \mathbb{R}^2 : x < 0, y > 0\},$$
$$\mathcal{R}_3 = \{(x, y) \in \mathbb{R}^2 : x < 0, y < 0\}, \qquad \mathcal{R}_4 = \{(x, y) \in \mathbb{R}^2 : x > 0, y < 0\}.$$

More precisely, we consider the discontinuous piecewise polynomial system

$$\dot{x} = -y\left(\frac{x^2+y^2}{2}\right)^m + \sum_{i=1}^{\infty} \varepsilon^i f_i^k(x, y), \qquad \dot{y} = x\left(\frac{x^2+y^2}{2}\right)^m + \sum_{i=1}^{\infty} \varepsilon^i g_i^k(x, y) \tag{2}$$

if $(x, y) \in \mathcal{R}_k$, where $\varepsilon \in \mathbb{R}$ is a small perturbation parameter, f_i^k and g_i^k are real polynomials of degree $n \geq 1$. Our main goal is to study the maximum number of crossing limit cycles of system (2) bifurcating from the unperturbed periodic annulus. The following theorem is our main result.

Theorem 1. *Let $\mathcal{M}_N(m, n)$ be the maximum number of crossing limit cycles of system* (2) *with $m \geq 0, n \geq 1$ which can be obtained using the averaging theory of order N for $|\varepsilon| > 0$ sufficiently small. We have the following statements.*

(i) *If $n \geq 2m + 1$, then $\mathcal{M}_N(m, n) \leq Nn$, and this upper bound is reached for $N = 1, 2$.*
(ii) *If $n \leq 2m$, then $\mathcal{M}_N(m, n) \leq (2m + 1)(N - 1) + n$. Moreover, this upper bound is reached for $N = 1$, and $N = 2$ and $n \geq m$, while for $N = 2$ and $n \leq m - 1$, it can be reduced to $3n + 1$ and the new upper bound is reached.*

We remark that Theorem 1 improves the results of [9,29,31]. In [29], using the first order Melnikov method for system (2) with $m = 0$ and any fixed n, Wang, Han and Constantinescu proved that n crossing limit cycles can bifurcate from the unperturbed periodic annulus. Thus n is a lower bound for the maximum number of crossing limit cycles bifurcating from the periodic annulus of the linear center $\dot{x} = -y, \dot{y} = x$, when we perturb it inside discontinuous piecewise polynomial systems with four zones separated by Σ. After this, the lower bound is updated as $2n - 1$ in [31] by introducing multiple small parameters into the considered systems and deriving a generalized first order Melnikov function. According to statement (i) of Theorem 1, we can further update the lower bound as $2n$ and thus improve the result of [31] providing one more crossing limit cycle.

On the other hand in [9] the authors considered discontinuous piecewise polynomial perturbations of the center (1) with $m \geq 1$ in k zones separated by k rays originating at the origin. According to [9, Theorem 1], for given m, k and degree n the first order averaging theory at most provides n crossing limit cycles and this number is reached. Therefore if we restrict the perturbations considered in [9] to system (2), i.e. $k = 4$ and the four rays are exactly the coordinate axes, all the results of Theorem 1 with $N = 1$ is just [9, Theorem 1]. Although system (2) is special one, we obtain new results using the higher order averaging theory as stated in Theorem 1 and thus we improve the work [9].

The paper is organized as follows. In section 2 we recall the averaging theory for non-autonomous discontinuous piecewise smooth differential systems with many zones. Then we prove Theorem 1 in section 3.

2. Averaging theory

It is well known that the averaging theory can be used to study the number of limit cycles of smooth differential systems, see [28]. With the averaging theory we can obtain some information on limit cycles bifurcating from a periodic annulus through studying the zeros of the so-called averaged functions. In recent years, stimulated by both theoretical development and real applications, the classical averaging theory developed for smooth differential systems has been extended to piecewise smooth differential systems, e.g. [17,22,23,30]. In this section we summarize the averaging theory for studying the crossing limit cycles of the discontinuous piecewise polynomial system (2) following [23].

Using the change to polar coordinates (r, θ) where $x = r\cos\theta$ and $y = r\sin\theta$ we write system (2) as

$$\frac{dr}{dt} = \sum_{i=1}^{\infty} \varepsilon^i p_i^k(\theta, r), \quad \frac{d\theta}{dt} = \frac{r^{2m}}{2^m} - \frac{1}{r}\sum_{i=1}^{\infty} \varepsilon^i q_i^k(\theta, r) \qquad \text{if } \theta \in \left[\frac{(k-1)\pi}{2}, \frac{k\pi}{2}\right], \tag{3}$$

where $k = 1, 2, 3, 4$ and

$$\begin{aligned} p_i^k(\theta, r) &= \cos\theta f_i^k(r\cos\theta, r\sin\theta) + \sin\theta g_i^k(r\cos\theta, r\sin\theta), \\ q_i^k(\theta, r) &= \sin\theta f_i^k(r\cos\theta, r\sin\theta) - \cos\theta g_i^k(r\cos\theta, r\sin\theta). \end{aligned}$$

Taking θ as the new independent variable, system (3) is transformed into

$$\frac{dr}{d\theta} = \sum_{i=1}^{N} \varepsilon^i F_i(\theta, r) + \varepsilon^{N+1} R(\theta, r, \varepsilon), \tag{4}$$

with

$$F_i(\theta,r)=\sum_{k=1}^{4}\chi_{[\frac{(k-1)\pi}{2},\frac{k\pi}{2}]}(\theta)F_i^k(\theta,r),\quad R(\theta,r)=\sum_{k=1}^{4}\chi_{[\frac{(k-1)\pi}{2},\frac{k\pi}{2}]}(\theta)R^k(\theta,r,\varepsilon),\tag{5}$$

and

$$\begin{aligned}F_i^k(\theta,r)=&\frac{2^m p_i^k(\theta,r)}{r^{2m}}+\sum_{l=0}^{i-1}p_l^k(\theta,r)\Bigg(\frac{2^{2m}q_{i-l}^k(\theta,r)}{r^{4m+1}}+\sum_{\substack{j_1+j_2=i-l\\1\le j_1,j_2\le i-l}}\frac{2^{3m}q_{j_1}^k(\theta,r)q_{j_2}^k(\theta,r)}{r^{6m+2}}+\cdots\\&+\sum_{\substack{j_1+j_2+\cdots+j_{i-l}=i-l\\1\le j_1,j_2,\ldots,j_{i-l}\le i-l}}\frac{2^{(i-l+1)m}q_{j_1}^k(\theta,r)q_{j_2}^k(\theta,r)\cdots q_{j_{i-l}}^k(\theta,r)}{r^{(2m+1)(i-l)+2m}}\Bigg),\end{aligned}\tag{6}$$

where we are assuming that $p_0^k(\theta,r)\equiv 0$ for the sake of convenience, $\chi_A(\theta)$ is the characteristic function on the interval A, i.e.

$$\chi_A(\theta)=\begin{cases}1 & \text{if } \theta\in A,\\ 0 & \text{if } \theta\notin A.\end{cases}$$

Let $\mathbb{S}^1\equiv\mathbb{R}/(2\pi\mathbb{Z})$. Clearly, $F_i^k:\mathbb{S}^1\times(0,+\infty)\to\mathbb{R}$ and $R^k:\mathbb{S}^1\times(0,+\infty)\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}$ are $\mathcal{C}^{N+1}$ functions which are 2π-periodic in the variable θ.

