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CONFIGURATION OF ZEROS OF ISOCHRONOUS VECTOR FIELDS OF DEGREE 5

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Abstract. In this paper we give the algebraic conditions that a configuration of 5 points in the plane must satisfy in order to be the configuration of zeros of an polynomial isochronous vector field. We use the obtained results to analyze configurations having some of its zeros satisfying some particular geometric conditions.

1. Introduction

We start defining an isochronous vector field and we express its general associated $1-$form, with its respective residues.

An *isochronous vector field* X is as a complex polynomial vector field on $\mathbb{C}$ whose zeros are all isochronous centers. A center is *isochronous* if the periods of the trajectories surrounding it are constant.

Let X be a complex polynomial vector field on $\mathbb{C}$ of degree $n \geqslant 1$, non–identically zero, as follows

$$X = (b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0)\frac{\partial}{\partial z} = \frac{1}{\lambda}(z-p_1)\dots(z-p_n)\frac{\partial}{\partial z}. \tag{1}$$

An isochronous vector field X is characterized by their associated $1-$form

$$\eta = \frac{dz}{b_n z^n + b_{n-1} z^{n-1} + \cdots + b_1 z + b_0} = \frac{\lambda\, dz}{(z-p_1)\dots(z-p_n)}$$

which has a unique zero at infinity of multiplicity $n-2$ and simple poles with non–zero pure imaginary residues. For $n \geqslant 2$, the residue of η at p_j is

$$r_j = \frac{\lambda}{(p_j - p_1)\dots\widehat{(p_j - p_j)}\dots(p_j - p_n)},$$

the hat $\widehat{(p_j - p_j)}$ means that the factor $(p_j - p_j)$ is omitted (see [3]).

The following well known result characterizes the polynomial isochronous vector fields.

Theorem 1. [3, 4] *Let X be a complex polynomial vector field on $\mathbb{C}$ of degree $n \geqslant 2$ defined as in* (1)*, then the following statements are equivalent.*

(a) *X has n isochronous centers.*
(b) *The zeros of X satisfy*

$$X'(p_j) = \frac{1}{\lambda}(p_j - p_1)\dots\widehat{(p_j - p_j)}\dots(p_j - p_n) \in i\mathbb{R}^* \textit{ for all } i = 1,\dots,\widehat{j},\dots n.$$

(c) *Their residues satisfy*

$$r_j \in i\mathbb{R}^* \textit{ for all } i = 1,\dots,\widehat{j},\dots n.$$

Here $i\mathbb{R}^*$ is the set of the pure imaginary complex numbers different from zero.

Next we characterize the polynomial isochronous vector fields in terms of the quotients of its residues.

Definition 2. *We say that the collection of zeros $[p_1, \ldots, p_n]$ is an* isochronous configuration *if there exists a rotation $e^{i\theta_0}$ such that the vector field $e^{i\theta_0}X$ is isochronous.*

If the residues $[r_1, \ldots, r_n]$ belong to the line $z = \rho\, e^{i\,\theta_0}$ for some θ_0, then the configuration $[p_1, \ldots, p_n]$ is isochronous. In short the configuration $[p_1, \ldots, p_n]$ is isochronous if and only if

$$\frac{r_j}{r_k} = -\frac{(p_k - p_1)(p_k - p_2)\ldots\widehat{(p_k - p_k)}\ldots\widehat{(p_k - p_j)}\ldots(p_k - p_n)}{(p_j - p_1)(p_j - p_2)\ldots\widehat{(p_j - p_j)}\ldots\widehat{(p_j - p_k)}\ldots(p_j - p_n)} \in \mathbb{R}^*, \tag{2}$$

for all $j, k = 1, \ldots, n$ with $j \neq k$. Here $\mathbb{R}^*$ is the set of real numbers different from zero.

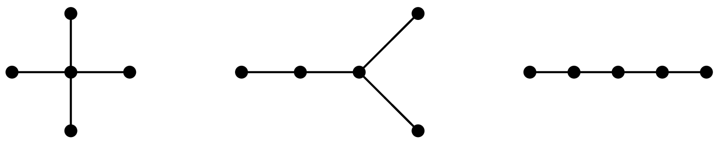
We can associate to each isochronous vector field X a weighted n–tree in the following way. The n vertices correspond to the n zeros of X, two vertices are connected with an edge if the basins of the corresponding centers are adjacent and the weights are the periods [3, 7]. We know that each embedded $n-$tree (without wights) is realized by an isochronous vector field X and that if the phase portraits of two different isochronous vector fields are topologically equivalent, then they have the same embedded n–tree (see [7]).

A topological classification of the isochronous vector fields of degree 2 can be found in [1, 2, 3, 9]. In [3] the authors characterize the isochronous vector fields of degree n in terms of the shape of the configuration of zeros when $n = 3, 4$ and they give partial results for $n \geqslant 5$ when the zeros present some symmetries. The known results for $n = 4$ and $n = 5$ are summarized in the next section.

The aim of this paper is to characterize the isochronous vector fields of degree 5 in terms of the configurations of zeroes without imposing any condition of symmetry.

If a polynomial vector field X over Riemann sphere is given, we can change the coordinates in such way X has a zero at 0 and another zero at 1 [3, 7]. Then, if the complex polynomial vector field X has degree 5, it has 8 free real parameters six for zeros two for main coefficient. Assuming that the position of two zeros is fixed, it will be proved that there can be up to seven different three–parameter families of isochronous configurations, see Theorem 9. Notice that the family of 5-degree isochronous vector fields has only three real parameters. There are still too much parameters to give a result providing all the possible shapes of the zeros of the isochronous configurations for $n = 5$. Nevertheless we can reduce the parameter space fixing either the position or the shape of some zeros. At the end of the paper we will analyze the shape of the isochronous configurations in some particular cases where the zeros present some symmetries and we will complete some of the cases studied in [3].

The paper is structured as follows. In Section 2 we give a summary of the known results on isochronous vector fields of degree 5. In Section 3 we describe how we solve the set of equations (2) for an arbitrary vector field of degree 5. In Section 4 we give explicitly the isochronous configurations in some particular cases: when four zeros are at the vertices of either a parallelogram or an isosceles trapezoid, when two zeros are on the line orthogonal to the line passing through other two zeros, and



(a) The star topology (b) The fork topology (c) The line topology

FIGURE 1. The planar 5–trees.

when three zeros are at the vertices of an equilateral triangle. We give numerical examples of some isochronous configurations with three zeros at the vertices of two isosceles triangles, when $p_1 = 0$, $p_2 = 1$ and $p_3 = 1/2 + i1/25$ and when $p_1 = 0$, $p_2 = 1$ and $p_3 = 1/2+i9/10$. Finally we give examples of isochronous configurations where the zeros do not satisfy any symmetry. We also analyze the phase portrait of the isochronous vector fields associated to the configurations. In particular, we find either the star or fork topology (Figure 1) in all cases except when we have all the zeros in the line. We have not found examples of the line topology when the zeros are not all in a line.

2. Some known results

In this section we summarize some known results on isochronous configurations of vector fields of degree $n \geqslant 4$ given in [3].

Theorem 3. [3] *A polynomial vector field X of degree 4 is isochronous if and only if the zeros p_1, p_2, p_3 and p_4 are either in a line or three are in the vertices of a triangle and one is at its orthocenter.*

The phase portrait of the isochronous vector fields of degree 5 can have as many different topologies as 5–trees, so they can have three different topologies (see Figure 1), namely the star topology, the fork topology and the line topology .

The next two theorems summarize the results isochronous configurations that are valid for vector fields of degree $n \geqslant 5$.

Theorem 4. [3] *The following statements hold.*

(a) *For each $n \geqslant 3$, if the zeros $p_1, p_2, \dots, p_n$ are in a line then X is isochronous and his phase portrait has the line topology.*
(b) *For each $n \geqslant 4$, if the zeros $p_2, p_3, \dots, p_n$ are at the vertices of a regular polygon and p_1 is at its center then X is isochronous and his phase portrait has the star topology.*
(c) *For each $n \geqslant 4$ there exist isochronous vector fields with the zeros $p_1, \dots, p_{n-2}$ in a line and the zeros p_{n-1} and p_n in new line orthogonal to the previous one.*

In addition, for $n = 5$, the following statements hold.

