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# COMPLETE INTEGRABILITY OF VECTOR FIELDS IN $\mathbb{R}^N$

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Abstract. We give necessary and sufficient conditions for the complete integrability of first order $N$–dimensional differential systems.

We propose a new method to determine in the Jacobi Theorem the last $N-1$ first integral for the complete integrability of an $N$–dimensional differential system with $N-2$ independent first integrals and with a Jacobi multiplier.

As an application we study the complete integrability of some 3–dimensional differential systems, more precisely the complete integrability of the asymmetric and symmetric May–Leonard differential systems.

## 1. Introduction

For the $N$–dimensional nonlinear differential systems the existence of $K < N-1$ independent first integrals means that these systems are *partially integrable*. The existence of $N-1$ independent first integrals means that the system is *completely integrable*, i.e. for such systems the intersection of the $N-1$ hypersurfaces obtained fixing the $N-1$ first integrals provide the trajectories of the differential system.

We give necessary and sufficient conditions under which the differential system

$$\dot{x}_j = X_j(x_1, \ldots, x_N), \quad \text{for} \quad j = 1, \ldots, N, \tag{1}$$

or its associated vector field

$$\mathcal{X} = X_1 \frac{\partial}{\partial x_1} + X_2 \frac{\partial}{\partial x_2} + \ldots + X_N \frac{\partial}{\partial x_N},$$

is completely integrable. Here $X_j : U \longrightarrow \mathbb{R}^N$ are $C^1$ functions defined in an open subset $U \subseteq \mathbb{R}^N$. Using these necessary and sufficient conditions we propose a new method to determine the last $N-1$ first integral in the Jacobi Theorem for the complete integrability of the differential system (1) having $N-2$ independent first integrals and a Jacobi multiplier.

This paper is organized as follows. In section 2 we state some basic definitions and results. In section 3 we give our main results. In section 4 we prove our results. Finally in section 5 we apply the obtained results to the study the completely integrability of the asymmetric and symmetric May–Leonard differential systems and Clebsch differential systems.

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## 2. Preliminary results and definitions

2.1. **Complete integrable vector fields.** For simplicity we shall assume that all the functions which appear below are of class $C^r$ for $r \geq 2$ although most of the results remain valid under weaker hypotheses.

Let $U$ be an open subset of $\mathbb{R}^N$. We say that a non-locally constant function $H : U \longrightarrow \mathbb{R}$ is a *first integral* of the differential system (1) if $H = H(x_1(t), \ldots, x_N(t))$ is constant for all values of $t$ for which the solution $(x_1(t), \ldots, x_N(t))$ is defined and contained in $U$. Clearly a $C^1$ function $H$ is a first integral of system (1) if and only if

$$\dot{H} = \frac{\partial H}{\partial x_1} X_1 + \frac{\partial H}{\partial x_2} X_2 + \ldots + \frac{\partial H}{\partial x_N} X_N \equiv 0 \quad \text{in } U.$$

If $H_r : U_r \longrightarrow \mathbb{R}$ for $r = 1, \ldots, K$ are $K$ first integrals of system (1), we say that they are *independent* in $\tilde{U}_K := U_1 \cap U_2 \ldots \cap U_K$ if their gradients are independent in all the points of $\tilde{U}_K$ except perhaps in a zero Lebesgue measure set.

We say that system (1) is *completely integrable* in an open set $\tilde{U}_{N-1}$ if it has $N-1$ independent first integrals. In this case the orbits of system (1) are contained in the curves

$$\Big\{H_1 = h_1\Big\} \cap \Big\{H_2 = h_2\Big\} \cap \ldots \cap \Big\{H_{N-1} = h_{N-1}\Big\},$$

when $h_1, h_2 \ldots, h_{N-1}$ vary in $\mathbb{R}$.

Let $J = J(x_1, \ldots, x_N)$ be a non–negative function non–identically zero on an open subset $U$ of $\mathbb{R}^N$. Then $J$ is a *Jacobi multiplier* of the differential system (1) if

$$\int_\Omega J(x_1, \ldots, x_N) dx_1 \ldots dx_N = \int_{\varphi_t(\Omega)} J(x_1, \ldots, x_N) dx_1 \ldots dx_N,$$

where $\Omega$ is any open subset of $U$, $\varphi_t$ is the flow defined by the differential system (1), and $\varphi_t(\Omega)$ is the image of the domain $\Omega$ under the flow $\varphi_t$.

The following result of Whittaker [20] plays a main role for detecting a Jacobi multiplier.

**Theorem 1.** *Let $J$ be a non–negative $C^1$ function non–identically zero defined on an open subset of $\mathbb{R}^N$. Then $J$ is a Jacobi multiplier of the differential system* (1) *if and only if the divergence of the vector field $J\mathcal{X}$ is zero, i.e.*

$$\text{div}(J\mathcal{X}) := \frac{\partial(JX_1)}{\partial x_1} + \ldots + \frac{\partial(JX_N)}{\partial x_N} = \mathcal{X}(J) + J\text{div}(\mathcal{X}) = 0. \tag{2}$$

Note that if $N = 2$ then the definition of Jacobi multiplier coincides with the definition of integrating factor.

The following result goes back to Jacobi, for a proof see Theorem 2.7 of [12].

**Theorem 2** (Jacobi Theorem)**.** *Consider the differential system* (1) *and assume that it has a Jacobi multiplier $J$ and $N-2$ independent first integrals $H_1, H_2, \ldots, H_{N-2}$. Then the system admits an additional first integral independent of the previous ones given by*

$$H_{N-1} = \int \frac{\tilde{J}}{\tilde{\Delta}} \left(\tilde{X}_2 dx_1 - \tilde{X}_1 dx_2\right), \tag{3}$$

*where* $\tilde{}$ *denotes quantities expressed in the variables* $(x_1, x_2, h_1, \ldots, h_{N-2})$ *with*

$$H_j(x_1, \ldots, x_N) = h_j, \tag{4}$$

*for* $j = 1, \ldots, N-2$ *and*

$$\Delta = \begin{vmatrix} \dfrac{\partial H_1}{\partial x_3} & \dfrac{\partial H_1}{\partial x_4} & \cdots & \dfrac{\partial H_1}{\partial x_N} \\ \dfrac{\partial H_2}{\partial x_3} & \dfrac{\partial H_2}{\partial x_4} & \cdots & \dfrac{\partial H_2}{\partial x_N} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial H_{N-2}}{\partial x_3} & \dfrac{\partial H_{N-2}}{\partial x_4} & \cdots & \dfrac{\partial H_{N-2}}{\partial x_N} \end{vmatrix}.$$

*Then system* (1) *is completely integrable.*

2.2. **Nambu bracket.** In the 1970s Nambu in [16] proposed a new approach to the classical dynamics based on an $N-$ dimensional Nambu–Poisson manifold replacing the even dimensional Poisson manifold and replacing a single Hamiltonian $H$ for $N-1$ Hamiltonian $H_1, \ldots, H_{N-1}$ In the canonical Hamiltonian formulation the equations of motion (Hamilton equations) are defined via the Poisson bracket. In Nambu‘s formalism the Poisson bracket is replaced by the *Nambu bracket.* Nambu had originally considered the case $N = 3$.

Although the *Nambu formalism* is a generalization of the *Hamiltonian formalism* its real applications are not as rich as the applications of this last one. In the monographs of Galliulin [9] used the Nambu formalism to study some inverse problems in ordinary differential equations. In this work there is also an extensive bibliography on the Nambu formalism.

Let $U$ be an open subset of $\mathbb{R}^N$. Let $H_j = H_j(x_1, \ldots, x_N)$ for $j = 1, 2, \ldots, N-1$ be independent functions defined in $U$.

Given functions $H_j$ for $j = 1, \ldots, N-1$ the *Nambu vector field* (see for instance [13]) associated to these functions is the $N$–dimensional vector field

$$\Big\{H_1, H_2, \ldots, H_{N-1}, *\Big\} := \begin{vmatrix} \dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \cdots & \dfrac{\partial H_1}{\partial x_N} \\ \dfrac{\partial H_2}{\partial x_1} & \dfrac{\partial H_2}{\partial x_2} & \cdots & \dfrac{\partial H_2}{\partial x_N} \\ \vdots & \vdots & \cdots & \vdots \\ \dfrac{\partial H_{N-1}}{\partial x_1} & \dfrac{\partial H_{N-1}}{\partial x_2} & \cdots & \dfrac{\partial H_{N-1}}{\partial x_N} \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \cdots & \dfrac{\partial}{\partial x_N} \end{vmatrix}.$$

When we apply a Nambu vector field to a function $F$, i.e. $\Big\{H_1, H_2, \ldots, H_{N-1}, F\Big\}$, the obtained function is called a *Nambu bracket,* (see [13, 16]). For properties on the Nambu braket see [13]. In particular from the property (iv) of [13] we get that the Nambu vector field has zero divergence i.e,

$$\text{div}\Big\{H_1, H_2, \ldots, H_{N-1}, *\Big\} = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\Big\{H_1, H_2, \ldots, H_{N-1}, x_j\Big\} \equiv 0. \tag{5}$$

In [13] we study the *inverse approach to the ordinary differential equations.* The inverse problem as the problem of finding the more general differential system of first order satisfying a set of given properties was stated by Erugin [7] and developed by Galiullin and his followers (see for instance [10, 11, 15]).

The new approach of the inverse problem which we proposed in [13] uses as an essential tool the Nambu bracket. We deduce new properties of this bracket which plays a very important role in the proof of all the results of this work and in its applications. In particular we prove the following properties of the Nambu bracket (see Proposition 1.2.2.)

**Proposition 3.** *We define*

$$\Omega\,(f_1 \ldots, f_{N-1}, g_1 \ldots, g_N, G) := -\{f_1 \ldots, f_{N-1}, G\}\{g_1 \ldots, g_N\} + \left(\sum_{n=1}^{N}\{f_1, \ldots, f_{N-1}, g_n\}\{g_1, \ldots, g_{n-1}, G, g_{n+1}, \ldots, g_N\}\right),$$

*and*

$$F_\lambda\,(f_1 \ldots, f_{N-1}, g_1, \ldots, g_N) := -\{f_1 \ldots, f_{N-1}, \lambda\{g_1 \ldots, g_N\}\} + \left(\sum_{n=1}^{N}\{g_1, \ldots, g_{n-1}, \lambda\{f_1 \ldots, f_{N-1}, g_n\}, g_{n+1}, \ldots, g_N\}\right),$$

*for arbitrary functions $f_1, \ldots, f_{N-1}$, $G$, $g_1, \ldots, g_N, \lambda$. Then the Nambu bracket satisfies the identities:*

(viii) $\Omega\,(f_1 \ldots, f_{N-1}, g_1 \ldots, g_N, G) = 0$, *and*

(ix) $F_\lambda\,(f_1 \ldots, f_{N-1}, g_1, \ldots, g_N) = 0$. *Note this identity is a generalization of the Filippov fundamental identity (see* [8]*) which is obtained when $\lambda = 1$.*

It is interesting to observe that the identity $\Omega\,(f_1 \ldots, f_{N-1}, g_1 \ldots, g_N, G) = 0$ is more basic, in the sense that for the identity $F_\lambda\,(f_1 \ldots, f_{N-1}, g_1, \ldots, g_N) = 0$, the following relation holds

$$F_\lambda\,(f_1, \ldots, f_{N-1}, g_1, \ldots, g_N) = \sum_{j=1}^{N} \frac{\partial}{\partial x_j}\Big(\lambda\Omega(f_1, \ldots, f_{N-1}, g_1, \ldots, g_N, x_j)\Big).$$

Let $H_j$ be for $j = 1, \ldots, N$ arbitrary $C^r$ independent functions with $r \geq 2$ satisfying

$$\{H_1, H_2, \ldots,\ H_N\} \neq 0, \tag{6}$$

in $\tilde{U} \subseteq U$ with $U \setminus \tilde{U}$ is a set of zero Lebesgue measure. In the solution of the inverse problem in ordinary differential equations play a fundamental roll the following vector field

$$
\mathcal{Y} = \frac{1}{\{H_1, H_2, \ldots, H_N\}} \begin{vmatrix} \frac{\partial H_1}{\partial x_1} & \frac{\partial H_1}{\partial x_2} & \cdots & \frac{\partial H_1}{\partial x_N} & \lambda_1 \\ \frac{\partial H_2}{\partial x_1} & \frac{\partial H_2}{\partial x_2} & \cdots & \frac{\partial H_2}{\partial x_N} & \lambda_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \frac{\partial H_N}{\partial x_1} & \frac{\partial H_N}{\partial x_2} & \cdots & \frac{\partial H_N}{\partial x_N} & \lambda_N \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \cdots & \frac{\partial}{\partial x_N} & 0 \end{vmatrix} \tag{7}
$$

$$
= -\sum_{j=1}^{N} \lambda_j \frac{\Big\{H_1, H_2, \ldots, H_{j-1}, *, H_{j+1} \ldots, H_N\Big\}}{\{H_1, \ldots, H_N\}},
$$

where $\lambda_k = \lambda_k(x_1, \ldots, x_N)$ are convenient functions satisfying

$$
\mathcal{Y}(H_k) = -\lambda_k, \quad \text{for} \quad k = 1, \ldots, N. \tag{8}
$$

In view of the identity (see identity (vi) of [13])

$$
\sum_{m=1}^{N} \frac{\partial f}{\partial x_m} \Big\{H_1, H_2, \ldots, H_{j-1}, x_m, H_{j+1}, \ldots, H_N\Big\} = \Big\{H_1, H_2, \ldots, H_{j-1}, f, H_{j+1}, \ldots, H_N\Big\}, \tag{9}
$$

we obtain that

$$
\begin{aligned}
\operatorname{div}(\mathcal{Y}) &= -\sum_{j=1}^{N} \sum_{m=1}^{N} \frac{\partial}{\partial x_m} \left( \lambda_j \, \frac{\Big\{H_1, H_2, \ldots, H_{j-1}, x_m, H_{j+1}, \ldots, H_N\Big\}}{\{H_1, H_2, \ldots, H_N\}} \right) \\
&= -\sum_{j=1}^{N} \Big\{H_1, H_2, \ldots, H_{j-1}, \tilde{\lambda}_j, H_{j+1}, \ldots, H_N\Big\},
\end{aligned}
$$

where $\tilde{\lambda}_j = \dfrac{\lambda_j}{\{H_1, H_2, \ldots, H_N\}}$.

