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ABSTRACT

We consider the four-dimensional hyperchaotic system $\dot{x} = a(y - x)$, $\dot{y} = bx + u - y - xz$, $\dot{z} = xy - cz$, and $\dot{u} = -du - jx + exz$, where a, b, c, d, j, and e are real parameters. This system extends the famous Lorenz system to four dimensions and was introduced in Zhou *et al.*, Int. J. Bifurcation Chaos Appl. Sci. Eng. **27**, 1750021 (2017). We characterize the values of the parameters for which their equilibrium points are zero-Hopf points. Using the averaging theory, we obtain sufficient conditions for the existence of periodic orbits bifurcating from these zero-Hopf equilibria and give some examples to illustrate the conclusions. Moreover, the stability conditions of these periodic orbits are given using the Routh–Hurwitz criterion.

I. INTRODUCTION

The chaos phenomenon is a complex dynamic behavior in a nonlinear dynamical system, which appears widely in nature. In 1963, meteorologist Lorenz[1] was the first to introduce the mathematical and physical chaotic model in $\mathbb{R}^3$, which is known as the Lorenz system. The Lorenz system planted the seed in chaos science. This system plays an important role in other areas such as in the modeling of lasers[2] and dynamos.[3] As one of the simplest models presenting chaos, the Lorenz system exhibits a rich range of dynamical properties, and it has been researched from different points of view, such as positive invariant,[4] integrability,[5–7] global dynamics,[8–10] and bifurcation.[11,12] After the Lorenz system, mathematicians and physicists from a physical or purely abstract mathematical point of view proposed various polynomial differential systems in $\mathbb{R}^3$, whose trajectories exhibit chaotic dynamics of the Lorenz system type. For example, one can refer to the Rikitake system,[13] Sprott A system,[14] Shimizu–Morioka system.[15]

Nowadays, three-dimensional nonlinear systems cannot provide adequate description of many phenomena in neural networks, social sciences, and engineering. To better describe the real world, we often necessitate to introduce high-dimensional (at least four dimensions) nonlinear systems. Recently, the hyperchaotic system has become a focus of research (see Refs. 16–23 and the references therein). The concept of hyperchaos was given by Rössler in Ref. 24. The precise definition of *hyperchaotic system* is as follows: (i) at least a four-dimensional autonomous differential system, (ii) a dissipative structure, and (iii) at least two unstable directions, of which at least one direction is nonlinear.[18] The hyperchaotic systems are very useful in secure communication due to the fact that the dynamic information of such systems is difficult to characterize and predict (see Ref. 25).

In this work, we use the classical averaging theory to investigate the zero-Hopf bifurcation of a hyperchaotic system. A zero-Hopf *equilibrium* is an equilibrium point of a four-dimensional autonomous differential system, which has a double zero eigenvalue and a pair of purely imaginary eigenvalues. There are rich works on three-dimensional zero-Hopf bifurcation (see Refs. 26–30). The zero-Hopf bifurcation of the hyperchaotic Lorenz system (i.e., four-dimensional) can be found in Refs. 17, 18, and 31. Actually, there are few results on the n-dimensional zero-Hopf bifurcation with $n > 3$.

In Ref. 32, Zhou *et al.* presented the following four-dimensional hyperchaotic system:

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= bx + u - y - xz, \\ \dot{z} &= xy - cz, \\ \dot{u} &= -du - jx + exz, \end{aligned} \tag{1}$$

where a, b, c, d, j, and e are real parameters. The hyperchaotic system (1) extends the Lorenz system to four dimensions and is invariant under the symmetry with respect to the z-axis, i.e., under the symmetry $\tau(x,y,z,u) = (-x,-y,z,-u)$. For the zero-Hopf bifurcation of system (1) at the origin, partial results are given by Yang *et al.* in Ref. 33. The objective of this paper is to study all the zero-Hopf bifurcations of system (1).

The equilibria and zero-Hopf equilibria of system (1) are described in the next two results.