(d) *If $p_1 = 1/2$, $p_2 = -1/2$, $p_3 = iy_3$, $p_4 = iy_4$ and $p_5 = iy_5$, then X is isochronous if and only if $y_3 + y_4 + y_5 = 4y_3y_4y_5$.*
(e) *For $n = 5$, if $p_2, \dots, p_5$ are at the vertices of a rhombus and p_1 is at its center, if the residues of X satisfy $r_1 = -2(r_2 + r_3)$, $r_2 = r_4$, and $r_3 = r_5$ then X is isochronous, moreover its phase portrait has the star topology.*

(e) *Assume that X is isochronous and p_3, p_4, and p_5 are in the bisector line of the segment with endpoints p_1 and p_2*
   (i) *if p_4 is in the convex hull of p_1, p_2, p_3 and p_5 and the residues pf X satisfy $r_4 = r_1 + r_2 + r_3 + r_5$ and $|r_1| = |r_2|$, then X is isochronus, moreover the phase portrait of X has the star topology;*
   (ii) *if p_3 and p_4 are in the convex hull of p_1, p_2, p_5 and the residues of X satisfy $|r_3| < |r_1| + |r_2| + |r_4|$ and $|r_4| < |r_5|$, then X is isochronous, moreover its phase portrait has the fork topology.*

3. Characterization of the isochronous vector fields of degree 5

Without loss of generality we can consider that the position of two of the zeros of X are fixed to 0 and 1 (see Section 1).

Assume that $p_1 = 0$, $p_2 = 1$, and let $p_3 = x_3 + i\, y_3$, $p_4 = x_4 + i\, y_4$, $p_5 = x_5 + i\, y_5$. From (2), the configuration $[p_1, \ldots, p_5]$ is isochronous if and only if $r_j/r_k \in \mathbb{R}^*$ for all $j, k = 1, \ldots, n$ with $j \neq k$. Note that we only are interested in configurations with $p_i \neq p_j$ for $i \neq j$. We define

$$
\begin{aligned}
&e_1 = Im(r_1/r_2), \quad e_2 = Im(r_1/r_3), \quad e_3 = Im(r_1/r_4), \quad e_4 = Im(r_1/r_5),\\
&e_5 = Im(r_2/r_3), \quad e_6 = Im(r_2/r_4), \quad e_7 = Im(r_2/r_5), \quad e_8 = Im(r_3/r_4),\\
&e_9 = Im(r_3/r_5), \quad e_{10} = Im(r_4/r_5).
\end{aligned}
$$

The denominators of the functions e_i are defined when $p_i \neq p_j$ for $i \neq j$ so we can drop them. Let f_i be the numerator of the factorization of e_i for all $i = 1, \ldots, 10$. Then $[p_1, \ldots, p_5]$ is isochronous if and only if $(x_3, y_3, x_4, y_4, x_5, y_5)$ is a solution of the set of polynomial equations

$$f_i = 0, \qquad i = 1, \ldots, 10. \tag{3}$$

Next we will provide information about the solutions of (3). We only are interested in solutions of (3) satisfying $(x_i, y_i) \neq (x_j, y_j)$ for $i \neq j$. In what follows these solutions will be called *valid solutions*.

The first two equations of system (3) can be written as

$$a_1\, x_4^2 + a_1\, y_4^2 - a_1\, x_4 + c_1\, y_4 = 0, \qquad a_2\, x_4^2 + a_2\, y_4^2 + b_2\, x_4 + c_2\, y_4 = 0, \tag{4}$$

where

$$
\begin{aligned}
a_1 &= x_5 y_3 - x_5^2 y_3 + x_3 y_5 - x_3^2 y_5 - y_3^2 y_5 - y_3 y_5^2,\\
c_1 &= -x_3 x_5 + x_3^2 x_5 + x_3 x_5^2 - x_3^2 x_5^2 + x_5 y_3^2 - x_5^2 y_3^2 + y_3 y_5 + x_3 y_5^2 - x_3^2 y_5^2 - y_3^2 y_5^2,\\
a_2 &= x_5 y_3 - 2 x_3 x_5 y_3 + x_5^2 y_3 - x_3 y_5 + x_3^2 y_5 - y_3^2 y_5 + y_3 y_5^2,\\
b_2 &= -2 x_3 x_5 y_3 + 3 x_3^2 x_5 y_3 + x_5^2 y_3 - 2 x_3 x_5^2 y_3 - x_5 y_3^3 + x_3^2 y_5 - x_3^3 y_5 - y_3^2 y_5\\
&\quad + 3 x_3 y_3^2 y_5 + y_3 y_5^2 - 2 x_3 y_3 y_5^2,\\
c_2 &= x_3^2 x_5 - x_3^3 x_5 - x_3 x_5^2 + x_3^2 x_5^2 - x_5 y_3^2 + 3 x_3 x_5 y_3^2 - x_5^2 y_3^2 + 2 x_3 y_3 y_5 - 3 x_3^2 y_3 y_5\\
&\quad + y_3^3 y_5 - x_3 y_5^2 + x_3^2 y_5^2 - y_3^2 y_5^2.
\end{aligned}
$$

Systems of the form (4) will appear often in our analysis. We give the solution of this kind of systems in Section 3.1.

Many solutions of (3) satisfy that $y_i = 0$ for some $i = 3, 4, 5$, these solutions correspond to isochronous configurations with at least three zeros aligned. In order to simplify our computations, this case is treated separately in Section 3.2. From now on the solutions of (3) with $y_i \neq 0$ for all $i = 3, 4, 5$ that are valid solutions will be called *admissible solutions*.

3.1. **The resolution of systems of the form** (4)**.** Let us consider a generic system of the form (4)

$$A_1 x^2 + A_1 y^2 - A_1 x + C_1 y = 0, \qquad A_2 x^2 + A_2 y^2 + B_2 x + C_2 y = 0. \tag{5}$$

Note that the two equations of (5) correspond to the equations of two circles (eventually degenerated to a line) passing through the point $(0, 0)$. So $(x, y) = (0, 0)$ is always a solution of (5).

After tedious but not difficult computations we see that when $D_1 = A_1(A_2+B_2)$ and $D_2 = A_2 C_1 - A_1 C_2$ are not simultaneously zero then the solutions of system (5) are $(x, y) = (0, 0)$ and $(x, y) = (x_g, y_g)$ with

$$(x_g, y_g) = \left(-\frac{(A_2C_1 - A_1C_2)(B_2C_1 + A_1C_2)}{D_1^2 + D_2^2}, -\frac{A_1(A_2 + B_2)(B_2C_1 + A_1C_2)}{D_1^2 + D_2^2} \right), \tag{6}$$

Solving system $D_1 = 0$ and $D_2 = 0$ we get the following conditions: $\kappa_1 : \{A_1 = 0, C_1 = 0\}$, $\kappa_2 : \{A_1 = 0, A_2 = 0\}$, $\kappa_3 : \{B_2 = -A_2, A_2C_1 - A_1C_2 = 0\}$. Then we can prove the following result.

Lemma 5. *Let $\kappa_1 : \{A_1 = 0, C_1 = 0\}$, $\kappa_2 : \{A_1 = 0, A_2 = 0\}$, $\kappa_3 : \{B_2 = -A_2, A_2C_1 - A_1C_2 = 0\}$. Then the following statements hold.*

(a) $(x, y) = (0, 0)$ *is a solution of* (5) *for all A_1, C_1, A_2, B_2 and C_2.*

(b) *If neither of conditions κ_1, κ_2 and κ_3 is satisfied, then the solutions of* (5) *are $(x, y) = (0, 0)$ and $(x, y) = (x_g, y_g)$ given in* (6).

(c) *If condition κ_1 is satisfied, then the solutions of* (5) *are the solutions of equation $A_2 x^2 + A_2 y^2 + B_2 x + C_2 y = 0$.*

(d) *If condition κ_1 is not satisfied and condition κ_2 is satisfied, then the solution of* (5) *is $(x, y) = (0, 0)$ when $B_2 \neq 0$ and $(x, y) = (x, 0)$ when $B_2 = 0$.*

(e) *If condition κ_1 is not satisfied and condition κ_3 is satisfied, then either the second equation is identically 0 or the two equations of* (5) *are linearly dependent.*

3.2. **Isochronous configurations with three zeros in a line.** Without loss of generality we can assume that the zeros that are aligned are p_1, p_2 and p_3, so we assume that $y_3 = 0$.

