

SOME OPEN PROBLEMS IN LOW DIMENSIONAL DYNAMICAL SYSTEMS

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Abstract. The aim of this paper is to share with the mathematical community a list of 33 problems that I have found along the years during my research. I believe that it is worth to think about them and, hopefully, it will be possible either to solve some of the problems or to make some substantial progress. Many of them are about planar differential equations but there are also questions about other mathematical aspects: Abel differential equations, difference equations, global asymptotic stability, geometrical questions, problems involving polynomials or some recreational problems with a dynamical component.

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1. INTRODUCTION

There are several famous well-known conjectures and open problems, like for instance Jacobian conjecture, Riemann’s conjecture, $3x + 1$ conjecture or Collatz problem, Goldbach’s conjecture, or Hilbert XVI problem, that almost all mathematicians know. Also a very interesting list of 18 open problems, covering many different branches of mathematics, has been published by Smale, see [122]. The aim of this work is much more modest. I will list several concrete problems that I have found along the years. I hope that, at least for some of them, it is possible either to solve or to make some substantial progress.

The problems will be classified in seven categories: periodic orbits, period function, piecewise linear systems, Markus-Yamabe and La Salle problems, geometrical problems, questions involving polynomials, and recreational questions with a dynamical flavour. Next we briefly describe them but without precise definitions. In the corresponding next sections they are contextualized and stated with more precision.

In Section 2 we will propose some questions about the maximum number of limit cycles of some low dimensional differential equations, including rigid systems, homogeneous type differential systems, Liénard systems, Riccati and Abel differential equations, and a new point of view of Hilbert’s XVI problem. Some other related questions considered in this section are on a second order singular differential equation, about the maximum number of centers for polynomial differential systems and on the characterization of some rational periodic difference equations.

In Section 3 we propose several problems for the period function of some families of planar systems: a Hamiltonian one, a system with homogenous components, a third one about the maximum number of critical periods for planar polynomial differential systems, and we end with the problem proposed by Chicone about the maximum number of critical periods for quadratic reversible centers and with a related one about the period function of a family of reversible equivariant planar differential systems.

Section 4 is devoted to planar piecewise linear systems. We state some problems about their number and type of limit cycles.

In Section 5 we present some questions related with global asymptotic stability: two problems inspired on the works of Markus, Yamabe, and La Salle and a third one dealing with linear random differential or difference equations.

In Section 6 we state three questions with a geometric component. The first one is well-known and it is about triangular billiards and the second one is about the extension of classical Poncelet's theorem for ellipses to more general algebraic ovals. The third question is the Loewner's conjecture.

Section 7 includes several problems involving polynomials. We start with a moments type problem, somehow related with the Jacobian conjecture, we recall the counterexamples of Kouchnirenko's conjecture about the number of solutions of fewnomials systems and propose an alternative question, and we end stating the Casas-Alvero's conjecture.

Finally, in Section 8 we collect three known conjectures with some dynamical flavour: the conjecture of multiplicative persistence, the 196 conjecture and Singmaster's conjecture.

2. PERIODIC ORBITS

The celebrated Hilbert XVIth problem, about the number of limit cycles of planar polynomials differential systems, has been extensively studied during the last century and also the beginning of the current one, see for instance the surveys [88, 100]. In this section, we present some related problems for some particular families of differential systems. We hope that advancing in these simpler cases can give some light to tackle the general question. Some related and complementary papers, collecting also some open problems are [25, 26, 76, 106]. This section also contains some questions about periodic orbits on different contexts.

2.1. Low degree rigid systems. Rigid systems are planar systems such that in polar coordinates their associated angular differential equation is $\dot{\theta} = 1$. The origin is their only equilibrium point, and their limit cycles, if exist are all nested. Moreover, centers are also isochronous centers. They were introduced by Conti ([49]) and afterwards they have been studied by many authors. They write as

$$(1) \quad \begin{cases} \dot{x} = -y + xF(x, y), \\ \dot{y} = x + yF(x, y), \end{cases}$$

where F is an arbitrary smooth function. Moreover, when F is a polynomial of degree n , $F = F_0 + F_1 + \dots + F_n$, where F_j are homogeneous polynomials of degree j , and in polar coordinates they write as

$$(2) \quad \frac{dr}{d\theta} = r' = \sum_{j=0}^n F_j(\cos \theta, \sin \theta)r^{j+1}.$$

Notice that this last expression is a 2π -periodic non-autonomous differential equation of Abel type. Its positive 2π -periodic solutions are precisely the periodic solutions of (1).

It is not difficult to see that when $n = 1$, that is $F = F_0 + F_1$, system (1) has not limit cycles. Let us prove this assertion by contradiction. Let γ be a periodic orbit of system (1). This periodic orbit is transformed into a positive 2π periodic solution of the Riccati differential equation (2), $r = r(\theta)$. Dividing (2) by r^2 and writing $F_1(x, y) = bx + cy$

we get that

$$\frac{r'(\theta)}{r^2(\theta)} = \frac{F_0}{r(\theta)} + b \cos \theta + c \sin \theta.$$

By integrating between $\theta = 0$ and $\theta = 2\pi$,

$$0 = \frac{1}{r(0)} - \frac{1}{r(2\pi)} = \int_0^{2\pi} \frac{r'(\theta)}{r^2(\theta)} d\theta = \int_0^{2\pi} \frac{F_0}{r(\theta)} d\theta + \int_0^{2\pi} (b \cos \theta + c \sin \theta) d\theta = \int_0^{2\pi} \frac{F_0}{r(\theta)} d\theta.$$

Therefore we get a contradiction, unless $F_0 = 0$. Hence we have proved that when $F_0 \neq 0$ system (2) with $F = F_0 + F_1$ has not periodic orbits. When $F_0 = 0$ it can have periodic orbits, but not limit cycles. This is so because, in this case, Riccati equation (2) is of separable variables and it can be easily integrated.

Equation (2) when $n = 2$ is precisely an Abel differential equation and it is the case we are interested. It writes as

$$(3) \quad \begin{cases} \dot{x} = -y + x(a + bx + cy + dx^2 + exy + fy^2), \\ \dot{y} = x + y(a + bx + cy + dx^2 + exy + fy^2). \end{cases}$$

In [73], examples with two limit cycles are given. For instance a way for obtaining two limit cycles is by a degenerate Andronov-Hopf bifurcation, because the first Lyapunov constants for system (3) are

$$V_1 = e^{2\pi a} - 1, \quad V_3 = \pi(d + f), \quad V_5 = \pi((c^2 - b^2)d - bce)/2.$$

Moreover the system has a center if and only if $V_1 = V_3 = V_5 = 0$.

Associated to F_2 and following again [73] we define the discriminant $\Delta := e^2 - 4df$. Then it holds that when $\Delta \leq 0$ system (3) has at most one limit cycle and when it exists it is hyperbolic. This is so because under this hypothesis the coefficient of r^3 , $F_2(\cos \theta, \sin \theta)$, of the Abel differential equation (2) when $n = 2$, does not change sign and when $d^2 + e^2 + f^2 \neq 0$ is not identically zero. Then, following [71, 103] it holds that this Abel equation has at most three periodic orbits, taking into account their multiplicities. Finally, since $r = 0$ is always one of these periodic orbits and, by symmetry of the equation, if $r(\theta)$ is one periodic orbit then $-r(\theta + \pi)$ it is also another one, we get that equation (2), when $n = 2$, has at most one positive periodic orbit, which has multiplicity one. This fact implies that when $\Delta \leq 0$ system (3) has at most one (hyperbolic) limit cycle, as we wanted to prove.

Hence, the left open problem reduces to the case $\Delta > 0$. In this case and without loss of generality, parameter f can be taken as 0 via a linear change of variables. Moreover, it is not restrictive to take $d \in \{0, 1\}$.

Problem 1. Consider the family of planar cubic rigid systems

$$\begin{cases} \dot{x} = -y + x(a + bx + cy + dx^2 + exy), \\ \dot{y} = x + y(a + bx + cy + dx^2 + exy). \end{cases}$$

Is 2 its maximum number of limit cycles?

2.2. Systems with homogeneous components. We start presenting next result given in [38], with a slightly different proof.

Theorem 2.1. Consider system

$$(4) \quad \dot{x} = P_n(x, y), \quad \dot{y} = Q_m(x, y),$$

where P_n and Q_m are homogeneous polynomials of degrees n and m , respectively. If it has limit cycles then $n \neq m$ and both n and m are odd. Moreover, in this case, there are polynomials P_n and Q_m such that system (4) has at least $(n + m)/2$ limit cycles.

Proof. Because P_n and Q_m are homogeneous, if (x_0, y_0) is a equilibrium point of (4) different of the origin then, all the real line $\lambda(x_0, y_0)$, $\lambda \in \mathbb{R}$, is full of equilibrium points. Therefore, if (4) has some periodic orbit, the origin must be the only equilibrium point of the system. Moreover, by using [60] we know that its index $\text{ind}(0, 0)$ satisfies

$$\text{ind}(0, 0) \equiv nm \pmod{2}.$$

It is well-known that if a periodic orbit surrounds a unique equilibrium point then its index must be 1. Hence, $nm \equiv 1 \pmod{2}$ and as a consequence n and m must be odd as we wanted to prove.

It is also well-known that if $n = m$ then (4) can have periodic orbits, but not limit cycles. This is so, because by homogeneity, if γ is a periodic orbit of the systems all orbits homothetic to γ are as well periodic orbits and hence γ is not an isolated periodic orbit, see also Section 3.2.

Hence the first part of the theorem is already proved. To prove the second part, recall that for general \mathcal{C}^1 perturbed Hamiltonian systems,

$$(5) \quad \begin{cases} \dot{x} = \frac{\partial H(x, y)}{\partial y} + \varepsilon R(x, y, \varepsilon), \\ \dot{y} = -\frac{\partial H(x, y)}{\partial x} + \varepsilon S(x, y, \varepsilon), \end{cases}$$

where ε is an small parameter, its associated Melnikov–Poincaré–Pontryagin function is

$$M(h) = \int_{\gamma(h)} S(x, y, 0) dx - R(x, y, 0) dy = \sigma \iint_{G(h)} \frac{\partial R(x, y, 0)}{\partial x} + \frac{\partial S(x, y, 0)}{\partial y} dx dy,$$

where σ is ± 1 according the orientation of the time parameterization of $\gamma(h)$. Here, the curves $\gamma(h)$ form a continuum of ovals contained in $\{H(x, y) = h, \text{ for } h \in (h_0, h_1)\}$, and the second expression is only valid if for $h = h_0$ the oval reduces to a point and $G(h)$ is the region surrounded by $\gamma(h)$, see for instance [29, 58]. It is known that each simple zero $h^* \in (h_0, h_1)$ of M gives rise to a limit cycle of (5) that tends, when $\varepsilon \rightarrow 0$, to $\gamma(h^*)$.