According to [23], the *averaged function of order* i associated to system (4) is

$$\mathcal{F}_i(r)=\frac{y_i(2\pi,r)}{i!}\tag{7}$$

where the functions $y_i(\theta,r):\mathbb{S}^1\times(0,\infty)\to\mathbb{R}$ for $i=1,2,\ldots,N$ are given recurrently by

$$\begin{aligned}y_1(\theta,r)&=\int_0^\theta F_1(\varphi,r)d\varphi,\\ y_i(\theta,r)&=i!\int_0^\theta\left(F_i(\varphi,r)+\sum_{l=1}^{i}\sum_{S_l}\frac{1}{K}\frac{\partial^L F_{i-l}(\varphi,r)}{\partial r^L}\prod_{j=1}^{l}y_j(\varphi,r)^{b_j}\right)d\varphi.\end{aligned}\tag{8}$$

Here

$$\frac{\partial^L F_i(\theta,r)}{\partial r^L}=\sum_{k=1}^{4}\chi_{[(k-1)\pi/2,k\pi/2]}(\theta)\frac{\partial^L F_i^k(\theta,r)}{\partial r^L},\tag{9}$$

where S_l is the set of all l-tuples of non-negative integers $(b_1,b_2,\ldots,b_l)$ satisfying $b_1+2b_2+\cdots+lb_l=l$, $L=b_1+b_2+\cdots+b_l$ and $K=b_1!b_2!2!^{b_2}\cdots b_l!l!^{b_l}$. Moreover we are assuming that $F_0=0$ in (7) for convenience.

The next theorem was proved in [23].

Theorem 2. *Consider the non-autonomous discontinuous piecewise smooth differential system (4). Suppose that i_0 is the first positive integer such that $\mathcal{F}_i=0$ for $1\le i\le i_0-1$ and $\mathcal{F}_{i_0}\neq 0$. If $\mathcal{F}_{i_0}(r^*)=0$ and $\mathcal{F}'_{i_0}(r^*)\neq 0$ for some $r^*\in(0,\infty)$, then for $|\varepsilon|>0$ sufficiently small there exists a 2π-periodic solution $r(\theta,\varepsilon)$ of system (4) such that $r(0,\varepsilon)\to r^*$ as $\varepsilon\to 0$.*

Theorem 2 implies that a simple positive real zero of the first non-vanishing averaged function provides a crossing limit cycle of system (2) bifurcating from some unperturbed periodic orbit of the period annulus of the center (1).

We state the Descartes Theorem for studying the number of positive real zeros of polynomials in the following, for a proof see [3].

Theorem 3. *Consider the real polynomial* $p(x) = a_{i_1 1}x^{i_1} + a_{i_2}x^{i_2} + ... + a_{i_r}x^{i_r}$ *with* $0 = i_1 < i_2 < ... < i_r$ *and* $r > 1$. *If* $a_{i_j}a_{i_{j+1}} < 0$, *we say that* a_{i_j} *and* $a_{i_{j+1}}$ *have a variation of sign. If the number of variations of signs is* $r_0 \in \{0, 1, 2, ..., r-1\}$, *then the polynomial* $p(x)$ *has at most* r_0 *positive real roots. Furthermore, we can choose the coefficients of the polynomial* $p(x)$ *in such a way that* $p(x)$ *has exactly* $r-1$ *positive real roots.*

3. Proof of Theorem 1

The goal of this section is to prove Theorem 1. To do this, from section 2 we only need to study the maximum number of simple positive real zeros of the averaged functions associated to system (4) satisfying (5) and (6). We start with the following lemma.

Lemma 4. *Consider the functions* $F_i(\theta, r)$ *and* $F_i^k(\theta, r)$, $i = 1, 2, ..., N$ *and* $k = 1, 2, 3, 4$, *given in* (5) *and* (6) *respectively. Let*

$$\widetilde{F}_i^k(\theta, r) = F_i^k(\theta, r)r^{(2m+1)i-1}, \qquad \widetilde{F}_i(\theta, r) = F_i(\theta, r)r^{(2m+1)i-1}. \tag{10}$$

If $n \geq 2m+1$ (*resp.* $n \leq 2m$), *then both* $\widetilde{F}_i^k(\theta, r)$ *and* $\widetilde{F}_i(\theta, r)$ *are polynomials of degree in* (*resp.* $(2m+1)(i-1)+n$) *in the variable* r.

Proof. Since $f_i^k(x, y)$ and $g_i^k(x, y)$ are polynomials of degree $n \geq 1$, the functions $p_i^k(\theta, r)$ and $q_i^k(\theta, r)$ defined in (3) are polynomials of degree $n \geq 1$ in the variable r whose coefficients are polynomials in the variables $\cos\theta$ and $\sin\theta$. Thus, if $n \geq 2m+1$ (resp. $n \leq 2m$), it follows from (6) that $\widetilde{F}_i^k(\theta, r) = F_i^k(\theta, r)r^{(2m+1)i-1}$ is a polynomial of degree in (resp. $(2m+1)(i-1)+n$) in the variable r whose coefficients are polynomials in the variables $\cos\theta$ and $\sin\theta$. Joining (5) and (10), we have

$$\widetilde{F}_i(\theta, r) = \sum_{k=1}^{4} \chi_{[\frac{(k-1)\pi}{2}, \frac{k\pi}{2}]}(\theta)F_i^k(\theta, r)r^{(2m+1)i-1} = \sum_{k=1}^{4} \chi_{[\frac{(k-1)\pi}{2}, \frac{k\pi}{2}]}(\theta)\widetilde{F}_i^k(\theta, r).$$

Furthermoe $\widetilde{F}_i(\theta, r)$ is a polynomial of degree in (resp. $(2m+1)(i-1)+n$) in the variable r whose coefficients are polynomials in the variables $\cos\theta$ and $\sin\theta$ if $n \geq 2m+1$ (resp. $n \leq 2m$). □

Having Lemma 4 we now obtain the following result.