We substitute $y_3 = 0$ into equations $f_1 = 0$ and $f_2 = 0$ and we get the equations

$$-(x_3 - 1)x_3h_1 := -(x_3 - 1)x_3(A_1 x_4^2 + A_1 y_4^2 - A_1 x_4 + C_1 y_4) = 0,$$
$$(x_3 - 1)x_3h_2 := (x_3 - 1)x_3(A_2 x_4^2 + A_2 y_4^2 + B_2 x_4 + C_2 y_4) = 0,$$

where

$$A_1 = y_5, \qquad C_1 = -x_5 + x_5^2 + y_5^2,$$
$$A_2 = y_5, \qquad B_2 = -x_3y_5, \qquad C_2 = -x_3x_5 + x_5^2 + y_5^2.$$

The solutions with $x_3 = 1$ and $x_3 = 0$ are not valid because they correspond to $p_3 = p_2$ and $p_3 = p_1$ respectively. Now, we analyze the solutions of system $h_1 = 0$, $h_2 = 0$. This is a system of the form (5), hence from Lemma 5 if conditions κ_i with $i = 1, 2, 3$ are not satisfied, then the non–trivial solution of system $h_1 = 0$, $h_2 = 0$ is

$$x_4 = x_5, \qquad y_4 = -y_5.$$

Condition κ_1 does not provide valid solutions because it is satisfied when either $(x_5, y_5) = (0, 0)$ or $(x_5, y_5) = (1, 0)$ (i.e. when either $p_5 = p_1$ or $p_5 = p_2$). Condition κ_2 is satisfied when $y_5 = 0$, under this condition $B_2 = 0$ so from Lemma 5 the solution of system $h_1 = 0$ and $h_2 = 0$ is $(x_4, y_4) = (x, 0)$ with $x \in \mathbb{R}$. Finally condition κ_3 is satisfied either when $x_3 = 1$, which does not provide a valid solution, or when $y_5 = 0$. This last case satisfies condition κ_2, so it has already been studied.

In short, system $f_1 = 0$, $f_2 = 0$ with $y_3 = 0$ has only two valid solutions:

$$s_{21} = \{y_3 = 0, x_4 = x_5, y_4 = -y_5, y_5 \neq 0\}, \qquad s_{22} = \{y_3 = 0, y_4 = 0, y_5 = 0\}.$$

By substituting s_{21} into equation $f_3 = 0$ we get

$$2x_3y_5(-x_3x_5 + x_5^2 + x_3x_5^2 - x_5^3 - y_5^2 - x_3y_5^2 + 3x_5y_5^2) = 0.$$

The solutions $x_3 = 0$ and $y_5 = 0$ of this equation are not valid, and the solutions of the last factor of the equation are

$$x_3 = \frac{x_5^3 - x_5^2 - 3x_5y_5^2 + y_5^2}{x_5^2 - x_5 - y_5^2}, \tag{7}$$

when $x_5^2 - x_5 - y_5^2 \neq 0$ and either $x_5 = 1$, $x_5 = 0$ or $x_5 = 1/2$ when $x_5^2 - x_5 - y_5^2 = 0$ (or equivalently when $y_5 = \pm\sqrt{x_5^2 - x_5}$). It is easy to check that the solutions with $x_5^2 - x_5 - y_5^2 = 0$ do not provide valid solutions. On the other hand, the solution s_{21} with x_3 given in (7) and the solution s_{22} satisfy all equations $f_i = 0$. Therefore they provide isochronous configurations. In short we have proved the following theorem.

Theorem 6. *If the zeros p_1, p_2 and p_3 are in a line, then the configuration $[p_1, \ldots, p_5]$ is isochronous if and only if it satisfies one of the following statements*

(a) *p_1, p_2, p_3, p_4 and p_5 are in a line.*

(b) *p_1, p_2, p_3 are in the bisector line of the segment with endpoints p_4 and p_5.*

In particular, if $p_1 = 0$, $p_2 = 1$, $p_3 = x_3$, $p_4 = x_4 + iy_4$ and $p_5 = x_5 + iy_5$, then X is isochronous if and only if one of the following statements holds

(c) $y_4 = y_5 = 0$,

(d) $x_4 = x_5$, $y_4 = -y_5$ *and* $x_3 = x_3^\ell = \dfrac{x_5^3 - x_5^2 - 3x_5y_5^2 + y_5^2}{x_5^2 - x_5 - y_5^2}$.

Theorem 6 completes in some sense the results in Theorem 4.

From now on we only will consider solutions of system (3) with $y_3, y_4, y_5 \neq 0$.

3.3. **Admissible solutions of system** (3)**.** Now we analyze the solutions of (4) that provide solutions of (3) with $p_i \neq p_j$ for $i \neq j$ and $y_3, y_4, y_5 \neq 0$; that is, that provide admissible solutions. System (4) is of the form (5) with $(x, y) = (x_4, y_4)$, $A_1 = a_1$, $A_2 = a_2$, $B_2 = b_2$, $C_1 = c_1$ and $C_2 = c_2$ so we will apply Lemma 5. Here we use the notation

$$K_1 : \{a_1 = 0, c_1 = 0\}, \quad K_2 : \{a_1 = 0, a_2 = 0\}, \quad K_3 : \{b_2 = -a_2, a_2c_1 - a_1c_2 = 0\}. \tag{8}$$

In order to simplify our computations we will use resultants theory in some cases. Next we summarize the basic properties of the resultants.

Let P and Q be two polynomials in the variable x with leading coefficient one. Let a_i, $i = 1, 2, \ldots, n$ be the roots of P and b_j, $j = 1, 2, \ldots, m$ be the roots of Q. The *resultant* of P and Q, $\mathrm{Res}[P, Q]$, is the expression formed by the product of all the differences $a_i - b_j$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, m$, see for instance [5] and

[8]. The main property of the resultant is that if P and Q have a common solution then necessarily $\mathrm{Res}[P, Q] = 0$.

Let now P and Q be polynomials in the variables (x, y). These polynomials can be considered as polynomials in x with polynomial coefficients in y, then the resultant with respect to x, $\mathrm{Res}[P, Q, x]$, is a new polynomial in the variable y with the following property. If P and Q have a common solution (x_0, y_0) then $\mathrm{Res}[P, Q, x](y_0) = 0$, and similarly for the variable y.

3.3.1. *Solutions of* (4) *satisfying condition* K_1. System $a_1 = 0$, $c_1 = 0$ (condition K_1) can be written in the form (5) with $(x, y) = (x_5, y_5)$, $A_1 = C_2 = y_3$, and $B_2 = C_1 = -A_2 = -(x_3 - x_3^2 - y_3^2)$ and can be solved by applying Lemma 5 again. Conditions κ_i for $i = 1, 2, 3$ provide solutions with either $p_3 = p_2$ or $p_3 = p_1$. The solution (x_g, y_g) in Lemma 5 becomes $(x_5, y_5) = (1, 0)$ (i.e. $p_2 = p_5$). Therefore, neither of the solutions of (4) satisfying condition K_1 can provide valid solutions of (3).

3.3.2. *Solutions of* (4) *satisfying condition* K_2. System $a_1 = 0$, $a_2 = 0$ (condition K_2) can be written again in the form (5) with $(x, y) = (x_5, y_5)$, $A_1 = A_2 = y_3$, $B_2 = y_3 - 2x_3y_3$, $C_1 = -(x_3 - x_3^2 - y_3^2)$ and $C_2 = -(x_3 - x_3^2 + y_3^2)$. All conditions κ_i in Lemma 5 provide solutions with $y_3 = 0$. Therefore conditions κ_i cannot provide admissible solutions of (3).

The solution (x_g, y_g) in Lemma 5 becomes

$$(x_5, y_5) = \left(x_3, \frac{x_3 - x_3^2}{y_3}\right).$$

We substitute this solution into equation $f_1 = 0$ and we get an equation equivalent to

$$(1 - x_3)x_3y_4\left(x_3^2 + y_3^2\right)\left((x_3 - 1)^2 + y_3^2\right) = 0.$$

Clearly neither of the solutions of this equation can provide admissible solutions of (3). In short, neither of the solutions of (4) satisfying condition K_2 can provide valid solutions of (3).

3.3.3. *Admissible solutions of* (3) *satisfying condition* K_3. Finally we analyze the solutions of system $F_1 = b_2 + a_2 = 0$ and $F_2 = a_2c_1 - a_1c_2 = 0$ (condition K_3). System $F_1 = 0$, $F_2 = 0$ cannot be written in the form (5). We will analyze the solution of this system by using the properties of resultants.