We write $n = 2k - 1 \geq 1$ and $m = 2\ell - 1 \geq 1$, where without loss of generality $k > \ell$, and consider,

$$(6) \quad \begin{cases} \dot{x} = y^{2k-1} + \varepsilon P_{2k-1}(x, y) = y^{2k-1} + \varepsilon \sum_{j=1}^{2k-1} \frac{a_j}{j} y^{2k-1-j} x^j, \\ \dot{y} = -x^{2\ell-1} + \varepsilon Q_{2\ell-1}(x, y) = -x^{2\ell-1} + \varepsilon \sum_{j=1}^{2\ell-1} \frac{b_j}{j} x^{2\ell-1-j} y^j. \end{cases}$$

Then $H(x, y) = \frac{x^{2\ell}}{2\ell} + \frac{y^{2k}}{2k}$, $(h_0, h_1) = (0, \infty)$ and

$$M(h) = \sum_{j=1}^{2k-1} a_j \iint_{G(h)} y^{2k-1-j} x^{j-1} dx dy + \sum_{j=1}^{2\ell-1} b_j \iint_{G(h)} x^{2\ell-1-j} y^{j-1} dx dy.$$

By symmetry of the sets $G(h) = \{x^{2\ell}/(2\ell) + y^{2k}/(2k) \leq h\}$, when j is even all the above integrals identically vanish. So we can write $j = 2i + 1$. Hence,

$$M(h) = \sum_{i=0}^{k-1} a_{2i+1} \iint_{G(h)} y^{2(k-i-1)} x^{2i} dx dy + \sum_{i=0}^{\ell-1} b_{2i+1} \iint_{G(h)} x^{2(\ell-i-1)} y^{2i} dx dy.$$

We introduce w such that $h = w^{2k\ell}$. Then, by using the change of variables $x = w^k X$ and $y = w^\ell X$, we get that

$$\iint_{G(h)} x^{2r} y^{2s} dx dy = w^{2(kr+\ell s)+k+\ell} \iint_{G(1)} x^{2r} y^{2s} dx dy =: I_{r,s} w^{2(kr+\ell s)+k+\ell}.$$

Hence,

$$\begin{aligned} M(h) &= \sum_{i=0}^{k-1} a_{2i+1} I_{i,k-i} w^{2k\ell+(2i+1)(k-\ell)} + \sum_{i=0}^{\ell-1} b_{2i+1} I_{\ell-i,i} w^{2k\ell+(2i+1)(\ell-k)} \\ &= w^{2k\ell} \left(\sum_{i=0}^{k-1} a_{2i+1} I_{i,k-i} \rho^{2i+1} + \sum_{i=0}^{\ell-1} b_{2i+1} I_{\ell-i,i} \rho^{-(2i+1)} \right) \\ &= w^{2k\ell+k+\ell} \rho^{1-2\ell} \left(\sum_{i=0}^{k-1} a_{2i+1} I_{i,k-i} \rho^{2(\ell+i)} + \sum_{i=0}^{\ell-1} b_{2i+1} I_{\ell-i,i} \rho^{2(\ell-1-i)} \right) \\ &=: w^{2k\ell+k+\ell} \rho^{1-2\ell} \sum_{j=0}^{k+\ell-1} c_j (\rho^2)^j, \end{aligned}$$

where $\rho = w^{2(k-\ell)}$ and c_j are arbitrary constants, given in terms of the parameters a_r and b_s . Therefore, taking suitable values of these parameters we get any polynomial of degree $k + \ell - 1$ in ρ^2 . Choosing it with all its roots positive and simple we obtain that M has $k + \ell - 1$ simple positive roots and as a consequence a system of the form (6), which clearly belong to family (5), such that for ε small enough has $k + \ell - 1 = (n + m)/2$ limit cycles. \square

From the above result a natural question is:

Problem 2. *Is $(n + m)/2$ the maximum number of limit cycles of*

$$\dot{x} = P_n(x, y), \quad \dot{y} = Q_m(x, y)$$

where $n \neq m$ and P_n and Q_m are homogeneous polynomials of odd degrees n and m , respectively?

A first challenge in the above problem is to deal with the simplest case, that corresponds to $n = 1$ and $m = 3$.

Problem 3. (i) Consider the cubic family

$$(7) \quad \begin{cases} \dot{x} = ax + by, \\ \dot{y} = cx^3 + dx^2y + exy^2 + fy^3. \end{cases}$$

Is 2 its maximum number of limit cycles?

(ii) Give a simple proof, if it is true, of the uniqueness of the limit cycle for equation

$$(8) \quad \begin{cases} \dot{x} = y, \\ \dot{y} = -x^3 + dx^2y + y^3. \end{cases}$$

Because of the difficulty to deal with (7) some efforts have been spend with the even more concrete system (8). For it is known that:

- It has not limit cycles for $d \geq 0$ and $d < -2.679$, see [66].
- It has at most one limit cycle $-2.381 < d < 0$, see [66].
- It has at least one limit cycle for $-2.110 < d < 0$, see [75].
- A numerically study seems to reduce the range of existence of limit cycles to $-2.198 < d < 0$.

The results of the first two items are obtained by using suitable Dulac functions, see next section for more details about this approach. The system is also studied with the same tool for some values of d in [28].

2.3. Low degree classical Liénard systems. Liénard equations $\ddot{x} + f(x)\dot{x} + x = 0$ are a subject of continuous study and for many functions f present isolated oscillations. Maybe the most famous one is the van der Pol equation, for which f is a cubic polynomial. These oscillations can be seen as limit cycles of the associated planar system:

$$(9) \quad \begin{cases} \dot{x} = y - F(x), \\ \dot{y} = -x, \end{cases}$$

with $F'(x) = f(x)$ and $F(0) = 0$.

When F is a polynomial of degree n its maximum number of limit cycles, say $\text{Lie}(n)$, is not known in general. During many years people tried to prove the conjecture of Lins, de Melo and Pugh ([102]) that asserted that $\text{Lie}(n) = [(n - 1)/2]$, where $[\]$ denotes the integer part function. This conjecture has been proved to be false for $n = 7$ in [59] and later, counterexamples for any $n \geq 6$, with 2 more limit cycles that the conjectured number, have been given in [52]. These counterexamples were found by studying slow-fast Liénard systems. Nowadays, no upper bound for arbitrary n is neither known nor conjectured. For more detailed information, see also the survey paper [107].

In any case, in [102] it is proved that $\text{Lie}(2) = 0$ and $\text{Lie}(3) = 1$, and in [99] that $\text{Lie}(4) = 1$. In particular, the proof of this last result is not easy at all. The first not known number is $\text{Lie}(5) \geq 2$.

There is a classical tool, based on the construction of the so-called Dulac functions that usually gives elegant proofs of the upper bound of the number of limit cycles. It is the Bendixson–Dulac theorem. We state a particular version of it, which is useful in many cases to prove uniqueness (and hyperbolicity) of limit cycles.

Theorem 2.2 (A very particular version of Bendixson–Dulac theorem, [66]). *Let $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ a C^1 function such that ∇V vanishes on $\{V(x, y) = 0\}$ at finitely many points and the set $\mathbb{R}^2 \setminus \{V(x, y) = 0\}$ has finitely many connected components, all them are*

simply connected but eventually one, that might have a hole (i.e., its fundamental group is \mathbb{Z}). Assume there exists $s \in \mathbb{R}$ such that

$$M_s = \frac{\partial V}{\partial x} P + \frac{\partial V}{\partial y} Q + s \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) V$$

does not change sign and vanishes only on a set of zero measure and, moreover, that the set $\{V(x, y) = 0\}$ does not contain periodic orbits of (10). Then, the \mathcal{C}^1 differential system

$$(10) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

has:

- (i) not periodic orbits when either $s \geq 0$ or no special region with a hole exists,
- (ii) at most one periodic orbit when $s < 0$ that, when exists is a hyperbolic limit cycle.

The idea of its proof when $s \neq 0$ is to show that the possible periodic orbits can not cut the set $\{V(x, y) = 0\}$ and later to apply the Dulac theorem to each of the connected components of $\mathbb{R}^2 \setminus \{V(x, y) = 0\}$ with the Dulac function $|V|^{1/s}$. When $s = 0$ it is easier to be proved, simply observing that $M_0 = \dot{V}$. To see a detailed proof of a more general version of the above result, and several examples of application, see [67] and their references. We detail a couple of examples of application for systems of the form (9).

Consider system (9) with $F(x) = cx^3 + x^5$. To prove that it has at most 1 limit cycle we will apply Theorem 2.2 with $F(x, y) = y^2 - F(x)y + x^2 + 2c/5$ and $s = -1$. Then $M_{-1}(x, y) = 2x^2(10x^4 + 10cx^2 + 3c^2)/5$. It is easy to see that $M_{-1}(x, y) \geq 0$ and vanishes only on the line $x = 0$. Moreover, since F is quadratic on y , $\mathbb{R}^2 \setminus \{V(x, y) = 0\}$ has at most one connected component that can have a hole. Hence, the corresponding system (9) has at most one limit cycle which is hyperbolic when it exists. In fact, the divergence of the vector field associated to the system is $3cx^2 + 5x^4$. Since when $c \geq 0$ it is always greater or equal that zero, in this case the system has no limit cycles by the classical Bendixson theorem. When $c < 0$ it is not difficult to see that the limit cycle exists. A similar approach can be used to prove the uniqueness and hyperbolicity of the limit cycle when $F(x) = cx^{2k+1} + x^{2m+1}$, for any natural numbers $k < m$, see [65].

In [117] it is proved that taking $F(x) = \frac{x(1-cx^2)}{1+cx^2}$ with $c > 0$, system (9) has at most 1 limit cycle and that a limit cycle exists for some values of c . By using Theorem 2.2 with $V(x, y) = y^2 - F(x)y + x^2$ and again $s = -1$ a simple and algebraic proof, by using the above theorem, of the uniqueness and hyperbolicity of the limit cycles is given in [67]. The theorem applies because

$$M_{-1}(x, y) = -\frac{4cx^4}{(1+cx^2)^2} \leq 0$$

and it only vanishes on the line $x = 0$, and $\mathbb{R}^2 \setminus \{V(x, y) = 0\} = \mathbb{R}^2 \setminus \{(0, 0)\}$ has only one connected component, which has a hole. This last assertion follows because the discriminant of V with respect to y is

$$\text{dis}_y(V(x, y)) = F^2(x) - 4x^2 = -\frac{x^2 (cx^2 + 3) (3cx^2 + 1)}{(cx^2 + 1)^2} \leq 0$$

and only vanishes at $x = 0$.

In next problem we propose to apply this method for low degree Liénard systems.

Problem 4. Find a proof using Dulac functions that $\text{Lie}(3) = 1$ and $\text{Lie}(4) = 1$.

We remark that a proof that $\text{Lie}(2) = 0$, follows easily from the above theorem. When $F(x) = ax + bx^2$, if we consider $V(x, y) = e^{-2by}$ and $s = 1$, it holds that $M_1(x, y) = -ae^{-2by}$. Hence if $a \neq 0$ this Liénard system has not periodic orbits. When $a = 0$ it has a reversible center at the origin, so it has periodic orbits but not limit cycles.

2.4. Riccati and Abel differential equations. Abel differential equations appear in the study of some planar vector fields, see for instance Section 2.1, but they are also interesting by themselves. In general, they write as

$$(11) \quad \frac{dx}{dt} = A_3(t)x^3 + A_2(t)x^2 + A_1(t)x + A_0(t),$$

where all the functions A_j are \mathcal{C}^1 and $T > 0$ periodic. We are interested on finding conditions for these functions to control their number of T periodic solutions. Usually the T periodic solutions that are isolated among all the T periodic solutions are also called limit cycles.

It is remarkable that while when $A_3 = 0$ (the Riccati differential equation) the maximum number of limit cycles is two, there is no upper bound for the number of limit cycles for general Abel differential equations (11), even when the functions A_j are trigonometrical polynomials, see [103].

The upper bound for Riccati differential equation follows for instance from the fact that, on its interval of definition, the solution of this differential equation, when $A_3 = 0$ and satisfying $\varphi(0; \rho) = \rho$ is

$$x = \varphi(t; \rho) = \frac{B(t)\rho + C(t)}{D(t)\rho + E(t)},$$

where B, C, D, E are smooth functions that depend on $A_j, j = 0, 1, 2$, see for instance [86]. Hence, for each fixed t , it is a Möbius map. Therefore its number of periodic solutions is given by the number of solutions of the quadratic equation obtained from the condition $\varphi(T, \rho) = \rho$. Moreover, limit cycles correspond to isolated solutions of the quadratic equation. Nevertheless we only know how to obtain explicitly these four functions when a particular solution of the Riccati equation is known, see for instance [32]. Hence the following problem remains:

Problem 5. *For a general T periodic Riccati differential equation*

$$\frac{dx}{dt} = A_2(t)x^2 + A_1(t)x + A_0(t)$$

give effective criteria to know when it has a continuum of periodic solutions, or it has exactly 2, 1 or 0 limit cycles.

Two useful results to obtain upper bounds on the number of limit cycles for Abel differential equation (11) are:

- (i) If $A_3 \neq 0$ and does not change sign, then the maximum number of limit cycles is 3, see [71].
- (ii) If $A_0 = A_1 = 0$ and there exist $a, b \in \mathbb{R}$ such that $aA_3 + bA_2 \neq 0$ and does not change sign, then the maximum number of limit cycles is 3, see [3]. Notice that one of them is $x = 0$.

One of the simplest natural open questions for Abel equations is:

Problem 6. Consider the family of trigonometric Abel differential equations

$$\frac{dx}{dt} = (a_0 + a_1 \sin t + a_2 \cos t)x^3 + (b_0 + b_1 \sin t + b_2 \cos t)x^2.$$

Is 3 its maximum number of 2π periodic limit cycles?

Notice that for the above differential equation $x = 0$ is always a periodic solution. So, if it is isolated from other 2π periodic solutions, it is one of these limit cycles. In [3] the problem is introduced, the above two general results are applied to this particular case, obtaining some particular positive answers, and the existence of examples with at least 3 limit cycles is established. In [16] further complementary results are obtained. In particular, it is proved that the answer is yes when $a_0 b_0 = 0$.