The function $g : U \longrightarrow \mathbb{R}$ and the set $\{(x_1, \ldots, x_N) \in U : \quad g(x_1, \ldots, x_N) = 0\}$ are called the *partial integral* and the *invariant hypersurface* of a vector field $\mathcal{Y}$, respectively, if $\mathcal{Y}(g)|_{g=0} = 0$.

The differential system generated by $\mathcal{Y}$ can be written as follows

$$
\dot{x}_m = \mathcal{Y}(x_m) = -\sum_{j=1}^{N} \lambda_j \, \frac{\Big\{H_1, H_2, \ldots, H_{j-1}, x_m, H_{j+1}, \ldots, H_N\Big\}}{\{H_1, H_2, \ldots, H_N\}} := Y_m, \tag{10}
$$

for $m = 1, \ldots, N$. Clearly that if $H_k$ is a first integral of $\mathcal{Y}$ then $\lambda_k = 0$, and if $H_k$ is a partial integral then $\lambda_k|_{H_k=0} = 0$, and if $H_k = J_k$ is a Jacobi multiplier then $\lambda_k = -J_k \operatorname{div}\mathcal{X}$, see formula (2).

As we shall see in the next two theorems (see Theorem 1.3,1 and 1.4.1 of [13]) by choosing properly the functions $H_1, H_2, \ldots, H_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ we can obtain the most general autonomous differential system (1) in $U \subset \mathbb{R}^N$ having the set of partial integrals $H_j = g_j$ for $j = 1, 2, \ldots, M_1$, and the given set of first integrals $H_k$ for $k = M_1 + 1, M_1 + 2, \ldots, M_1 + M_2 = M$, with $M = M_1 + M_2 \le N$ defined in $U$ and such that (6) holds

The first result characterizes the differential systems (1) having a given set of $M = M_1$ partial integrals with $M \le N$.

**Theorem 4.** *Let $H_j = g_j = g_j(x_1, \ldots, x_N)$ for $j = 1, 2, \ldots, M$ with $M \le N$ be a given set of independent functions defined in an open set $U \subset \mathbb{R}^N$. Then any differential system defined in $U$ which admits the set of partial integrals $g_j$ for $j = 1, 2, \ldots, M$ can be written as*

$$\begin{aligned} \dot{x}_j = & \sum_{k=1}^{M} \Phi_k \frac{\{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\}}{\{g_1, g_2, \ldots, g_N\}} + \\ & \sum_{k=M+1}^{N} \lambda_k \frac{\{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\}}{\{g_1, g_2, \ldots, g_N\}} = \mathcal{Y}(x_j), \end{aligned}$$

*where $H_{M+j} = g_{M+j} = g_{M+j}(x_1, \ldots, x_N)$ for $j = 1, \ldots, N - M$, are arbitrary functions defined in $U$ which we choose in such a way that the Jacobian $\{g_1, \ldots, g_N\} \ne 0$, in the set $\tilde{U} \subseteq U$ and the functions $\lambda_j = \Phi_j = \Phi_j(x_1, \ldots, x_N)$, for $j = 1, 2, \ldots, M$ and $\lambda_{M+k} = \lambda_{M+k}(x_1, \ldots, x_N)$ for $k = 1, 2, \ldots N - M$ are arbitrary functions such that $\Phi_j|_{g_j=0} = 0$, for $j = 1, \ldots, M$.*

The second main result characterizes the differential systems (1) having a given set of $M_1$ partial integrals and $M_2$ first integrals with $1 \le M_2 < N$ and $M_1 + M_2 \le N$.

**Theorem 5.** *Let $H_l = g_l = g_l(x_1, \ldots, x_N)$ for $l = 1, 2, \ldots, M_1$ and $H_k = H_k(x_1, \ldots, x_N)$ for $k = 1, 2, \ldots, M_2$ with $M_1 + M_2 = M \le N$ be independent functions defined in the open set $U$. Then the most general differential systems in $U$ which admits the partial integrals $g_l$ for $j = 1, \ldots, M_1$ and the first integrals $H_k$ for $k = 1, \ldots, M_2$ are*

(11)

$$\begin{aligned} \dot{x}_j = & \sum_{k=1}^{M_1} \Phi_k \frac{\{g_1, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_{M_1}, H_1, \ldots, H_{M_2}, g_{M+1} \ldots g_N\}}{\{g_1, \ldots g_{M_1}, H_1, \ldots, H_{M_2}, g_{M+1}, \ldots, g_N\}} + \\ & \sum_{k=M+1}^{N} \lambda_k \frac{\{g_1, \ldots, g_{M_1}, H_1, \ldots, H_{M_2}, g_{M+1}, \ldots, g_{k-1}, x_j, g_{k+1}, \ldots, g_N\}}{\{g_1, \ldots, g_{M_1}, H_1, \ldots, H_{M_2}, g_{M+1}, \ldots, g_N\}}, \end{aligned}$$

*for $j = 1, 2, \ldots, N$, where $H_{M+j} = g_{M+j}$ for $j = 1, \ldots, N - M$ are arbitrary functions satisfying $\{g_1, \ldots g_{M_1}, H_1, \ldots, H_{M_2}, g_{M+1}, \ldots, g_N\} \ne 0$ in the set $\tilde{U} \subseteq U$ and the functions $\lambda_j = \Phi_j = \Phi_j(x_1, \ldots, x_N)$, for $j = 1, 2, \ldots, M_1$ and $\lambda_{M+k} = \lambda_k(x_1, \ldots, x_N)$ for $k = M+1, 2, \ldots N$ are arbitrary functions such that $\Phi_j|_{g_j=0} = 0$, for $j = 1, \ldots, M$.*

*Moreover if* $M_1 = 0$ *and* $M_2 = N-2$ *then system* (11) *becomes*

(12)
$$\begin{aligned}\dot{x}_j = \;& Y_j(x_j)\\ = \;& -\lambda_{N-1}\frac{\{H_1,\ldots,H_{N-2},x_j,g_N\}}{\{H_1,\ldots,H_{N-2},g_{N-1},g_N\}} - \lambda_N\frac{\{H_1,\ldots,H_{N-2},g_{N-1},x_j\}}{\{H_1,\ldots,H_{N-2},g_{N-1},g_N\}},\end{aligned}$$

*for* $j = 1,2,\ldots,N$. *for* $j = 1,\ldots,N$, *and if* $M_1 = 0$ *and* $M_2 = N-1$ *then system* (11) *becomes*

$$\dot{x}_j = -\lambda_N\frac{\{H_1,\ldots,H_{N-1},\,x_j\}}{\{H_1,\ldots,H_{N-1},\,g_N\}}. \tag{13}$$

## 3. Statement of the main results

The first result is related with the completely integrability of vector field $\mathcal{Y}$ see [13, 14, 18]

**Theorem 6.** *Differential system* (4) *is completely integrable if and only if*

$$\lambda_k = -\mu\{F_1,\ldots,F_{N-1},H_k\}, \tag{14}$$

*for* $k = 1,\ldots,N$ *where* $\mu$, $F_1,\ldots,F_{N-1}$ *are convenient independent functions in* $U$. *Moreover if* (14) *holds then vector field* $\mathcal{Y}$ *becomes*

$$\mathcal{Y} = \mu\{F_1,...,F_{N-1},*\}, \tag{15}$$

*i.e. it is completely integrable with independent first integrals* $F_1,\ldots,F_{N-1}$ *and the Jacobi multiplier* $J = 1/\mu$.

The following classical result is well known: If a 2–dimensional differential system

$$\dot{x}_1 = X_1(x_1,x_2), \qquad \dot{x}_2 = X_2(x_1,x_2), \tag{16}$$

has an integrating factor $J$, then it can be written as

$$\dot{x}_1 = -\frac{1}{J}\frac{\partial H_1}{\partial x_2} = X_1, \qquad \dot{x}_2 = \frac{1}{J}\frac{\partial H_1}{\partial x_1} = X_2, \tag{17}$$

being $H_1$ is a first integral of system (16). The next theorem will extend this classical result to $N$–dimensional differential systems.

**Corollary 7.** *A differential system* (1) *is completely integrable first integrals* $H_j$ *for* $j = 1,\ldots,N-1$ *if and only if it can be written as*

$$\dot{x}_j = \frac{1}{J}\Big\{H_1,H_2,\ldots,H_{N-1},x_j\Big\} = Y_j, \quad \textit{for} \quad j = 1,\ldots,N, \tag{18}$$

*where* $J = J(x_1,x_2,\ldots,x_N)$ *is a Jacobi multiplier.*

We note that Corollary 7 is the natural extension of (17) because

$$\dot{x}_1 = \frac{1}{J}\{H_1,x_1\} = Y_1, \qquad \dot{x}_2 = \frac{1}{J}\{H_1,x_2\} = Y_2,$$

where $\{H_1,x_1\}$ and $\{H_1,x_2\}$, are Nambu brackets, which in this case coincides with the Poisson bracket.

We observe that Corollary 7 is a simple consequence of Theorem 6. Another proof of Corollary 7 appeared in [19, 4].

The proofs of Theorem 6 and Corollary 7 are given in section 4.

From Corollary 7 it is immediate to prove the following result.

**Corollary 8.** *Assume that differential systems* (1) *has* $N-2$ *independent first integrals* $H_1$, $H_2, \ldots, H_{N-2}$ *and a Jacobi multiplier* $J$, *then another independent first integral* $H_{N-1}$ *can be obtained as a solution of the partial differential equation*

$$\{H_1, H_2, \ldots, H_{N-2},\, H_{N-1}, x_j\Big\} = JX_j, \qquad \textit{for} \quad j = 1, \ldots, N.$$

Note that in order to compute the integral (3) by applying the Jacobi Theorem we need to solve the system of equations (3) for $x_3, \ldots, x_N$. In general to get $x_k(x_1, x_2, h_1, \ldots, h_{N-2})$ for $k = 3, \ldots, N$ cap be a very difficult problem. Additionally, in general the explicit computation of the integral (3) is practically impossible to obtain.

Corollary 8 provide a new method for computing a first integral $H_{N-1}$ which avoid the necessity of the inversion of (3).

In the next theorem we provide an extension of the classical result which goes back to Jacobi (see Theorem 2).

**Theorem 9.** *Consider the differential equations* (1) *with* $N-r$ *independent first integrals* $H_1, \ldots, H_{N-r}$ *and* $r-1$ *distinct Jacobi multipliers* $J_{N-r+1}, \ldots, J_{N-1}$ *such that the functions* $H_n =: \dfrac{J_n}{J_{N-1}}$ *are not locally constants for* $n = N-r+1, \ldots, N-2$. *If the functions* $H_1, H_2, \ldots, H_{N-2}$ *are independent, then system* (1) *can be written as*

(19)

$$\begin{aligned}\dot{x}_j = \quad & \nu\, \frac{\{H_1, \ldots, H_{N-r}, \log H_{N-r+1}, \ldots, \log H_{N-2},\, x_j, g_N\}}{\{H_1, \ldots, H_{N-r}, \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, \log J_{N-1}, g_N\}} \\ & -\lambda_N \frac{\{H_1, \ldots, H_{N-r}, \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, \log J_{N-1}, x_j\}}{\{H_1, \ldots, H_{N-r}, \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, \log J_{N-1}, g_N\}} = Y_j,\end{aligned}$$

*for* $j = 1, 2, \ldots, N$, *where* $g_N$, $\nu$ *and* $\lambda_N$ *are functions which satisfy the first order partial differential equation*

$$\begin{aligned}& \{H_1, \ldots, H_{N-r} \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, \tilde{\nu}, g_N\} \\ & +\{H_1, \ldots, H_{N-r} \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, J_{N-1}, \tilde{\lambda}_N\} = 0.\end{aligned} \tag{20}$$

*where*

$$\begin{aligned}\tilde{\nu} = \quad & \frac{J_{N-1}\nu}{\{H_1, \ldots, H_{N-r}, \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, \log\, J_{N-1}, g_N\}}, \\ \tilde{\lambda}_N = \quad & \frac{J_{N-1}\lambda_N}{\{H_1, \ldots, H_{N-r}, \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, \log\, J_{N-1}, g_N\}}.\end{aligned}$$

*Moreover functions* $H_{N-r+1}, \ldots, H_{N-2}$ *are first integrals, independent with* $H_1, \ldots, H_{N-r}$. *and using the Jacobi multiplier* $J_{N-1}$, *the differential system* (19) *is completely integrable.*