We compute the resultant of F_1 and F_2 with respect to x_5 and we get

$$\mathrm{Res}(F_1, F_2, x_5) = -2y_3^3y_5^2\left(x_3^2 + y_3^2\right)\left((x_3^2 - 1)^2 + y_3^2\right)^4(y_3 - y_5)F_a,$$

where

$$\begin{aligned}F_a = {} & 9x_3^6 - 27x_3^5 + 3x_3^4y_3^2 + 18x_3^4y_3y_5 + 27x_3^4 - 6x_3^3y_3^2 - 36x_3^3y_3y_5 - 9x_3^3 - 5x_3^2y_3^4 \\ & + 20x_3^2y_3^3y_5 + 18x_3^2y_3y_5 + 5x_3y_3^4 - 20x_3y_3^3y_5 + 3x_3y_3^2 + y_3^6 + 2y_3^5y_5 - 3y_3^4 + 2y_3^3y_5\end{aligned}$$

By the properties of the resultant we know that if (x_3, y_3, x_5, y_5) is a solution of system $F_1 = 0$ and $F_2 = 0$, then the coordinates (x_3, y_3, y_5) satisfy equation $\mathrm{Res}(F_1, F_2, x_5) = 0$. Thus in order to solve system $F_1 = 0$ and $F_2 = 0$ it is sufficient to find the solutions (x_3, y_3, y_5) of $\mathrm{Res}(F_1, F_2, x_5) = 0$ that satisfy system $F_1 = 0$ and $F_2 = 0$.

Clearly the first four factors of $\mathrm{Res}(F_1, F_2, x_5)$ do not provide admissible solutions of (3). Thus we only will consider solutions with either $y_3 = y_5$ or $F_a = 0$.

Case $y_3 = y_5$
We substitute $y_3 = y_5$ into equation $F_1 = 0$ and we get

$$y_5(x_5 - x_3)\left(x_3^2 - 2x_3x_5 - 2x_3 + 2x_5 - y_5^2 + 1\right) = 0.$$

The first two factors of this equation do no provide admissible solutions of (3). From the last factor of the equation we have

$$y_5 = \pm y_5^* = \pm\sqrt{(1-x_3)(1-x_3+2x_5)}.$$

We substitute $y_5 = \pm y_5^*$ into equation $F_2 = 0$ and we get the following solutions

$$x_5 = x_3, \qquad x_5 = \frac{x_3-1}{2}, \qquad x_3 = 1, \qquad x_5 = x_3 - 1, \qquad x_5 = x_5^* = \frac{1}{2}(4x_3 - 1).$$

The first solution corresponds to $p_3 = p_5$, the second and the third imply $y_5 = 0$, and the fourth corresponds to $p_3 = p_2$. Therefore, the unique solutions of system $F_1 = 0$ and $F_2 = 0$ that can provide admissible solutions of (3) are

$$x_5 = x_5^* = \frac{1}{2}(4x_3 - 1), \qquad y_5 = \pm y_5^* = \pm\sqrt{3}\sqrt{x_3 - x_3^2}.$$

Notice that y_5^* is defined for $x_3 \in [0,1]$ and $y_5^* = 0$ when either $x_3 = 0$ or $x_3 = 1$.

We substitute $x_5 = x_5^*$ and $y_5 = \pm y_5^*$ into (3). Since condition K_3 is satisfied, equations $f_1 = 0$ and $f_2 = 0$ are linearly dependent, so we will work with equations $f_1 = 0$ and $f_4 = 0$ instead of equations $f_1 = 0$ and $f_2 = 0$. If $x_5 = x_5^*$ and $y_5 = \pm y_5^*$, then system $f_1 = 0$, $f_4 = 0$ can be written as

$$\pm\frac{1}{4}(2x_3 - 3)(2x_3 + 1)F_1^* = 0, \qquad \pm\frac{1}{8}(2x_3 - 3)(2x_3 - 1)F_2^* = 0, \tag{9}$$

where

$$F_1^* = A_1^*x_4^2 + A_1^*y_4^2 - A_1^*x_4 + C_1^*y_4 = 0, \quad F_2^* = A_2^*x_4^2 + A_2^*y_4^2 + B_2^*x_4 + C_2^*y_4 = 0.$$

Here

$$\begin{array}{lll} A_1^* = y_5^*, & C_1^* = \mp 2(x_3 - x_3^2), & \\ A_2^* = 2y_5^*, & B_2^* = (1 - 6x_3)y_5^*, & C_2^* = \mp(7 - 10x_3)x_3, \end{array}$$

the upper sign corresponds to the positive value y_5^* and the lower sign to the negative value y_5^*. Notice that in order to obtain the expressions F_1^* and F_2^* substituted into system $f_1 = 0$ and $f_4 = 0$ not only $x_5 = x_5^*$ and $y_5 = \pm y_5^*$ but also $y_5^2 = 3(x_3 - x_2^2)$, $y_5^3 = \pm y_5^* y_5^2$, $y_5^4 = 9(x_3 - x_2^2)^2$.

Next, we analyze the solutions of (9) providing admissible solutions of (3). The factors $(2x_3 - 3)$ and $(2x_3 + 1)$ do not provide solutions in the domain of definition of y_5^* and the factor $(2x_3 - 1)$ provides solutions with $p_5 = p_3$. We analyze the solutions of $F_1^* = 0$ and $F_2^* = 0$ by applying again Lemma 5. Conditions κ_1, κ_2 in Lemma 5 give solutions with either $p_3 = p_2$ or $p_3 = p_1$ and condition κ_3 gives solutions with either $p_3 = p_2$, $p_3 = p_1$ or $p_5 = p_3$. The solution given by (x_g, y_g) in Lemma 5 becomes

$$(x_4, y_4) = (x_3, \pm\sqrt{3}\sqrt{x_3 - x_3^2}),$$

and corresponds to $p_3 = p_4$. Therefore the factor $y_3 = y_5$ does not provide admissible solutions of (3).

Case $F_a = 0$
From equation $F_a = 0$ we get

$$y_5 = y_5^\star = -\frac{1}{2y_3\, d}\left(y_3^2 - 3x_3^2\right)\left(-3x_3^2 + 6x_3 + y_3^2 - 3\right)\left(x_3^2 - x_3 + y_3^2\right),$$

where

$$d = 9x_3^4 - 18x_3^3 + 9x_3^2 + \left(10x_3^2 - 10x_3 + 1\right)y_3^2 + y_3^4.$$

Note that $y_5^\star$ is defined when $y_3 \neq 0$ and $d \neq 0$.

Solutions with $d = 0$.
First we analyze the solutions with $d = 0$. Solving equation $d = 0$ with respect to y_3 we get $y_3 = \pm\sqrt{d_1}$ and $y_3 = \pm\sqrt{d_2}$ where

$$d_1 = \frac{1}{2}\left(-10x_3^2 + 10x_3 - 1 - \sqrt{(2x_3-1)^2\left(16x_3^2 - 16x_3 + 1\right)}\right),$$
$$d_2 = \frac{1}{2}\left(-10x_3^2 + 10x_3 - 1 + \sqrt{(2x_3-1)^2\left(16x_3^2 - 16x_3 + 1\right)}\right).$$

The domain of definition of d_1 and d_2 is the set

$$D = (-\infty, \frac{1}{4}\left(2-\sqrt{3}\right)] \cup \{1/2\} \cup [\frac{1}{4}\left(2+\sqrt{3}\right), +\infty).$$

On the other hand, analyzing the functions d_1 and d_2 we can see that $d_1 \leqslant -3/16$ when $x_3 \in D \setminus \{1/2\}$, $d_1 = 3/4$ when $x_3 = 1/2$; $d_2 = 0$ when either $x_3 = 0$ or $x_3 = 1$, $d_2 < 0$ when $x_3 \in D \setminus \{1/2, 0, 1\}$, and $d_2 = 3/4$ when $x_3 = 1/2$. Thus, the solutions $y_3 = \pm\sqrt{d_1}$ and $y_3 = \pm\sqrt{d_2}$ are defined only when $x_3 = 1/2$. Therefore the denominator of $y_5^\star$ is zero when either $y_3 = 0$ or $(x_3, y_3) = (1/2, \pm\sqrt{3}/2)$. The solution $y_3 = 0$ does not provide admissible solutions of system (3).

Now we analyze the solutions of (3) with $(x_3, y_3) = (1/2, \pm\sqrt{3}/2)$. In this case, F_a is identically 0. By substituting $(x_3, y_3) = (1/2, \pm\sqrt{3}/2)$ into equation $F_1 = 0$ and solving the resulting equation we get the solutions $y_5 = \pm y_{51}$ and $y_5 = \pm y_{52}$ where

$$y_{51} = \frac{1 + \sqrt{-12x_5^2 + 12x_5 + 1}}{2\sqrt{3}}, \qquad y_{52} = \frac{1 - \sqrt{-12x_5^2 + 12x_5 + 1}}{2\sqrt{3}}.$$

These solutions also satisfy equation $F_2 = 0$. We substitute the solution with $(x_3, y_3) = (1/2, \sqrt{3}/2)$ and $y_5 = y_{51}$ into equations $f_1 = 0$ and $f_4 = 0$ and we obtain a system of equations of the form (4) in the variables $(x, y) = (x_4, y_4)$ where the coefficients A_1, C_1, A_2, B_2, C_2 depend on x_5. We solve this system by applying Lemma 5 as we have done in the previous cases and we see that it has two unique solutions, one satisfying $p_4 = p_5$ and the other one satisfying $p_3 = p_5$. The same occurs with the solution $(x_3, y_3) = (1/2, -\sqrt{3}/2)$ and $y_5 = -y_{51}$. Analyzing the cases $(x_3, y_3) = (1/2, \pm\sqrt{3}/2)$ and $y_5 = \pm y_{52}$ in a similar way we get three unique solutions, which satisfy $p_4 = p_5$, $p_1 = p_5$ and $p_2 = p_5$, respectively. Therefore system (3) has no valid solutions when $d = 0$.