In fact, the above problem can be extended to next one:

Problem 7. Given two integer numbers $p > q \geq 2$, and $m, n \in \mathbb{N}$, find the maximum number of 2π periodic limit cycles for next family of Abel type differential equations

$$\frac{dx}{dt} = A_m(t)x^p + B_n(t)x^q,$$

where A_m and B_n are 2π -trigonometric polynomials with respective degrees m and n .

This question was introduced in [5], where some lower bounds of the number of limit cycles were given. A recent improvement of these bounds has been obtained in [87].

In fact, the class of Abel type equations is very interesting and intriguing. For instance, the following result proved in [68] extends the results of previous item (i). We remark that the result when n is odd was also obtained in [113].

Theorem 2.3. Consider the \mathcal{C}^1 , T periodic Abel type differential equation

$$\frac{dx}{dt} = A_3(t)x^3 + A_2(t)x^2 + A_1(t)x + A_0(t),$$

where $n \geq 3$ and $0 \neq A_n$ does not change sign. Then:

- (i) If n is odd, it has at most 3 limit cycles and the upper bound is sharp.
- (ii) If n is even, there is no upper bound for its number of limit cycles.

2.5. A new Hilbert XVith type problem. For each $m \in \mathbb{N}$ fixed, consider the following family of polynomials differential equations:

- Family \mathcal{M}_m given by

$$(\dot{x}, \dot{y}) = \sum_{j=1}^m a_j X_j(x, y), \quad \text{with} \quad X_j(x, y) = \begin{cases} (x^{n_j} y^{k_j}, 0), & \text{or,} \\ (0, x^{n_j} y^{k_j}), \end{cases}$$

where $(a_1, a_2, \dots, a_m) \in \mathbb{R}^m$ and the couples $(n_j, k_j) \in \mathbb{N}^2$ vary among all the possible values. Varying m , this family covers all polynomial differential equations.

The letter \mathcal{M} is chosen because the important point is to count the number of involved monomials. We define $\mathcal{H}^M[m] \in \mathbb{N} \cup \{\infty\}$ to be the maximum number of limit cycles that systems of the family \mathcal{M}_m can have. This point of view is similar to the one of counting the number of real solutions of planar fewnomial systems, see Section 7.2. In the recent preprint [20] we prove:

Theorem 2.4. *It holds that $\mathcal{H}^M[m] = 0$ for $m = 1, 2, 3$ and for $m \geq 4$, $\mathcal{H}^M[m] \geq m - 3$. Moreover, there exists a sequence of values of m tending to infinity such that $\mathcal{H}^M[m] \geq N(m)$, where*

$$N(m) = \left(\frac{\binom{m-3}{2} \log\left(\frac{m-3}{2}\right)}{\log 2} \right) (1 + o(1)).$$

The proof of the first part for $m = 1, 2, 3$ follows by a case by case study. In particular, we prove that systems $(\dot{x}, \dot{y}) = (ax^p y^q, bx^i y^j + cx^k y^l)$, where $(a, b, c) \in \mathbb{R}^3$ and $(p, q, i, j, k, l) \in \mathbb{N}_0^6$, with $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, have no limit cycle. Currently we are working to try to prove that $\mathcal{H}^M[4] = 1$.

The fact that $\mathcal{H}^M[m] \geq m - 3$ for $m > 3$ is a straightforward consequence of known results about classical Liénard systems. In fact, an example with this number of limit cycles is already provided by the one given in [102] to prove that $\text{Lie}(n) \geq [(n - 1)/2]$, see Section 2.3, simply taking F odd with $n = 2m - 5$. Then, the Liénard system (9) has m monomials and $m - 3$ limit cycles.

The second part is a direct corollary of the recent paper [2] where the authors study limit cycles for generalized slow-fast Liénard systems.

It is worth to mention that the celebrated examples of quadratic systems that prove that $\mathcal{H}(2) \geq 4$ are given by systems with $m = 8$ monomials and so they have $m - 4$ limit cycles, see [27, 114]. The slow-fast example of Liénard equation of degree 6 and 4 limit cycles, that gives a counterexample of Lins, de Melo and Pugh's conjecture, has also 8 monomials, see [52]. The cubic system given [98] that shows that $\mathcal{H}(3) \geq 13$ has $m = 9$ monomials and at least $m + 4$ limit cycles.

Under the light of the above results, some natural problems are:

Problem 8. (i) *Find upper and lower bounds for $\mathcal{H}^M[m]$.*

(ii) *Find the minimal m such that there exists a planar polynomial differential system with m monomials having at least $m + 1$ limit cycles: the simplest polynomial differential system with more limit cycles than monomials.*

2.6. A second order differential equation. In [17, 126, 127] there are several motivations to study the T periodic solutions of the second order singular, T periodic differential equation $x^p(t)x''(t) = f(t)$, $0 < p \in \mathbb{R}$. From their results the following interesting problem can be formulated:

Problem 9. *Let $f(t)$ be a continuous T periodic function, $f(t) \not\equiv 0$ and $0 < p \in \mathbb{R}$. Find necessary and sufficient conditions on the function f that ensure the existence of positive T periodic solutions of $x^p(t)x''(t) = f(t)$.*

Notice that a simple necessary condition is that f changes sign, because if $x(t)$ is a T periodic positive solution, then

$$\int_0^T \frac{f(t)}{x^p(t)} dt = \int_0^T x''(t) dt = x'(T) - x'(0) = 0.$$

A second necessary condition is that $\int_0^T f(t) dt < 0$. This can be easily proved by using integration by parts. If $x(t)$ is a T periodic positive solution, then

$$\begin{aligned} \int_0^T f(t) dt &= \int_0^T x^p(t)x''(t) dt = x^p(t)x'(t) \Big|_{t=0}^{t=T} - p \int_0^T x^{p-1}(t)(x'(t))^2 dt \\ &= -p \int_0^T x^{p-1}(t)(x'(t))^2 dt < 0. \end{aligned}$$

In fact, in [126] it is proved that if $p \geq 2$ and $f(t)$ has only nondegenerate zeroes, meaning that it is continuously differentiable, with nonvanishing derivative, in a neighbourhood of each of its zeroes, the above two necessary conditions are also sufficient. On the other hand, the same author proves in [127] that when $p = 5/3$ there is a function f , of class C^∞ , satisfying both necessary conditions and such that the corresponding differential equation has no positive periodic solution.

2.7. Number of centers. By Bezout's theorem, a planar polynomial differential systems of degree $n > 0$, with finitely many equilibrium points, has at most n^2 equilibrium points. Moreover, at most $(n^2 + n)/2$ can have index $+1$, see for instance [44, 92]. Therefore, if we define \mathcal{C}_n as the maximum number of centers for this class of polynomials systems, it holds that $\mathcal{C}_n \leq (n^2 + n)/2$, because, remember that all centers have index $+1$. It is also clear that $\mathcal{C}_1 = 1$.

Moreover, by using the beautiful Euler–Jacobi's formula, it was proved in [37] that for $n \geq 2$ not all points of index $+1$ can lie on the same algebraic curve of degree at most $n - 1$. Since all centers are on the algebraic curve given by the divergence of the vector field equal zero, which has at most degree $n - 1$, we get that $\mathcal{C}_n \leq (n^2 + n)/2 - 1$.

Recall that this formula, for the planar case, asserts that if a polynomial system $P(x, y) = 0, Q(x, y) = 0$, with P and Q with respective degrees n and m , has exactly nm solutions (hence all them are finite and simple) then it holds that

$$\sum_{\{(u,v): P(u,v)=Q(u,v)=0\}} \frac{R(u,v)}{\det(D(P,Q))(u,v)} = 0,$$

for any polynomial $R(x, y)$ of degree smaller than $n + m - 2$, see for instance [80]. Here, $D(P, Q)$ denotes the differential of the map (P, Q) .

On the other hand, in [36] it is proved that planar polynomial Hamiltonian differential systems of degree n have at most $[(n^2 + 1)/2]$ centers and that this upper bound is attained. Hence $[(n^2 + 1)/2] \leq \mathcal{C}_n$. In fact, examples of Hamiltonian systems with this number of centers are not difficult to be obtained. It suffices to consider

$$\dot{x} = F(y), \quad \dot{y} = -F(x), \quad \text{with} \quad F(u) = \prod_{j=1}^n (u - j).$$

For these systems, centers and saddles are located like white and black squares on an $n \times n$ chessboard. In short, for $n \geq 2$ it is known that

$$\left\lceil \frac{n^2 + 1}{2} \right\rceil \leq \mathcal{C}_n \leq \frac{n^2 + n}{2} - 1.$$

Hence $\mathcal{C}_2 = 2, \mathcal{C}_3 = 5$, and $8 \leq \mathcal{C}_4 \leq 9$.

Problem 10. Determine the maximum number of centers, \mathcal{C}_n , for planar polynomial differential systems of degree $n \geq 4$.

Once the number \mathcal{C}_n is determined, it is also interesting to know the different possible phase portraits that systems having these maximal number of centers can have, see for instance [14], where the Hamiltonian case when $n = 3$ is studied. One of the reasons is that these systems are good candidates to have after perturbation different configurations with many limit cycles.

2.8. On some rational difference equations. A classical problem for a given family of planar vector fields is the so-called center-focus problem. It consists on the distinction between these two types of monodromic equilibrium points: center or focus, or shortly into determining all centers of the family. Next we present a similar question for some rational difference equations. The goal here will be to determine all difference equations that are periodic. It is proved in [32] that this periodicity is also strongly related with the complete integrability of the discrete dynamical system associated to the difference equation.

Let us introduce some definitions and state precisely the problem. Consider the family of k -th order difference equations

$$(12) \quad x_{n+k} = \frac{A_0 + A_1x_n + A_2x_{n+1} + \cdots + A_kx_{n+k-1}}{B_0 + B_1x_n + B_2x_{n+1} + \cdots + B_kx_{n+k-1}},$$

with $\sum_{i=0}^k A_i > 0$, $\sum_{i=0}^k B_i > 0$, $A_i \geq 0$, $B_i \geq 0$, and $A_1^2 + B_1^2 \neq 0$. For every initial condition $(x_1, x_2, \dots, x_k) \in (0, \infty)^n$, they define a sequence $\{x_i\}_{i \geq 1}$ of positive real numbers. One of these difference equations is called p -periodic if for all these initial conditions it holds that $x_n = x_{n+p}$, for all $1 \leq n \in \mathbb{N}$, and this value $0 < p \in \mathbb{N}$ is the smallest number with this property. That is, all the sequences with positive initial conditions generated by (12) are p -periodic.

The following examples of p -periodic difference equations of the form (12) are known:

$$(13) \quad \begin{aligned} x_{n+1} = x_n \quad \text{with } p = 1, \quad & x_{n+1} = \frac{1}{x_n} \quad \text{with } p = 2, \\ x_{n+2} = \frac{x_{n+1}}{x_n} \quad \text{with } p = 6, \quad & x_{n+2} = \frac{1 + x_{n+1}}{x_n} \quad \text{with } p = 5, \\ x_{n+3} = \frac{1 + x_{n+1} + x_{n+2}}{x_n} \quad & \text{with } p = 8. \end{aligned}$$

Moreover, every p -periodic k -th order difference equation produces in a natural way, and for each $\ell \in \mathbb{N}$, another one which is $p\ell$ -periodic and of $k\ell$ -th order. For instance, the one of second order given in (13) gives

$$x_{n+2\ell} = \frac{x_{n+\ell}}{x_n} \quad \text{with } p = 6\ell, \quad x_{n+2\ell} = \frac{1 + x_{n+\ell}}{x_n} \quad \text{with } p = 5\ell.$$

Similarly, every p -periodic k -th order difference equation can be unfold into a 1-parametric family with the same property. It suffices to consider for any $n \in \mathbb{N}$, $y_n = ax_n$, with $0 \neq a \in \mathbb{R}$. For instance, the above ones give rise to

$$y_{n+2\ell} = \frac{ay_{n+\ell}}{y_n} \quad \text{with } p = 6\ell, \quad y_{n+2\ell} = \frac{a^2 + ay_{n+\ell}}{y_n} \quad \text{with } p = 5\ell.$$

For short, all these new difference equations are called equivalent to (13).

Problem 11. *Are there rational difference equations of the form (12) that are not equivalent to the five ones given in (13)?*

The answer to the above question for $k \in \{1, 2, 3, 4, 5, 7, 9, 11\}$ is no, see [41]. We remark that when the condition of non-negativeness of the coefficients of (12) is removed much more periods and periodic difference equations appear. For instance, when $k = 1$ there are periodic Möbius maps with all the periods. To see more information about related problems, see [35] and its references.