*The additional first integral $H_{N-1}$ can be determine as a solution of the first order partial differential equation*

$$\begin{aligned}&\{H_1,\ldots,H_{N-r},\log H_{N-r+1},\ldots,\log\, H_{N-2},\, H_{N-1},x_j\}\\ &=\tilde{\nu}\,\frac{\{H_1,\ldots,H_{N-r},\log H_{N-r+1},\ldots,\log\, H_{N-2},\, x_j,g_N\}}{\{H_1,\ldots,H_{N-r},\log H_{N-r+1},\ldots,\log\, H_{N-2},\,\log\, J_{N-1},g_N\}}\\ &-\tilde{\lambda}_N\,\frac{\{H_1,\ldots,H_{N-r},\log H_{N-r+1},\ldots,\log\, H_{N-2},\,\log\, J_{N-1},x_j\}}{\{H_1,\ldots,H_{N-r},\log H_{N-r+1},\ldots,\log\, H_{N-2},\,\log\, J_{N-1},g_N\}}.\end{aligned}\tag{21}$$

**Corollary 10.** *Consider the differential equations* (1) *with $N-r$ first integrals $H_1,\ldots,H_{N-r}$ and $r$ distinct Jacobi multipliers $J_{N-r+1},\ldots,J_N$ such that $H_n =: \dfrac{J_n}{J_N}$ are not locally constant for $n = N-r+1,\ldots,N-1$. If the functions $H_1,\, H_2,\ldots,H_{N-1}$ are independent, then system* (1) *can be written as*

$$\dot{x}_j=\nu\,\frac{\{H_1,\ldots,H_{N-r},\log H_{N-r+1},\ldots,\log\, H_{N-2},\log\, H_{N-1},x_j\}}{\{H_1,\ldots,H_{N-r},\log H_{N-r+1},\ldots,\log\, H_{N-2},\,\log\, H_{N-1},\log J_N\}}=Y_j,$$

*and it is completely integrable. Moreover if $r = N$ then system* (1) *can be written as*

$$\dot{x}_j=\nu\,\frac{\{\log H_1,\ldots,\log\, H_{N-1},x_j\}}{\{\log H_1,\ldots,\log\, H_{N-1},\log J_N\}}=Y_j,$$

## 4. Proofs of the results

*Proof of Theorem 6.* Assume that the vector field $\mathcal{Y}$ associated to differential system (4) is completely integrable, with the $N-1$ independent first integrals $F_1,\ldots,F_{N-1}$, and consequently

$$\begin{vmatrix}\dfrac{\partial F_1}{\partial x_1} & \ldots & \dfrac{\partial F_1}{\partial x_N}\\ \vdots & & \vdots\\ \vdots & & \vdots\\ \dfrac{\partial F_{N-1}}{\partial x_1} & \ldots & \dfrac{\partial F_{N-1}}{\partial x_N}\end{vmatrix}=\{F_1,\ldots,F_{N-1},x_N\}\neq 0,$$

in $\tilde{U}\subseteq U$. Thus from the equations $\mathcal{Y}(F_k)=\displaystyle\sum_{j=1}^{N}\frac{\partial F_k}{\partial x_j}Y_j=0$ for $k=1,\ldots,N-1$ or, equivalent

$$\sum_{j=1}^{N-1}\frac{\partial F_k}{\partial x_j}Y_j=-Y_N\frac{\partial F_k}{\partial x_N},$$

for $j = 1, \ldots, N-1$. Solving this linear system in the unknown $Y_1, \ldots, Y_{N-1}$ we get

$$Y_i = \frac{Y_N \begin{vmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_{i-1}} & \frac{\partial F_1}{\partial x_N} & \frac{\partial F_1}{\partial x_{i+1}} & \cdots & \frac{\partial F_1}{\partial x_{N-1}} \\ \frac{\partial F_2}{\partial x_1} & \cdots & \frac{\partial H_2}{\partial x_{i-1}} & \frac{\partial F_2}{\partial x_N} & \frac{\partial F_2}{\partial x_{i+1}} & \cdots & \frac{\partial F_2}{\partial x_{N-1}} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \frac{\partial F_{N-1}}{\partial x_1} & \cdots & \frac{\partial F_{N-1}}{\partial x_{i-1}} & \frac{\partial F_{N-1}}{\partial x_N} & \frac{\partial F_{N-1}}{\partial x_{i+1}} & \cdots & \frac{\partial F_{N-1}}{\partial x_{N-1}} \end{vmatrix}}{\Big\{F_1, \ldots, F_{N-1}, x_N\Big\}}$$

for $i = 1, \ldots, N-1$. Consequently

$$\mathcal{Y} = Y_1 \frac{\partial}{\partial x_1} + Y_2 \frac{\partial}{\partial x_2} + \ldots + Y_N \frac{\partial}{\partial x_N} = \mu \Big\{F_1, F_2, \ldots, F_{N-1}, *\Big\},$$

where $\mu = Y_N \left(\{F_1, \ldots, F_{N-1}, x_N\}\right)^{-1}$. So the "only if" part of the theorem follows. Now we shall prove the "if" part.

We suppose that in system (4) we have that $\lambda_l = -\mu\{F_1, ..., F_{N-1}, H_l\}$. Thus the vector field associated to differential system (4) takes the form

$$\begin{aligned} \mathcal{Y}(x_j) = & \quad -\sum_{n=1}^{N} \lambda_n \frac{\{H_1, \ldots, H_{n-1}, x_j, H_{n+1}, \ldots, H_N\}}{\{H_1, H_2, \ldots, H_N\}} \\ = & \quad \mu \sum_{n=1}^{N} \{F_1, ..., F_{N-1}, H_n\} \frac{\{H_1, \ldots, H_{n-1}, x_j, H_{n+1}, \ldots, H_N\}}{\{H_1, H_2, \ldots, H_N\}}. \end{aligned}$$

In view of the identity $\Omega\left(f_1 \ldots, f_{N-1}, g_1 \ldots, g_N, G\right) = 0$ with $G = x_j$ $f_j = F_j$ for $j = 1, \ldots, N-1$, and $H_j = g_j$ for $j = 1, \ldots, N$ we get that

$$\begin{aligned} \mathcal{Y}(x_j) = \mu \sum_{n=1}^{N} & \quad \{F_1, ..., F_{N-1}, H_n\} \frac{\{H_1, \ldots, H_{n-1}, x_j, H_{n+1}, \ldots, H_N\}}{\{H_1, H_2, \ldots, H_N\}} \\ = & \quad \mu \{F_1, ..., F_{N-1}, x_j\} \frac{\{H_1, H_2, \ldots, H_N\}}{\{H_1, H_2, \ldots, H_N\}} = \mu \{F_1, ..., F_{N-1}, x_j\}. \end{aligned}$$

Thus $\mathcal{Y} = \mu\{F_1, ..., F_{N-1}, *\}$. Consequently $\mathcal{Y}(F_k) = 0$ for $k = 1, \ldots, N-1$ i.e. $F_1, \ldots, F_{N-1}$ are first integral. Hence the vector field $\mathcal{Y}$ is completely integrable.

Finally we prove that $1/\mu$ is a Jacobi multiplier. Indeed in view of relation (5) we get that

$$\operatorname{div}\left(\frac{\mathcal{Y}}{\mu}\right) == \frac{\partial(\frac{Y_1}{\mu})}{\partial x_1} + \frac{\partial(\frac{Y_2}{\mu})}{\partial x_2} + \ldots + \frac{\partial(\frac{Y_N}{\mu})}{\partial x_N} = \operatorname{div}\left(\{F_1, \ldots, F_{N-1}, *\}\right) = 0.$$

Hence from Theorem 1 we obtain that $1/\mu$ is a Jacobi multiplier. Thus the theorem is proved. □

*Proof of Corollary 7.* It follows from Theorem 6. Indeed in this case from differential system (13) which is the most general differential equations which have $N-1$

independent first integral, we have that this system is completely integrable if and only if $\lambda_N = -\mu\{H_1,\ldots,H_{N-1}, g_N\}$ where $J = 1/\mu$ is a Jacobi multiplier. $\square$

*Proof of Theorem 9.* First by considering that

$$\mathrm{div}(J_j\mathcal{Y}) = Y_1\frac{\partial J_j}{\partial x_1} + Y_2\frac{\partial J_j}{\partial x_2} + \ldots + Y_N\frac{\partial J_j}{\partial x_N} + J_j\mathrm{div}\mathcal{Y} = \mathcal{Y}(J_j) + J_j\mathrm{div}\mathcal{X} = 0.$$

Hence $\mathcal{Y}(J_j) = -J_j\mathrm{div}\mathcal{Y} := -\nu\, J_j = \lambda_j$, for $j = N-r+1,\ldots,N-1$.

Now we shall construct the most general differential system (1) with $N-r$ independent first integrals $H_1,\ldots,H_{N-r}$ and $r-1$ Jacobi multipliers $J_{N-r+1},\ldots,J_{N-1}$. We use the vector field $\mathcal{Y}$ given in formula (4).

Since $H_j$ are first integrals, from (8) it follows that $\lambda_j = 0$ for $j = 1,\ldots,N-r$ and $H_j = J_j,\ \lambda_j = -\nu J_j$ for $j = N-r+1,\ldots,N-1$, we get that vector field (7) becomes

$$\mathcal{Y} = \frac{\begin{vmatrix} \dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \ldots & \dfrac{\partial H_1}{\partial x_N} & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \dfrac{\partial H_{N-r}}{\partial x_1} & \dfrac{\partial H_{N-r}}{\partial x_2} & \ldots & \dfrac{\partial H_{N-r}}{\partial x_N} & 0 \\ \dfrac{\partial J_{N-r+1}}{\partial x_1} & \dfrac{\partial J_{N-r+1}}{\partial x_2} & \ldots & \dfrac{\partial J_{N-r+1}}{\partial x_N} & -\nu\, J_{N-r+1} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \dfrac{\partial J_{N-1}}{\partial x_1} & \dfrac{\partial J_{N-1}}{\partial x_2} & \ldots & \dfrac{\partial J_{N-1}}{\partial x_N} & -\nu\, J_{N-1} \\ \dfrac{\partial H_N}{\partial x_1} & \dfrac{\partial H_N}{\partial x_2} & \ldots & \dfrac{\partial H_N}{\partial x_N} & \lambda_N \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \ldots & \dfrac{\partial}{\partial x_N} & 0 \end{vmatrix}}{\{H_1,\ldots,\, H_{N-r},\, J_{N-r+1},\ldots,J_{N-1}, H_N\}}.$$

Multiplying the numerator and the denominator of the vector field $\mathcal{Y}$ by $1/(J_{N-r+1}\ldots\ J_{N-1})$ we get that $\mathcal{Y}$ can be written as

$$
\mathcal{Y}=\frac{\begin{vmatrix}
\dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \ldots & \dfrac{\partial H_1}{\partial x_N} & 0\\
\vdots & \vdots & \ldots & \vdots & \vdots\\
\dfrac{\partial H_{N-r}}{\partial x_1} & \dfrac{\partial H_{N-r}}{\partial x_2} & \ldots & \dfrac{\partial H_{N-r}}{\partial x_N} & 0\\
\dfrac{\partial \log J_{N-r+1}}{\partial x_1} & \dfrac{\partial \log\ J_{N-r+1}}{\partial x_2} & \ldots & \dfrac{\partial \log\ J_{N-r+1}}{\partial x_N} & -\nu\\
\vdots & \vdots & \ldots & \vdots & \vdots\\
\dfrac{\partial \log\ J_{N-1}}{\partial x_1} & \dfrac{\partial \log\ J_{N-1}}{\partial x_2} & \ldots & \dfrac{\partial \log\ J_{N-1}}{\partial x_N} & -\nu\\
\dfrac{\partial H_N}{\partial x_1} & \dfrac{\partial H_N}{\partial x_2} & \ldots & \dfrac{\partial H_N}{\partial x_N} & \lambda_N\\
\dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial\, x_2} & \ldots & \dfrac{\partial}{\partial x_N} & 0
\end{vmatrix}}{\{H_1,\ldots,\ H_{N-r},\ \log J_{N-r+1},\ldots,\log\ J_{N-1},H_N\}} \tag{22}
$$

Subtracting the file $N-1$ of the determinant of (22) to the files $N-r+1,\ \ldots, N-2$ we obtain

$$
\mathcal{Y}=\frac{\begin{vmatrix}
\dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \ldots & \dfrac{\partial H_1}{\partial x_N} & 0\\
\vdots & \vdots & \ldots & \vdots & \vdots\\
\dfrac{\partial H_{N-r}}{\partial x_1} & \dfrac{\partial H_{N-r}}{\partial x_2} & \ldots & \dfrac{\partial H_{N-r}}{\partial x_N} & 0\\
\dfrac{\partial \log H_{N-r+1}}{\partial x_1} & \dfrac{\partial \log\ H_{N-r+1}}{\partial x_2} & \ldots & \dfrac{\partial \log\ H_{N-r+1}}{\partial x_N} & 0\\
\vdots & \vdots & \ldots & \vdots & \vdots\\
\vdots & \vdots & \ldots & \vdots & \vdots\\
\dfrac{\partial \log H_{N-2}}{\partial x_1} & \dfrac{\partial \log\ H_{N-2}}{\partial x_2} & \vdots & \dfrac{\partial \log\ H_{N-2}}{\partial x_N} & 0\\
\dfrac{\partial \log\ J_{N-1}}{\partial x_1} & \dfrac{\partial \log\ J_{N-1}}{\partial x_2} & \ldots & \dfrac{\partial \log\ J_{N-1}}{\partial x_N} & -\nu\\
\dfrac{\partial H_N}{\partial x_1} & \dfrac{\partial H_N}{\partial x_2} & \ldots & \dfrac{\partial H_N}{\partial x_N} & \lambda_N\\
\dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial\, x_2} & \ldots & \dfrac{\partial}{\partial x_N} & 0
\end{vmatrix}}{\{H_1,\ldots,\ H_{N-r},\ \log J_{N-r+1},\ldots,\log\ J_{N-1},H_N\}} \tag{23}
$$

where $H_j = \dfrac{J_n}{J_{N-1}}$ for $j = N-r+1, \ldots, N-2$. Hence (19) follows.