Solutions with $d \neq 0$.
Assume that $(x_3, y_3) \neq (1/2, \pm\sqrt{3}/2)$. We substitute $y_5 = y_5^\star$ into equation $F_1 = 0$

and we solve the resulting equation obtaining in this way the solutions

$$
\begin{aligned}
&y_3 = 0,\\
&x_5 = x_5^{(\star,1)} = \frac{1}{2d}(12x_3^5 - 21x_3^4 + 8x_3^3y_3^2 + 6x_3^3 - 2x_3^2y_3^2 + 3x_3^2 - 4x_3y_3^4 - 2x_3y_3^2 + 3y_3^4 - y_3^2),\\
&x_5 = x_5^{(\star,2)} = \frac{1}{2(x_3-1)d}(15x_3^6 - 57x_3^5 + 13x_3^4y_3^2 + 81x_3^4 - 42x_3^3y_3^2 - 51x_3^3 - 3x_3^2y_3^4\\
&\quad + 44x_3^2y_3^2 + 12x_3^2 - x_3y_3^4 - 15x_3y_3^2 - y_3^6 + 3y_3^4).
\end{aligned}
$$

Note that the last solution is defined only when $x_3 \neq 1$.

First, we analyze the solution $(x_5, y_5) = (x_5^{(\star,1)}, y_5^\star)$. It is easy to check that equation $F_2 = 0$ is always satisfied when $(x_5, y_5) = (x_5^{(\star,1)}, y_5^\star)$. We substitute this solution into equations $f_1 = 0$ and $f_4 = 0$ and we get

(10)

$$
\begin{aligned}
&\left(y_3^2 - 3x_3^2\right)\left(-3x_3^2 + 6x_3 + y_3^2 - 3\right)\left(x_3^2 + y_3^2\right)\left(x_3^2 - 2x_3 + y_3^2 + 1\right)F_1^\star = 0,\\
&\left(3x_3^2 - y_3^2\right)\left(-3x_3^2 + 6x_3 + y_3^2 - 3\right)\left(x_3^2 + y_3^2\right)\left(3x_3^4 - 6x_3^3 + 6x_3^2y_3^2 + 3x_3^2 - 6x_3y_3^2 + 3y_3^4 - y_3^2\right)F_2^\star = 0,
\end{aligned}
$$

where

$$
F_1^\star = A_1^\star x_4^2 + A_1^\star y_4^2 - A_1^\star x_4 + C_1^\star y_4 = 0 \qquad F_2^\star = A_2^\star x_4^2 + A_2^\star y_4^2 + B_2^\star x_4 + C_2^\star y_4.
$$

Here

$$
\begin{aligned}
A_1^\star &= y_3,\\
C_1^\star &= -x_3^2 + x_3 - y_3^2,\\
A_2^\star &= -2y_3^2(x_3^2 + y_3^2 - 1)d,\\
B_2^\star &= 27x_3^9 - 135x_3^8 + 72x_3^7y_3^2 + 270x_3^7 - 288x_3^6y_3^2 - 270x_3^6 + 54x_3^5y_3^4 + 450x_3^5y_3^2\\
&\quad + 135x_3^5 - 166x_3^4y_3^4 - 342x_3^4y_3^2 - 27x_3^4 + 202x_3^3y_3^4 + 126x_3^3y_3^2 - 8x_3^2y_3^6\\
&\quad - 114x_3^2y_3^4 - 18x_3^2y_3^2 - 9x_3y_3^8 + 22x_3y_3^6 + 23x_3y_3^4 + 5y_3^8 - 10y_3^6 + y_3^4,\\
C_2^\star &= -y_3\left(-27x_3^8 + 108x_3^7 - 80x_3^6y_3^2 - 162x_3^6 + 244x_3^5y_3^2 + 108x_3^5 - 78x_3^4y_3^4\right.\\
&\quad - 254x_3^4y_3^2 - 27x_3^4 + 164x_3^3y_3^4 + 88x_3^3y_3^2 - 24x_3^2y_3^6 - 102x_3^2y_3^4 + 10x_3^2y_3^2\\
&\quad \left.+28x_3y_3^6 + 12x_3y_3^4 - 8x_3y_3^2 + y_3^8 - 10y_3^6 + 5y_3^4\right).
\end{aligned}
$$

The first factor of equations (10) gives solutions with $p_5 = p_1$, the second one gives solutions with $p_5 = p_2$ and the third factor does not give real solutions. The fourth factor of the first equation of (10) gives solutions with $p_3 = p_2$ and the fourth factor of the second equation gives solutions with $p_5 = p_3$. Therefore neither of them provides valid solutions.

We solve system $F_1^\star = 0$ and $F_2^\star = 0$ by applying Lemma 5 again and we get the solutions $(x_4, y_4) = (0, 0)$ and $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ with

$$
\begin{aligned}
x_4^{(\star,1)} &= -\frac{1}{d}\left(x_3^2 - 2x_3 + y_3^2\right)\left(3x_3^3 - 6x_3^2 + 3x_3y_3^2 + 3x_3 - 2y_3^2\right),\\
y_4^{(\star,1)} &= \frac{1}{y_3\,d}\left(3x_3^3 - 6x_3^2 + 3x_3y_3^2 + 3x_3 - 2y_3^2\right)\left(3x_3^3 - 3x_3^2 + 3x_3y_3^2 - y_3^2\right),
\end{aligned}
$$

when conditions κ_1, κ_2 and κ_3 are not satisfied. Conditions κ_1 and κ_2 provide solutions with $y_3 = 0$ and the condition κ_3 provides solutions with either $p_5 = p_2$, $p_5 = p_3$, $p_3 = p_1$, $p_3 = p_2$ or $d = 0$. Therefore these conditions cannot provide admissible solutions of (3). Finally, it is easy to check that for all $i = 1, \dots, 10$ the equations $f_i = 0$ are satisfied when $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ and $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$. In short, this solution will provide admissible solutions when (x_3, y_3) is such that $p_i \neq p_j$ for $i \neq j$ and $d \neq 0$.

Now we consider the solution $(x_5, y_5) = (x_5^{(\star,2)}, y_5^{\star})$ (assuming that $(x_3, y_3) \neq (1/2, \pm\sqrt{3}/2)$) and we proceed in a similar way. First we substitute the solution $(x_5, y_5) = (x_5^{(\star,2)}, y_5^{\star})$ into equation $F_2 = 0$ and we get equation

$$-\left(-3x_3^2 + 6x_3 + y_3^2 - 3\right)^2 \left(x_3^2 + y_3^2\right) \left(x_3^2 - 2x_3 + y_3^2 + 1\right) \left(x_3^2 - x_3 + y_3^2\right)^2 \left(3x_3^2 - 3x_3 + y_3^2\right) G^{\star} = 0,$$

where

$$G^{\star} = -3x_3^6 + 24x_3^5 - 5x_3^4y_3^2 - 54x_3^4 + 32x_3^3y_3^2 + 48x_3^3 - x_3^2y_3^4 - 44x_3^2y_3^2 - 15x_3^2 + 8x_3y_3^4 + 16x_3y_3^2 + y_3^6 - 6y_3^4 + y_3^2.$$

Clearly, the second factor of equation $F_2 = 0$ does not provide real solutions, the first and fourth factors provide solutions with $p_5 = p_1$, the third provides solutions with $p_3 = p_2$, and the fifth provides solutions with $p_5 = p_3$. Therefore the unique factor that can provide admissible solutions of (3) is $G^{\star}$.

It is not difficult to check that

$$x_5^{(\star,2)} = x_5^{(\star,1)} - \frac{G^{\star}}{2(x_3 - 1)d}.$$

So if $x_3 \neq 1$ and $G^{\star} = 0$, then $x_5^{(\star,2)} = x_5^{(\star,1)}$ and this is the case we have just been studied. Furthermore, if $x_3 = 1$, then equation $F_1 = 0$ has only the two solutions $y_3 = 0$ and $x_5 = x_5^{(\star,1)}$. Since the solution $x_5^{(\star,2)} = x_5^{(\star,1)}$ is defined when $x_3 = 1$ we do not need to consider this case separately.