3. PERIOD FUNCTION

Let $\gamma(s)$, with s in a real open interval, be a smooth parameterized continua of periodic orbits of a smooth planar autonomous differential system. Usually, if the system is Hamiltonian the parameter s is taken to be the energy of the system. When the continuum of periodic orbits ends in a critical point, then the maximal set covered by them is called period annulus of the point. The function that assigns to each s the minimal period of $\gamma(s)$ is called period function and it is usually denoted by $T(s)$. The zeroes of $T'(s)$ are called critical periods and determine them is a key point to know the behaviour of $T(s)$. To know properties of this function (monotonicity, number of oscillations, . . .) is interesting from a theoretical point of view, as well as for applications for instance in physics or ecology ([48, 116, 129]).

3.1. A class of Hamiltonian systems. From the results of [47, 50, 69] it is known that, on the period annulus of the origin, the period function has at most one critical period for the family of Hamiltonian systems with Hamiltonian

$$H(x, y) = \frac{1}{2}(x^2 + y^2) + H_m(x, y),$$

where $H_m(x, y)$ is a homogeneous polynomial of degree $m \geq 3$. Next question proposes to study if the same result holds for a more general class of Hamiltonian systems.

Problem 12. *Consider a Hamiltonian system with a center at the origin and Hamiltonian*

$$H(x, y) = H_{2n}(x, y) + H_m(x, y), \quad m > 2n,$$

where H_{2n} and H_m are homogeneous polynomials of degrees $2n$ and m , respectively. Has the period annulus of the origin at most 1 critical period?

In [4] it is proved that the answer is yes when $m \geq 4n - 2$. So it remains to study the cases $2n < m < 4n - 2$. Notice that the simplest open question corresponds to the Hamiltonian $H(x, y) = H_4(x, y) + H_5(x, y)$.

3.2. Systems with homogeneous components. We consider again systems

$$\begin{cases} \dot{x} = P_{2k+1}(x, y), \\ \dot{y} = Q_{2\ell+1}(x, y), \end{cases}$$

where P_{2k+1} and $Q_{2\ell+1}$ are homogeneous polynomials of degrees $2k + 1$ and $2\ell + 1$, respectively.

Problem 13. *(i) Characterize the centers of the above family.*

(ii) Which is the maximum number of oscillations of the period function for the centers of the above family?

When $k = \ell$ both questions have a simple answer. The centers can be characterized studying their expression in polar coordinates $(x, y) = (r \cos \theta, r \sin \theta)$, because they are easily integrable, see [9]. In fact, they write as

$$\dot{r} = f(\theta)r^{2k+1}, \quad \dot{\theta} = g(\theta)r^{2k},$$

where

$$\begin{aligned} f(\theta) &= P_{2k+1}(\cos \theta, \sin \theta) \cos \theta + Q_{2k+1}(\cos \theta, \sin \theta) \sin \theta, \\ g(\theta) &= Q_{2k+1}(\cos \theta, \sin \theta) \sin \theta - P_{2k+1}(\cos \theta, \sin \theta) \cos \theta. \end{aligned}$$

Hence the center conditions are that g does not vanish (otherwise the system would have invariant lines through the origin) and

$$\int_0^{2\pi} \frac{f(\theta)}{g(\theta)} d\theta = 0,$$

where we have obtained this last equality because the solution of $\frac{dr}{d\theta} = \frac{f(\theta)}{g(\theta)}r$ with initial condition $r(0) = s > 0$ is

$$r(\theta; s) = s \exp\left(\int_0^\theta \frac{f(\psi)}{g(\psi)} d\psi\right).$$

Hence by imposing that $r(0) = r(2\pi)$ the condition follows.

The period function can also be obtained from the above equations. In fact,

$$T(s) = \int_0^{2\pi} \frac{1}{|g(\theta)|r^{2k}(\theta; s)} d\theta = \left(\int_0^{2\pi} \frac{\exp\left(-2k \int_0^\theta \frac{f(\psi)}{g(\psi)} d\psi\right)}{|g(\theta)|} d\theta\right) \frac{1}{s^{2k}} =: \frac{A_k}{s^{2k}},$$

and it is constant for $k = 0$ (these are the linear centers) and decreasing for $k \geq 1$.

In [22] the authors studied all the phase portraits when $k = 0$ and $\ell = 1$, modulus the number of limit cycles, which recall that at least is 2, see Section 2.2.

3.3. About the maximum number of critical periods. Let $\mathcal{H}(n)$ denote the maximum number of limit cycles that planar polynomial systems of degree n can have. From [31] it is known that

$$\mathcal{H}(n) \geq Kn^2 \log(n), \quad \text{for some } K > 0.$$

On the other hand, if we denote as $\mathcal{T}(n)$ the maximum number of critical periods that planar polynomial systems of degree n can have, from the results of [70] it is also known that

$$\mathcal{T}(n) \geq \frac{1}{4}n^2.$$

Recently, this lower bound has been essentially doubled in [24, 53]. A natural question is:

Problem 14. *Is it true that $\mathcal{T}(n) \geq Cn^2 \log(n)$ for some $C > 0$?*

3.4. Reversible quadratic systems. Although there are some subsequent results, in [108] there is an excellent source of information about one of the most famous open problems about critical periods. It was proposed by Chicone in 1994 in a review of MathSciNet and it reads as follows:

Problem 15. *Consider the family of reversible quadratic centers*

$$(14) \quad \begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x + Dx^2 + Fy^2. \end{cases}$$

Is 2 its maximum number of critical periods?

The above systems are sometimes called Loud's systems, because this author studied them in 1964, see [26, 108].

3.5. Some reversible equivariant planar differential systems. Any planar analytic system, $(\dot{x}, \dot{y}) = (f(x, y), g(x, y))$, can be written in complex coordinates as $\dot{z} = F(z, \bar{z})$, where $z = x + iy$. Moreover, when the origin is a weak focus, after a constant rescaling of time, it writes as $\dot{z} = iz + G(z, \bar{z})$, where G starts at least with second order terms.

Recall that, in real coordinates, one of the simplest criteria to know that the origin is a center is the so-called Poincaré's reversibility criterion. It simply says that if a equilibrium point (the origin) is monodromic and the system is invariant by the change of variables and time $(x, y, t) \rightarrow (x, -y, -t)$ then it is a center. This is so because if $(x(t), y(t))$ is a solution of the system, then the same happens for $(x(-t), -y(-t))$ and by the monodromy condition and the uniqueness of solutions, both trajectories intersect and, hence, they coincide. This proves that this solution is a periodic orbit which is symmetric with respect the line $y = 0$. Notice that in complex variables this criterion works when the differential equation is invariant by the change of variables and time $(z, t) \rightarrow (\bar{z}, -t)$. Simply by considering symmetries with respect arbitrary straight lines passing through the origin we obtain the following well-known general result: if the origin of a differential equation $\dot{z} = F(z, \bar{z})$ is a monodromic critical point and, for some $\alpha \in \mathbb{R}$, this equation is invariant by the change of variables and time $(z, t) \rightarrow (e^{i\alpha}\bar{z}, -t)$, then the origin is a (reversible) center.

If we consider the origin of $\dot{z} = F(z, \bar{z})$ to be a weak focus, and we write this equation as

$$\dot{z} = iz + \sum_{m+n \geq 2} A_{m,n} z^m \bar{z}^n, \quad A_{m,n} \in \mathbb{C},$$

then the condition for this equilibrium point to be a reversible center is simply that there exists some $\alpha \in \mathbb{R}$ such that $A_{m,n} = -\bar{A}_{m,n} e^{i(1-m+n)\alpha}$, for all $m, n \in \mathbb{N}$.

Another remarkable class of planar systems are the so-called \mathbb{Z}_k -equivariant differential equations, see for instance [100, Sec. 7] and their references. They are differential equations $\dot{z} = F(z, \bar{z})$ that are invariant by a rotation through $2\pi/k$ about the origin, or in other words, such that the change of variable $z \rightarrow e^{i\beta}z$, for $\beta = 2\pi/k, k \in \mathbb{N}$, leaves them invariant. For these differential equations, the phase portrait on each sector centered at the origin and width $2\pi/k$, is repeated k times.

Consider the following family of polynomial \mathbb{Z}_k -equivariant differential equations

$$(15) \quad \dot{z} = iz + (z\bar{z})^n z^{k+1},$$

with $n \in \mathbb{N}$ and k a positive integer. It has a reversible center at the origin because of Poincaré's extended result with $\alpha = \pi/k$. We are interested on the behaviour of the

period function associated to this center. Notice that when $n = 0$ the differential equation is holomorphic and so it has an isochronous center at the origin ([26]).

Proposition 3.1. *Consider the period function associated to the origin for (15). Then its behaviour and number of critical periods coincide with the one of the period function of the origin of the quadratic reversible center (14) with $F = 1 + D$ and $D = -k/(2(k + n)) \in [-1/2, 0)$.*

Proof. Equation (15) in polar coordinates $z = re^{i\theta}$ writes as

$$\dot{r} = r^{2n+k+1} \cos(k\theta), \quad \dot{\theta} = 1 + r^{2n+k} \sin(k\theta).$$

Taking $R = r^{2n+k}$, $\Theta = k\theta$, and reparametrizing the time by the constant factor k , the above system of equations is converted into

$$R' = bR^2 \cos \Theta, \quad \theta' = 1 + R \sin \Theta, \quad \text{where } b = 1 + \frac{2n}{k}.$$

Introducing again real coordinates $X + iY = Re^{i\Theta}$ this last system of equations writes as

$$X' = -Y + bX^2 - Y^2, \quad Y' = X + (1 + b)XY.$$

Now, taking $x = -(1 + b)Y$, $y = -(1 + b)X$, and a change of sign of the time, we arrive to

$$\begin{cases} \dot{x} = -y + xy, \\ \dot{y} = x - \frac{1}{1+b}x^2 + \frac{b}{1+b}y^2, \end{cases}$$

which is precisely a system of the form (14) with $F = 1 + D$ and $D = -1/(1 + b) \in [-1/2, 0)$.

Finally, notice that the period of a periodic orbit surrounding the origin for this last system is proportional to the time spend by a periodic orbit of system (15) for going from $\theta = 0$ to $\theta = 2\pi/k$. Since the system is \mathbb{Z}_k -equivariant, the total period of this periodic orbit is k times this last time, and the result follows. \square

Notice that when $n = 0$ we recover one of the quadratic isochronous centers, $(D, F) = (-1/2, 1/2)$, see [26]. Hence we have reduced our problem to a similar one, but for the quadratic reversible centers (14) on the line $D - F + 1 = 0$ and $-1/2 < D < 0$. Unfortunately, despite all the efforts done to study this quadratic family, the behaviour of the period function on this line is not yet know, although it is believed that it is monotonous decreasing, see [108]. Hence the following question arises:

Problem 16. *Is the period function associated to the period annulus of the origin of the differential equation $\dot{z} = iz + (z\bar{z})^n z^{k+1}$, with n and k a positive integers, monotonous decreasing?*

In fact, the above differential equation has other centers whose period functions also deserve to be studied.

4. PIECEWISE LINEAR SYSTEMS

In non-smooth dynamics the differential equations appearing in the simplest models are piecewise linear. Moreover, the discontinuity curve is often given by a straight line. These models have attracted the attention of many scientists not only because its simplicity, but also for the accuracy of the results obtained by using them, compared with the real observations, see more details for instance in [1, 18, 94]. We present a couple of questions concerning their number of limit cycles.

4.1. Algebraic limit cycles and related questions. In this section we compare some results about limit cycles for quadratic systems with similar ones for piecewise linear systems with a straight line of separation to highlight the parallelism between both settings.

The existence of examples with 4 limit cycles for quadratic systems has been already revealed in Section 2.5 and an example with 3 limit cycles for piecewise linear systems is given in [105]. We remark that when we consider limit cycles for piecewise linear systems we only refer to crossing limit cycles, see [55, 62]. This means that the two sides of the limit cycle cut transversally the line of discontinuity and at these points of cutting both vector fields point to the same half-plane, see Figure 1. In other words, the crossing limit cycles never follow the discontinuity line, avoiding the so-called sliding motion, see again Figure 1.