Proposition 5. *The averaged function of order* N *associated to system* (4) *satisfying* (5) *and* (6) *has at most* Nn (*resp.* $(2m+1)(N-1)+n$) *simple positive real zeros if* $n \geq 2m+1$ (*resp.* $n \leq 2m$).

Proof. Let

$$\tilde{y}_i(\theta, r) = y_i(\theta, r)r^{(2m+1)i-1}, \tag{11}$$

where $i = 1, 2, ..., N$ and $y_i(\theta, r)$ is the function defined in (8). We claim that $\tilde{y}_i(\theta, r)$ is a polynomial of degree in (resp. $(2m+1)(i-1)+n$) in the variable r if $n \geq 2m+1$ (resp. $n \leq 2m$). In fact, from (8), (10) and Lemma 4 it follows that

$$\tilde{y}_1(\theta, r) = y_1(\theta, r)r^{2m} = \int_0^\theta F_1(\phi, r)d\phi r^{2m} = \int_0^\theta \widetilde{F}_1(\phi, r)d\phi$$

is a polynomial of degree n in r, i.e. the claim holds for $i = 1$.

Assuming that this claim also holds for $i = 2, 3, ..., i_0 - 1$ with $2 \le i_0 \le N$, by induction we only need to prove it for $i = i_0$ in the following. For $1 \le i_0 - l \le i_0 - 1$, using (9), (10) and the derivative method for the composition of functions, we have

$$\begin{aligned} \frac{\partial^L F_{i_0-l}(\theta,r)}{\partial r^L} &= \sum_{k=1}^{4} \chi_{[\frac{(k-1)\pi}{2},\frac{k\pi}{2}]}(\theta) \frac{\partial^L F^k_{i_0-l}(\theta,r)}{\partial r^L} \\ &= \sum_{k=1}^{4} \chi_{[\frac{(k-1)\pi}{2},\frac{k\pi}{2}]}(\theta) \sum_{s=0}^{L} \frac{L!}{s!(L-s)!} \frac{\partial^s \widetilde{F}^k_{i_0-l}(\theta,r)}{\partial r^s} \frac{d^{L-s} r^{-2(i_0-l)m-i_0+l+1}}{dr^{L-s}} \\ &= \frac{P_{i_0-l}(\theta,r)}{r^{(2m+1)(i_0-l)+L-1}}, \end{aligned} \tag{12}$$

where $P_{i_0-l}(\theta,r)$ is a polynomial in the variable r such that the above equality holds. By Lemma 4 and a direct computation, it is not difficult to get that the degree of $P_{i_0-l}(\theta,r)$ is $(i_0-l)n$ (resp. $(2m+1)(i_0-l-1)+n$) if $n \ge 2m+1$ (resp. $n \le 2m$). We do not provide the specific expression of $P(\theta,r)$ because it is not necessary for the rest of the proof.

Since $b_1 + 2b_2 + \cdots + lb_l = l$ and $b_1 + b_2 + \cdots + b_l = L$, for $l \le i_0 - 1$ we have

$$\prod_{j=1}^{l} y_j(\theta,r)^{b_j} = \prod_{j=1}^{l} \left(\frac{\tilde{y}_j(\theta,r)}{r^{(2m+1)j-1}} \right)^{b_j} = \frac{1}{r^{(2m+1)l-L}} \prod_{j=1}^{l} \tilde{y}_j(\theta,r)^{b_j}, \tag{13}$$

and $\prod_{j=1}^{l} \tilde{y}_j(\theta,r)^{b_j}$ is a polynomial of degree ln (resp. $(2m+1)(l-L)+Ln$) in the variable r if $n \ge 2m+1$ (resp. $n \le 2m$). Here we used the assumption that the claim holds for $i = 1, 2, 3, ..., i_0 - 1$.

Thus, combining (8), (12), (13) and $F_0 = 0$ in (8), we get

$$\begin{aligned} y_{i_0}(\theta,r) &= i_0! \int_0^\theta \left(F_{i_0}(\phi,r) + \sum_{l=1}^{i_0-1} \sum_{S_l} \frac{1}{K} \frac{\partial^L F_{i_0-l}(\phi,r)}{\partial r^L} \prod_{j=1}^{l} y_j(\phi,r)^{b_j} \right) d\phi \\ &= \frac{i_0!}{r^{(2m+1)i_0-1}} \int_0^\theta \left(\widetilde{F}_{i_0}(\phi,r) + \sum_{l=1}^{i_0-1} \sum_{S_l} \frac{1}{K} P_{i_0-l}(\phi,r) \prod_{j=1}^{l} \tilde{y}_j(\phi,r)^{b_j} \right) d\phi. \end{aligned} \tag{14}$$

With the properties of $P_{i_0-l}(\theta,r)$ and $\prod_{j=1}^{l} \tilde{y}_j(\theta,r)^{b_j}$ obtained above, i.e.

$$P_{i_0-l}(\theta,r) \prod_{j=1}^{l} \tilde{y}_j(\theta,r)^{b_j}$$

is a polynomial of degree $i_0 n$ (resp. $(2m+1)(i_0-L-1)+(L+1)n$) in r if $n \ge 2m+1$ (resp. $n \le 2m$). Moreover, from Lemma 4 we know that $\widetilde{F}_{i_0}(\theta,r)$ is a polynomial of degree $i_0 n$ (resp. $(2m+1)(i_0-1)+n$) in the variable r if $n \ge 2m+1$ (resp. $n \le 2m$). Consequently, if $n \ge 2m+1$ we obtain that $\tilde{y}_{i_0}(\theta,r) = y_{i_0}(\theta,r) r^{(2m+1)i_0-1}$ is a polynomial of degree $i_0 n$ in the variable r from (14), while if $n \le 2m$, we have

$$(2m+1)(i_0-L-1)+(L+1)n < (2m+1)(i_0-1)+n$$

and then $\tilde{y}_{i_0}(\theta,r)$ is a polynomial of degree $(2m+1)(i_0-1)+n$ in the variable r by (14) again from (14). This completes the proof of the claim.