This differential system has $N-2$ first integrals $H_j$ for $j = 1, \ldots, N-2$. Note that if $\mathcal{Y}$ is a vector field associated to the differential system (19), then it follows that $\mathcal{Y}(H_j) = 0$ for $j = N-r+1, \ldots, N-2$, so these $H_j$'s are first integrals. In fact it is well known that the quotient of two Jacobi multipliers is a first integral. Thus in view of Jacobi Theorem 2 we get that system (19) is completely integrable.

The relation (20) holds. Indeed, the vector field

(24)
$$\begin{aligned} J_{N-1}\mathcal{Y}(x_j) == \quad & J_{N-1}\nu \frac{\{H_1, \ldots, H_{N-r} \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, x_j, g_N\}}{\{H_1, \ldots, H_{N-r} \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, \log J_{N-1}, g_N\}} \\ & -J_{N-1}\lambda_N \frac{\{H_1, \ldots, H_{N-r} \log H_{N-r+1}, \ldots, \log\, H_{N-2},\, \log J_{N-1}, x_j\}}{\{H_1, \ldots, H_{N-r}, \log H_{N-r+1}, \ldots \log\, H_{N-2},\, \log J_{N-1}, g_N\}}, \end{aligned}$$

has zero divergence, i.e. $\operatorname{div}(J_{N-1}\mathcal{Y}) = 0$. In view of identity (34) we get after some computations (20).

The proof of relation (21) is obtained by considering that the constructed vector field is completely integrable with Jacobi multiplier $J_{N-1}$. Consequently there exists an additional first integral $H_{N-1}$. Thus the vector field $\mathcal{Y}$ can be written as (18) with $J = J_{N-1}$. i.e.

$$J_{N-1}\mathcal{Y}(x_j) = \{H_1,\, H_2, \ldots, H_{N-2},\, H_{N-1}, x_j\}\,.$$

Hence in view of (24) we obtain the relations (21). In short the theorem is proved. □

*Proof of Corollary 10.* It follows from Theorem 9 taking $g_N = J_N$ and $\lambda_N = -\nu\, J_N$, from (4) we obtain

$$\mathcal{Y} = \frac{\begin{vmatrix} \dfrac{\partial H_1}{\partial x_1} & \dfrac{\partial H_1}{\partial x_2} & \cdots & \dfrac{\partial H_1}{\partial x_N} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \dfrac{\partial H_{N-r}}{\partial x_1} & \dfrac{\partial H_{N-r}}{\partial x_2} & \cdots & \dfrac{\partial H_{N-r}}{\partial x_N} & 0 \\ \dfrac{\partial \log H_{N-r+1}}{\partial x_1} & \dfrac{\partial \log H_{N-r+1}}{\partial x_2} & \cdots & \dfrac{\partial \log H_{N-r+1}}{\partial x_N} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \dfrac{\partial \log H_{N-2}}{\partial x_1} & \dfrac{\partial \log H_{N-2}}{\partial x_2} & \cdots & \dfrac{\partial \log H_{N-2}}{\partial x_N} & 0 \\ \dfrac{\partial \log H_{N-1}}{\partial x_1} & \dfrac{\partial \log H_{N-1}}{\partial x_2} & \cdots & \dfrac{\partial \log H_{N-1}}{\partial x_N} & 0 \\ \dfrac{\partial H \log J_N}{\partial x_1} & \dfrac{\partial \log J_N}{\partial x_2} & \cdots & \dfrac{\partial \log J_N}{\partial x_N} & -\nu \\ \dfrac{\partial}{\partial x_1} & \dfrac{\partial}{\partial x_2} & \cdots & \dfrac{\partial}{\partial x_N} & 0 \end{vmatrix}}{\{H_1, \ldots, H_{N-r}, \log J_{N-r+1}, \ldots, \log H_{N-1}, \log J_N\}}.$$

Hence $\mathcal{Y} = \nu \dfrac{\{H_1, \ldots, H_{N-r}, \log J_{N-r+1}, \ldots, \log H_{N-1}, *\}}{\{H_1, \ldots, H_{N-r}, \log J_{N-r+1}, \ldots, \log J_{N-1}, \log J_N\}}$ where $H_{N-1} = J_{N-1}/J_N$. □

4.1. **Construction of differential systems with given first integrals and Jacobi multiplers.** We shall construct differential system (1) for $N = 3$ with a given first integral and Jacobi multipliers.

First we study the case when the functions $H_1$, $H_2$ and $H_3$ and $\lambda_1$, $\lambda_2$ and $\lambda_3$ are

$$\begin{aligned} H_1 &= \frac{|z|}{|y||1-x-y-z|^{-1+b_3}}, \ H_2 = \frac{1}{|xyz(x+y+z-1)|}, \ H_3 = \frac{|x|}{|y||1-x-y-z|^{1-b_2}}, \\ \lambda_1 &= a\kappa(x,y,z)\, H_1, \qquad \lambda_2 = b\kappa(x,y,z)\, H_2, \qquad \lambda_3 = c\kappa(x,y,z)\, H_3, \end{aligned}$$

where $\kappa = \kappa(x, y, z)$ is a convenient function and $a, b, c$ are constant. The vector field $\mathcal{Y}$ given in (7) can be written as

$$\mathcal{Y} = \frac{\begin{vmatrix} \dfrac{\partial \log H_1}{\partial x} & \dfrac{\partial \log H_1}{\partial y} & \dfrac{\partial \log H_1}{\partial z} & a\kappa \\ \dfrac{\partial \log H_2}{\partial x} & \dfrac{\partial \log H_2}{\partial y} & \dfrac{\partial \log H_2}{\partial z} & b\kappa \\ \dfrac{\partial \log H_3}{\partial x} & \dfrac{\partial \log H_3}{\partial y} & \dfrac{\partial \log H_3}{\partial z} & c\kappa \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} & 0 \end{vmatrix}}{\{\log H_1, \log H_2, \log H_3\}}.$$

If $abc \neq 0$ then $\mathcal{Y} = \kappa \dfrac{\{\log F_1, \log F_2, *\}}{\{\log H_1, \log H_2, \log H_3\}}$, where $F_1$ and $F_2$ are the first integrals such that $F_1 = H_1^c/H_3^a, \qquad F_2 = H_2^c/H_3^b$.

Choosing properly the function $\kappa$ we obtain, after some computations, that the differential equations generated by the vector field $\mathcal{Y}$ are

$$(25) \qquad \begin{aligned} \dot{x} = \; & x\Big((a+b-2c)(x-1) + ((2-b_2)(a+b) - (b_2+2)c)\, y \\ & + ((b_2+1)(a-b) + (b_3-4)(c-b))\, z\Big) = \tilde{Y}_1, \\ \dot{y} = \; & y\Big((a+b+c)(y-1) + (b_2(a+b) + (b_3+1)c)x \\ & +((3-b_2)a + (2-b_3)(c+b)z\Big) = \tilde{Y}_2, \\ \dot{z} = \; & z\Big((-2a+b+c)(z-1) + ((b_2+2)(b-a) + (b_3-3)(b-c))x \\ & +(b_3(b+c) - (b_3+b_2+1)a)y\Big) = \tilde{Y}_3, \end{aligned}$$

which are the completely integrable Lotka–Volterra systems. Hence we get that

$$(26) \qquad \tilde{\mathcal{Y}}(H_1) - a\, H_1 \mathrm{div}\mathcal{Y}, \quad \tilde{\mathcal{Y}}(H_2) = -b\, H_2 \mathrm{div}\tilde{\mathcal{Y}}, \qquad \tilde{\mathcal{Y}}(H_3) = -c\, H_3 \mathrm{div}\tilde{\mathcal{Y}}.$$

where $\tilde{\mathcal{Y}} = \tilde{Y}_1 \dfrac{\partial}{\partial x} + \tilde{Y}_2 \dfrac{\partial}{\partial y} + \tilde{Y}_3 \dfrac{\partial}{\partial z}$ and

$$\mathrm{div}\tilde{\mathcal{Y}} = b\Big((2b_1 + b_3 + 1)x + (4 + b_3 - b_2)y + (7 - b_2 - 2b_3)z - 3b\Big).$$

From (26) we get that if $a = b = c = 1$ then $H_1, H_2$ and $H_3$ are Jacobi multipliers. If $a = c = 0$ and $b = 1$ then $H_1$, $H_3$ are first integrals and $H_2$ is a Jacobi multiplier.

Second we study differential system (19) for the case when $N = 3$ with the first integral $H_1$ and the Jacobi multipliers $J_2 = H_2$ and $J_3 = H_3$.

We study differential system (12) for the case when $N = 3$ with first integral

$$H_1 = \log\left(\left|\frac{x}{z}\right|^{b_3-1} \left|\frac{x}{z}\right|^{b_1-1}\right),$$

and two Jacobi multiplier $J_2$, $J_3$.

$$J_2 = \frac{1}{xyz|x+y+z-1|}, \qquad J_3 = \left(\frac{y}{z}\right)^{\frac{b_1}{b_3-1}} y^{-1}z^{-2},$$

with $\lambda_2 = -\nu\, J_2$, $\quad \lambda_3 = -\nu\, J_3$, where $\nu$ is a convenient function.

After some computations we get that

$$\begin{aligned}
\{H_1, J_2, J_3\} = & \Big(7 - 2b_1 - b_3)x + (b_1 + 2b_3 + 1)y + (b_1 - b_3 + 4)z + 3\Big) J_2^2 J_3 := \nu_1 J_2^2 J_3,\\
\{H_1, x, J_3\} = & \frac{(b_3 + 2b_1 - 4)}{yz} J_3, \quad \{H_1, y, J_3\} = \frac{(2 - b_1 - 2b_3)}{xz} J_3,\\
\{H_1, J_2, x\} = & \Big((2b_1 + 3b_3 - 5)x + (b_1 + 2b_3 - 3)y + (b_1 + 3b_3 - 4)z + 3 - b_1 - 2b_3\Big) J_2^2,\\
\{H_1, J_2, y\} = & \Big((3 - 2b_1 - b_3)x + (5 - 2b_3 - 3b_1)y + (4 - 3b_1 - b_3)z + 2b_1 + b_3 - 3\Big) J_2^2,\\
\{H_1, J_2, z\} = & \Big((2b_1 - b_3 - 1)x + (b_1 - 2b_3 + 1)y + (b_1 - b_3)z + b_3 - b_1\Big) J_2^2.
\end{aligned}$$

Hence the equation (12) for $N = 3$ becomes

$$\begin{aligned}
\dot{x} = & -\lambda_2 \frac{\{H_1, x, J_3\}}{\{H_1, J_2, J_3\}} - \lambda_3 \frac{\{H_1, J_2, x\}}{\{H_1, J_2, J_3\}} = \frac{\nu}{\nu_1 J_2} \Big( \{H_1, x, \log J_3\} + \{H_1, \log J_2, x\} \Big)\\
= & \frac{\nu}{\nu_1 J_2} \left\{ H_1, \log \frac{J_2}{J_3}, x \right\} = \mathcal{Y}(x),\\
\dot{y} = & -\lambda_2 \frac{\{H_1, y, J_3\}}{\{H_1, J_2, J_3\}} - \lambda_3 \frac{\{H_1, J_2, y\}}{\{H_1, J_2, J_3\}} = \frac{\nu}{\nu_1 J_2} \Big( \{H_1, y, \log J_3\} + \{H_1, \log J_2, y\} \Big)\\
= & \frac{\nu}{\nu_1 J_2} \left\{ H_1, \log \frac{J_2}{J_3}, y \right\} = \mathcal{Y}(y), ,\\
\dot{z} = & -\lambda_2 \frac{\{H_1, z, J_3\}}{\{H_1, J_2, J_3\}} - \lambda_3 \frac{\{H_1, J_2, z\}}{\{H_1, J_2, J_3\}} = \frac{\nu}{\nu_1 J_2} \Big( \{H_1, z, \log J_3\} + \{H_1, \log J_2, z\} \Big)\\
= & \frac{\nu}{\nu_1 J_2} \left\{ H_1, \log \frac{J_2}{J_3}, z \right\} = \mathcal{Y}(z).
\end{aligned}$$

Taking $\nu = \nu_1$ we obtain the following differential system

$$\begin{aligned}
\dot{x} = & \; x\,(1 - x - (b_1 + b_3 - 1)y - b_1 z) = \frac{1}{J_2} \left\{ H_1, \log \frac{J_2}{J_3}, x \right\} = \mathcal{Y}(x),\\
\dot{y} = & \; y\,(1 - y - (2 - b_3)z - b_2 x) = \frac{1}{J_2} \left\{ H_1, \log \frac{J_2}{J_3}, y \right\} = \mathcal{Y}(y),\\
\dot{z} = & \; z\,(1 - z - (2 - b_1)x - b_3 y) = \frac{1}{J_2} \left\{ H_1, \log \frac{J_2}{J_3}, z \right\} = Y(z).
\end{aligned} \tag{27}$$

This differential system has the first integral $H_1$ and two Jacobi multiplier $J_2$ and $J_3$ and consequently $J_2/J_3$ is a first integral. It is easy to show that $\mathrm{div}\mathcal{Y} = -\nu_1$. We observe that the obtained differential system (27) is a particular case of the asymmetric May–Leonard system (see for instance [5])

## 5. New method for determining the additional first integral in the Jacobi Theorem 2

The problem on the determination the $(N-1)$–th first integral in the Jacobi Theorem 2 was study in several papers. In particular for $N=3$ the following result is given in [1].