Proposition 1. Let $\Delta = c(bd-d-j)/(d-e)$, *with* $d \neq e$. *The hyperchaotic system (1) has the following equilibria:*

(i) *If* $c = 0$, *system (1) has a straight line of equilibria* $(0,0,z,0)$.
(ii) *If* $\Delta \le 0$ *and* $c \neq 0$, *system (1) has an unique equilibrium point* $E_0 = (0,0,0,0)$.
(iii) *If* $\Delta > 0$ *and* $c \neq 0$, *system (1) has three equilibria* $E_0 = (0,0,0,0)$,

$$E_1 = \left(\sqrt{\Delta}, \sqrt{\Delta}, \frac{bd-d-j}{d-e}, -\frac{(e+j-be)\sqrt{\Delta}}{(d-e)}\right),$$

and

$$E_2 = \left(-\sqrt{\Delta}, -\sqrt{\Delta}, \frac{bd-d-j}{d-e}, \frac{(e+j-be)\sqrt{\Delta}}{d-e}\right).$$

Proposition 1 follows easily by direct computations.

Theorem 2. *For the hyperchaotic system (1), the following statements hold:*

(i) *There is a two-parameter family of systems (1) for which the origin of coordinates is a zero-Hopf equilibrium point. Specifically,* $c = 0$, $d = -a-1$, $b = -(1+a+a^2+\omega^2)/a$, and $j = ((1+a)^3 + (1+a)\omega^2)/a$.
(ii) *There is a three-parameter family of systems (1) for which the equilibria* $E_{1,2}$ *are zero-Hopf equilibrium points. Specifically,* $a = 0$, $j = bd, c = -d-1$, *and* $(d^2e + de + e - d^3)(d-e) > 0$.
(iii) *When* $c = 0$, *there is a three-parameter family of system (1) for which the equilibria* $(0,0,z_0,0)$ *are zero-Hopf equilibrium points. Specifically,* $a = -1-d, j = (b-1)d + z_0(e-d)$, *and* $(b-d-z_0)(d+1) > 1$.

Theorem 2 is proved in Sec. III.

In the following theorem, we characterize the periodic orbits bifurcating from the zero-Hopf equilibrium E_0 of system (1).

Theorem 3. *Let*

$$\begin{aligned} b &= -\frac{a^2+a+1+\omega^2}{a} + \varepsilon b_1, \\ c &= \varepsilon c_1, \\ d &= -a-1+\varepsilon d_1, \\ j &= \frac{(a+1)^3 + (a+1)\omega^2}{a} + \varepsilon j_1, \end{aligned}$$

with $\omega > 0$ *and* $\varepsilon > 0$ being *sufficiently small parameters. If* $\eta = (a+1)^2 d_1 + a(a+1)b_1 + aj_1$, $d_1 \neq 0$, $c_1\eta a(a+e+1) > 0$, *and* $c_1(\omega^2 d_1 + \eta)a(a+e+1) > 0$, *then for* $\varepsilon > 0$ *sufficiently small, the hyperchaotic system (1) has a zero-Hopf bifurcation at the equilibrium point located at* E_0, *and at most four periodic orbits can bifurcate from this equilibrium when* $\varepsilon = 0$ *and two of them are stable if either* $c_1 < 0, d_1 < 0, \eta < 0$, *or* $\eta > 0$, $c_1 < 0$, $-\frac{\eta}{\omega^2} < d_1 < 0$. *Moreover, there are systems (1) for which this zero-Hopf bifurcation exhibits the four periodic orbits [see Example 1].*

The proofs of Theorem 3 and Example 1 are given in Sec. IV and use the averaging theory of first order (see Subsection II A).

Example 1. The hyperchaotic system,

$$\dot{x} = x - y,\ \dot{y} = 2x + u - y - xz,\ \dot{z} = xy,\ \dot{u} = -xz, \tag{2}$$

has four small periodic orbits bifurcating from the equilibrium point $(0,0,0,0)$ *and two of them are stable.*

After the proof of Example 1, we show that the averaging theory of first order described in Sec. II does not provide any information about the possible periodic orbits, which can bifurcate from the zero-Hopf equilibria of the family of statement (ii) of Theorem 2.

In Sec. II, we present some basic results that we shall need for proving our theorems.

II. PRELIMINARIES

A. Averaging theory

In this subsection, we present the results on averaging theory that we need for proving our results. Consider the following differential equation:

$$\dot{\mathbf{x}} = \varepsilon \mathbf{F}(t,\mathbf{x}) + \varepsilon^2 \mathbf{G}(t,\mathbf{x},\varepsilon), \quad (t,\mathbf{x},\varepsilon) \in [0,\infty) \times \Omega \times (0,\varepsilon_0], \tag{3}$$

where Ω is an open subset of $\mathbb{R}^n$ and $\mathbf{F}(t,\mathbf{x})$ and $\mathbf{G}(t,\mathbf{x},\varepsilon)$ are T-periodic in t. We introduce the averaged function

$$\mathcal{F}(\mathbf{x}) = \frac{1}{T}\int_0^T \mathbf{F}(t,\mathbf{x})dt. \tag{4}$$