According to the definitions (7) and (11), the averaged function of order N associated to system (4) satisfying (5) and (6) is

$$\mathcal{F}_N(r) = \frac{y_N(2\pi, r)}{N!} = \frac{\tilde{y}_N(2\pi, r)}{N! r^{(2m+1)N-1}}.$$

Thus $\mathcal{F}_N(r)$ and $\tilde{y}_N(2\pi, r)$ have the same simple positive real zeros. By the above claim, $\tilde{y}_N(2\pi, r)$ is a polynomial of degree Nn (resp. $(2m+1)(N-1)+n$) if $n \geq 2m+1$ (resp. $n \leq 2m$), so that Proposition 5 follows. $\square$

Proposition 5 gives a unified upper bound for the maximum number of simple positive real zeros of the averaged function of order N. Note that the upper bound for $N = 1$ is always n whatever $n \geq 2m+1$ or $n \leq 2m$. But for $N = 2$ and $n \leq m-1$ analyzing the second order averaged function we can reduce the upper bound as it is stated in the next proposition.

Proposition 6. *If $n \leq m-1$, the second order averaged function associated to system* (4) *satisfying* (5) *and* (6) *has at most $3n+1$ simple positive real zeros.*

Proof. Since $n \leq m-1 \leq 2m$, by Lemma 4 and the proof of Proposition 5 we know that

(a) $P_1(\theta, r) = r^{2m+1}\partial F_1(\theta, r)/\partial r$ is a polynomial of degree n in the variable r,
(b) $\tilde{y}_1(\theta, r) = y(\theta, r) r^{2m}$ is a polynomial of degree n in the variable r,
(c) $\widetilde{F}_2(\theta, r) = F_2(\theta, r) r^{4m+1}$ is a polynomial of degree $2m+n+1$ in the variable r.

On the other hand, it follows from (6) that

$$\widetilde{F}_2^k(\theta, r) = F_2^k(\theta, r) r^{4m+1} = 2^m r^{2m+1} p_2^k(\theta, r) + 2^{2m} p_1^k(\theta, r) q_1^k(\theta, r),$$

for $k = 1, 2, 3, 4$, where p_1^k, q_1^k and p_2^k are given in (3). Since $n \leq m-1$ and both $p_1^k(\theta, r)$ and $q_1^k(\theta, r)$ are polynomial of degree n in the variable r as stated in the proof of Lemma 4, all the terms of $\widetilde{F}_2^k(\theta, r)$ from r^{2n+1} to r^{2m} vanish for $k = 1, 2, 3, 4$, i.e. $\widetilde{F}_2^k(\theta, r)$ has at most $3n+2$ terms. Because of (5), hence all the terms of $\widetilde{F}_2(\theta, r)$ from r^{2n+1} to r^{2m} also vanish.

Finally, using (7) and statements (a), (b) and (c) we have

$$\mathcal{F}_2(r) = \int_0^{2\pi} F_2(\theta, r) + \frac{\partial F_1(\theta, r)}{\partial r} y_1(\theta, r) d\theta = \frac{1}{r^{4m+1}} \int_0^{2\pi} \widetilde{F}_2(\theta, r) + P_1(\theta, r)\tilde{y}_1(\theta, r) d\theta,$$

and then $\mathcal{F}_2(r) r^{4m+1}$ is a polynomial of degree $2m+n+1$ having at most $3n+2$ terms. Thus $\mathcal{F}_2(r) r^{4m+1}$ has at most $3n+1$ simple positive real zeros due to Theorem 3. This concludes the proof of Proposition 6 because $\mathcal{F}_2(r)$ and $\mathcal{F}_2(r) r^{4m+1}$ have the same simple positive real zeros. $\square$

Next we study the realization of the upper bounds obtained in Propositions 5 and 6.

Proposition 7. *Consider the discontinuous piecewise polynomial system*

$$(15) \quad \begin{cases} \dot{x} = -y\left(\dfrac{x^2+y^2}{2}\right)^m + \varepsilon \displaystyle\sum_{i=0}^{n} a_i y^i, & \dot{y} = x\left(\dfrac{x^2+y^2}{2}\right)^m \quad \text{if } (x, y) \in \mathcal{R}_1, \\ \dot{x} = -y\left(\dfrac{x^2+y^2}{2}\right)^m, & \dot{y} = x\left(\dfrac{x^2+y^2}{2}\right)^m \quad \text{if } (x, y) \in \mathcal{R}_2 \cup \mathcal{R}_3 \cup \mathcal{R}_4. \end{cases}$$

For any given $m \geq 0$ and $n \geq 1$ there exists a choice of parameters a_i, $i = 0, 1, 2, ..., n$, such that the corresponding first order averaged function has exactly n simple positive real zeros.

Proof. Writing system (15) into the form (4) in the polar coordinates (r,θ), we get

$$F_1(\theta,r)=\begin{cases}\dfrac{2^m}{r^{2m}}\cos\theta\displaystyle\sum_{i=0}^{n}a_i\sin^i\theta r^i & \text{if } \theta\in[0,\dfrac{\pi}{2}],\\ 0 & \text{if } \theta\in[\dfrac{\pi}{2},2\pi].\end{cases}$$

Thus the first order averaged function is

$$\mathcal{F}_1(r)=\int_0^{\frac{\pi}{2}}F_1^1(\theta,r)d\theta=\int_0^{\frac{\pi}{2}}\frac{2^m}{r^{2m}}\cos\theta\sum_{i=0}^{n}a_i\sin^i\theta r^i d\theta=\frac{2^m}{r^{2m}}\sum_{i=0}^{n}\frac{1}{i+1}a_ir^i$$

by (7) and (8). Clearly, $\mathcal{F}_1(r)$ and $\mathcal{F}_1(r)r^{2m}$ have the same simple positive real zeros and $\mathcal{F}_1(r)r^{2m}$ is a complete polynomial of degree n, i.e. all coefficients of $\mathcal{F}_1(r)r^{2m}$ can be chosen arbitrarily. Consequently, there exists a choice of parameters a_i, $i=0,1,2,...,n$, such that $\mathcal{F}_1(r)$ has n simple positive real zeros. This ends the proof of Proposition 7. $\square$