Recall that a limit cycle of a smooth differential system is called algebraic if it is an oval of an irreducible algebraic curve. The degree of the limit cycle is the one of the curve. Similarly, a (piecewise crossing) algebraic limit cycle for a piecewise linear system is given also by a topological oval such that all its points are contained, on each of the sides of the discontinuity, in an irreducible algebraic curve in any of the two sides. Then the degree of this limit cycle is a couple $(m, n) \in \mathbb{N}^2$ being each one of these numbers the degrees of each one of the invariant algebraic curves.

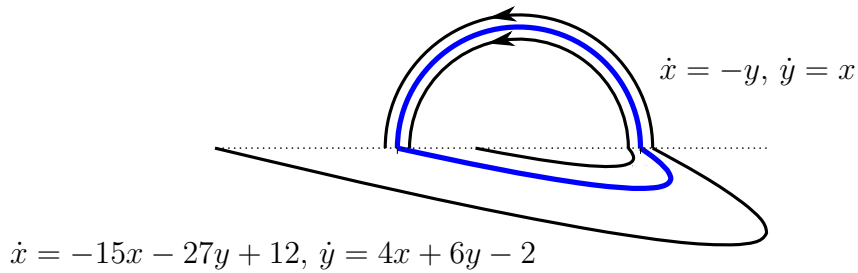


FIGURE 1. Example of algebraic (crossing) limit cycle of a piecewise linear system.

In Figure 1 an example of piecewise algebraic limit cycle of degree $(2, 2)$ for a piecewise linear system is showed, see [21]. It is formed by a piece of the parabola $(x+3y)^2 - 3y - 1 = 0$ and a piece of the circumference $x^2 + y^2 - 1 = 0$. We remark, that although algebraic limit cycles with sliding do exist, see for instance [21], we do not consider them.

The results about algebraic limit cycles for quadratic systems have been obtained along the years in several papers, see for instance [30] and the references therein. Until today, irreducible algebraic limit cycles of degrees 2,4,5 and 6 are known and it is also proved that there are not limit cycles of degree 3. It is not known if these degrees are the only possible ones.

On the other hand, in [21] it is proved that for piecewise linear systems, given any couple of natural numbers $(m, n) \in \mathbb{N}^2$, with $n \geq 2$ and $m \geq 2$, there are algebraic limit cycles of degree (m, n) . Moreover, when $n = m$ it is also known that the two pieces of the piecewise algebraic limit cycle do not correspond to a single algebraic curve. In that paper also appear examples with 2 algebraic and hyperbolic limit cycles and examples with a double semi-stable algebraic limit cycle.

Examples of quadratic systems with 2 critical periods are given for the quadratic reversible centers (14) introduced in Section 3.4. See also [79] for other examples. We do not know that the number of critical periods with piecewise linear systems has been studied when the separation curve is a straight line. The case where this curve is more general is studied in [130].

The table in next problem collects the above results and shows the best known lower bound for the objects described in the left column. When a question mark appears it means that it is not proved that the given lower bound is the actual value. It is believed that the table should be as it is but without question marks, but as we have already explained, the only known result is that algebraic and non-algebraic limit cycles never coexist for piecewise linear systems with a straight line of separation ([21]).

Problem 17. *Improve this table:*

	<i>Quadratic sys.</i>	<i>Piecewise linear sys.</i>
<i>Limit cycles (l.c.)</i>	$4?$	$3?$
<i>Algebraic limit cycles</i>	$1?$	$2?$
<i>Non hyperbolic algebraic l.c.</i>	$0?$	$1?$
<i>Coex. of algebraic and non-algebraic l.c.</i>	$NO?$	NO
<i>Critical periods</i>	$2?$	$?$

4.2. Another Hilbert's XVI type problem. Let $\mathcal{L}(n)$ denote the maximum number of (crossing) limit cycles of planar piecewise linear differential systems with two zones separated by a branch of an algebraic curve of degree n . A branch is an unbounded curve diffeomorphic to \mathbb{R} and that defines a closed set. We also stress that although the commonly used name is linear, indeed both vector fields are affine. In principle, although it seems improvable, we admit that some of the numbers $\mathcal{L}(n)$ could be infinity. Recall that $\mathcal{H}(n)$ denotes the maximum number of limit cycles that planar polynomial systems of degree n can have. With these notations in mind we propose the following problem.

Problem 18. *Improve, if possible, the lower bounds of this table:*

<i>Polynomial case</i>	<i>Linear piecewise case</i>
$\mathcal{H}(2) \geq 4$	$\mathcal{L}(1) \geq 3$
$\mathcal{H}(3) \geq 13$	$\mathcal{L}(2) \geq 4$
$\mathcal{H}(n) \geq Kn^2 \log(n)$	$\mathcal{L}(n) \geq \lceil n/2 \rceil$

The lower bounds for the values of $\mathcal{H}(n)$ given in the above table have already appeared in this paper, see Sections 2.5 and 3.3. Recently, very good lower bounds for $\mathcal{H}(n)$ and n small are given in [115]. For instance, $\mathcal{H}(4) \geq 28$ or $\mathcal{H}(5) \geq 37$.

The lower bounds for $\mathcal{L}(n)$ and $n = 1, 2$ are given in [105] and [74], respectively. In [11] it is proved that $\mathcal{L}(3) \geq 7$ and in the recent preprint [7] that $\mathcal{L}(3) \geq 8$. Also in [74] the general lower bound for $\mathcal{L}(n)$ given above is proved with the aim of showing that $\mathcal{L}(n)$ tends to infinity when n does. We believe that there is room for improving it. Next, we include an idea of its proof.

Define $f_n(x, \varepsilon) = \varepsilon T_n(x)$ with $\varepsilon > 0$ suitable small, where $T_n(x)$ is the Chebyshev polynomial of the first kind, i.e. for $|x| \leq 1$, $T_n(x) = \cos(n \arccos x)$, and for $|x| > 1$, its analytic extension. It is known that $T_n(x)$ is a polynomial of degree n and all its roots are real and in $[-1, 1]$.

The curves of degree n , $y = f_n(x, \varepsilon)$, have a single branch and separate the plane in two zones, Ω^+ when $y \geq f_n(x, \varepsilon)$ and Ω^- when $y \leq f_n(x, \varepsilon)$. We consider the piecewise

linear differential systems

$$(16) \quad (\dot{x}, \dot{y}) = \begin{cases} (x - 4y - 2, \frac{1}{2}x - y), & \text{on } \Omega^+, \\ (-y + 1, x), & \text{on } \Omega^-. \end{cases}$$

Some easy calculations show that the first integrals of each one of the linear systems are

$$H^+(x, y) = 8y + x^2 - 4xy + 8y^2 \quad \text{and} \quad H^-(x, y) = -2y + x^2 + y^2,$$

respectively. If instead of the separation curve $y = f_n(x, \varepsilon)$ we consider the separation line $y = 0$, all the solutions are periodic orbits because $H^\pm(x, 0) = x^2$. The important point is that the ones that pass through the points $(\pm x_k, 0)$, where $x_k \neq 0$ is a zero of $T_n(x)$, are the ones that remain as limit cycles, for ε small enough, see Figure 2.

More specifically, let $m = [(n - 2)/2]$, and $\pm x_0, \dots, \pm x_m$ be the $2(m + 1)$ zeros of $f_n(x, \varepsilon)$ which are not zero. Then

$$x_k = \cos\left(\frac{2k + 1}{2n}\pi\right), \quad k = 0, 1, \dots, m.$$

Then, for each $k \in \{0, 1, \dots, m\}$ and ε small enough,

$$\Gamma_k := \{(x, y) \mid H^+(x, y) = H^+(P_{\pm k}), y \geq 0\} \cup \{(x, y) \mid H^-(x, y) = H^-(P_{\pm k}), y \leq 0\},$$

is a periodic orbit of our piecewise system (16), where $P_{\pm k} = (\pm x_k, 0)$ for $k \in \{0, 1, \dots, m\}$.

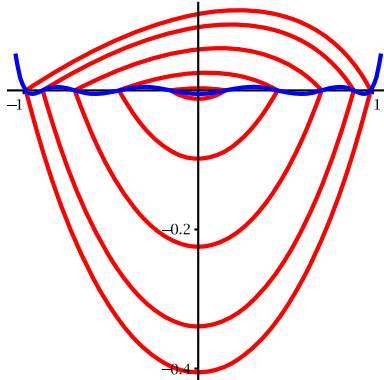


FIGURE 2. Separation curve defined by a Chebyshev polynomial of degree 10 and 5 limit cycles.

To prove that Γ_k is a hyperbolic limit cycle we compute the derivative of the Poincaré map, which is a composition of two maps and prove that it is not 1. This can be done by using the nice formula ([8])

$$\Pi'(0) = \frac{\langle X(0), (\gamma'_0(0))^\perp \rangle}{\langle X(T), (\gamma'_1(0))^\perp \rangle} \exp\left(\int_0^T \operatorname{div} X(\varphi(t)) dt\right),$$

where $\langle \cdot, \cdot \rangle$ is the inner product of two vectors, the superscript \perp denotes the orthogonal of a two dimensional vector, that is $(u, v)^\perp = (-v, u)$, and $\gamma_0(s)$ and $\gamma_1(s)$ are the local expressions of two transversal sections Σ_0 and Σ_1 to a vector field X , and T is the time moving from $P = \gamma_0(0)$ to $\Pi(P) = \gamma_1(0)$, see Figure 3.

5. ON GLOBAL ASYMPTOTIC STABILITY

We will present three problems about global asymptotic stability of dynamical systems.

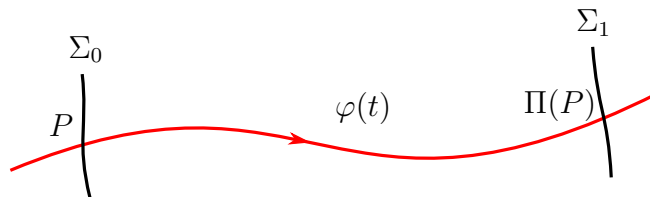


FIGURE 3. Poincaré map.

5.1. A Markus–Yamabe problem for differential equations. Markus–Yamabe conjecture for ordinary differential equations was stated by L. Markus and H. Yamabe in 1960, and it said:

Let $\dot{x} = F(x)$, $x \in \mathbb{R}^n$, be a smooth differential equation such that $F(0) = 0$ and for all $x \in \mathbb{R}^n$, all the eigenvalues of the matrix $DF(x)$ have negative real part. Then the origin is globally asymptotically stable.

Nowadays it is known that it is true in dimensions 1 (it has a very simple proof) and 2 (see [61, 77, 81]), and it is false in higher dimensions. A smooth counterexample in dimension 4 was given in 1996 in [12]. It was obtained by perturbing a continuous linear piecewise differential system defined in \mathbb{R}^4 having a hyperbolic periodic orbit and so it also has a periodic orbit. Polynomial counterexamples in dimension 3 and higher were given in 1997 in [45]. These counterexamples are

$$\dot{x} = -x + z_1(x + yz_1)^2, \quad \dot{y} = -y - (x + yz_1)^2, \quad \dot{z}_i = -z_i,$$

$i = 1, 2, \dots, n - 2$. It can be seen that at any point all the eigenvalues of the differential matrix DF are -1 , and that it has the particular solution

$$(x, y, z_1, \dots, z_{n-2}) = (18e^t, -12e^{2t}, e^{-t}, \dots, e^{-t}).$$

Notice that this solution tends to infinity when t increases. Therefore, it appears the following natural question:

Problem 19. *Are there smooth vector fields in \mathbb{R}^3 under the hypotheses of the Markus–Yamabe’s conjecture and having periodic orbits?*

5.2. A Markus–Yamabe/La Salle problem for DDS. J. P. La Salle in 1976 proposed some possible sufficient conditions for discrete dynamical systems with a fixed point, $x_{m+1} = F(x_m)$, $x \in \mathbb{R}^n$, to be globally asymptotically stable, see [96]. Some of them are discrete versions of the Markus–Yamabe conditions given in previous section and have been studied in [39, 40]. Two of these conditions are:

$$(C_1) \text{ For all } x \in \mathbb{R}^n, \rho(DF(x)) < 1,$$

$$(C_2) \text{ For all } x \in \mathbb{R}^n, \rho(|DF(x)|) < 1,$$

where ρ is the spectral radius of the differential matrix and, given a square matrix $A = (a_{i,j})$, $|A|$ denotes the new matrix with all its entries $|a_{i,j}|$. It is known that $\rho(A) \leq \rho(|A|)$, see [64]. Hence Condition C_2 is stronger than C_1 , because $\rho(|A|) < 1 \implies \rho(A) < 1$, but it can be easily seen that $\rho(A) < 1 \not\implies \rho(|A|) < 1$.