**Theorem 11.** *Suppose that the polynomial vector field*

$$\mathcal{Y} = x(\lambda + ax + by + cz)\frac{\partial}{\partial x} + y(\mu + dx + ey + fz)\frac{\partial}{\partial y} + z(\nu + gx + hy + kz)\frac{\partial}{\partial z}$$

*has an analytic first integral $F_1 = x^\alpha y^\beta z^\gamma(1 + O(x, y, z))$ with at least one of $\alpha, \beta, \gamma \neq 0$, and a Jacobi multiplier $J = x^r y^s z^t(1 + O(x, y, z))$ if the cross product of $(r - i - 1, s - j - 1, t - k - 1)$ and $(\alpha, \beta, \gamma)$ is bounded away from zero for any integers $i, j, k \geq 0$, then the system has a second analytic first integral of the form $F_2 = x^{1-r}y^{1-s}z^{1-t}(1 + O(x, y, z))$, and hence $\mathcal{Y}$ is completely integrable in a neighborhood of the origin.*

Jacobi (see Theorem 2) proposed a constructive method to build an extra first integral by knowing the multiplier $J$ and $N-2$ first integrals for an $N-$ dimensional differential system (1). In particular for $N=3$ Theorem 2 writes:

**Theorem 12.** *Consider a three dimensional vector field* (1) *and assume that it admits a Jacobi multiplier and one first integral $H_1(x_1, x_2, x_3) = h_1$. Then the system has the following extra first integral given by (see* (3) *for $N = 3$*

$$H_2 = \int \frac{\tilde{J}}{\tilde{\Delta}}\left(\tilde{X}_1 dx_2 - \tilde{X}_2 dx_1\right), \tag{28}$$

*where*

$$\tilde{\Delta} = \left.\frac{\partial H_1(x_1, x_2, x_3)}{\partial x_3}\right|_{x_3=\Psi(x_1,x_2,h_1)}, \qquad \tilde{J} = J(x_1, x_2, x_3)|_{x_3=\Psi(x_1,x_2,h_1)},$$

$$\tilde{X}_1 = X_1(x_1, x_2, x_3)|_{x_3=\Psi(x_1,x_2,h_1)}, \qquad \tilde{X}_2 = X_2(x_1, x_2, x_3)|_{x_3=\Psi(x_1,x_2,h_1)},$$

*and $x_3 = \Psi(x_1, x_2, h_1)$ is a solution of the equation $H_1 = h_1$.*

**As we observe above the inversion of $H_1 = h_1$, in terms of one of the variables is much more involved** and the resulting extra first integral is given by the integral (28) which in general is very difficult to compute. Below we propose a new method for computing a first integral $H_2$ which avoid the necessity of the inversion of $H_1 = h_1$.

**Corollary 13.** *Assume that differential systems* (1) *with $N = 3$ has a first integral $H_1$ and a Jacobi multiplier $J$, then a second first integral $H_2$ can be obtained as a solution of the partial differential equation $\{H_1, H_2, x_j\} = JX_j$, for $j = 1, 2, 3$ which is equivalent to*

$$\begin{aligned}
\frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial z} - \frac{\partial H_1}{\partial z}\frac{\partial H_2}{\partial y} &= JX_1,\\
\frac{\partial H_1}{\partial z}\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial z} &= JX_2,\\
\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y} - \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x} &= JX_3,
\end{aligned} \tag{29}$$

*where* $x = x_1$, $y = x_2$ *and* $z = x_3$.

*Proof.* The existence of the second first integral $H_2$ follows from Jacobi Theorem 2. To obtain the function $H_2$ we apply Theorem 7 with $N = 3$ from which we get equations (29). □

**Corollary 14.** *If*

$$x\frac{\partial H_1}{\partial x} + y\frac{\partial H_1}{\partial y} + z\frac{\partial H_1}{\partial z} = 0, \tag{30}$$

*then* $H_2$ *is a solution of the differential equations*

$$\begin{aligned} \left(x\frac{\partial H_2}{\partial x} + y\frac{\partial H_2}{\partial y} + z\frac{\partial H_2}{\partial z}\right)\frac{\partial H_1}{\partial z} &= J(xX_2 - yX_1), \\ \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y} - \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x} &= JX_3, \end{aligned} \tag{31}$$

*Moreover if* $\dfrac{J(xX_2 - yX_1)}{\dfrac{\partial H_1}{\partial z}} := L$ *is a homogenous function of degree one, then the solution of the first equation of* (31) *is*

$$H_2 = U(x/z, y/z)L(x, y, z). \tag{32}$$

*Proof.* From the two first equations of (29) it follows that

$$\left(x\frac{\partial H_1}{\partial x} + y\frac{\partial H_1}{\partial y} + z\frac{\partial H_1}{\partial z}\right)\frac{\partial H_2}{\partial z} - \left(x\frac{\partial H_2}{\partial x} + y\frac{\partial H_2}{\partial y} + z\frac{\partial H_2}{\partial z}\right)\frac{\partial H_1}{\partial z} = J(yX_1 - xX_2). \tag{33}$$

Hence if $H_1$ is a homogenous function of degree zero, i.e, (30) holds, then from equation (33) if follows the first equation of (31).

If the function $V$ is a homogenous function of degree one, i.e.

$$x\frac{\partial L}{\partial x} + y\frac{\partial L}{\partial y} + z\frac{\partial L}{\partial z} = L,$$

then after some computations it follows that the solution of (33) can be written as (32). □

**Example 15.** *The particular case of Lotka-Volterra system*

$$\begin{aligned} \dot{x} = &\quad x(b_3 + a_1x + a_2y + \frac{a_1b_4(a_2 - b_2)}{b_2(a_1 - b_1)}z = X_1, \\ \dot{y} = &\quad y(b_3 + b_1x + b_2y + b_4z) = X_2, \\ \dot{z} = &\quad b_3z = X_3 \end{aligned}$$

*is completely integrable with first integral* $H_1$ *and Jacobi multiplier* $J$ *such that*

$$\begin{aligned} H_1 = &\quad x^{b_2(b_1-a_1)}y^{a_1(a_2-b_2)}z^{(a_2-b_2)(a_1-b_1)}\,G^{a_1b_2-a_2b_1}, \\ J = &\quad \left(y^{b_1b_2-a_1a_2)}z^{(a_2-b_2)(a_1-b_1)}G^{b_1(a_2+b_2)-2a_1b_2}\right)^{1/(b_2(a_1-b_1)}, \end{aligned}$$

*where* $G = b_2(a_1 - b_1)x + (a_2 - b_2)b_2y + b_4(a_2 - b_2)\, z = 0$ *is an additional invariant plane. It is easy to show that* $H_1$ *and* $J$ *are homogenous functions of degree zero and three respectively. In this case the first equation of* (31) *becomes*

$$\left(x\frac{\partial H_2}{\partial x} + y\frac{\partial H_2}{\partial y} + z\frac{\partial H_2}{\partial z}\right)\frac{\partial H_1}{\partial z} = \frac{J\,x\,y\,z\,G}{a_1 b_4(b_2 - a_2)H_2}. \tag{34}$$

*By considering that*

$$L := \frac{J\,x\,y\,z\,G}{a_1 b_4(b_2 - a_2)H_2} = \tilde{L}(x/z, y/z)G$$

*is a homogenous function of degree one we get that the solution of equation* (34) *is*

$$H = \tilde{L}(x/z, y/z)G + U(x/z, y/z).$$

*Inserting it into the second equation of* (31) *with* $X_3 = b_3 z$ *we get the equation*

$$\frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y} - \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x} = b_3 z.$$

*After some computations we have that*

$$(a_2 - b_2)(b_1x + b_2y + b_4z)x\frac{\partial H}{\partial y} - \frac{b_2a_1(a_1 - b_1)x + a_2b_2(a_1 - b_1)y + b_4(a_2 - b_2)z}{b_2(a_1 - b_1)}y\frac{\partial H}{\partial x}$$
$$= \frac{1 + a_2b_4}{a_1}\frac{J\,x\,y\,z\,G}{a_1b_4(b_2 - a_2)H_2}.$$

*In view of relation*

$$(a_2 - b_2)(b_1x + b_2y + b_4z)\frac{\partial L}{\partial y} - \frac{b_2a_1(a_1 - b_1)x + a_2b_2(a_1 - b_1)y + b_4(a_2 - b_2)z}{b_2(a_1 - b_1)}\frac{\partial L}{\partial x} \equiv 0,$$

*we obtain that the function* $U(x/z, y/z)$ *satisfies the equation*

$$(a_2 - b_2)(b_1x + b_2y + b_4z)\frac{\partial U}{\partial y} - \frac{b_2a_1(a_1 - b_1)x + a_2b_2(a_1 - b_1)y + b_4(a_2 - b_2)z}{b_2(a_1 - b_1)}\frac{\partial U}{\partial x}$$
$$= \frac{1 + a_2b_4}{a_1}\frac{J\,x\,y\,z\,G}{a_1b_4(b_2 - a_2)H_2}.$$

*After the change* $\xi = x/z$ *and* $\eta = y/z$ *we obtain*

$$(a_2 - b_2)(b_1\xi + b_2\eta + b_4)\xi\frac{\partial \tilde{U}}{\partial \eta} - \frac{b_2a_1(a_1 - b_1)\xi + a_2b_2(a_1 - b_1)\eta + b_4(a_2 - b_2)}{b_2(a_1 - b_1)}\frac{\partial \bar{U}}{\partial \xi}$$
$$= \frac{1 + a_2b_4}{a_1}\tilde{L}(\xi, \eta)G(\xi, \eta, 1).$$

*where* $\tilde{U} = \tilde{U}(\xi, \eta) := U|_{x = z\xi,\, y = z\eta}$ .

5.1. **Applications of Corollary 13.** Now we shall illustrated the applications of Corollary 13 in the determination of the second first integral in three particular cases of 3–dimensional Lotka-Volterra differential systems. More precisely, we study the existence of a second first integral $H_2$

(i) of the integrable asymmetric May–Leonard model,
(ii) of the integrable symmetric May–Leonard model.
(iii) of some integrable cases for the special Lotka–Volterra systems studied in [1].

5.2. **Integrability of the asymmetric May–Leonard model. Determination of the second first integral.** A particular 3–dimensional Lotka–Volterra system is

$$\begin{aligned} \dot{x} =&\quad x\,(1-x-a_1y-b_1z)=X_1,\\ \dot{y} =&\quad y\,(1-y-a_2z-b_2x)=X_2, \\ \dot{z} =&\quad z\,(1-z-a_3x-b_3y)=X_3. \end{aligned} \tag{35}$$

This system is known as the *asymmetric May–Leonard model* (see for instance [5]). This model describes the competitions between three species and depending on six non–negative parameters $a_j$ and $b_j$ for $j=1,2,3$. The state space is the set

$$\mathbb{R}^3_+=\left\{(x,y,z)\in\mathbb{R}^3: x\geq 0 \quad y\geq 0 \text{and} \quad z\geq 0,\right\}.$$

We shall study the integrability of system (35) under the conditions that it has an additional invariant plane and the first integral $H_1$ satisfying (30), consequently $H_1=H_1(x/y,\,z/y)$. From (33) it follows that the additional first integral $H_2$ is a solution of the first order partial differential system

$$\begin{aligned} \left(x\frac{\partial H_2}{\partial x}+y\frac{\partial H_2}{\partial y}+z\frac{\partial H_2}{\partial z}\right)\frac{\partial H_1}{\partial z} =&\quad -Jxy((b_2-1)x+(1-a_1)y+(a_2-b_1)z),\\ \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y}-\frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x} =&\quad JX_3, \end{aligned}$$

**Proposition 16.** *Differential system* (35) *under the conditions*

$$b_2=\frac{a_3(a_2-1)-a_2b_3+1}{1-b_3},\qquad a_1=\frac{a_3b_1-b_1b_3+b_3-1}{a_3-1}, \tag{36}$$

*i.e. differential system*

$$\begin{aligned} \dot{x} =&\quad x\left(1-x-\frac{a_3b_1-b_1b_3+b_3-1}{a_3-1}y-b_1z\right)=\tilde{X}_1,\\ \dot{y} =&\quad y\left(1-y-a_2z-\frac{a_3(a_2-1)-a_2b_3+1}{1-b_3}x\right)=\tilde{X}_2,\\ \dot{z} =&\quad z\,(1-z-a_3x-b_3y)=\tilde{X}_3. \end{aligned} \tag{37}$$

*has the additional invariant plane*

$$g:=g(x,y,z)=(1-a_3)(a_2-1)x+(1-b_1)(b_3-1)y+(a_2-1)(b_1-1)z=0,$$

*with cofactor* $K=1-x-y-z$. *Moreover this differential system is completely integrable with first integral* $H_1$ *and the Jacobi multiplier* $J$ *given by*

$$H_1=\log\left(\left|\frac{x}{g}\right|^{\alpha_1}\left|\frac{y}{g}\right|^{\alpha_2}\left|\frac{z}{g}\right|^{\alpha_3}\right),\qquad J=\left|\frac{y}{g}\right|^{\beta_2}\left|\frac{z}{g}\right|^{\beta_3}|g|^{-3},$$

*where*

$$\begin{aligned} \alpha_1 =&\quad (a_3-1)(b_3-1)(a_2-1),\quad \alpha_2=(1-a_3)(b_3-1)(b_1-1),\quad \alpha_3=(1-a_2)(b_1-1)(a_3-b_3),\\ \beta_2 =&\quad \frac{1-a_2-b_1}{a_2-1},\qquad \beta_3=\frac{a_3(b_1-b_3-1)-b_1b_3}{(1-b_3)(1-a_3)}, \end{aligned}$$