Proposition 8. *Consider the discontinuous piecewise polynomial system*

$$(16)\quad\begin{cases}\dot{x}=-y\left(\dfrac{x^2+y^2}{2}\right)^m+\varepsilon\displaystyle\sum_{i=0}^{n}a_iy^i, & \dot{y}=x\left(\dfrac{x^2+y^2}{2}\right)^m+\varepsilon\displaystyle\sum_{i=0}^{n}b_ix^i \quad \text{if } (x,y)\in\mathcal{R}_1,\\ \dot{x}=-y\left(\dfrac{x^2+y^2}{2}\right)^m, & \dot{y}=x\left(\dfrac{x^2+y^2}{2}\right)^m \quad \text{if } (x,y)\in\mathcal{R}_2,\\ \dot{x}=-y\left(\dfrac{x^2+y^2}{2}\right)^m+\varepsilon\displaystyle\sum_{i=0}^{n}c_iy^i, & \dot{y}=x\left(\dfrac{x^2+y^2}{2}\right)^m \quad \text{if } (x,y)\in\mathcal{R}_3,\\ \dot{x}=-y\left(\dfrac{x^2+y^2}{2}\right)^m+\varepsilon^2\displaystyle\sum_{i=0}^{n}d_iy^i, & \dot{y}=x\left(\dfrac{x^2+y^2}{2}\right)^m \quad \text{if } (x,y)\in\mathcal{R}_4,\end{cases}$$

with $c_i=(-1)^i(a_i+b_i)$ *for* $i=0,1,2,...,n$. *For any given* $m\geq 0$ *and* $n\geq 1$ *the corresponding first order averaged function vanishes, and there exists a choice of parameters* a_i, b_i *and* d_i *such that the corresponding second order averaged function has exactly* $2n$ (*resp.* $2m+n+1, 3n+1$) *simple positive real zeros if* $n\geq 2m+1$ (*resp.* $m\leq n\leq 2m, n\leq m-1$).

Proof. Writing system (16) into the form (4) in the polar coordinates (r,θ) and using the condition $c_i=(-1)^i(a_i+b_i)$ for $i=0,1,2,...,n$, we get

$$F_1(\theta,r)=\begin{cases}\dfrac{2^m}{r^{2m}}\displaystyle\sum_{i=0}^{n}(a_i\cos\theta\sin^i\theta+b_i\sin\theta\cos^i\theta)r^i & \text{if } \theta\in[0,\dfrac{\pi}{2}],\\ 0 & \text{if } \theta\in[\dfrac{\pi}{2},\pi],\\ \dfrac{2^m}{r^{2m}}\displaystyle\sum_{i=0}^{n}(-1)^i(a_i+b_i)\cos\theta\sin^i\theta r^i & \text{if } \theta\in[\pi,\dfrac{3\pi}{2}],\\ 0 & \text{if } \theta\in[\dfrac{3\pi}{2},2\pi],\end{cases}$$

and

$$F_2(\theta,r)=\begin{cases}\dfrac{2^{2m}}{r^{4m+1}}\displaystyle\sum_{i=0}^{n}(a_i\cos\theta\sin^i\theta+b_i\sin\theta\cos^i\theta)r^i\sum_{i=0}^{n}(a_i\sin^{i+1}\theta-b_i\cos^{i+1}\theta)r^i & \text{if } \theta\in[0,\dfrac{\pi}{2}],\\ 0 & \text{if } \theta\in[\dfrac{\pi}{2},\pi],\\ \dfrac{2^{2m}}{r^{4m+1}}\displaystyle\sum_{i=0}^{n}(-1)^i(a_i+b_i)\cos\theta\sin^i\theta r^i\sum_{i=0}^{n}(-1)^i(a_i+b_i)\sin^{i+1}\theta r^i & \text{if } \theta\in[\pi,\dfrac{3\pi}{2}],\\ \dfrac{2^{m}}{r^{2m}}\displaystyle\sum_{i=0}^{n}d_i\cos\theta\sin^i\theta r^i & \text{if } \theta\in[\dfrac{3\pi}{2},2\pi].\end{cases}$$

Therefore, by (7) and (8) the first order averaged function is

$$\begin{aligned}\mathcal{F}_1(r)=&\frac{2^m}{r^{2m}}\int_0^{\frac{\pi}{2}}\sum_{i=0}^{n}(a_i\cos\theta\sin^i\theta+b_i\sin\theta\cos^i\theta)r^id\theta\\&+\frac{2^m}{r^{2m}}\int_{\pi}^{\frac{3\pi}{2}}\sum_{i=0}^{n}(-1)^i(a_i+b_i)\cos\theta\sin^i\theta r^id\theta=0,\end{aligned}$$

and the second order averaged function is
(17)

$$\begin{aligned}\mathcal{F}_2(r)=&\frac{2^m}{r^{2m}}\int_{\frac{3\pi}{2}}^{2\pi}\sum_{i=0}^{n}d_i\cos\theta\sin^i\theta r^id\theta\\&+\frac{2^{2m}}{r^{4m+1}}\int_0^{\frac{\pi}{2}}\sum_{i=0}^{n}(a_i\cos\theta\sin^i\theta+b_i\sin\theta\cos^i\theta)r^i\sum_{i=0}^{n}(a_i\sin^{i+1}\theta-b_i\cos^{i+1}\theta)r^id\theta\\&+\frac{2^{2m}}{r^{4m+1}}\int_{\pi}^{\frac{3\pi}{2}}\sum_{i=0}^{n}(-1)^i(a_i+b_i)\cos\theta\sin^i\theta r^i\sum_{i=0}^{n}(-1)^i(a_i+b_i)\sin^{i+1}\theta r^id\theta\\&+\frac{2^{2m}}{r^{4m+1}}\int_0^{\frac{\pi}{2}}\sum_{i=0}^{n}(a_i\cos\theta\sin^i\theta+b_i\sin\theta\cos^i\theta)(i-2m)r^i\sum_{i=0}^{n}\frac{a_i\sin^{i+1}\theta-b_i\cos^{i+1}\theta+b_i}{i+1}r^id\theta\\&+\frac{2^{2m}}{r^{4m+1}}\int_{\pi}^{\frac{3\pi}{2}}\sum_{i=0}^{n}(-1)^i(a_i+b_i)\cos\theta\sin^i\theta(i-2m)r^i\sum_{i=0}^{n}(a_i+b_i)\frac{(-1)^i\sin^{i+1}\theta+1}{i+1}r^id\theta.\end{aligned}$$

We can write $\mathcal{F}_2(r)$ as $\mathcal{F}_2(r)=\widetilde{\mathcal{F}}_2(r)/r^{4m+1}$ where

$$\begin{aligned}\widetilde{\mathcal{F}}_2(r)=&2^m\sum_{i=0}^{n}\frac{(-1)^i}{i+1}d_ir^{i+2m+1}\\&+2^{2m}\sum_{i=0}^{2n}\left(\sum_{\substack{i_1+i_2=i\\0\le i_1\le i_2\le n}}\alpha_{i_1,i_2}a_{i_1}a_{i_2}+\sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le n}}(\beta_{i_1,i_2}b_{i_1}b_{i_2}+\gamma_{i_1,i_2}a_{i_1}b_{i_2})\right)r^i.\end{aligned}$$