In fact, in [39] the authors already gave an example due to W. Szlenk of a rational map satisfying condition C_1 , for $n = 2$, for which the origin is not globally asymptotically stable, because it has a 4-periodic orbit. Similarly, polynomial maps satisfying condition C_1 ,

for $n \geq 3$, for which the origin is not globally asymptotically stable are presented in [45]. These maps have orbits that tend to infinity. The following question remains open:

Problem 20. *Let $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a smooth map having a fixed point and such that*

$$\rho(|DF(x)|) < 1, \quad \text{for all } x \in \mathbb{R}^2.$$

Is it true that the fixed point is globally asymptotically stable?

5.3. Random linear differential or difference equations. Given a n -th order linear homogeneous differential equation it is natural to wonder which is the probability of the zero solution of being a global attractor. Let us formalize this question.

Consider for instance the 3-rd order linear differential equation

$$Ax'''(t) + Bx''(t) + Cx'(t) + Dx(t) = 0,$$

where A, B, C, D are real continuous random variables. It is natural to require that all these random variables are independent and identically distributed (i.i.d.). Also it seems reasonable to impose that they are such that the random vector (A, B, C, D) has uniform distribution in \mathbb{R}^4 . But such a distribution is impossible for unbounded probability spaces. Anyway, let us see that there is a natural election for it.

It is clear that the solutions of the above differential equation do not vary if we multiply the equation by a positive constant. This means that in the space of parameters, \mathbb{R}^4 , all the differential equations with parameters belonging to the same half-straight line passing through the origin are the same. Hence, we can ask for a probability distribution density f of the coefficients such that the random vector

$$(17) \quad \left(\frac{A}{S}, \frac{B}{S}, \frac{C}{S}, \frac{D}{S} \right), \quad \text{with } S = \sqrt{A^2 + B^2 + C^2 + D^2},$$

has a uniform distribution on the sphere $\mathbb{S}^3 \subset \mathbb{R}^4$, that is a compact set. In [34] it is proved the following result:

Theorem 5.1. *Let X_1, X_2, \dots, X_n be i.i.d. one-dimensional real random variables with a continuous positive density function f . The random vector*

$$\left(\frac{X_1}{S}, \frac{X_2}{S}, \dots, \frac{X_n}{S} \right), \quad \text{with } S = \left(\sum_{i=1}^n X_i^2 \right)^{1/2},$$

has a uniform distribution in $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ if and only if each X_i is a normal random variable with zero mean.

Hence it is natural to consider linear random homogeneous differential equations of order n

$$(18) \quad A_n x^{(n)}(t) + A_{n-1} x^{(n-1)}(t) + \dots + A_2 x''(t) + A_1 x'(t) + A_0 x(t) = 0,$$

where all A_j are i.i.d. random variables with $N(0, 1)$ distribution.

To know the probability that the zero solution is globally asymptotically stable is equivalent to know the probability, say p_n , that all the roots of its associated random characteristic polynomial

$$Q(\lambda) = A_n \lambda^n + A_{n-1} \lambda^{n-1} + \dots + A_1 \lambda + A_0$$

have negative real part. Recall that the conditions among the coefficients that imply this property are given by algebraic relations among them and can be obtained via the Routh–Hurwitz criterion.

In [34] it is proved that $p_n \leq 1/2^n$, so $\lim_{n \rightarrow \infty} p_n = 0$. Furthermore, it is also shown that $p_1 = 1/2$, $p_2 = 1/4$, $p_3 = 1/16$, $p_4 < 1/32$, and, by using Monte Carlo simulation, that $p_4 \simeq 0.0092$ and $p_5 \simeq 0.0007$. Hence the following questions arise:

Problem 21. *Let p_n be the probability that the zero solution of the n -th order linear random differential equation (18) is globally asymptotically stable. Find the asymptotic expansion of p_n at $n = \infty$. Is it true that the sequence p_n is strictly decreasing?*

Similar problems can be consider for the random difference equations of order n of type

$$(19) \quad A_n x_{k+n} + A_{n-1} x_{k+n-1} + \cdots + A_1 x_{k+1} + A_0 x_k = 0,$$

where all the coefficients are again i.i.d. random variables with $N(0, 1)$ distribution. In this situation, the global asymptotic stability happens when all the zeros of the associated random characteristic polynomial $Q(\lambda)$ have modulus smaller than 1. Recall that this property is characterized by the so-called Jury criterion. In fact, it is possible to get Jury conditions from Routh–Hurwitz conditions and viceversa by using the Möbius transformation that sends the left hand part of the complex plane into the complex ball of radius 1 and its inverse.

If we call q_n the probability of the zero solution of (19) to be globally asymptotically stable, for instance, $q_1 = 1/2$, $q_2 = \arctan(\sqrt{2})/\pi \simeq 0.304$, $q_3 \simeq 0.172$, $q_4 \simeq 0.103$, and $q_5 \simeq 0.059$, see again [34].

6. SOME GEOMETRICAL PROBLEMS

In this section we present three problems with a geometric flavour.

6.1. Triangular billiards. Consider a mathematical ideal convex billiard with a smooth \mathcal{C}^1 boundary. A punctual ball moves on it alternating between free motion (following a straight line) and specular reflections from its boundary. When the particle hits the boundary it reflects from it without loss of speed with an elastic collision. Then, it is known that there are always periodic trajectories ([91]). The number of times that a periodic trajectory touches the boundary before closing is called its period.

If we consider a convex billiard, but with a polygonal boundary it is natural to wonder if periodic trajectories also always exist. For this type of billiards if the punctual ball arrives to a corner the trajectory stops and, of course, it is not periodic. In fact, even for triangular billiards this question is an open problem.

Problem 22. *Do all triangular billiards have some periodic trajectory?*

For many triangular billiards the answer is yes, see [10, 85]. For instance, this is the case when the boundary of the billiard is an acute triangle. In this case there is always a periodic trajectory of period 3. It is formed by the triangle that has as vertices the basis points at the boundaries of the three heights, see Figure 4. Sometimes this trajectory is called Fagnano’s trajectory, because he found it in 1775 for solving another problem: *find the inscribed triangle to an acute triangle with smaller length*, see Figure 4. The answer is also affirmative for rectangular triangles, isosceles triangles ([46]), for obtuse triangles

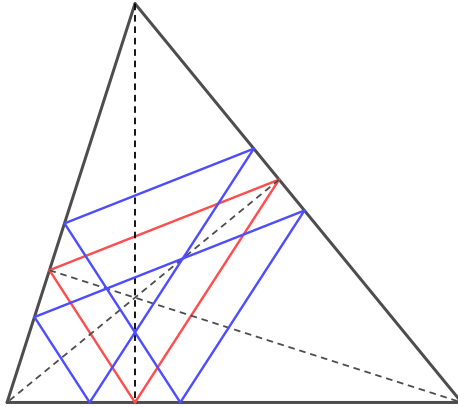


FIGURE 4. Trajectories with periods 3 and 6 for an acute triangular billiard.

with no angle larger than 100 degrees ([118]), and also for rational triangles ([15]). Recall that a triangle is called rational if all its angles are rational multiples of π .

6.2. An extended Poncelet's problem. Given two convex algebraic ovals, γ and Γ , as in Figure 5, consider the Poncelet's map P from the exterior curve Γ into itself, also introduced in this figure. The iteration of this procedure is sometimes called Poncelet's procedure.

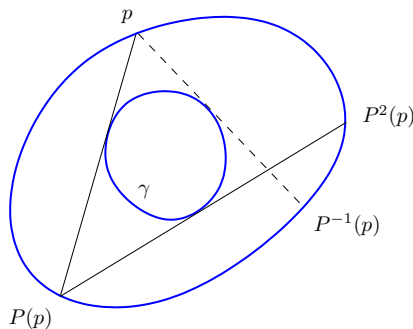


FIGURE 5. Poncelet's map.

The name for this map is introduced in [33] because Poncelet, a French engineer and a mathematician, considered it for first time when both ovals are ellipses, while he was prisoner in Saratov (Russia) during 1812-1814. He proved the following nice theorem that is illustrated in Figure 6 when $n = 3$.

Theorem 6.1 (Poncelet's theorem). *If given an initial point p on the exterior ellipse Γ the Poncelet's procedure closes for first time after n steps, then the same happens for any other initial condition.*

There are several proofs of the above theorem, see [123]. In fact, in the language of dynamical systems the above result can be extended giving the following theorem:

Theorem 6.2. *Let P be the Poncelet's map between two ellipses. Then P is conjugated with a rotation R of the circle. In particular,*

- (i) *if the rotation number of R is rational then all points are periodic for P and with the same period (original Poncelet's theorem).*

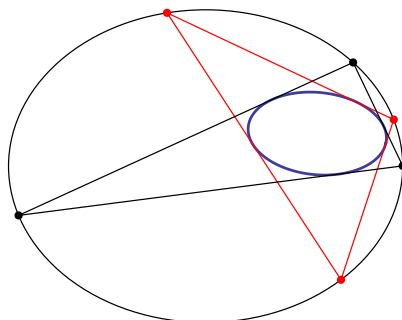


FIGURE 6. Two 3 steps closed Poncelet's trajectories.

- (ii) if the rotation number of R is irrational then all points of any orbit of P are dense on the exterior ellipse.

In [33] it is proved the following result, that shows that Poncelet's result is not true for all ovals.

Proposition 6.3. Consider $\gamma = \{x^{2n} + y^{2n} = 1\}$ and $\Gamma = \{x^{2m} + y^{2m} = 2\}$ with $n, m \in \mathbb{N}$. Then their associated Poncelet's map has rotation number $1/4$ and it is conjugated to a rotation if and only if $n = m = 1$.

A natural question is the following:

Problem 23. Are there two irreducible algebraic curves of degrees n and m , with $n+m > 4$, having each one of them an oval, for which the Poncelet's map P is well defined and it is conjugated to a rotation of the circle?

This problem is somehow reminiscent of the classical Birkhoff's conjecture. Recall that it claims that the boundary of a strictly convex integrable billiard table is necessarily an ellipse (or a circle as a special case). Recently, in [90] it is proved a local version of this conjecture: a small integrable perturbation of an ellipse must be an ellipse.

Inspired on the above point of view, we propose next local version of the above question:

Problem 24. Consider the ovals $\gamma = \{x^2 + y^2 - 1 = 0\}$ and $\Gamma_\varepsilon = \{p_2(x, y) + \varepsilon p_m(x, y) = 0\}$ where Γ_0 is an ellipse that surrounds γ , $p_2(x, y) + \varepsilon p_m(x, y) = 0$ is an irreducible curve, with p_m a polynomial of degree $m \geq 3$, and ε is a small parameter. Is it true that the Poncelet's map associated to both ovals is conjugated to a rotation if and only if $\varepsilon = 0$?

6.3. Loewner's conjecture. By using the complex notation introduced in Section 3.5 we consider the Cauchy–Riemann operator

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Then

$$\frac{\partial^2}{\partial \bar{z}^2} = \frac{1}{4} \left(\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} + 2i \frac{\partial^2}{\partial x \partial y} \right).$$

Similarly, given $1 < n \in \mathbb{N}$, we can define $\frac{\partial^n}{\partial \bar{z}^n}$. Given a neighbourhood of the origin $\mathcal{U} \subset \mathbb{R}^2$ and a class \mathcal{C}^{n+1} function $f : \mathcal{U} \rightarrow \mathbb{R}$ such that $f(0, 0) = 0$, we look at the planar

differential equation

$$(20) \quad \dot{x} = 2^n \operatorname{Re} \left(\frac{\partial^n}{\partial \bar{z}^n} f(x, y) \right), \quad \dot{y} = 2^n \operatorname{Im} \left(\frac{\partial^n}{\partial \bar{z}^n} f(x, y) \right).$$

For instance for $n = 1$ and 2 we have

$$\begin{aligned} (\dot{x}, \dot{y}) &= (f_x(x, y), f_y(x, y)), & n = 1, \\ (\dot{x}, \dot{y}) &= (f_{xx}(x, y) - f_{yy}(x, y), 2f_{x,y}(x, y)), & n = 2. \end{aligned}$$

Recall that the index is an integer number associated to any isolated equilibrium point of a planar differential equation that measures the number of turns of its associated vector field near it, see [58] for a precise definition. When this isolated equilibrium point admits a finite sectorial decomposition (this always happens for instance in the analytic case, for non-monodromic singularities, see [89]) and e, h , and p denote its number of elliptic, hyperbolic, and parabolic sectors, respectively, then the index is $1 + \frac{e-h}{2}$, due to Poincaré's index formula.