*Moreover the first integral $H_2$ becomes*

(38)

$$H_2 = -\lambda\left(\frac{y}{|g|}\right)^{1+\beta_2}\left(\frac{z}{|g|}\right)^{1+\beta_3} x + \Lambda(y/x, z/x) = -\lambda\left(\frac{y}{x}\right)^{1+\beta_2}\left(\frac{z}{x}\right)^{1+\beta_3}|g| + \Lambda(y/x, z/x),$$

*where $\lambda$ is a convenient constan and $\Lambda = \Lambda(y/x, z/x)$ is a function which satisfies the first order partial differential equation*

$$T_2\,(1,\xi,\eta))\,\eta\frac{\partial\Lambda}{\partial\eta} - T_3\,(1,\xi,\eta)\,\xi\frac{\partial\Lambda}{\partial\xi} = \left|\frac{\xi}{g(1,\xi,\eta)}\right|^{1+\beta_2}\left|\frac{\eta}{g(1,\xi,\eta)}\right|^{1+\beta_3} \tag{39}$$

*where $\xi = y/x$ and $\eta = z/x$, and*

$$xg\frac{\partial H_1}{\partial x} = T_1(x,y,z), \qquad yg\frac{\partial H_1}{\partial y} = T_2(x,y,z), \qquad zg\frac{\partial H_1}{\partial z} = T_3(x,y,z),$$

*Proof.* After some computations we can check that

$$\mathcal{X}(g) = (1-x-y-z)g, \quad \mathcal{X}(H_1) = 0, \qquad \mathrm{div}(J\mathcal{X}) = 0.$$

**This completes the proof of the first statement**.

The partial differential system (29) for system (37) becomes

$$\begin{aligned} T_3 y\frac{\partial H_2}{\partial y} - T_2 z\frac{\partial H_2}{\partial z} &= \omega_1,\\ T_1 z\frac{\partial H_2}{\partial z} - T_3 x\frac{\partial H_2}{\partial x} &= \omega_2,\\ T_2 x\frac{\partial H_2}{\partial x} - T_1 y\frac{\partial H_2}{\partial y} &= \omega_3, \end{aligned} \tag{40}$$

where $\omega_1 = Jyz\tilde{X}_1, \quad \omega_2 = Jxz\tilde{X}_2, \quad \omega_3 = Jxy\tilde{X}_3$.

By considering that $T_1 + T_2 + T_3 = 0$ from the two first equations of (40) we get

$$\begin{aligned} x\frac{\partial H_2}{\partial x} + y\frac{\partial H_2}{\partial y} + z\frac{\partial H_2}{\partial z} &= -\frac{yz\tilde{X}_1 - \lambda xz\tilde{X}_2}{T_3}\\ &= -\lambda xyzgJ = -\left|\frac{y}{g}\right|^{1+\beta_2}\left|\frac{z}{g}\right|^{1+\beta_3} x\\ &= -\lambda\left|\frac{y}{x}\right|^{1+\beta_2}\left|\frac{z}{x}\right|^{1+\beta_3} g = F(y/x, z/x)\,x \end{aligned} \tag{41}$$

Hence we get that $H_2 = F(y/x, z/x)\,x + \Lambda(y/x, z/x)$.

Inserting $H_2$ into the third equation of (40) and by considering that

$$T_1 y\frac{\partial F(y/x,z/x)\,x}{\partial y} - T_2 x\frac{\partial F(y/x,z/x)\,x}{\partial x} - gJxyR = F(y/x,z/x)\,x \tag{42}$$

we have that function $\Lambda = \Lambda(y/x, z/x)$ is a solution of the equation

$$T_1 y\frac{\partial H_2}{\partial y} - T_2 x\frac{\partial H_2}{\partial x} = xyzJg$$

In view of the relations

$$y\frac{\partial H_2}{\partial y} = \xi\frac{\partial H_2}{\partial \xi}, \quad x\frac{\partial H_2}{\partial x} = -\left(\xi\frac{\partial H_2}{\partial \xi} + \eta\frac{\partial H_2}{\partial \eta}\right)$$

we deduce that (42) becomes

$$(T_1+T_2)\xi\frac{\partial H_2}{\partial \xi}+T_2\eta\frac{\partial H_2}{\partial \eta}=x\left|\frac{\xi}{g(1,\xi,\eta)}\right|^{1+\beta_2}\left|\frac{\eta}{g(1,\xi,\eta)}\right|^{1+\beta_3}$$

Hence by considering that $T_1+T_2=-T_3$ and dividing by $x$ after some computations we obtain (39). In short the proposition is proved. □

Now we shall study integrability of another particular cases of differential system (35).

**Proposition 17.** *Differential system* (35) *under the conditions*

$$a_1=2-b_2,\ a_3=2-b_1,\ a_2=2-b_3, \tag{43}$$

*i.e. differential equations* (27) *have the following five invariant planes with the corresponding cofactors:*

$$\begin{aligned} g_1 = &\quad x, && K_1=1-x-(2-b_2)y-b_1z,\\ g_2 = &\quad y, && K_2=1-y-(2-b_3)z-b_2x,\\ g_3 = &\quad z, && K_3=1-z-(2-b_1)x-b_3y,\\ g_4 = &\quad 1-x-y-z, && K_4=-x-y-z,\\ g_5 = &\quad x+y+z, && K_5=1-x-y-z. \end{aligned}$$

*Proof.* The fourth and firth invariant planes come from the relations

$$\frac{d}{dt}(1-x-y-z)=-(x+y+z)(1-x-y-z),\qquad \frac{d}{dt}(x+y+z)=(1-x-y-z)(x+y+z).$$

This completes the proof. □

Under the assumptions of Proposition 17 and Proposition

**Proposition 18.** *Differential system* (27) *is completely integrable. More precisely, system* (27) *has the first integral* $H_1$ *and Jacobi multiplier* $J$ *where*

$$\begin{aligned} H_1 = &\quad \log\left(\left(\frac{|x|}{|x+y+z|}\right)^{b_3-1}\left(\frac{|y|}{|x+y+z|}\right)^{b_1-1}\left(\frac{|z|}{|x+y+z|}\right)^{b_2-1}\right),\\ J = &\quad \frac{1}{|xyz(x+y+z-1)|}. \end{aligned}$$

*A second first integral* $H_2$ *is*

$$H_2=\log\frac{e^{\Psi(y/x,z/x)}}{|x+y+z-1|}, \tag{44}$$

*where* $\Psi(y/x,z/x)$ *is a solution of the partial differential equation*

$$\begin{aligned} &\Big((2-b_1-b_3)(1+\xi+\eta)+(b_1+b_2+b_3-3)(1+\eta)\Big)\xi\frac{\partial\Psi}{\partial\xi}\\ &+\Big((1-b_3)(1+\xi+\eta)+(b_1+b_2+b_3-3)\eta\Big)\eta\frac{\partial\Psi}{\partial\eta}=-(1+\xi+\eta), \end{aligned} \tag{45}$$

*with* $\xi=x/y$ *and* $\eta=z/y$.

*If* $b_1+b_2+b_3=3$ *then* $H_1$ *and* $H_2$ *becomes*

$$
\begin{aligned}
H_1 = & \quad \log\left(\left|\frac{z}{y}\right|^{b_2-1}\left|\frac{x}{y}\right|^{b_3-1}\right)=(b_2-1)F_1+(b_3-1)F_2,\\
H_2 = & \quad \log\Big(\left(\frac{z}{y|x+y+z-1|^{b_3-1}}\right)^{1-b_2}\left(\frac{x}{y|x+y+z-1|^{1-b_2}}\right)^{b_3-1}\Big)\\
= & \quad (1-b_2)F_1+(b_3-1)F_2,
\end{aligned}
\tag{46}
$$

*where* $F_1=\log\left(\frac{|x|}{|y||x+y+z-1|^{1-b_2}}\right)$, *and* $F_2=\log\left(\frac{|z|}{|y||x+y+z-1|^{b_3-1}}\right)$.. *Moreover differential system* (27) *can be written as*

$$
\dot{x}=\frac{1}{J}\{F_1,F_2,x\},\qquad \dot{y}=\frac{1}{J}\{F_1,F_2,y\},\qquad \dot{z}=\frac{1}{J}\{F_1,F_2,z\}. \tag{47}
$$

*Proof.* Partial differential system (29) can be rewritten as (40) with $T_j=T_j(x,y,z)$, and $\omega_j$ for $j=1,2,3$ are given by

$$
\begin{aligned}
T_1 = & \quad (1-b_3)(x+y+z)+(b_1+b_2+b_3-3)x,\\
T_2 = & \quad (1-b_1)(x+y+z)+(b_1+b_2+b_3-3)y,\\
T_3 = & \quad (1-b_2)(x+y+z)+(b_1+b_2+b_3-3)z,\\
\omega_1 = & \quad -\frac{(x+y+z)}{(x+y+z-1)}\Big((b_1-1)z+(1-b_2)y+x+y+z-1\Big),\\
\omega_2 = & \quad -\frac{x+y+z}{(x+y+z-1)}\Big((b_2-1)x+(1-b_3)z+x+y+z-1\Big),\\
\omega_3 = & \quad -\frac{x+y+z}{(x+y+z-1)}\Big((b_3-1)y+(1-b_1)x+x+y+z-1\Big).
\end{aligned}
$$

Hence by considering that $T_1+T_2+T_3=0$, and $\omega_1-\omega_2=-\frac{x+y+z}{x+y+z-1}T_3$, then from the two first equations and the first equation of (41) we get that

$$
x\frac{\partial H_2}{\partial x}+y\frac{\partial H_2}{\partial y}+z\frac{\partial H_2}{\partial z}=-\frac{x+y+z}{(x+y+z-1)}.
$$

By considering the equation

$$
x\frac{\partial H_1}{\partial x}+y\frac{\partial H_1}{\partial y}+z\frac{\partial H_1}{\partial z}=0,
$$

we obtain that $H_2$ satisfies the equation

$$
x\frac{\partial H_2}{\partial x}+y\frac{\partial H_2}{\partial y}+z\frac{\partial H_2}{\partial z}=-\frac{x+y+z}{x+y+z-1}.
$$

A particular solution of this partial differential equation is $-\log(x+y+z-1)$. Consequently $H_2$ can be written as

$$
H_2=-\log(x+y+z-1)+\Psi(x,y,z),
$$

where$\Psi=\Psi(x,y,z)$ is a solution of the equation

$$
x\frac{\partial \Psi}{\partial x}+y\frac{\partial \Psi}{\partial y}+z\frac{\partial \Psi}{\partial z}=0.
$$

Thus $\Psi = \Psi(x/y, z/y)$. Hence $H$ becomes (44). Inserting into the third equation of (40) and after some computations we obtain the equation

$$T_1\eta\frac{\partial\Psi}{\partial\eta} + (T_1+T_2)\xi\frac{\partial\Psi}{\partial\xi} = -(x+y+z),$$

where $\xi = x/y$ and $\eta = z/y$. By considering that $T_1 + T_2 = (x+y)(b_1+b_2+b_3-3) + (2-b_1-b_3)(x+y+z)$, then we get the equation

$$\Big((2-b_1-b_3)(x+y+z) + (b_1+b_2+b_3-3)(x+y)\Big)\frac{x}{y}\frac{\partial\Psi}{\partial\xi}$$
$$+\Big((1-b_3)(x+y+z) + (b_1+b_2+b_3-3)z\Big)\frac{z}{y}\frac{\partial\Psi}{\partial\eta} = -(x+y+z),$$

and by dividing this equation by $y$ and introducing the notations $\xi = x/y$ and $\eta = z/y$ after some computations we get equation (45).

Now we assume that $b_1+b_2+b_3 = 3$. From (45) we obtain that

$$(b_2-1))\xi\frac{\partial\Psi}{\partial\xi} - (b_3-1)\eta\frac{\partial\Psi}{\partial\eta} = -1.$$

A particular solution of this equation is $\log\left(\eta^{1/(2-2b_2)}\xi^{1/(2b_3-2)}\right)$. Thus the general solution of the previous equation is

$$\Psi = \log\left(\eta^{1/(2-2b_2)}\xi^{1/(2b_3-2)}\right) + \Theta(\xi^{1-b_3}\eta^{1-b_2}),$$

where $\Theta = (\xi^{1-b_3}\eta^{1-b_2})$ is an arbitrary function. Taking $\Theta = 0$ and equation (44) becomes equation (46). The representation (46) and deduction of differential system (47) are easy to obtain. In short the proposition is proved. □

5.3. **Determination of the second first integral $H_2$ of the symmetric May–Leonard model.** The differential system

$$\begin{aligned} \dot{x} &= x\left(1-x-ay-bz\right), \\ \dot{y} &= y\left(1-y-bx-az\right), \\ \dot{z} &= z\left(1-z-ax-by\right), \end{aligned} \tag{48}$$

is well known as the symmetric May–Leonard model (see for instance [2]), where $a$ and $b$ are positive parameters. Clearly differential system (48) is obtained from (35) under the conditions $b_j = b$ and $a_j = a$ for $j = 1,2,3$.

**Proposition 19.** *Consider the symmetric May–Leonard model.*

(i) *If $a+b=2$ then this system has the first integral $H_1$ and the Jacobi multiplier $J$, where*

$$H_1 = \log\left|\frac{xyz}{(x+y+z)^3}\right|, \quad \textit{and } J = \frac{1}{|xyz(x+y+z-1)|}.$$

*Moreover a second independent first integral is*

$$H_2 = (b-1)\log|x+y+z-1| + \Lambda\left(\frac{y}{x},\frac{z}{x}\right),$$

*where* $\Lambda = \Lambda\left(\frac{y}{x}, \frac{z}{x}\right)$ *is a solution of the partial differential equation*

$$-(2\eta - \xi - 1)\xi\frac{\partial\tilde{\Lambda}}{\partial\xi} + (2\xi - \eta - 1)\eta\frac{\partial\tilde{\Lambda}}{\partial\eta} = -(1 + \xi + \eta), \tag{49}$$

*where* $\tilde{\Lambda} = \Lambda(\xi, \eta)$, $\xi = y/x$, *and* $\eta = z/x$.