Theorem 4. *Assume that* $\mathbf{F}$*, its Jacobian* $\partial\mathbf{F}/\partial\mathbf{x}$*, and its Hessian* $\partial^2\mathbf{F}/\partial\mathbf{x}^2$*;* $\mathbf{G}$*, its Jacobian* $\partial\mathbf{G}/\partial\mathbf{x}$ *are defined, continuous and bounded by a constant independent of* ε *in* $[0,\infty)\times\Omega\times(0,\varepsilon_0]$ *and that the period* T *is a constant independent of* ε*. Then, the following statements hold:*

(i) *If* $\mathbf{p}$ *is the zero of the averaged function* $\mathcal{F}(\mathbf{x})$ *such that the Jacobian*

$$\det\left(\frac{\partial\mathcal{F}}{\partial\mathbf{x}}\right)\bigg|_{\mathbf{x}=\mathbf{p}} \neq 0, \tag{5}$$

then there exists a T*-periodic solution* $\mathbf{x}(t,\varepsilon)$ *of* Eq. (3) *such that* $\mathbf{x}(0,\varepsilon)\to\mathbf{p}$ *as* $\varepsilon\to 0$*.*

(ii) *The stability of the periodic solution* $\mathbf{x}(t,\varepsilon)$ *is determined by the eigenvalues of the Jacobian matrix* $(\partial\mathcal{F}/\partial\mathbf{x})|_{\mathbf{x}=\mathbf{p}}$*.*

For more details about the Proof of Theorem 4, see Ref. 34.

We consider the problem of the bifurcation of T-periodic solutions from differential systems of the form

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}) + \varepsilon F_1(t,\mathbf{x}) + \varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{6}$$

with $\varepsilon = 0$ to $\varepsilon \neq 0$ being sufficiently small. Here, the functions $F_0, F_1 : \mathbb{R}\times\Omega\to\mathbb{R}^n$ and $F_2 : \mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ are $\mathcal{C}^2$ functions, T-periodic in the first variable, and Ω is an open subset of $\mathbb{R}^n$. The main assumption is that the unperturbed system,

$$\dot{\mathbf{x}} = F_0(t,\mathbf{x}), \tag{7}$$

has a submanifold of periodic solutions. A solution of this problem is given using the averaging theory.

Let $\mathbf{x}(t,\mathbf{z},\varepsilon)$ be the solution of system (7) such that $\mathbf{x}(0,\mathbf{z},\varepsilon) = \mathbf{z}$. We write the linearization of the unperturbed system along a periodic solution $\mathbf{x}(t,\mathbf{z},0)$ as

$$\dot{\mathbf{y}} = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}. \tag{8}$$

In the following, we denote by $M_{\mathbf{z}}(t)$ some fundamental matrix of the linear differential system (8) and by $\xi : \mathbb{R}^k\times\mathbb{R}^{n-k}\to\mathbb{R}^k$ the projection of $\mathbb{R}^n$ onto its first k coordinates, i.e., $\xi(x_1,\ldots,x_n) = (x_1,\ldots,x_k)$.

We assume that there exists a k-dimensional submanifold $\mathcal{Z}$ of Ω filled with T-periodic solutions of (7). Then, an answer to the problem of bifurcation of T-periodic solutions from the periodic solutions contained in $\mathcal{Z}$ for system (6) is given in the following result.

Theorem 5. *Let* W *be an open and bounded subset of* $\mathbb{R}^k$*, and let* $\beta : \mathrm{Cl}(W)\to\mathbb{R}^{n-k}$ *be a* $\mathcal{C}^2$ *function. We assume that*

(i) $\mathcal{Z} = \{\mathbf{z}_\alpha = (\alpha,\beta(\alpha)),\quad \alpha\in\mathrm{Cl}(W)\}\subset\Omega$ *and that for each* $\mathbf{z}_\alpha\in\mathcal{Z}$*, the solution* $\mathbf{x}(t,\mathbf{z}_\alpha)$ *of* (7) *is* T*-periodic;*