According to [124], next conjecture was proposed by Loewner around 1950, see also [104].

Problem 25 (Loewner's conjecture). *Assume that the origin is an isolated equilibrium point of the differential equation (20) and that f is analytic at this point. Then the index of the associated vector field at the origin is not greater than n .*

Another related conjecture is Carathéodory's conjecture, that asserts that every smooth convex embedding of a 2-sphere in \mathbb{R}^3 , i.e. an ovaloid, must have at least two umbilics. Recall that for any surface in \mathbb{R}^3 , the eigenspaces of the second fundamental form define two orthogonal line fields (principal directions) whose singularities are exactly the so-called umbilics. It is known that if Loewner's conjecture is true when $n = 2$ the Carathéodory's conjecture is also true in the analytic case, see for instance [6, 83].

Loewner's conjecture is true for $n = 1$ and several authors have proved it for $n = 2$, although there are several wrong proofs, see some comments in [82, 83]. Anyway, it would be very interesting to have simple proofs of Loewner's conjecture for $n = 2$ and to further investigate it, also for functions of class \mathcal{C}^{n+1} .

7. PROBLEMS INVOLVING POLYNOMIALS

7.1. A moments problem. Arno van den Essen, the author of the interesting monograph [128] about the Jacobian conjecture, which recall that asserts the bijectivity of the complex polynomial maps with constant Jacobian, call our attention to the following question:

Problem 26. *Let $f(x_1, x_2, \dots, x_n)$ be a polynomial in $\mathbb{C}[x_1, x_2, \dots, x_n]$ such that the following moment conditions hold*

$$M_m := \int_0^1 \int_0^1 \cdots \int_0^1 f^m(x_1, x_2, \dots, x_n) dx_1 dx_2 \cdots dx_n = 0, \quad m \geq 1.$$

Is it true that $f = 0$?

This question and some extensions where some especial weights are added to the above integrals, is very related with the Jacobian conjecture, see [54, 63]. Obviously it can also be extended to wider classes of maps f .

In fact, the answer when $n = 1$ is yes, see [112], but anyway, in this case it would be nice to get a simple direct proof. Notice that when $n = 1$ for $|t|$ small enough

$$\frac{1}{1 - tf(x)} = 1 + \sum_{m=1}^{\infty} (tf(x))^m,$$

and since the convergence is uniform, we get that under the above hypotheses,

$$\int_0^1 \frac{1}{1 - tf(x)} dx = 1 + \sum_{m=1}^{\infty} t^m \int_0^1 f^m(x) dx = 1 + \sum_{m=1}^{\infty} M_m t^m = 1.$$

Hence the answer of the above question when $n = 1$ is equivalent to prove that if $f(x)$ be a polynomial in $\mathbb{C}[x]$ such that for all $|t|$ small enough:

$$\int_0^1 \frac{1}{1 - tf(x)} dx = 1,$$

then $f = 0$.

Also an interesting related question is the following:

Problem 27. (i) Let $f(x)$ be a polynomial in $\mathbb{C}[x]$ with k monomials. Is there a value $N(k)$ such that if the following finite set of moment conditions hold:

$$M_n := \int_0^1 f^n(x) dx = 0, \quad 1 \leq n \leq N(k),$$

then $f = 0$?

(ii) If the answer is yes, find $N(k)$ or a good upper bound of this number.

This type of questions also appear in some classical problems for Abel differential equations, where the vanishing of certain moments imply the solution of the center problem, see for instance [42].

7.2. Around Kouchnirenko's conjecture. Descartes' rule implies that a 1-variable real polynomial with m monomials has at most $m - 1$ simple positive real roots.

The Kouchnirenko's conjecture was posed as an attempt to extend this rule to the several variables context. In the 2-variables case this conjecture said that:

A real polynomial system $f_1(x, y) = f_2(x, y) = 0$ would have at most $(m_1 - 1)(m_2 - 1)$ simple solutions with positive coordinates, where m_i is the number of monomials of each polynomial f_i , $i = 1, 2$.

This conjecture was stated by A. Kouchnirenko in the late 70's, and published in [93] in 1980. In 2002, in [84] a family of counterexamples given by two trinomials, being their minimal degree 106, was constructed. In 2003 a much simpler family of counterexamples was presented in [101] again formed by two trinomials, but of degree 6. Both have exactly 5 simple solutions with positive coordinates instead of the 4 predicted by the conjecture. A similar counterexample is:

Proposition 7.1 ([72]). *The bivariate trinomial system*

$$(21) \quad \begin{cases} P(x, y) := x^6 + \frac{61}{43}y^3 - y = 0, \\ Q(x, y) := y^6 + \frac{61}{43}x^3 - x = 0, \end{cases}$$

has 5 real simple solutions with positive entries.

It is not difficult to find numerically 5 approximated solutions of the system. They are $(\tilde{x}_1, \tilde{x}_5), (\tilde{x}_2, \tilde{x}_4), (\tilde{x}_3, \tilde{x}_3), (\tilde{x}_4, \tilde{x}_2), (\tilde{x}_5, \tilde{x}_1)$, where $\tilde{x}_1 = 0.59679166, \tilde{x}_2 = 0.68913517, \tilde{x}_3 = 0.74035310, \tilde{x}_4 = 0.77980435$ and $\tilde{x}_5 = 0.81602099$. A proof that these solutions actually exist follows by using Poincaré-Miranda theorem, see [72]. Recall that this theorem is an extension of the classical Intermediate Value theorem (or Bolzano's theorem) to higher dimensions. It was stated by H. Poincaré in 1883 and 1884, and proved by himself in 1886. In 1940, C. Miranda ([110]) re-obtained the result as an equivalent formulation of Brouwer fixed point theorem:

Theorem 7.2 (Poincaré-Miranda ([109])). *Set $\mathcal{B} = \{\mathbf{x} = (x_1, \dots, x_n) \in \mathbb{R}^n : L_i < x_i < U_i, 1 \leq i \leq n\}$. Suppose that $f = (f_1, f_2, \dots, f_n) : \overline{\mathcal{B}} \rightarrow \mathbb{R}^n$ is continuous, $f(\mathbf{x}) \neq \mathbf{0}$ for all $\mathbf{x} \in \partial\mathcal{B}$, and for $1 \leq i \leq n$,*

$$\begin{aligned} f_i(x_1, \dots, x_{i-1}, L_i, x_{i+1}, \dots, x_n) &\leq 0 \text{ and} \\ f_i(x_1, \dots, x_{i-1}, U_i, x_{i+1}, \dots, x_n) &\geq 0. \end{aligned}$$

Then, there exists $\mathbf{s} \in \mathcal{B}$ such that $f(\mathbf{s}) = \mathbf{0}$.

In Figure 7 we illustrate the hypotheses of the theorem for $n = 2$.

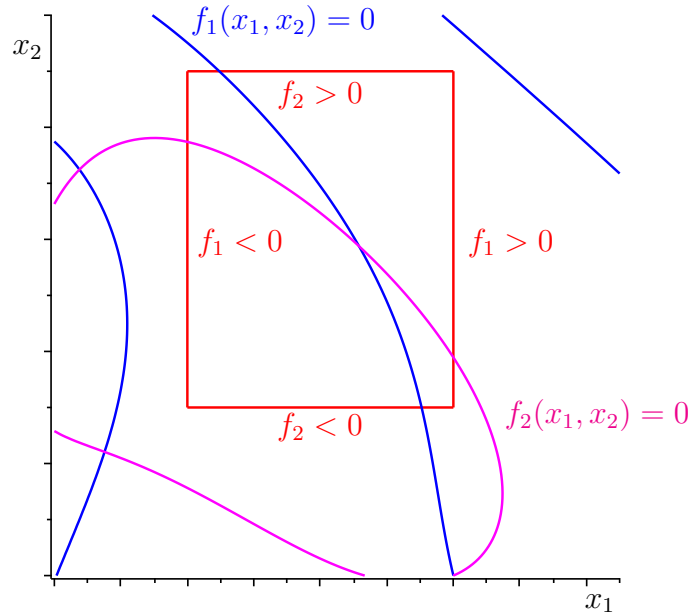


FIGURE 7. A Poincaré-Miranda box.

To prove Proposition 7.1 we consider the following 5 intervals, with $\tilde{x}_i \in I_i$,

$$I_1 = \left[\frac{1}{2}, \frac{1619}{2500} \right], I_2 = \left[\frac{1619}{2500}, \frac{18}{25} \right], I_3 = \left[\frac{18}{25}, \frac{75857}{100000} \right],$$

$$I_4 = \left[\frac{75857}{100000}, \frac{4}{5} \right], I_5 = \left[\frac{4}{5}, \frac{83}{100} \right].$$

and prove that our system has 5 actual solutions (x_1, x_5) , (x_2, x_4) , (x_3, x_3) , (x_4, x_2) , (x_5, x_1) , with $x_i \in I_i$. By Descartes' rule we know that there is exactly one simple positive real root of $P(x, x)$. The corresponding (x_3, x_3) is in $I_3 \times I_3$. By the symmetry of the system, if (x^*, y^*) is one of its solutions then (y^*, x^*) also is. Finally, we can prove the existence of two more solutions (and so, their symmetric ones) by using the Poincaré-Miranda theorem in the boxes $I_1 \times I_5$ and $I_2 \times I_4$, see again [72].

In [56, 101] the authors prove that any bivariate trinomial system $m_1 = m_2 = 3$ has at most 5 real simple solutions with positive entries and henceforth this bound is sharp. A very interesting problem is:

Problem 28. Find a reasonable (or sharp) upper bound in terms of m_i for the maximum number of simple solutions with positive coordinates, for a real polynomial system $f_1(x, y) = f_2(x, y) = 0$, where $m_i, i = 1, 2$, is the number of monomials of each f_i .

Not sharp upper bounds are known from the nice approach of Khovanskiĭ who was a pioneer in 1980 in the study of the so called fewnomials ([93]). His general upper bound is as follows: given a system of n real polynomial equations in n variables with a total of $n + k + 1$ distinct monomials possesses at most $2^{\binom{n+k}{2}}(n+1)^{n+k}$ nondegenerate solutions with positive entries. This upper bound has been improved in [13] decreasing it until $\frac{e^2+3}{4}2^{\binom{k}{2}}n^k$. In fact, in [13] when $n > k$, an example with $\lfloor \frac{n+k}{k} \rfloor^k$ nondegenerate solutions with positive entries is given, showing that for k fixed and n big enough this last upper bound is almost asymptotically sharp.

To illustrate that the above results are not sharp enough, let us apply them to the planar trinomial situation ($n = 2$) where the sharp upper bound is 5. In principle, it seems that $k = 3$, because there are 6 involved monomials, but notice that the number of solutions in the first quadrant remains unchanged when we multiply any of the equations by any monomial $x^i y^j$. Hence, before applying the given bounds, we can do this modification in a convenient way to force to coincide two of them. In this way the value k can be assumed to be $k = 2$. Hence, Khovanskiĭ's upper bound gives $2^6 3^4 = 5184$ and its improvement gives 20.

Recall again that by using Descartes' rule it is easy to answer this last problem in one variable. Moreover, the $m - 1$ corresponding to positive solutions implies a global upper bound of $2m - 1$ solutions: $m - 1$ positive roots, $m - 1$ negative ones and, eventually, the root 0, that can be multiple, with any multiplicity.

It is natural to wonder if the following modified Kouchnirenko's bound works.

Problem 29. Is $(2m_1 - 1)(2m_2 - 1)$ the maximum number of simple solutions of a real polynomial system $f_1(x, y) = f_2(x, y) = 0$, where m_i is the number of monomials of each f_i ?

It is very easy to find examples of uncoupled systems having $(2m_1 - 1)(2m_2 - 1)$ simple solutions. For instance, for $m_1 = m_2 = 3$, then $(2m_1 - 1)(2m_2 - 1) = 25$. Consider $(x^2 - 1)(x^2 - 4)x = x^5 - 5x^3 - x$. Then the system

$$\begin{cases} x^5 - 5x^3 - x = 0, \\ y^5 - 5y^3 - y = 0, \end{cases}$$

has the 25 simple solutions (x_i, x_j) with $x_i, x_j \in \{-2, -1, 0, 1, 2\}$. Similarly, the system

$$\begin{cases} x^{5+r} - 5x^{3+r} - x^{1+r} = 0, \\ y^{5+s} - 5y^{3+s} - y^{1+s} = 0, \quad s > 0, r > 0, \end{cases}$$

has 16 simple solutions and 9 multiple ones.