To determine the number of simple positive real zeros of $\widetilde{\mathcal{F}}_2(r)$, or equivalently $\mathcal{F}_2(r)$, we next compute $\alpha_{i_1,i_2}, \beta_{i_1,i_2}$ and γ_{i_1,i_2}. Collecting all terms of $a_{i_1}a_{i_2}$ from (17), we obtain

$$\begin{aligned}\alpha_{i_1,i_2} =& \int_0^{\frac{\pi}{2}} \cos\theta \sin^{i_1}\theta \sin^{i_2+1}\theta d\theta + \int_\pi^{\frac{3\pi}{2}} (-1)^{i_1+i_2} \cos\theta \sin^{i_1}\theta \sin^{i_2+1}\theta d\theta\\ &+ \int_0^{\frac{\pi}{2}} (i_1-2m)\cos\theta \sin^{i_1}\theta \frac{\sin^{i_2+1}\theta}{i_2+1} d\theta\\ &+ \int_\pi^{\frac{3\pi}{2}} (-1)^{i_1}(i_1-2m)\cos\theta \sin^{i_1}\theta \frac{(-1)^{i_2}\sin^{i_2+1}\theta+1}{i_2+1} d\theta\\ &+ \int_0^{\frac{\pi}{2}} \cos\theta \sin^{i_2}\theta \sin^{i_1+1}\theta d\theta + \int_\pi^{\frac{3\pi}{2}} (-1)^{i_1+i_2} \cos\theta \sin^{i_2}\theta \sin^{i_1+1}\theta d\theta\\ &+ \int_0^{\frac{\pi}{2}} (i_2-2m)\cos\theta \sin^{i_2}\theta \frac{\sin^{i_1+1}\theta}{i_1+1} d\theta\\ &+ \int_\pi^{\frac{3\pi}{2}} (-1)^{i_2}(i_2-2m)\cos\theta \sin^{i_2}\theta \frac{(-1)^{i_1}\sin^{i_1+1}\theta+1}{i_1+1} d\theta\\ =&\frac{1}{i_1+1}+\frac{1}{i_2+1}\end{aligned}$$

if $i_1 < i_2$, while if $i_1 = i_2 = \frac{i}{2}$,

$$\begin{aligned}\alpha_{\frac{i}{2},\frac{i}{2}} =& \int_0^{\frac{\pi}{2}} \cos\theta \sin^{\frac{i}{2}}\theta \sin^{\frac{i}{2}+1}\theta d\theta + \int_\pi^{\frac{3\pi}{2}} (-1)^{i} \cos\theta \sin^{\frac{i}{2}}\theta \sin^{\frac{i}{2}+1}\theta d\theta\\ &+ \int_0^{\frac{\pi}{2}} \left(\frac{i}{2}-2m\right)\cos\theta \sin^{\frac{i}{2}}\theta \frac{\sin^{\frac{i}{2}+1}\theta}{\frac{i}{2}+1} d\theta\\ &+ \int_\pi^{\frac{3\pi}{2}} (-1)^{\frac{i}{2}}\left(\frac{i}{2}-2m\right)\cos\theta \sin^{\frac{i}{2}}\theta \frac{(-1)^{\frac{i}{2}}\sin^{\frac{i}{2}+1}\theta+1}{\frac{i}{2}+1} d\theta\\ =&\frac{1}{\frac{i}{2}+1}.\end{aligned}$$

Collecting all terms of $b_{i_1}b_{i_2}$ and $a_{i_1}b_{i_2}$ from (17) respectively, we obtain

$$\begin{aligned}\beta_{i_1,i_2} =& \int_0^{\frac{\pi}{2}} \sin\theta \cos^{i_1}\theta(-\cos^{i_2+1}\theta) d\theta + \int_\pi^{\frac{3\pi}{2}} (-1)^{i_1+i_2} \cos\theta \sin^{i_1}\theta \sin^{i_2+1}\theta d\theta\\ &+ \int_0^{\frac{\pi}{2}} (i_1-2m)\sin\theta \cos^{i_1}\theta \frac{1-\cos^{i_2+1}\theta}{i_2+1} d\theta\\ &+ \int_\pi^{\frac{3\pi}{2}} (-1)^{i_1}(i_1-2m)\cos\theta \sin^{i_1}\theta \frac{(-1)^{i_2}\sin^{i_2+1}\theta+1}{i_2+1} d\theta\\ =&0,\end{aligned}$$

and

$$
\begin{aligned}
\gamma_{i_1,i_2} =& \int_0^{\frac{\pi}{2}} \cos\theta \sin^{i_1}\theta(-\cos^{i_2+1}\theta) + \sin\theta\cos^{i_2}\theta\sin^{i_1+1}\theta d\theta \\
&+ \int_\pi^{\frac{3\pi}{2}} (-1)^{i_1+i_2}\cos\theta\sin^{i_1}\theta\sin^{i_2+1}\theta + (-1)^{i_1+i_2}\cos\theta\sin^{i_2}\theta\sin^{i_1+1}\theta d\theta \\
&+ \int_0^{\frac{\pi}{2}} (i_1-2m)\cos\theta\sin^{i_1}\theta\frac{1-\cos^{i_2+1}\theta}{i_2+1} + (i_2-2m)\sin\theta\cos^{i_2}\theta\frac{\sin^{i_1+1}\theta}{i_1+1}d\theta \\
&+ \int_\pi^{\frac{3\pi}{2}} (-1)^{i_1}(i_1-2m)\cos\theta\sin^{i_1}\theta\frac{(-1)^{i_2}\sin^{i_2+1}\theta+1}{i_2+1} \\
&+ (-1)^{i_2}(i_2-2m)\cos\theta\sin^{i_2}\theta\frac{(-1)^{i_1}\sin^{i_1+1}\theta+1}{i_1+1}d\theta \\
=& \int_0^{\frac{\pi}{2}} -\sin^{i_1}\theta\cos^{i_2+2}\theta + \cos^{i_2}\theta\sin^{i_1+2}\theta d\theta + \int_0^{\frac{\pi}{2}} 2\cos\theta\sin^{i_1+i_2+1}\theta d\theta \\
&+ \int_0^{\frac{\pi}{2}} (i_1-2m)\cos\theta\sin^{i_1}\theta\frac{1-\cos^{i_2+1}\theta}{i_2+1} + (i_2-2m)\sin\theta\cos^{i_2}\theta\frac{\sin^{i_1+1}\theta}{i_1+1}d\theta \\
&+ \int_0^{\frac{\pi}{2}} (i_1-2m)\cos\theta\sin^{i_1}\theta\frac{\sin^{i_2+1}\theta-1}{i_2+1} + (i_2-2m)\cos\theta\sin^{i_2}\theta\frac{\sin^{i_1+1}\theta-1}{i_1+1}d\theta \\
=& \int_0^{\frac{\pi}{2}} -\frac{i_1+i_2+1}{i_2+1}\sin^{i_1}\theta\cos^{i_2+2}\theta + \frac{i_1+i_2+1}{i_1+1}\cos^{i_2}\theta\sin^{i_1+2}\theta d\theta + \frac{1}{i_2+1} \\
=& \frac{1}{i_2+1},
\end{aligned}
$$