3.3.4. *Admissible solutions of* (3) *that do not satisfy any condition* K_i. From Lemma 5 when neither of conditions K_i is satisfied the solutions of system $f_1 = 0$ and $f_2 = 0$ are $(x_4, y_4) = (0, 0)$ and $(x_4, y_4) = (x_{40}, y_{40})$ with

$$(x_{40}, y_{40}) = (x_g, y_g), \tag{11}$$

where (x_g, y_g) is defined in Lemma 5 and $A_1 = a_1$, $A_2 = a_2$, $B_2 = b_2$, $C_1 = c_1$ and $C_2 = c_2$ are defined in (4). We substitute the solution $(x_4, y_4) = (x_{40}, y_{40})$ into equations $f_i = 0$ for $i = 3, \dots, 10$, we factorize the resulting equations and we drop the denominators obtaining a new system of polynomial equations

$$g_i = 0, \qquad i = 3, \dots, 10, \tag{12}$$

where the function g_5 is identically zero. Next we analyze the solutions of (12).

The factorization of g_7 consist of two factors

$$g_7 = 2\widetilde{g}_{71}\widetilde{g}_{72},$$

where

$$
\begin{aligned}
\widetilde{g}_{71} ={}& x_3^3 y_5 - 3x_3^2 x_5 y_3 - 2x_3^2 y_5 + 2x_3 x_5^2 y_3 + 4x_3 x_5 y_3 - 3x_3 y_3^2 y_5 + 2x_3 y_3 y_5^2 + x_3 y_5 - 2x_5^2 y_3 \\
&+ x_5 y_3^3 - x_5 y_3 + 2y_3^2 y_5 - 2y_3 y_5^2, \\
\widetilde{g}_{72} ={}& (2x_3 - 1) y_3^3 y_5^7 + y_3^2 \left(3x_3^3 + 3x_5 x_3^2 - 6x_3^2 - y_3^2 x_3 - 3x_5 x_3 + 3x_3 - 3x_5 y_3^2 + 2y_3^2\right) y_5^6 \\
&+ y_3 \left(6x_5 x_3^4 - 3x_3^4 - 4y_3^2 x_3^3 - 12x_5 x_3^3 + 6x_3^3 - 6x_5 y_3^2 x_3^2 + 9y_3^2 x_3^2 + 6x_5 x_3^2 - 3x_3^2 \right. \\
&\left. - 4y_3^4 x_3 - 2x_5^2 y_3^2 x_3 + 8x_5 y_3^2 x_3 - 6y_3^2 x_3 + 4x_5 y_3^4 + x_5^2 y_3^2\right) y_5^5 + \left(-x_3^7 + 3x_5 x_3^6 + 2x_3^6 \right. \\
&+ y_3^2 x_3^5 - 9x_5 x_3^5 - 9x_5 y_3^2 x_3^4 + 2y_3^2 x_3^4 + 9x_5 x_3^4 - 2x_3^4 + 5y_3^4 x_3^3 - 9x_5^2 y_3^2 x_3^3 + 27x_5 y_3^2 x_3^3 \\
&- 9y_3^2 x_3^3 - 3x_5 x_3^3 + x_3^3 - 11x_5 y_3^4 x_3^2 - 2y_3^4 x_3^2 + 5x_5^3 y_3^2 x_3^2 + 6x_5^2 y_3^2 x_3^2 - 18x_5 y_3^2 x_3^2 \\
&+ 6y_3^2 x_3^2 + 3y_3^6 x_3 + 11x_5^2 y_3^4 x_3 + 3y_3^4 x_3 - 5x_5^3 y_3^2 x_3 + 3x_5^2 y_3^2 x_3 + x_5 y_3^6 - 2y_3^6 - 5x_5^3 y_3^4 \\
&\left. + 2x_5^2 y_3^4 - 3x_5 y_3^4\right) y_5^4 + y_3 \left(6x_5 x_3^6 - 3x_3^6 - 18x_5^2 x_3^5 + 4x_3^5 + 8x_5^3 x_3^4 + 33x_5^2 x_3^4 \right. \\
&+ 10x_5 y_3^2 x_3^4 - 5y_3^2 x_3^4 - 18x_5 x_3^4 + x_3^4 - 16x_5^3 x_3^3 - 12x_5^2 x_3^3 - 12x_5^2 y_3^2 x_3^3 - 8x_5 y_3^2 x_3^3 \\
&+ 4y_3^2 x_3^3 + 12x_5 x_3^3 - 2x_3^3 + 2x_5 y_3^4 x_3^2 - y_3^4 x_3^2 + 8x_5^3 x_3^2 - 3x_5^2 x_3^2 + 12x_5^3 y_3^2 x_3^2 \\
&+ 12x_5 y_3^2 x_3^2 - 3y_3^2 x_3^2 + 6x_5^2 y_3^4 x_3 - 8x_5 y_3^4 x_3 - 10x_5^4 y_3^2 x_3 + 8x_5^3 y_3^2 x_3 - 12x_5^2 y_3^2 x_3 \\
&\left. - 2x_5 y_3^6 + y_3^6 + 4x_5^3 y_3^4 - 9x_5^2 y_3^4 + 6x_5 y_3^4 + 5x_5^4 y_3^2 - 4x_5^3 y_3^2 + 3x_5^2 y_3^2\right) y_5^3 \\
&+ (x_5 - 1) x_5 \left(x_3^7 - x_5 x_3^6 - 3x_3^6 - y_3^2 x_3^5 + 3x_5 x_3^5 + 3x_3^5 + 11x_5 y_3^2 x_3^4 - 3y_3^2 x_3^4 - 3x_5 x_3^4 \right. \\
&- x_3^4 - 5y_3^4 x_3^3 - 11x_5^2 y_3^2 x_3^3 - 11x_5 y_3^2 x_3^3 + 7y_3^2 x_3^3 + x_5 x_3^3 + 9x_5 y_3^4 x_3^2 + 3y_3^4 x_3^2 + x_5^3 y_3^2 x_3^2 \\
&+ 15x_5^2 y_3^2 x_3^2 - 3x_5 y_3^2 x_3^2 - 3y_3^2 x_3^2 - 3y_3^6 x_3 + 9x_5^2 y_3^4 x_3 - 18x_5 y_3^4 x_3 - x_5^3 y_3^2 x_3 - 4x_5^2 y_3^2 x_3 \\
&\left. + 3x_5 y_3^2 x_3 - 3x_5 y_3^6 + 3y_3^6 - x_5^3 y_3^4 - 3x_5^2 y_3^4 + 6x_5 y_3^4\right) y_5^2 - (x_5 - 1)^2 x_5^2 y_3 \left(2x_3^5 - 2x_5 x_3^4 \right. \\
&- 4x_3^4 + 8y_3^2 x_3^3 + 4x_5 x_3^3 + 2x_3^3 - 18x_5 y_3^2 x_3^2 - 3y_3^2 x_3^2 - 2x_5 x_3^2 + 6y_3^4 x_3 + 6x_5^2 y_3^2 x_3 \\
&\left. + 12x_5 y_3^2 x_3 - 3y_3^4 - 3x_5^2 y_3^2 - 2x_5 y_3^2\right) y_5 + (x_5 - 1)^3 x_5^3 y_3^2 \left(x_3^3 - x_5 x_3^2 - x_3^2 - 3y_3^2 x_3 \right. \\
&\left. + x_5 x_3 + x_5 y_3^2 + y_3^2\right).
\end{aligned}
$$

Then the solutions of (12) must satisfy either $\widetilde{g}_{71} = 0$ or $\widetilde{g}_{72} = 0$. We see that factor $\widetilde{g}_{72}$ is common to all the functions g_i for $i = 3, 4, 6, 7, 8, 9, 10$. Therefore all the solutions (x_3, y_3, x_5, y_5) of equation $\widetilde{g}_{72} = 0$ that do not satisfy conditions K_i with $i = 1, 2, 3$ provide isochronous configurations with $(x_4, y_4) = (x_{40}, y_{40})$.

It is easy to check that $g_{71} = -F_1$ (see Section 3.3.3). Therefore all the solutions of equation $\widetilde{g}_{71} = 0$ satisfy condition K_3 and the solution $(x_4, y_4) = (x_{40}, y_{40})$ is not defined in this case.