Another example with 25 solutions can be constructed from our counterexample (21). It is obtained by taking the equations $yP(x^2, y^2) = 0$ and $xQ(x^2, y^2) = 0$, giving

$$\begin{cases} \left(x^{12} + \frac{61}{43}y^6 - y^2 \right) y = x^{12}y + \frac{61}{43}y^7 - y^3 = 0, \\ \left(y^{12} + \frac{61}{43}x^6 - x^2 \right) x = y^{12}x + \frac{61}{43}x^7 - x^3 = 0, \end{cases}$$

which has $4 \times 5 = 20$ solutions, 5 in each quadrant, plus 5 more on the axes: $(0, 0)$, $(\pm x^*, 0)$ and $(0, \pm y^*)$, for some x^* and y^* . Again 25 solutions and here $(0, 0)$ is not a simple solution.

Of course there are also natural extensions to n equations and n variables of the above problems.

7.3. Casas-Alvero's conjecture. Casas-Alvero arrived to the next conjecture at the turn of this century, when he was working trying to obtain an irreducibility criterion for two variable power series with complex coefficients ([23]).

Problem 30 (Casas-Alvero's conjecture). *If a complex polynomial P of degree $n > 1$ shares roots with all its derivatives, $P^{(k)}$, $k = 1, 2, \dots, n - 1$, then there exist two complex numbers, a and $b \neq 0$, such that $P(z) = b(z - a)^n$.*

Notice that, in principle, the common root between P and each $P^{(k)}$ might depend on k . Several authors have got partial answers, but as far as we know, the conjecture remains open. For $n \leq 4$ the conjecture is a simple consequence of the wonderful Gauss–Lucas theorem, that asserts that the complex roots of $P'(z)$ are in the convex hull of the roots of $P(z)$. It is also known that the conjecture is true for low degrees and also when n is p^m , $2p^m$, $3p^m$, or $4p^m$, for some prime number p and $m \in \mathbb{N}$. Nowadays, the first cases left open are $n = 24, 28$, or 30 . See the nice survey [57] and its references.

It is also known that if the conjecture holds in \mathbb{C} , then it is true over all fields of characteristic 0. On the other hand, it is not true over all fields of characteristic p , see [78]. For instance, consider $P(x) = x^2(x^2 + 1)$ in characteristic 5 with roots $0, 0, 2$ and 3 . Then $P'(x) = 2x(2x^2 + 1)$, $P''(x) = 12x^2 + 2 = 2(x^2 + 1)$, and $P'''(x) = 4x$ and all them share roots with P .

Adding the hypotheses that P is a real polynomial and all its roots are real, the conjecture has a real counterpart, that also remains open. It says that $P(x) = b(x - a)^n$ for some real numbers a and $b \neq 0$. For this case, from Rolle's theorem it follows easily for $n \leq 4$. However, it is not difficult to see that this tool does not suffice to prove it for bigger n .

Remarkably, in the real case it is proved in [57] that if the condition for one of the derivatives of P is removed, then there exist polynomials, different from $b(x - a)^n$, satisfying the remaining $n - 2$ conditions. The construction presented in that paper of some of these polynomials is a nice consequence of the Brouwer's fixed point theorem in a suitable context.

Recently in [43] it is shown that the natural extension of this real conjecture to the smooth world is not true. There it is considered the following problem: Fix $1 < n \in \mathbb{N}$ and let F be a class \mathcal{C}^n real function such that $F^{(n)}(x) \neq 0$ for all $x \in \mathbb{R}$, having n real zeroes, taking into account their multiplicities. Assume that F shares zeroes with all its derivatives, $F^{(k)}$, $k = 1, 2, \dots, n - 1$. Is it true that $F(x) = b(f(x))^n$ for some $0 \neq b \in \mathbb{R}$ and some f , a class \mathcal{C}^n real function, that has exactly one simple zero?

The answer for the above problem is “yes” for $n \leq 4$ and “no” for $n = 5$. More concretely, it is proved that there exists $r > 1$ such that if we consider

$$F(x) = \int_0^x \int_0^u \int_1^w \int_c^z \int_1^y (r - \sin(t)) dt dy dz dw du$$

it holds that F has the five zeroes $0, 0, 1, c, d$ satisfying $0 < 1 < c < d$,

$$F'(0) = 0, F''(1) = 0, F'''(c) = 0, F^{(4)}(1) = 0, \text{ and } F^{(5)}(x) = r - \sin(x) > 0,$$

and F is not of the proposed form.

8. SOME CONJECTURES WITH A DYNAMICAL FLAVOUR

One of the most famous open problems is the so-called $3x + 1$ conjecture or Collatz problem ([95]). Recall that it assures that for any $x_0 \in \mathbb{N}$, the sequence defined by

$$x_{n+1} = g(x_n) = \begin{cases} 3x_n + 1, & \text{when } x_n \text{ is odd,} \\ x_n/2, & \text{when } x_n \text{ is even,} \end{cases}$$

arrives after finitely many steps to the 3-periodic behaviour $4, 2, 1, 4, 2, 1, \dots$

We end this paper with three similar but less known conjectures. The first one was proposed by N. Sloane ([121]) in a journal of recreational mathematics.

Problem 31 (Conjecture of multiplicative persistence). *Given $n \in \mathbb{N}$, let $\Pi(n) \in \mathbb{N}$ be the product of all its digits. Set $\text{Pm}(n) \in \mathbb{N}$ for the first positive integer such that $\Pi^{\text{Pm}(n)}(n) = \Pi^{\text{Pm}(n)+1}(n)$, where $\Pi^0 = \text{Id}$ i $\Pi^k(n) = \Pi(\Pi^{k-1}(n))$. Is it true that for all $n \in \mathbb{N}$, $\text{Pm}(n) \leq 11$?*

For instance, for $n = 68889$, $\Pi(68889) = 6 \times 8 \times 8 \times 8 \times 9 = 27648$ and

$$68889 \rightarrow 27648 \rightarrow 2688 \rightarrow 768 \rightarrow 336 \rightarrow 54 \rightarrow 20 \rightarrow 0 \rightarrow 0 \rightarrow 0 \rightarrow \dots$$

Then $\text{Pm}(6889) = 7$, because $\Pi^7(6889) = \Pi^8(6889) = 0$ is the first coincidence. The smallest numbers with respective multiplicative persistence $1, 2, \dots, 11$ are

$$10, 25, 39, 77, 679, 6788, 68889, 2677889, 26888999, 377888999, 277777888899.$$

There are not examples with higher multiplicative persistence for $n < 10^{233}$.

Notice that for instance $\Pi(M) = 2^{19}3^{47}6$. In general, a simple first observation already pointed out in [121] is that the prime factors of any $\Pi(n)$ with persistence bigger than 3 must be either $2^i3^j7^k$ or $3^i5^j7^k$. Therefore, it suffices to study the persistence of these numbers. This is so, because $\Pi(n) = 2^i3^j5^k7^m$, is a product of one digit prime numbers, with all the exponents greater or equal than zero, and moreover if 2 and 5 appear together

$2^i 3^j 5^k 7^m$ ends with zero and then $\Pi^2(n) = \Pi^3(n) = 0$. This problem has been recently extended to other basis and studied from a dynamical systems point of view in [51].

Problem 32 (196 conjecture). *Let $f : \mathbb{N} \rightarrow \mathbb{N}$ be defined as $f(n) = n + \text{rev}(n)$, where rev is the map that reverses the order of the digits of n . Then, there are infinitely many natural numbers n such that $f^k(n)$, for $0 < k \in \mathbb{N}$, where $f^0 = \text{Id}$ and $f^k(n) = f(f^{k-1}(n))$, is never a palindromic number. Moreover, the smallest of these numbers is 196.*

For instance, if $n = 183$, $f(183) = 183 + 381 = 564$, and

$$183 \rightarrow 564 \rightarrow 564 + 465 = 1029 \rightarrow 1029 + 9201 = 10230 \rightarrow 10230 + 3201 = 13431,$$

that is a palindromic number. Starting with $n = 89$, we need 24 iterations to arrive to a palindromic number, that is 8813200023188. Until today, starting with $n = 196$ no palindromic numbers have been found yet, see [111]. It is not clear the origin of this problem. The first reference goes back to Lehmer in 1938 ([97]). The question recovered some interest after the paper [125], published in 1967. Sometimes the numbers n such that $f^k(n)$ is never a palindromic number are called Lychrel's numbers (it is an acronym of the name Cheryl).

The first numbers that could be Lychrel's numbers are

$$196, 295, 394, 493, 592, 689, 691, 788, 790, 879, 887, 978, 986, \dots$$

In Figure 8 we plot the function h that assigns to each $n \in \mathbb{N}$ the minimum value $h(n) \leq 1000$ such that $f^{h(n)}$ is a palindromic number, or 1000 when none of the first 1000 iterates is a palindromic number. I thank Antoni Guillamon for sharing with me the Maple code that I have used to generate this figure. For the sake of clarity, we restrict the plot to the strip $1 \leq h(n) \leq 40$. Its spikes correspond to the 13 values of the above list. Notice also that for all the other values of n , the function $h(n)$ is at most 24.

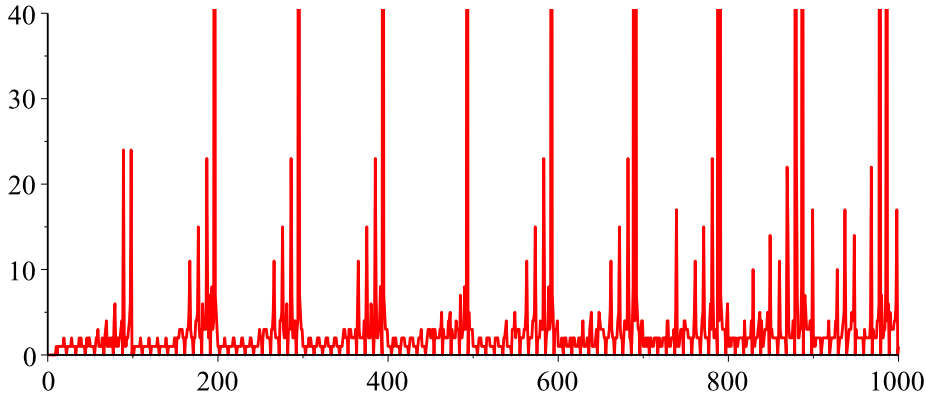


FIGURE 8. Possible Lychrel's numbers.

Although in basis 10 the existence of Lychrel's numbers is an open problem, it is not difficult to find some of them in other basis. For instance in basis 2, if we take $n = 10110_2$ is one of them. This is so, because $f^4(n) = 10110100_2$,

$$10110_2 \rightarrow 10110_2 + 01101_2 = 100011_2 \rightarrow 1010100_2 \rightarrow 1101001_2 \rightarrow 10110100_2,$$

$f^8(n) = 1011101000_2$, $f^{12}(n) = 101111010000_2$, and in general ([19]) after $4m$ iterates we arrive to a number that starts with 10, after has $m + 1$ ones, then 01 and it ends with $m + 1$ zeroes.

Problem 33 (Singmaster’s conjecture). *There is a value $S \in \mathbb{N}$ such that any number different from 1 appears in the Pascal’s triangle at most S times.*

This conjecture was proposed by David Singmaster in 1971, see [119]. He already proved in 1975 that there are infinitely many values that appear 6 times. One of them is 120,

$$120 = \binom{120}{1} = \binom{120}{119} = \binom{16}{2} = \binom{16}{14} = \binom{10}{3} = \binom{10}{7}.$$

In fact, it holds that

$$m = \binom{m}{1} = \binom{m}{m-1} = \binom{n+1}{k+1} = \binom{n+1}{n-k} = \binom{n}{k+2} = \binom{n}{n-k-2},$$

where for any $i \in \mathbb{N}$, $n = F_{2i+2}F_{2i+3} - 1$, $k = F_{2i}F_{2i+3} - 1$, $m = \binom{n+1}{k+1}$, and F_j is the j -th Fibonacci number, being $F_0 = 0$ and $F_1 = 1$, see [120]. The only known number that appears 8 times is

$$3003 = \binom{3003}{1} = \binom{3003}{3002} = \binom{78}{2} = \binom{78}{76} = \binom{15}{5} = \binom{15}{10} = \binom{14}{6} = \binom{14}{8}.$$

Hence, if the conjecture holds, $S \geq 8$. It seems that Singmaster thought that S could be 10 or 12, although many people starts thinking that $S = 8$. Notice that any $1 < m \in \mathbb{N}$ appears finitely many times because this value can only appear in the first $m + 1$ files.

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