where we have used the formula

$$
\int_0^{\frac{\pi}{2}} \sin^{i_1}\theta\cos^{i_2+2}\theta d\theta = \frac{i_2+1}{i_1+1}\int_0^{\frac{\pi}{2}} \cos^{i_2}\theta\sin^{i_1+2}\theta d\theta.
$$

The above computations reduce $\widetilde{\mathcal{F}}_2(r)$ to

$$
\begin{aligned}
\widetilde{\mathcal{F}}_2(r) =& 2^m\sum_{i=0}^{n}\frac{(-1)^i}{i+1}d_i r^{i+2m+1} \\
&+ 2^{2m}\sum_{i=0}^{2n}\left(\frac{1}{\frac{i}{2}+1}a_{\frac{i}{2}}a_{\frac{i}{2}} + \sum_{\substack{i_1+i_2=i\\0\le i_1<i_2\le n}}\left(\frac{1}{i_1+1}+\frac{1}{i_2+1}\right)a_{i_1}a_{i_2} + \sum_{\substack{i_1+i_2=i\\0\le i_1,i_2\le n}}\frac{1}{i_2+1}a_{i_1}b_{i_2}\right)r^i \\
=& 2^m\widetilde{\mathcal{F}}_{21}(r) + 2^{2m}\widetilde{\mathcal{F}}_{22}(r)\widetilde{\mathcal{F}}_{23}(r),
\end{aligned}
$$

where

$$
\widetilde{\mathcal{F}}_{21}(r) = \sum_{i=0}^{n}\frac{(-1)^i}{i+1}d_i r^{i+2m+1}, \quad \widetilde{\mathcal{F}}_{22}(r) = \sum_{i=0}^{n}a_i r^i, \quad \widetilde{\mathcal{F}}_{23}(r) = \sum_{i=0}^{n}\frac{1}{i+1}(a_i+b_i)r^i.
$$

Since both $\widetilde{\mathcal{F}}_{22}(r)$ and $\widetilde{\mathcal{F}}_{23}(r)$ are complete polynomials of degree n, there exists a choice of parameters a_i and b_i for $i = 0, 1, ..., n$ such that $\widetilde{\mathcal{F}}_{22}(r)$ and $\widetilde{\mathcal{F}}_{23}(r)$ have n simple positive real zeros respectively. Moreover, the independence of $\widetilde{\mathcal{F}}_{22}(r)$ and $\widetilde{\mathcal{F}}_{23}(r)$ allow to ensure that the n simple positive real zeros of $\widetilde{\mathcal{F}}_{22}(r)$ are different from the ones of $\widetilde{\mathcal{F}}_{23}(r)$. This means that $\widetilde{\mathcal{F}}_2(r)$ has $2n$ simple positive real zeros if we fix $d_i = 0$ and choose conveniently the parameters a_i and b_i for $i = 0, 1, 2, .., n$. Thus Proposition 8 holds directly if $n \geq 2m + 1$.

However, if $m \leq n \leq 2m$ we see that $\widetilde{\mathcal{F}}_{21}(r)$ contributes with $2m - n + 1$ additional monomials of higher degree from r^{2n+1} to r^{2m+n+1} to the degree of $\widetilde{\mathcal{F}}_{22}(r)\widetilde{\mathcal{F}}_{23}(r)$. Since $\widetilde{\mathcal{F}}_{21}(r)$ is a complete polynomial, fixing the parameters a_i and b_i for $i = 0, 1, ..., n$ obtained in the last paragraph we can perturb d_i for $i = 2n - 2m, 2n - 2m + 1, ..., n$ in such a way that $2m - n + 1$ simple positive real zeros bifurcate from the infinity. Hence, adding the $2n$ zeros obtained in the last paragraph we get $2m + n + 1$ simple positive real zeros of $\widetilde{\mathcal{F}}_2(r)$, i.e. Proposition 8 holds if $m \leq n \leq 2m$.

If $n \leq m-1$, then $\widetilde{\mathcal{F}}_{21}(r)$ contributes with $n+1$ additional monomials of higher degree from r^{2m+1} to r^{2m+n+1} to the degree of $\widetilde{\mathcal{F}}_{22}(r)\widetilde{\mathcal{F}}_{23}(r)$. Similarly we can perturb d_i for $i = 0, 1, ..., n$ in such a way that $n + 1$ simple positive real zeros bifurcate from the infinity. Hence, adding the $2n$ zeros obtained above we eventually get $3n + 1$ simple positive real zeros of $\widetilde{\mathcal{F}}_2(r)$ and thus Proposition 8 also holds if $n \leq m - 1$. □

Finally, we can prove Theorem 1.