Remark 7. *Equation $\widetilde{g}_{72} = 0$ has at most seven different solutions $y_5 = y_5(x_3, y_3, x_5)$ with $y_3 \neq 0$ when $x_3 \neq 1/2$, at most six different solutions $y_5 = y_5(y_3, x_5)$ with $y_3 \neq 0$ when $x_3 = 1/2$ and $x_5 \neq 1/2$, and the solution $(x_3, y_3) = (1/2, y_3)$, $(x_5, y_5) = (1/2, y_5)$ when $x_3 = x_5 = 1/2$.*

Indeed, if $(2x_3 - 1)y_3^3 \neq 0$, then $\widetilde{g}_{72}$ is a polynomial of degree 7 in the variable y_5 therefore there could exist up to seven different real solutions $y_5 = y_5(x_3, y_3, x_5)$ of $\widetilde{g}_{72} = 0$. The coefficient of degree 7 of $\widetilde{g}_{72}$, $(2x_3 - 1)y_3^3$, is equal to zero when either $x_3 = 1/2$ or $y_3 = 0$. Here we are not interested in solutions with $y_3 = 0$. So $\widetilde{g}_{72}$ becomes a polynomial of degree 6 with at most 6 solutions when $x_3 = 1/2$. If

$x_3 = 1/2$, then the coefficient of degree 6 of $\widetilde{g}_{72}$ is

$$-\frac{3}{8}(2x_5 - 1)y_3^2\left(4y_3^2 + 1\right),$$

which becomes zero when $x_5 = 1/2$. But if $x_3 = 1/2$ and $x_5 = 1/2$ then $\widetilde{g}_{72} = 0$.

We note that the solution with (x_3, y_3), $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$, $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$ always satisfies equation $\widetilde{g}_{72} = 0$, but in this case the solution $(x_4, y_4) = (x_{40}, y_{40})$ is not defined because $D_1^2 + D_2^2 = 0$ (or equivalently, because conditions $K_i = 0$ are satisfied for some $i = 1, 2, 3$). In short we have proved the following Theorem.

Theorem 8. *Let $p_1 = 0$, $p_2 = 1$, $p_3 = x_3 + iy_3$, $p_4 = x_4 + iy_4$ and $p_5 = x_5 + iy_5$. Then we can have the following families of solutions of* (3) *with $p_i \neq p_j$ for $i \neq j$ providing isochronous configurations with $y_i \neq 0$ for $i = 3, 4, 5$.*

(a) *Up to seven different three parameter families of solutions with $(x_4, y_4) = (x_{40}, y_{40})$ (see* (11)*) and $y_5 = y_5(x_3, y_3, x_5)$ satisfying equation $\widetilde{g}_{72} = 0$ when $x_3 \neq 1/2$.*

(b) *Up to six different two parameter families of solutions with $(x_4, y_4) = (x_{40}, y_{40})$ and $y_5 = y_5(y_3, x_5)$ satisfying equation $\widetilde{g}_{72} = 0$ when $x_3 = 1/2$ and $x_5 \neq 1/2$.*

(c) *The two parameter family of solutions with $(x_4, y_4) = (x_{40}, y_{40}) = (1/2, (y_3 + y_5)/(4y_3y_5 - 1))$, $(x_3, y_3) = (1/2, y_3)$ and $(x_5, y_5) = (1/2, y_5)$.*

(d) *The two parameter family with $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ and $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$ (see Section 3.3.3).*

Notice that the solution $(x_4, y_4) = (x_{40}, y_{40})$ is defined only for solutions of $\widetilde{g}_{72} = 0$ that do not satisfy any of conditions K_1, K_2, K_3 defined in (8)*. The solution in d) is defined when $(x_3, y_3) \neq (1/2, \pm\sqrt{3}/2)$.*

Theorem 6 gives a complete description of the isochronous configurations with $y_i = 0$ for some $i = 3, 4, 5$ and Theorem 8 gives a complete description of the isochronous configurations with $y_i \neq 0$ for all $i = 3, 4, 5$. Hence, we have a complete description of all the isochronous configurations of the vector fields of degree 5 which is summarized in Theorem 9.

We can see that all the zeros given in Theorem 6 satisfy equation $\widetilde{g}_{72} = 0$. The zeros given by Theorem 6(c) do not provide solutions with $(x_4, y_4) = (x_{40}, y_{40})$ because in this case $D_1^2 + D_2^2 = 0$. This does not happen with the zeros given by Theorem 6(d). Moreover we can see that if $y_3 = 0$ and $x_3 = x_3^{\ell}$, then $(x_{40}, y_{40}) = (x_5, -y_5)$. Hence statement (d) of Theorem 6 can be included in either statement (a), (b) or (c) of Theorem 8 if we do not consider the assumption $y_i \neq 0$ for all $i = 3, 4, 5$.

Theorem 9. *Let $p_1 = 0$, $p_2 = 1$, $p_3 = x_3 + iy_3$, $p_4 = x_4 + iy_4$ and $p_5 = x_5 + iy_5$. The only solutions of* (3) *with $p_i \neq p_j$ for $i \neq j$ providing isochronous configurations are the following.*

(a) $y_3 = y_4 = y_5 = 0$.

(b) $(x_4, y_4) = (x_{40}, y_{40})$ *(see* (11)*) and $y_5 = y_5(x_3, y_3, x_5)$ a solution of equation $\widetilde{g}_{72} = 0$ that does not satisfy any of conditions K_1, K_2, K_3 defined in* (8).

(c) $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ *and $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$ (see Section 3.3.3).*

Remark 10. *Using the symmetries of the configuration it is not difficult to see that if* $p_1 = 0$, $p_2 = 1$, $p_3 = x_3 + y_3\, i$, $p_4 = x_4 + y_4\, i$ *and* $p_5 = x_5 + y_5\, i$ *is an isochronous configuration then so is the configuration* $p_1 = 0$, $p_2 = 1$, $p_3 = x_3 - y_3\, i$, $p_4 = x_4 - y_4\, i$, $p_5 = x_5 - y_5\, i$, *the configuration* $p_1 = 0$, $p_2 = 1$, $p_3 = 1 - x_3 + y_3\, i$, $p_4 = 1 - x_4 + y_4\, i$ $p_5 = 1 - x_5 + y_5\, i$, *and the configuration* $p_3 = 1 - x_3 - y_3\, i$, $p_4 = 1 - x_4 - y_4\, i$ $p_5 = 1 - x_5 - y_5\, i$. *So it is sufficient to analyze the configurations with* $x_3 \leqslant 1/2$ *any* $y_3 \geqslant 0$.

4. Isochronous configurations with $n = 5$ in some particular cases

In the previous section, more precisely in Theorem 9 we have given a complete description of the isochronous configurations with $n = 5$. Here we will analyze the isochronous configurations for some particular configurations of zeros.

4.1. **Four zeros at the vertices of a parallelogram.**

Without loss of generality we can assume that the segment joining p_1 and p_2 is an edge of the parallelogram, then the remaining vertices of the parallelogram can be taken as $p_3 = x_3 + y_3\, i$, and $p_5 = x_3 + 1 + y_3\, i$ with $y_3 > 0$. The assumption $y_3 > 0$ is not restrictive by Remark 10.

Since $y_3 \neq 0$ all isochronous configurations of this type are given by Theorem 8. First we see that the solution with $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ and $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$ does not provide isochornous configurations with four zeros at the vertices of a parallelogram. Indeed, if the configuration of p_1, p_2, p_3, p_5 is a parallelogram then (x_3, y_3) satisfies equations $x_5^{(\star,1)} = x_3 + 1$, $y_5^{\star} = y_3$ and this system of equations does not have real solutions.

Now we substitute $(x_3, y_3, x_5, y_5) = (x_3, y_3, x_3 + 1, y_3)$ into equation $\widetilde{g}_{72} = 0$ and we get

$$-4y_3^2\left(x_3^2 + y_3^2 - 1\right)\left(x_3^2 + y_3^2\right)^2 = 0.$$

This equation has a unique real solution with $y_3 > 0$, $y_3 = \sqrt{1 - x_3^2}$. By substituting this solution into the expression of (x_{40}, y_{40}) we get

$$(x_4, y_4) = (x_{40}, y_{40}) = \left(\frac{x_3 + 1}{2}, \frac{\sqrt{1 - x_3^2}}{2}\right).$$

In short, we have a unique one parameter family of configurations with four zeros at the vertices of a parallelogram and $y_3 > 0$. It is given by the solution

$$(x_3, y_3) = \left(x_3, \sqrt{1 - x_3^2}\right), \quad (x_4, y_4) = \left(\frac{x_3 + 1}{2}, \frac{\sqrt{1 - x_3^2}}{2}\right), \quad (x_5, y_5) = \left(x_3 + 1, \sqrt{1 - x_3^2}\right).$$

It is easy to check that in this family of configurations p_1, p_2, p_3 and p_5 are at the vertices of a rhombus (degenerated to a square when $x_3 = 0$) and p_4 is at its center. These configurations correspond to the ones given by Theorem 4(e), so its phase portrait has the star topology.