(ii) *for each* $\mathbf{z}_\alpha\in\mathcal{Z}$*, there is a fundamental matrix* $M_{\mathbf{z}_\alpha}(t)$ *of* (8) *such that the matrix* $M_{\mathbf{z}_\alpha}^{-1}(0) - M_{\mathbf{z}_\alpha}^{-1}(T)$ *has in the upper right corner the* $k\times(n-k)$ *zero matrix and in the lower right corner a* $(n-k)\times(n-k)$ *matrix* Δ_α*, with* $\det(\Delta_\alpha)\neq 0$*.*

We consider the function $\mathcal{F} : \mathrm{Cl}(W)\to\mathbb{R}^k$*,*

$$\mathcal{F}(\alpha) = \xi\left(\frac{1}{T}\int_0^T M_{\mathbf{z}_\alpha}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z}_\alpha))dt\right). \tag{9}$$

If there exists $a\in W$ *with* $\mathcal{F}(a) = 0$ *and* $\det((d\mathcal{F}/d\alpha)(a))\neq 0$*, then there is a* T*-periodic solution* $\varphi(t,\varepsilon)$ *of system* (6) *such that* $\varphi(0,\varepsilon)\to\mathbf{z}_a$ *as* $\varepsilon\to 0$*.*

Theorem 5 goes back to the works of Malkin[35] and Roseau[36] (for a shorter proof, see Ref. 37).

B. Roots of cubic equation

The Routh–Hurwitz criterion gives necessary and sufficient conditions in order that all the roots of a polynomial $p(x) \in \mathbb{R}[x]$ have negative real parts (for more details, see page 231 of Ref. 38).

Theorem 6 (Routh–Hurwitz criterion). *All roots of the real polynomial* $p(x) = b_0x^n + b_1x^{n-1} + \cdots + b_{n-1}x + b_n$ $(b_0 > 0)$ *have negative real parts if and only if*

$$\Delta_1 > 0, \quad \Delta_2 > 0, \dots, \Delta_n > 0,$$

where

$$\Delta_i = \det\begin{pmatrix} b_1 & b_3 & b_5 & \cdots & & \\ b_0 & b_2 & b_4 & \cdots & & \\ 0 & b_1 & b_3 & \cdots & & \\ 0 & b_0 & b_2 & b_4 & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \\ & & & & & b_i \end{pmatrix} (b_k = 0 \text{ if } k > n)$$

is the Hurwitz determinant of order i $(i = 1, 2, \dots, n)$.

Corollary 7. All the roots of the real polynomial $b_0x^3 + b_1x^2 + b_2x + b_3 (b_0 > 0)$ *have negative real parts if and only if*

$$\Delta_1 = b_1 > 0, \Delta_2 = b_1b_2 - b_3b_0 > 0, b_3 > 0.$$

III. PROOF OF THEOREM 2

(i) The characteristic polynomial $p(\lambda)$ of the linearization of system (1) at the origin is

$$p(\lambda) = \lambda^4 + (a + c + d + 1)\lambda^3 + (a(1 - b + c + d) + cd + c + d)\lambda^2 + (a(c(1 - b + d) - bd + d + j) + cd)\lambda + ac(j - bd + d).$$

Since the origin of the hyperchaotic system (1) is a zero-Hopf equilibrium, $p(\lambda)$ must be of the form $p(\lambda) = \lambda^2(\lambda^2 + \omega^2)$, with $\omega \in \mathbb{R}^+$. Then, we obtain

$$c = 0, \ d = -a - 1, \ b = -\frac{a^2 + a + 1 + \omega^2}{a}, \ j = \frac{(a+1)^3 + (a+1)\omega^2}{a}.$$

(ii) Let $\Delta = c(bd - d - j)/(d - e)$. Then, $c = \Delta(d - e)/(bd - d - j)$. The characteristic polynomial of the linear part of system (1) at E_1 is given by

$$p(\lambda) = \lambda^4 + \left(a - \frac{\Delta(d-e)}{j - bd + d} + d + 1\right)\lambda^3 + \frac{\Delta(d(a+b+d) - e(a+d) - e - j)}{bd - d - j}\lambda^2 + \frac{a(b-1)e + (a+1)d^2 - (a+1)de - aj}{d - e}\lambda^2 + \frac{\Delta\big(a\big(d(2b - e - 2) + (b-1)e + d^2 - 3j\big) + (d-e)(bd - j)\big)}{bd - d - j}\lambda + 2a\Delta(d - e).$$

If the equilibrium E_1 is a zero-Hopf equilibrium, then $p(\lambda)$ must be of the form $p(\lambda) = \lambda^2(\lambda^2 + \omega^2)$, with $\omega \in \mathbb{R}^+$. Hence, we get that the parameters must satisfy

$$\text{either } a = -1 - d, \ c = 0, \ \text{and } j = \frac{d^3 + (b-1)e + (b - 1 - d)de + (d - e)\omega^2}{1 + d}$$
$$\text{or } a = 0, \ j = bd, \ \Delta = d^2 + d + \omega^2 + 1, \ \text{and } e = \frac{d(d^2 + \omega^2)}{d^2 + d + \omega^2 + 1}.$$

Clearly, $\Delta > 0$; otherwise, the equilibrium E_1 does not exist.