(ii) *If* $a = b \neq 1$ *then*

$$H_1 = \lg\left|\frac{y(x-z)}{x(y-z)}\right| \quad \textit{and} \quad J = \left(|x|^{b+1}|z|^{2b-1}|y-x|^{b-2}|z-x|^{-b-1}\right)^{\frac{1}{1-b}},$$

*where* $H_1$ *is a first integral and* $J$ *is a Jacobi multiplier, moreover a second independent first integral is*

$$H_2 = \log\left||b-1|(y-z)^2\left(|x|^{b-3}|z|^{1-2b}|y-x|^{2-b}|z-x|^{b+1}\right)^{\frac{1}{1-b}}\right| + \Lambda\left(\frac{y}{x}, \frac{z}{x}\right),$$

*where* $\Lambda$ *is a solution of the partial differential equation*

$$(\xi - 1)\xi\frac{\partial\tilde{\Lambda}}{\partial\xi} + \eta(\eta - 1)\frac{\partial\tilde{\Lambda}}{\partial\eta} = -\Phi_1(\xi, \eta),$$

*where* $\tilde{\Lambda} = \Lambda(\xi, \eta)$, $\xi = y/x$, $\eta = z/x$ *and* $\Phi_1(\xi, \eta) = \eta^{\frac{2b-1}{b-1}}(\eta - 1)^{\frac{b+1}{b-1}}(\xi - 1)^{\frac{2-b}{b-1}}(\xi - \eta)^2$.

(iii) *If* $a = b = 1$ *then*

$$H_1 = \log\left|\frac{y}{x}\right|, \quad \textit{and} \quad J = \frac{1}{|xyz(x+y+z-1)|},$$

*where* $H_1$ *is a first integral and* $J$ *is a Jacobi multiplier, moreover a second independent first integral is* $H_2 = \log\left|\frac{z}{x}\right|$.

*Proof.* For the case when $a + b = 2$ the symmetric May–Leonard systems has two Jacobi multiplier

$$J_1 = \frac{1}{|xyz(x+y+z-1)|}, \qquad J_2 = \frac{1}{|(x+y+z)^3(x+y+z-1)|},$$

hence $H_1 = \log\frac{J_2}{J_1} = \log\frac{|xyz|}{|x+y+z|^3}$ is a first integral.

The second independent first integral can be obtained from equations (29) which, after some computations can be written as (40) with

$$\begin{aligned}
T_1 =& \quad x + y + z - 3x, \quad T_2 = x + y + z - 3y, \quad T_3 = x + y + z - 3z,\\
\omega_1 =& \quad \frac{x+y+z}{(x+y+z-1)}\left(1 - x - (2-b)y - bz\right),\\
\omega_2 =& \quad \frac{x+y+z}{(x+y+z-1)}\left(1 - y - (2-b)z - bx\right),\\
\omega_3 =& \quad \frac{x+y+z}{(x+y+z-1)}\left(1 - z - (2-b)x - by\right).
\end{aligned}$$

By considering that $T_1 + T_2 + T_3 = 0$ we get that the first of equation of system (41) becomes

$$x\frac{\partial H_2}{\partial x} + y\frac{\partial H_2}{\partial y} + z\frac{\partial H_2}{\partial z} = \frac{x+y+z}{H_1(x+y+z-1)}(b-1).$$

and considering the equation

$$x\frac{\partial H_1}{\partial x}+y\frac{\partial H_1}{\partial y}+z\frac{\partial H_1}{\partial z}=0,$$

we obtain that $H_2$ satisfies the equation

$$x\frac{\partial H_2}{\partial x}+y\frac{\partial H_2}{\partial y}+z\frac{\partial H_2}{\partial z}=-\frac{x+y+z}{x+y+z-1}.$$

The general solution of this equation is

$$H_2=(b-1)\log|x+y+z-1|+\Lambda(x/y,z/y)=\log(\frac{e^{\Lambda(x/y,z/y)}}{|x+y+z-1|^{1-b}}.$$

Inserting it into the second equation and introducing the notations $\xi=x/y$ and $\eta=z/y$, we obtain

$$(T_1+T_2)\xi\frac{\partial\Lambda}{\partial\xi}+T_1\eta\frac{\partial\Lambda}{\partial\eta}=-(x+y+z).$$

Dividing this equation by $y$ we deduce (49). So the proof of statement (i) is done.

The proof of statement (ii) is the following. Symmetric May–Leonard system with $a=b\neq 1$ is completely integrable (see [2]). Moreover the Jacobi multiplier $J$ and first integral $H_1$ are

$$J=\left(|x|^{b+1}|z|^{2b-1}|y-x|^{b-2}|z-x|^{-b-1}\right)^{\frac{1}{1-b}},\quad H_1=\log\left|\frac{y(x-z)}{x(y-z)}\right|,$$

respectively.

The second independent first integral $H_2=H$ can be obtained from equations (40) with

$$T_1=y-z,\quad T_2=z-x,\quad T_3=y-x,$$

$$\omega_1=\lambda(x+b(y+z)-1),\quad \omega_2=\lambda(y+b(x+z)-1),\quad \omega_3=\lambda(z+b(y+x)-1),$$

where $\lambda=(y-z)^2\left(x^{b-3}z^{1-2b}(y-x)^{2-b}(z-x)^{b+1}\right)^{\frac{1}{1-b}}$.

The differential system (41) in this case becomes

$$
\begin{aligned}
x\frac{\partial H_2}{\partial x}+y\frac{\partial H_2}{\partial y}+z\frac{\partial H_2}{\partial z} &= (b-1)\lambda,\\
(x-z)x\frac{\partial H_2}{\partial x}+(y-z)y\frac{\partial H_2}{\partial y} &= -z\left(x\frac{\partial H_2}{\partial x}+y\frac{\partial H_2}{\partial y}+z\frac{\partial H_2}{\partial z}\right)\\
&\quad +x^2\frac{\partial H_2}{\partial x}+y^2\frac{\partial H_2}{\partial y}+z^2\frac{\partial H_2}{\partial z}\\
&= \lambda(b(x+y+z)-1-(b-1)z).
\end{aligned}
\tag{50}
$$

Hence in view of the first equation the second equation can be written as follows

$$x^2\frac{\partial H_2}{\partial x}+y^2\frac{\partial H_2}{\partial y}+z^2\frac{\partial H}{\partial z}=\lambda(b(x+y+z)-1).$$

By considering that the function $\lambda=\lambda(x,y,z)$ can be rewritten as

$$\lambda=x\Phi_1(\frac{y}{x},\frac{z}{x})=(xyz)^{\frac{b}{b-1}}\,\Phi_2\left(\frac{1}{x}-\frac{1}{y},\frac{1}{x}-\frac{1}{z}\right),$$

where

$$\Phi_1(\frac{y}{x},\frac{z}{x}) = \left(\left(\frac{z}{x}\right)^{2b-1}\left(\frac{y}{x}-1\right)^{2-b}\left(\frac{z}{x}-1\right)^{b+1}\right)^{1/(b-1)}\left(\frac{z}{x}-\frac{y}{x}\right)^2,$$

$$\Phi_2\left(\frac{1}{x}-\frac{1}{y},\frac{1}{x}-\frac{1}{z}\right) = \left(\left(\frac{1}{x}-\frac{1}{z}\right)^{b+1}\left(\frac{1}{x}-\frac{1}{y}\right)^{2-b}\right)^{\dfrac{1}{b-1}}\left(\frac{1}{z}-\frac{1}{y}\right)^2,$$

after some computations the functions $\Phi_1$ and $\Phi_2$ satisfies the equations

$$\begin{aligned} x\frac{\partial \Phi_1}{\partial x}+y\frac{\partial \Phi_1}{\partial y}+z\frac{\partial \Phi_1}{\partial z} &= 0,\\ x^2\frac{\partial \Phi_2}{\partial x}+y^2\frac{\partial \Phi_2}{\partial y}+z^2\frac{\partial \Phi_2}{\partial z} &= 0.\end{aligned}$$

Consequently the function $\lambda$ satisfies the following partial differential equations

$$\begin{aligned} x\frac{\partial \lambda}{\partial x}+y\frac{\partial \lambda}{\partial y}+z\frac{\partial \lambda}{\partial z} &= \lambda,\\ x^2\frac{\partial \lambda}{\partial x}+y^2\frac{\partial \lambda}{\partial y}+z^2\frac{\partial \lambda}{\partial z} &= \frac{b}{b-1}(x+y+z)\lambda.\end{aligned}$$

Thus the solution of the first equation of (50) is

$$H_2 = (b-1)\lambda(x,y,z)+\Lambda\left(\frac{y}{x},\frac{z}{x}\right),$$

where $\Lambda$ is a solution of the equation

$$\left(x^2\frac{\partial}{\partial x}+y^2\frac{\partial}{\partial y}+z^2\frac{\partial}{\partial z}\right)\Lambda\left(\frac{y}{x},\frac{z}{x}\right) = -x\,\Phi_1\left(\frac{y}{x},\frac{z}{x}\right) = -\lambda(x,y,z). \tag{51}$$

After the change $x=x$, $y=x\xi$ and $z=x\eta$ and by considering that

$$\frac{\partial \Lambda}{\partial x} = -\frac{y}{x^2}\frac{\partial \Lambda}{\partial \xi}-\frac{z}{x^2}\frac{\partial \Lambda}{\partial \eta},\qquad \frac{\partial \Lambda}{\partial y} = \frac{1}{x}\frac{\partial \Lambda}{\partial \xi},\qquad \frac{\partial \Lambda}{\partial z} = \frac{1}{x}\frac{\partial \Lambda}{\partial \eta},$$

and dividing equation (51) by $x$ we obtain, after some computations, that it becomes

$$(\xi-1)\xi\frac{\partial \tilde\Lambda}{\partial \xi}+\eta(\eta-1)\frac{\partial \tilde\Lambda}{\partial \eta} = -\Phi_1(\xi,\eta),$$

where $\tilde\Lambda = \Lambda|_{y=x\xi,\, z=x\eta}$ and $\Phi_1(\xi,\eta) = \eta^{\frac{2b-1}{b-1}}(\eta-1)^{\frac{b+1}{b-1}}(\xi-1)^{\frac{2-b}{b-1}}(\xi-\eta)^2$. Hence statement (ii) is proved.

Finally we prove statement (iii). Then the symmetric May–Leonard system with $a=b=1$ is completely integrable. Moreover a Jacobi multiplier and a first integral are

$$J = \frac{1}{|xyz(x+y+z-1)|},\qquad H_1 = \log\left|\frac{y}{x}\right|,$$

respectively.

The second independent first integral can be obtained from equations (29) which, after some computations, in this case becomes

$$\frac{1}{y}\frac{\partial H_2}{\partial z} = \frac{1}{yz},\quad \frac{1}{x}\frac{\partial H_2}{\partial z} = \frac{1}{xz},\quad -\frac{1}{x}\frac{\partial H_2}{\partial y}-\frac{1}{y}\frac{\partial H_2}{\partial x} = \frac{1}{yx},$$

where we have assumed that the absolute value of $J$ is positive. If it negative the first integrals only changes the sign.

The previous equations become

$$\frac{\partial H_2}{\partial z}=\frac{1}{z},\quad x\frac{\partial H_2}{\partial x}+y\frac{\partial H_2}{\partial y}=-1.$$

The solution of these equations is

$$H_2=\log\sqrt{\frac{z^2}{|xy|}}+\Phi_1(\frac{x}{y})=\log\left|\frac{z}{x}\right|+\log\sqrt{\left|\frac{x}{y}\right|}+\Phi_1(\frac{x}{y})=\log\left|\frac{z}{x}\right|+\Phi_2(H_1),$$

where $\Phi_j(\frac{x}{y})$ is an arbitrary function for $j=1,2$. Consequently a second independent first integral is $\log\left|\frac{z}{x}\right|$. In short the proposition is proved. □

5.4. **Determination of the second first integral $H_2$ in the Lotka–Volterra system which are integrable at the origin.** Now we shall apply Proposition 13 to some particular Lotka–Volterra systems having a first integral and Jacobi multiplier given in the paper [1]. Other results on the integrability of 3–dimensional Lotka-Volterra systems can be found in [3].

**Proposition 20.** *The Lotka–Volterra differential system*

$$\dot{x}=x(2+ax),\quad \dot{y}=y(-1+dx+hy+kz),\quad \dot{z}=z(1+gx+hy+kz),$$

*has the first integral* $H_1=z(2+ax)^{\frac{d+a-g}{a}}/(xy)$ *and the Jacobi multiplier* $J=x^{\frac{5}{2}}y^3(2+ax)^{\frac{2(g-2d)-a}{2a}}$. *Then it has the second first integral*

$$H_2=\frac{(2+ax)^{\frac{a+2d}{2a}}}{\sqrt{x}}\left(\frac{1}{y}+T(\frac{z}{y},x)\right). \tag{52}$$

*where the function* $T=T(\frac{z}{y},x)$ *is a solution of the partial differential equation*

$$(2+(g-d)x)\eta\frac{\partial T}{\partial \eta}+x(2+ax)\frac{\partial T}{\partial x}+(x\,d-1)T(\eta,x)=h+k\eta, \tag{53}$$

*with* $\eta=\frac{z}{y}$.