Proof of Theorem 1. By the averaging theory and the used change to polar coordinates, we know that a simple positive real zero of the first non-vanishing averaged function provides a crossing limit cycle of system (2) that bifurcate from a periodic orbit of the periodic annulus of the center $\dot{x} = -y((x^2 + y^2)/2)^m, \dot{y} = x((x^2 + y^2)/2)^m$ with $m \geq 0$. Consequently, statement (i) of Theorem 1 follows from Propositions 5, 7 and 8, and statement (ii) follows from Propositions 5, 6, 7 and 8. □

Acknowledgements

The author is supported by the grant CSC #201906240094 from the P.R. China. The second author is partially supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grants MTM2016-77278-P (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

References

[1] M.U. Akhmet, D. Aruğaslan, Bifurcation of a non-smooth planar limit cycle from a vertex, *Nonlin. Anal.* **71** (2009), e2723–e2733.
[2] V.I. Arnold, Ten problems, *Adv. Soviet Math.* **1** (1990), 1–8.
[3] I.S. Berezin, N.P. Zhidkov, *Computing Methods*, Reading, Mass.-London, 1965.
[4] M. di Bernardo, C.J. Budd, A.R. Champneys, P. Kowalczyk, *Piecewise-Smooth Dynamical systems: Theory and Applications*, Applied Mathematical Sciences, Vol.163 (Springer Verlag, London), 2008.
[5] M. di Bernardo, A.R. Champneys, C.J. Budd, Grazing, skipping and sliding: analysis of the nonsmooth dynamics of the DC/DC buck converter, *Nonlinearity* **11** (1998), 858–890.
[6] A. Buică, J. Giné, J. Llibre, Bifurcation of limit cycles from a polynomial degenerate center, *Adv. Nonlinear Stud.* **10** (2010), 597-609.
[7] C.A. Buzzi, M.F.S. Lima, J. Torregrosa, Limit cycles via higher order perturbations for some piecewise differential systems, *Physica D* **371** (2018), 28–47.
[8] C.A. Buzzi, C. Pessoa, J. Torregrosa, Piecewise linear perturbations of a linear center, *Discrete Contin. Dyn. Syst.* **9** (2013), 3915–3936.
[9] T. de Carvalho, J. Llibre, D. J. Tonon, Limit cycles of discontinuous piecewise polynomial vector fields, *J. Math. Anal. Appl.* **449** (2017), 572-579.

[10] G. Dong, C. Liu, Note on limit cycles for m-piecewise discontinuous polynomial Liénard differential equations, *Z. Angew. Math. Phys.* **68** (2017), No. 97.
[11] N. Hu, Z. Du, Bifurcation of periodic orbits emanated from a vertex in discontinuous planar systems, *Commun. Nonlinear Sci. Numer. Simul.* **18** (2013), 3436–3448.
[12] J. Fan, S. Xue, G. Chen, On discontinuous dynamics of a periodically forced double-belt friction oscillator, *Chaos, Solitons and Fractals* **109** (2018), 280–302.
[13] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer Academic Publishers, Dordrecht, 1988.
[14] A. Gasull, J. Torregrosa, A relation between small amplitude and big limit cycles, *Rocky. Mountain J. Math.* **31** (2001), 1277–1303.
[15] D. Hilbert, Mathematische Probleme, Lecture, Second Internat. Congr. Math. (Paris, 1900), *Nachr. Ges. Wiss. Gottingen Math. Phys. KL.* (1900), 253–297; English transl., *Bull. Amer. Math. Soc.* **8** (1902), 437–479; *Bull. (New Series) Amer. Math. Soc.* **37** (2000), 407–436.
[16] I.D. Iliev, The number of limit cycles due to polynomial perturbations of the harmonic oscillator, *Math. Proc. Cambridge Philos. Soc.* **127** (1999), 317–322.
[17] J. Itikawa, J. Llibre, D.D. Novaes, A new result on averaging theory for a class of discontinuous planar differential systems with applications, *Rev. Mat. Iberoam.* **33** (2017), 1247–1265.
[18] Yu. A. Kuznetsov, S. Rinaldi, A. Gragnani, One parameter bifurcations in planar Filippov systems, *Int. J. Bifur. Chaos* **13**(2003), 2157–2188.
[19] J. Li, Hilbert's 16th problem and bifurcations of planar polynomial vector fields, *Int. J. Bifur. Chaos* **13** (2003), 47–106.
[20] T. Li, H. Chen, J. Zhao, Harmonic solutions of a dry friction system, *Nonlin. Anal. Real World Appl.* **35** (2017), 30–44.
[21] A. Lins Neto, W. de Melo, C.C. Pugh, On Liénard equations, in: Proc. Symp. Geom. and topol, in: Lectures Notes in Math., vol. 597, Springer-Verlag, 1977, pp. 335–357.
[22] J. Llibre, A.C. Mereu, D.D. Novaes, Averaging theory for discontinuous piecewise differential systems, *J. Differential Equations* **258** (2015), 4007–4032.
[23] J. Llibre, D.D. Novaes, C.A.B. Rodrigues, Averaging theory at any order for computing limit cycles of discontinuous piecewise differential systems with many zones, *Physica D* **353-354** (2017), 1–10.
[24] J. Llibre, Y. Tang, Limit cycles of discontinuous piecewise quadratic and cubic polynomial perturbations of a linear center, *Discrete Contin. Dyn. Syst. Ser. B* **24** (2019), 1769–1784.
[25] J. Llibre, M.A. Teixeira, Limit cycles for m-piecewise discontinuous polynomial Liénard differential equations, *Z. Angew. Math. Phys.* **66** (2015), 51–66.
[26] J.C. Lucero, C.A. Gajo, Oscillation region of a piecewise-smooth model of the vocal folds, *Comm. Math. Sci.* **4** (2006), 453–469.
[27] O. Makarenkov, J.S.W. Lamb, Dynamics and bifurcations of nonsmooth systems: a survey, *Physica D* **241** (2012), 1826–1844.
[28] J. A. Sanders, F. Verhulst and J. Murdock, *Averaging Methods in Nonlinear Dynamical Systems*, Second edition, Applied Mathematical Sciences 59, Springer, New York, 2007.
[29] Y. Wang, M. Han, D. Constantinescu, On the limit cycles of perturbed discontinuous planar systems with 4 switching lines, *Chaos, Solitons and Fractals* **83** (2016), 158–177.
[30] L. Wei, X. Zhang, Averaging theory of arbitrary order for piecewise smooth differential systems and its application, *J. Dyn. Diff. Equat.* **30** (2018), 55–79.
[31] Y. Xiong, Limit cycle bifurcations by perturbing non-smooth Hamiltonian systems with 4 switching lines via multiple parameters, *Nonlin. Anal. Real World Appl.* **41** (2018) 384–400.
[32] J. Yang, Limit cycle bifurcations from a quadratic center with two switching lines, *Qual. Theory Dyn. Syst* **19** (2020), No. 21.

[1] Department of Mathematics, Sichuan University, 610064 Chengdu, Sichuan, P.R. China

Email address: `t.l.scumath@gmail.com`

[2] Departament de Matemàtiques, Universitat Autònoma de Barcelona, 08193 Bellaterra, Barcelona, Catalonia, Spain

Email address: `jllibre@mat.uab.cat`