(iii) The characteristic polynomial at the equilibrium point $(0,0,z_0,0)$ is

$$p(\lambda) = \lambda^4 + (a+d+1)\lambda^3 + (a(d+1-b) + az_0 + d)\lambda^2 + a(z_0(d-e) + d + j - bd)\lambda.$$

Since the $(0,0,z_0,0)$ is a zero-Hopf equilibrium, the parameters must be satisfied,

$$a = -1-d,\ b = \frac{d^2+d+\omega^2+1}{d+1} + z_0,\ j = \frac{d\left(d^2+\omega^2\right)}{d+1} + ez_0,$$

where $\omega \in \mathbb{R}^+$. This completes the Proof of Theorem 2.

IV. PROOF OF THEOREM 3

Let

$$(b,c,d,j) = \left(-\frac{a^2+a+1+\omega^2}{a} + \varepsilon b_1, \varepsilon c_1, -a-1+\varepsilon d_1, \frac{(a+1)^3+(a+1)\omega^2}{a} + \varepsilon j_1\right),$$

where $\omega > 0$ and $\varepsilon > 0$ are sufficiently small parameters. Then, the hyperchaotic system (1) becomes

$$\begin{aligned}
\dot{x} &= a(y-x),\\
\dot{y} &= u - xz - y + \left(-\frac{a^2+a+1+\omega^2}{a} + b_1\varepsilon\right)x,\\
\dot{z} &= xy - \varepsilon c_1 z,\\
\dot{u} &= exz - u(-a-1+\varepsilon d_1) - \left(\frac{(a+1)\omega^2+(a+1)^3}{a} + \varepsilon j_1\right)x.
\end{aligned} \tag{10}$$

Performing the rescaling of variables

$$(x,y,z,u) \mapsto (\varepsilon x, \varepsilon y, \varepsilon z, \varepsilon u), \tag{11}$$

system (10) can be written as

$$\begin{aligned}
\dot{x} &= a(y-x),\\
\dot{y} &= u - y - \frac{a^2+a+1+\omega^2}{a}x + \varepsilon x(b_1 - z),\\
\dot{z} &= \varepsilon(xy - c_1 z),\\
\dot{u} &= u + au - \frac{(a+1)^3+(a+1)\omega^2}{a}x + \varepsilon(exz - d_1 u - j_1 x).
\end{aligned} \tag{12}$$

After the linear change in variables $(x,y,z,u) \mapsto (X,Y,Z,U)$,

$$\begin{aligned}
x &= \frac{a\omega Y + Z}{\omega^2}, \qquad y = \frac{a\omega Y - \omega^2 X + Z}{\omega^2},\\
z = U, \quad u &= \frac{\omega a(a+1)(aY - \omega X + Y) + \left((a+1)^2 + \omega^2\right)Z}{a\omega^2},
\end{aligned} \tag{13}$$

the linear part at the origin of system (12) for $\varepsilon = 0$ can be transformed into its real Jordan normal form,

$$\begin{pmatrix} 0 & \omega & 0 & 0\\ -\omega & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

Under the change in variable (13), system (12) can be written as

$$\begin{aligned}
\dot{x} &= \omega y + \frac{\varepsilon(u-b_1)(a\omega y+z)}{\omega^2},\\
\dot{y} &= -\omega x + \frac{\varepsilon(a\omega y+z)A}{a\omega^2},\\
\dot{z} &= d_1\varepsilon(a(a+1)x-z) - \frac{\varepsilon(a\omega y+z)A}{\omega},\\
\dot{u} &= \varepsilon\left(\frac{(a\omega y+z)(\omega(ay-\omega x)+z)}{\omega^4} - c_1 u\right),
\end{aligned} \tag{15}$$

where we have written (x, y, z, u) instead of (X, Y, Z, U) and

$$A = \frac{ab_1(a+1) + d_1(a+1)^2 + \big(j_1 - (a+e+1)u\big)a}{\omega}. \tag{16}$$