*Proof.* In this case the partial differential system (29) becomes

$$\begin{aligned} y\frac{\partial H_2}{\partial y}+z\frac{\partial H_2}{\partial z}+\frac{(2+ax)^{\frac{a+2d}{2a}}}{y\sqrt{x}} &= 0,\\ x(2+ax)\frac{\partial H_2}{\partial x}+z(2+(g-d)x)\frac{\partial\, H}{\partial z}-(-1+dx+hy+kz)\frac{(2+ax)^{\frac{a+2d}{2a}}}{y\sqrt{x}} &= 0,\\ x(2+ax)\frac{\partial H_2}{\partial x}-y(2+(g-d)x)\frac{\partial\, H_2}{\partial y}-(1+gx+hy+kz)\frac{(2+ax)^{\frac{a+2d}{2a}}}{y\sqrt{x}} &= 0, \end{aligned}$$

The solution of the first equation is $H_2$ given by the formula $H_2 = \dfrac{(2+ax)^{\frac{a+2d}{2a}}}{\sqrt{x}}\Lambda(x,y)$, where $\Lambda(x,y)$ is a solution of the partial differential equation

$$y\frac{\partial \Lambda}{\partial y} + z\frac{\partial \Lambda}{\partial z} = \frac{1}{y}.$$

Consequently $\Lambda = 1/y + T(z/y, x)$. Hence

$$H_2 = \frac{(2+ax)^{\frac{a+2d}{2a}}}{\sqrt{x}}\left(\frac{1}{y} + T(z/y,x)\right) := \lambda(x)\left(\frac{1}{y} + T(z/y,x)\right).$$

Inserting this expression into the second equation and by considering that $\lambda'(x) = \dfrac{\lambda(x)}{x(2+ax)}$, and introducing the notation $\eta = \dfrac{z}{y}$ we obtain the differential equation

$$\lambda(x)(ax-1)(\frac{1}{y}+T(z/y,x))+x(2+ax)\frac{\partial T}{\partial x}+(2+(g-d)x)\eta\frac{\partial T}{\partial \eta} = \frac{(-1+ax+hy+gz)\lambda(x)}{y},$$

thus after simplification we get differential equation (53). □

**Proposition 21.** *The Lotka–Volterra differential system*

$$\dot{x} = x(1+gx+by+kz), \quad \dot{y} = y(-2+ey), \quad \dot{z} = z(1+gx+hy+kz),$$

*has the first integral* $H_1 = x\left(1-ey/2\right)^{\frac{h-b}{e}}/z$, *and the Jacobi multiplier* $J = \left(z\sqrt{y}\right)^{-3}\left(1-ey/2\right)^{-\frac{2b-4h+e}{e}}$. *Then it has the second first integral*

$$H_2 = \frac{2\left(1-\dfrac{ey}{2}\right)^{\frac{2h+e}{2e}}}{\sqrt{y}}\left(-\frac{1}{z} + T(\frac{z}{x}, y)\right),$$

*where the function* $T = T(z/x, y)$ *is a solution of the partial differential equation*

$$(2-ey)y\eta\frac{\partial T}{\partial y} + (b-h)\eta^2 y\frac{\partial T}{\partial \eta} = (1+hy)\eta T + \eta k + g, \tag{54}$$

*with* $\eta = \dfrac{z}{x}$.

*Proof.* The d1ifferential system (29) in this case becomes

$$\begin{aligned}
(2-ey)\frac{\partial H}{\partial y} - (h-b)z\frac{\partial H}{\partial z} &= \frac{2\left(1-\dfrac{ey}{2}\right)^{\frac{2h+e}{2e}}}{zy^{\frac{3}{2}}}(1+gx+by+kz),\\
x\frac{\partial H}{\partial x} + z\frac{\partial H}{\partial z} &= \frac{2\left(1-\dfrac{ey}{2}\right)^{\frac{2h+e}{2e}}}{zy^{\frac{1}{2}}},\\
(2-ey)\frac{\partial H}{\partial y} + (h-b)x\frac{\partial H}{\partial x} &= \frac{2\left(1-\dfrac{ey}{2}\right)^{\frac{2h+e}{2e}}}{zy^{\frac{3}{2}}}(1+gx+hy+kz),
\end{aligned}$$

From the second equation and in a similar way to the determination of function (52) we get the function $H_2$. Inserting this function into the first and third equation and in analogous way to the proof of Proposition 20 we get that the function $T$ satisfies equation (54). □

**Proposition 22.** *The Lotka–Volterra differential system*

$$\dot{x} = x(1+ax+by+2kz), \quad \dot{y} = y(-2+ey-kz), \quad \dot{z} = z(1+kz),$$

*has the first integral* $H_1 = yz^2/(1-ey/2+kz)$ *and the Jacobi multiplier* $J = (x/z)^{-2}(1-ey/2+kz)^{\frac{b}{e}-2}(1+kz)^{1-\frac{b}{e}}$. *Then the system has the second first integral*

$$H_2 = z\left(1-\frac{ey}{2}+kz\right)^{\frac{b}{e}}(1+kz)^{1-\frac{b}{e}}\left(-\frac{1}{x}+T(y,z)\right), \tag{55}$$

*where* $T = T(y,z)$ *is a solution of the partial differential equation*

$$y(ey-kz-2)\frac{\partial T}{\partial y} + z(1+kz)\frac{\partial H}{\partial z} + (1+by+2kz)T = a. \tag{56}$$

*Proof.* System (29) in this case becomes

$$\begin{aligned} y(ey-kz-2)\frac{\partial H}{\partial y} + z(1+kz)\frac{\partial H}{\partial z} &= -z\left(1-\frac{ey}{2}+kz\right)^{\frac{b}{e}}(1+kz)^{1-\frac{b}{e}} \\ &\quad \left(\frac{1+ax+by+2kz)}{x}\right), \\ \frac{\partial H}{\partial x} &= \frac{z\left(1-\frac{ey}{2}+kz\right)^{\frac{b}{e}}(1+kz)^{1-\frac{b}{e}}}{x^2}. \end{aligned} \tag{57}$$

The solution of the second equation is $H_2$ given in (55). Inserting this function into the first equation of (57) and by considering that

$$y(ey-kz-2)\frac{\partial \lambda}{\partial y} + z(1+kz)\frac{\partial \lambda}{\partial z} = -(1+by+2kz)\lambda,$$

where $\lambda = z(1-ey/2+kz)^{\frac{b}{e}}(1+kz)^{1-\frac{b}{e}}$, we obtain differential equation

$$\begin{aligned} &\left(y(ey-kz-2)\frac{\partial \lambda}{\partial y} + z(1+kz)\frac{\partial \lambda}{\partial z}\right)(-\frac{1}{x}+T)+ \\ &\lambda\left(y(ey-kz-2)\frac{\partial T}{\partial y} + z(1+kz)\frac{\partial T}{\partial z}\right) = \frac{1+ax+by+2kz}{x}\lambda. \end{aligned}$$

Hence after some computations we get (56). □

**Remark 23.** *Propositions* 16, 17, 18, 19, 20, 21 *and* 22 *illustrate the possibilities of the method which we propose to determine the complementary first integral in the Jacobi Theorem. To determine the final expression for* $H_2$ *it is necessary to solve a partial differential equation, which in general is a non trivial problem.*

5.5. **On the Clebsch vector fields.** In physics and mathematics the vector field

$$\mathcal{X} = X_1\frac{\partial}{\partial x_1} + X_2\frac{\partial}{\partial x_2} + X_3\frac{\partial}{\partial x_3} \tag{58}$$

is called *solenoidal* if there exists a function $J$ such that (2) holds. Any solenoidal vector field can be expressed as *curl* of some other vector field $\mathcal{Y} = A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} +$

$C\dfrac{\partial}{\partial z}$ (see for instance [17]), i.e.

$$JX = curl(\mathcal{Y}), \tag{59}$$

or equivalently

$$JX_1 = \frac{\partial B}{\partial z} - \frac{\partial C}{\partial y}, \quad JX_2 = \frac{\partial C}{\partial x} - \frac{\partial A}{\partial z}, \quad JX_3 = \frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}, \tag{60}$$

In physics and mathematics the *Clebsch representation* of the three-dimensional vector field $\mathcal{Y}$ is

$$\mathcal{Y} = \mathbf{grad}\varphi + \mu\mathbf{grad}\psi, \tag{61}$$

where the functions $\varphi = \varphi(x, y, z)$, $\mu = \mu(x, y, z)$ and $\psi = \psi(x, y, z)$ are known as *Clebsch potentials* (see for instance [6]).

**Proposition 24.** *The 3–dimensional solenoidal vector field* (58) *satisfying* (59) *with $\mu$ and $\psi$ independent functions is completely integrable if and only if $\mathcal{Y}$ is a Clebsch vector field.*

*Proof.* Clearly if (61) holds then $J\mathcal{X} = curl(\mathcal{Y}) = \mathbf{grad}\mu \wedge \mathbf{grad}\psi$, where $\mathbf{a} \wedge \mathbf{b}$ is the cross product of the vectors $\mathbf{a}$ and $\mathbf{b}$. Hence

$$\begin{aligned} J\mathcal{X}(\mu) = & \quad J\left(X_1\frac{\partial \mu}{\partial x} + X_2\frac{\partial \mu}{\partial y} + X_3\frac{\partial \mu}{\partial z}\right) = 0, \\ J\mathcal{X}(\psi) = & \quad J\left(X_1\frac{\partial \psi}{\partial x} + X_2\frac{\partial \psi}{\partial y} + X_3\frac{\partial \psi}{\partial z}\right) = 0, \end{aligned}$$

then $\mu$ and $\psi$ are independent first integrals, i.e. differential system (58) for $N = 3$ is completely integrable.

The reciprocity is obtained as follows. Assume that (58) is completely integrable with first integral $H_1$ and $H_2$, i.e. admits the representation (18) for $N = 3$, and (29) holds. By considering that

$$\begin{aligned} \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial z} - \frac{\partial H_1}{\partial z}\frac{\partial H_2}{\partial y} &= \frac{\partial}{\partial z}\left(H_2\frac{\partial H_1}{\partial y}\right) - \frac{\partial}{\partial y}\left(H_2\frac{\partial H_1}{\partial z}\right) = JX_1, \\ \frac{\partial H_1}{\partial z}\frac{\partial H_2}{\partial x} - \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial z} &= \frac{\partial}{\partial x}\left(H_2\frac{\partial H_1}{\partial z}\right) - \frac{\partial}{\partial z}\left(H_2\frac{\partial H_1}{\partial x}\right) = JX_2, \\ \frac{\partial H_1}{\partial x}\frac{\partial H_2}{\partial y} - \frac{\partial H_1}{\partial y}\frac{\partial H_2}{\partial x} &= \frac{\partial}{\partial y}\left(H_2\frac{\partial H_1}{\partial x}\right) - \frac{\partial}{\partial x}\left(H_2\frac{\partial H_1}{\partial y}\right) = JX_3. \end{aligned}$$

Consequently from (60) we get that the functions $A$, $B$ and $C$ are such that

$$\begin{aligned} A = & \quad H_1\frac{\partial H_2}{\partial x} + \frac{\partial\left(\Phi - \frac{1}{2}H_1H_2\right)}{\partial x} = -H_2\frac{\partial H_1}{\partial x} + \frac{\partial\left(\Phi + \frac{1}{2}H_1H_2\right)}{\partial x}, \\ B = & \quad H_1\frac{\partial H_2}{\partial y} + \frac{\partial\left(\Phi - \frac{1}{2}H_1H_2\right)}{\partial y} = -H_2\frac{\partial H_1}{\partial y} + \frac{\partial\left(\Phi + \frac{1}{2}H_1H_2\right)}{\partial y}, \\ C = & \quad H_1\frac{\partial H_2}{\partial x} + \frac{\partial\left(\Phi - \frac{1}{2}H_1H_2\right)}{\partial x} = -H_2\frac{\partial H_1}{\partial x} + \frac{\partial\left(\Phi + \frac{1}{2}H_1H_2\right)}{\partial x}, \end{aligned}$$

where $H_1$ and $H_2$ are independent first integrals, and $\Phi = \Phi(x, y, z)$ is an arbitrary function. Hence the vector field $\mathcal{Y}$ becomes

$$\begin{aligned}\mathcal{Y} = & \quad H_2 \,\mathbf{grad}\, H_1 + \mathbf{grad}\left(\Phi - \frac{1}{2} H_1 H_2\right)\\ = & \quad -H_1 \,\mathbf{grad}\, H_2 + \mathbf{grad}\left(\Phi + \frac{1}{2} H_1 H_2\right).\end{aligned}$$

Thus the vector field $\mathcal{Y}$ is a Clebsh vector field with Clebsh potentials $H_2$ (or -$H_1$,) $H_1$ (or $H_2$) and $\Phi - \frac{1}{2} H_1 H_2$ (or $\Phi + \frac{1}{2} H_1 H_2$). In short the proposition is proved. □

**Example 25.** *The vector field* $\mathcal{X} = x(y - z)\dfrac{\partial}{\partial x} + y(z - x)\dfrac{\partial}{\partial y} + z(x - y)\dfrac{\partial}{\partial z}$ *is solenoidal. The vector field* $\mathcal{Y}$ *in this case is* $\mathcal{Y} = xyz\,\boldsymbol{grad}(x + y + z) + \boldsymbol{grad}\varphi$, *where* $\varphi$ *is an arbitrary* $C^1$ *function. It is easy to show that* $xyz$ *and* $x + y + z$ *are first integral of* $\mathcal{X}$.

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