Performing the cylindrical change of variables,

$$(x, y, z, u) \mapsto (r\cos\theta, r\sin\theta, z, u), \tag{17}$$

system (15) becomes

$$\begin{aligned}
\frac{dr}{d\theta} &= \varepsilon\Bigg(\frac{\sin\theta\big(a^2(b_1-u)r\cos\theta + d_1(a(a+1)r\cos\theta - z)\big)}{a\omega^2}\\
&\quad + \frac{(b_1-u)z\cos\theta}{\omega^3} - \frac{A(a\omega r\sin\theta + z)\sin\theta}{a\omega^3}\Bigg) + O(\varepsilon^2)\\
&= \varepsilon F_1(\theta, r, z, u) + O(\varepsilon^2),\\
\frac{dz}{d\theta} &= \varepsilon\left(\frac{d_1(z - a(a+1)r\cos\theta)}{\omega} + \frac{A(a\omega r\sin\theta + z)}{\omega^2}\right) + O(\varepsilon^2)\\
&= \varepsilon F_2(\theta, r, z, u) + O(\varepsilon^2),\\
\frac{du}{d\theta} &= \frac{\varepsilon\big(c_1\omega^4 u - (a\omega r\sin\theta + z)(\omega(a\sin\theta - \omega\cos\theta)r + z)\big)}{\omega^5} + O(\varepsilon^2)\\
&= \varepsilon F_3(\theta, r, z, u) + O(\varepsilon^2).
\end{aligned} \tag{18}$$

System (18) is written in the normal form (3) for applying the averaging theory and satisfies all the assumptions of Theorem 4. Then, using the notations of the averaging theory described in Theorem 4, we have $t = \theta$, $T = 2\pi$, $\mathbf{x} = (r, z, u)$,

$$\mathbf{F}(\theta, r, z, u) = \begin{pmatrix} F_1(\theta, r, z, u)\\ F_2(\theta, r, z, u)\\ F_3(\theta, r, z, u)\end{pmatrix}, \text{ and } \mathcal{F}(r, z, u) = \begin{pmatrix} \mathcal{F}_1(r, z, u)\\ \mathcal{F}_2(r, z, u)\\ \mathcal{F}_3(r, z, u)\end{pmatrix},$$

where

$$\begin{aligned}
\mathcal{F}_1(r, z, u) &= \frac{1}{2\pi}\int_0^{2\pi} F_1(\theta, r, z, u)d\theta = -\frac{rA}{2\omega^2},\\
\mathcal{F}_2(r, z, u) &= \frac{1}{2\pi}\int_0^{2\pi} F_2(\theta, r, z, u)d\theta = \frac{(\omega d_1 + A)z}{\omega^2},\\
\mathcal{F}_3(r, z, u) &= \frac{1}{2\pi}\int_0^{2\pi} F_3(\theta, r, z, u)d\theta = -\frac{a^2\omega^2 r^2 - 2c_1\omega^4 u + 2z^2}{2\omega^5}.
\end{aligned}$$

The system $\mathcal{F}_1(r, z, u) = \mathcal{F}_2(r, z, u) = \mathcal{F}_3(r, z, u) = 0$ has the following five solutions:

$$\begin{aligned}
\mathbf{s}_0 &= (0, 0, 0),\\
\mathbf{s}_{1,2} &= \left(\mp\frac{\omega}{a}\sqrt{\frac{2c_1\eta}{a(a+e+1)}}, 0, \frac{\eta}{a(a+e+1)}\right),
\end{aligned}$$

$$\mathbf{s}_{3,4} = \left(0, \mp\omega^2\sqrt{\frac{c_1(\omega^2 d_1+\eta)}{a(a+e+1)}}, \frac{\omega^2 d_1+\eta}{a(a+e+1)}\right),$$

where $\eta = (a+1)^2 d_1 + a(a+1)b_1 + aj_1$. The first solution $\mathbf{s}_0$ corresponds to the equilibrium at the origin, so it is not a good solution. For other four solutions, we get

$$\det\left(\frac{\partial\mathcal{F}}{\partial\mathbf{x}}(\mathbf{s}_1)\right) = \det\left(\frac{\partial\mathcal{F}}{\partial\mathbf{x}}(\mathbf{s}_2)\right) = \frac{c_1 d_1\eta}{\omega^5},$$
$$\det\left(\frac{\partial\mathcal{F}}{\partial\mathbf{x}}(\mathbf{s}_3)\right) = \det\left(\frac{\partial\mathcal{F}}{\partial\mathbf{x}}(\mathbf{s}_4)\right) = -\frac{c_1 d_1(\omega^2 d_1+\eta)}{\omega^5}.$$

Since by assumptions $d_1 \neq 0$, $c_1\eta a(a+e+1) > 0$, and $c_1(\omega^2 d_1+\eta)a(a+e+1) > 0$, the solutions $\mathbf{s}_i$ exist and $\det(\partial\mathcal{F}(\mathbf{s}_i)/\partial\mathbf{x}) \neq 0$ for $i = 1,2,3,4$. From Theorem 4, it follows that system (18) for $\varepsilon > 0$ sufficiently small has four 2π-periodic orbits, $\gamma_i = (r_i(\theta,\varepsilon), z_i(\theta,\varepsilon), u_i(\theta,\varepsilon))$, such that $(r_i(0,\varepsilon), z_i(0,\varepsilon), u_i(0,\varepsilon)) \to \mathbf{s}_i$ as $\varepsilon \to 0$, with $i = 1,2,3,4$.

The Jacobian matrices $\partial\mathcal{F}(\mathbf{s}_1)/\partial\mathbf{x}$ and $\partial\mathcal{F}(\mathbf{s}_2)/\partial\mathbf{x}$ have the same characteristic equation,

$$\lambda^3 - \frac{c_1+d_1}{\omega}\lambda^2 + \frac{c_1(\eta+\omega^2 d_1)}{\omega^4}\lambda - \frac{c_1 d_1\eta}{\omega^5} = 0. \tag{19}$$

By Corollary 7, all the roots of Eq. (19) have negative real parts if

$$-\frac{c_1+d_1}{\omega} > 0,\ -\frac{c_1(c_1\eta + d_1(c_1+d_1)\omega^2)}{\omega^5} > 0,\ -\frac{c_1 d_1\eta}{\omega^5} > 0$$

or, equivalently, if $c_1 < 0, d_1 < 0, \eta < 0$. Thus, the periodic orbits γ_1 and γ_2 are stable if $c_1 < 0, d_1 < 0, \eta < 0$.

The Jacobian matrices $\partial\mathcal{F}(\mathbf{s}_3)/\partial\mathbf{x}$ and $\partial\mathcal{F}(\mathbf{s}_4)/\partial\mathbf{x}$ have the same characteristic equation,

$$\lambda^3 - \frac{2c_1+d_1}{2\omega}\lambda^2 - \frac{c_1(4\eta+3\omega^2 d_1)}{2\omega^4}\lambda + \frac{c_1 d_1(\eta+\omega^2 d_1)}{\omega^5} = 0. \tag{20}$$

Using Corollary 7, all the roots of Eq. (20) have negative real parts if

$$-\frac{2c_1+d_1}{2\omega} > 0,\ \frac{c_1(8c_1(\eta+d_1\omega^2) - (2c_1+d_1)d_1\omega^2)}{4\omega^5} > 0,\ \frac{c_1 d_1(\eta+\omega^2 d_1)}{\omega^5} > 0$$

or, equivalently, $\eta > 0$, $c_1 < 0$, $-\frac{\eta}{\omega^2} < d_1 < 0$. This implies that the periodic orbits γ_3 and γ_4 are stable if one of the three previous conditions holds. This completes the Proof of Theorem 3.

We can apply the averaging theory for studying the zero-Hopf bifurcation at the equilibria $(0,0,z_0,0)$ for all $z_0 \in \mathbb{R}$, after writing it in the normal form (3) and doing similar changes in variables to the ones of the Proof of Theorem 3. However, the determinant (5) evaluated at the zeros of the averaged function becomes zero, so the averaging theory of Theorem 4 does not provide any information on the periodic orbits that could exist in the zero-Hopf bifurcation at the equilibria $(0,0,z_0,0)$.

Proof of Example 1. Taking $a = e = -1$, $b = 2$, and $c = d = j = 0$, system (1) becomes system (2). Since the origin of system (2) has a double zero eigenvalue and a pair of purely imaginary eigenvalues $\pm i$, the origin is a zero-Hopf equilibrium point. Let $c_1 = d_1 = j_1 = -1$ and $\omega = 1$. Consider the perturbation of Theorem 3, that is, $b = 2+\varepsilon b_1$, $j = \varepsilon$, and $c = d = -\varepsilon$ in system (2), with $\varepsilon > 0$ being a sufficiently small parameter.

By the steps of averaging theory, we have the following functions:

$$\mathcal{F}_1(r,z,u) = \frac{r(u+1)}{2},\quad \mathcal{F}_2(r,z,u) = -(u+2)z,\quad \mathcal{F}_3(r,z,u) = -\frac{r^2}{2} - u - z^2. \tag{21}$$

System (21) has five solutions $\mathbf{s}_0 = (0,0,0)$, $\mathbf{s}_{1,2} = (0,\pm\sqrt{2},-2)$, and $\mathbf{s}_{3,4} = (\pm\sqrt{2},0,-1)$. Since the determinants

$$\det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)}{\partial(r,z,u)}\right|_{\mathbf{s}_{1,2}}\right) = 2 \quad\text{and}\quad \det\left(\left.\frac{\partial(\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3)}{\partial(r,z,u)}\right|_{\mathbf{s}_{3,4}}\right) = -1,$$

four periodic orbits can bifurcate from the zero-Hopf equilibrium at the origin. The eigenvalues of $\mathbf{s}_{3,4}$ are -1 and $(-1 \pm i\sqrt{3})/2$. For the solutions $\mathbf{s}_{1,2}$, the associated eigenvalues are $-1/2$ and $(-1 \pm \sqrt{17})/2$. Therefore, two of the four periodic orbits are stable. □

In order to study the zero-Hopf bifurcation at the equilibria E_1 and E_2, it is sufficient to study it for the equilibrium point E_1 due to the symmetry exhibited by system (1).

For the family of statement (ii) of Theorem 2, we translate the equilibrium E_1 at the origin of coordinates. After that, we take the values

$$j = bd + \varepsilon^2 j_1, \quad a = \varepsilon a_1, \quad c = -1 - d + \varepsilon^2 c_1.$$

Thus, when $\varepsilon = 0$, we have at the origin the zero-Hopf equilibrium of statement (ii) of Theorem 2. Now, we perform the linear change in variables,

$$\begin{pmatrix} x \\ y \\ z \\ u \end{pmatrix} = A \begin{pmatrix} X \\ Y \\ Z \\ U \end{pmatrix},$$

where

$$A = \begin{pmatrix} 0 & 0 & 0 & \frac{1}{2} \\ \frac{1}{-d+e+1} & \frac{(d-e)^{3/2}}{(d-e-1)\sqrt{e+d(e+d(e-d))}} & \frac{d(d+1)(d-e)}{d^3-(d^2+d+1)e} & \frac{d(db+b+d+2)-(b+(b-1)d)e}{2d^3-2(d^2+d+1)e} \\ \frac{\sqrt{1+\frac{1}{d}}\sqrt{d-e}}{-d+e+1} & \frac{\sqrt{1+\frac{1}{d}}\left(d^2-(d+1)e\right)}{(d-e-1)\sqrt{e+d(e+d(e-d))}} & \frac{\sqrt{1+\frac{1}{d}}d^2\sqrt{d-e}}{d^3-(d^2+d+1)e} & \frac{\sqrt{1+\frac{1}{d}}d(d(b+d+2)-(b+d+1)e)}{2\sqrt{d-e}(d^3-(d^2+d+1)e)} \\ \frac{e}{d-e-1} & \frac{\sqrt{d-e}e}{(-d+e+1)\sqrt{e+d(e+d(e-d))}} & \frac{d(d+1)e}{e+d(e+d(e-d))} & -\frac{d(bd(d-e)+2\,de+e)}{2d^3-2(d^2+d+1)e} \end{pmatrix}.$$

This linear change in variables writes the linear part of the differential system at the origin in its real Jordan normal form,

$$\begin{pmatrix} 0 & \frac{\sqrt{(d^2+d+1)e-d^3}}{\sqrt{d-e}} & 0 & 0 \\ -\frac{\sqrt{(d^2+d+1)e-d^3}}{\sqrt{d-e}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In the differential system obtained similar to system (15), we do the change to cylindrical variables (17). After that, we write the new differential system obtained, taking the variable θ as an independent variable, and we obtain a system similar to system (18). This differential system is written in the normal form (6), but when we try to apply to that differential system the averaging theory described in Theorem 4, we note that the condition (ii) of that theorem is not satisfied. Hence, the averaging theory does not provide any information about the possible periodic orbits that can bifurcate from the zero-Hopf equilibrium point of the second family of statement (ii) of Theorem 2.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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