In short we have proved the following result, which in some sense can be thought as a generalization of the statement (b) of Theorem 4.

Lemma 11. *Assume that the configuration* $[p_1, \ldots, p_5]$ *is isochronous. The following statements hold.*

(a) *If the zeros* p_1, p_2, p_3, p_5 *are at the vertices of a parallelogram, then this parallelogram is a rhombus and* p_4 *is at the center of the rhombus.*

(b) *There exist a unique configuration with four zeros at the vertices of a given rhombus. Assuming that the vertices of the rhombus are* $p_1 = 0$, $p_2 = 1$, $p_3 = x_3 + i\sqrt{1 - x_3^2}$, $p_5 = x_3 + 1 + i\sqrt{1 - x_3^2}$, *then the configuration satisfies*

$$p_4 = \frac{x_3 + 1}{2} + i\,\frac{\sqrt{1 - x_3^2}}{2}.$$

Notice that the rhombus is a square when $x_3 = 0$.

(c) *The phase portrait associated to the configuration* $[p_1, \ldots, p_5]$ *has the star topology.*

4.2. **Four zeros at the vertices of an isosceles trapezoid.** Without loss of generality we can assume that the segment joining p_1 and p_2 is an edge of the isosceles trapezoid, then the remaining vertices of the trapezoid can be taken as $p_3 = x_3 + y_3\, i$, and $p_5 = 1 - x_3 + y_3\, i$. As in the previous case, here we have assumed that $y_3 > 0$, this is not restrictive by Remark 10.

As in the previous section, since $y_3 \neq 0$ all isochronous configurations of this type are given by Theorem 8. First we analyze the solution with $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ and $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$. By imposing that the configuration is an isosceles trapezoid we get the following equations

$$x_5^{(\star,1)} = 1 - x_3, \qquad y_5^{\star} = y_3.$$

Solving this system of equations we get the solutions

$$(x_3, y_3) = \left(\frac{1}{2}, \pm\frac{1}{2\sqrt{3}}\right),$$

which is not a valid solution because $p_3 = p_5$.

Now we substitute $(x_3, y_3, x_5, y_5) = (x_3, y_3, 1 - x_3, y_3)$ into equation $\widetilde{g}_{72} = 0$ and we get

$$-4(2x_3 - 1)^3 y_3^4 \left(3x_3^4 - 6x_3^3 + 2x_3^2 y_3^2 + 3x_3^2 - 2x_3 y_3^2 - y_3^4 + y_3^2\right) = 0. \tag{13}$$

The solution with $x_3 = 1/2$ is not possible because it corresponds to $p_3 = p_5$. The last factor in (13) provides a unique real solution with $y_3 > 0$, the solution

$$y_3 = y_3^t = \sqrt{x_3^2 - x_3 + \frac{1}{2}\beta + \frac{1}{2}}, \tag{14}$$

where

$$\beta = \sqrt{16x_3^4 - 32x_3^3 + 20x_3^2 - 4x_3 + 1}.$$

Substituting this solution into (x_{40}, y_{40}) we get

$$(x_{40}, y_{40}) = (x_{40}^t, y_{40}^t) = \left(\frac{1}{2}, \frac{(1 + 4(x_3 - 1)x_3 - \beta)\sqrt{1 + 2(x_3 - 1)x_3 + \beta}}{4\sqrt{2}x_3(x_3 - 1)}\right). \tag{15}$$

So we have a unique one parameter family of isosceles trapezoid configurations which is given by

$$(x_3, y_3) = (x_3, y_3^t), \quad (x_4, y_4) = (x_{40}^t, y_{40}^t), \quad (x_5, y_5) = (1 - x_3, y_3^t),$$

defined for $x_3 \neq 1/2$. It is not difficult to check that $y_3^t \geqslant y_{40}^t > 0$ for all $x_3 \in \mathbb{R} \setminus \{0, 1\}$ and that $y_3^t = y_{40}^t$ when $x_3 = 1/2$. In short we have proved the following result.

Lemma 12. *Assume that the configuration $[p_1,\dots,p_5]$ is isochronous. The following statements hold.*

(a) *If the zeros p_1, p_2, p_3, p_5 are at the vertices of an isosceles trapezoid, then p_4 is at the interior of the trapezoid on its axis of symmetry.*

(b) *There exist a unique configuration with four zeros at the vertices of a given isosceles trapezoid. Assuming that the vertices of the trapezoid are $p_1 = 0$, $p_2 = 1$, $p_3 = x_3 + i\,y_3$, $p_5 = 1 - x_3 + i\,y_3$, then the configuration satisfies $y_3 = y_3^t$ with y_3^t given in* (14)*, and $p_4 = x_{40}^t + i\,y_{40}^t$ with (x_{40}^t, y_{40}^t) given in* (15)*.*

We have plotted the phase portrait of the isochronous vector fields associated to the configurations given by Lemma 12(b) for many values of x_3. After analyzing the obtained results we conjecture that all the isochronous configurations given by Lemma 12(b) have a phase portrait with the star topology.

4.3. **Two zeros on the line orthogonal to the line passing through other two zeros.** In Theorem 6 we have proved that if p_1, p_2 and p_3 are in a line L, then p_4 and p_5 are either in the same line L or L is the bisector line of the segment with endpoints p_4 and p_5. Now we prove the following more generic result.

Lemma 13. *Assume that the configuration $[p_1,\dots,p_5]$ is isochronous. If p_1, p_2 are in a line L and p_3, p_5 are in a line orthogonal to L, L', then either p_4 is in L and L is the bisector line of the segment with endpoints p_3 and p_5; or p_4 is in L' and L' is the bisector line of the segment with endpoints p_1 and p_2.*

Proof. Assume that $p_1 = 0$, $p_2 = 1$, $p_3 = x_3 + y_3\,i$, and $p_5 = x_3 + y_5\,i$ with $y_3 > 0$. This is not restrictive by Remark 10. Let L be the line passing through p_1 and p_2 and L' the line passing through p_3 and p_5.

Since $y_3 \neq 0$ all the isochronous configurations of this type are given by Theorem 8. The equation $\widetilde{g}_{72} = 0$ evaluated at $x_5 = x_3$ becomes

$$(1-2x_3)(y_3-y_5)^3(y_3+y_5)\left(x_3^2 - x_3 + y_3y_5\right)^3 = 0. \tag{16}$$

We note that the solution of (16) with $y_3 = y_5$ does not provide valid solutions of (3) because it corresponds to $p_3 = p_5$. If $y_5 = (x_3 - x_3^2)/y_3$, then condition K_2 is satisfied and therefore the solution of (3) with $(x_4, y_4) = (x_{40}, y_{40})$ is not defined. The solutions of (16) $x_3 = 1/2$ and $y_5 = -y_3$ provide valid solutions of (3) with $(x_4, y_4) = (x_{40}, y_{40})$ which are given by

(17)
$$\begin{aligned}
&(x_3,y_3) = \left(\frac{1}{2}, y_3\right), \quad (x_4,y_4) = \left(\frac{1}{2}, \frac{y_3+y_5}{4y_3y_5-1}\right), \qquad (x_5,y_5) = \left(\frac{1}{2}, y_5\right),\\
&(x_3,y_3) = (x_3,y_3), \quad (x_4,y_4) = \left(\frac{x_3^3 - x_3^2 - 3x_3y_3^2 + y_3^2}{x_3^2 - x_3 - y_3^2}, 0\right), \quad (x_5,y_5) = (x_3,-y_3),
\end{aligned}$$

respectively. In the first solution p_4 is on L' and L' is the bisector line of the segment with endpoints p_1 and p_2. In the second solution p_4 is on L and L is the bisector line of the segment with endpoints p_3 and p_5. Notice that the solutions given in (17) corresponds to the configuration given in Theorem 4(d).

Finally we analyze the solutions of (3) with $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ and $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$. When $x_5 = x_3$ equation $x_5^{(\star,1)} = x_3$ is equivalent to

$$(1-2x_3)\left(3x_3^4 - 6x_3^3 + 6x_3^2y_3^2 + 3x_3^2 - 6x_3y_3^2 + 3y_3^4 - y_3^2\right) = 0.$$

We can see that all the solutions of the second factor provide solutions of (3) with $y_5 = y_3$ which are not valid. The solution $x_3 = 1/2$ provides the solution

$$(x_3, y_3) = \left(\frac{1}{2}, y_3\right), \quad (x_4, y_4) = \left(\frac{1}{2}, -\frac{1}{4y_3}\right), \quad (x_5, y_5) = \left(\frac{1}{2}, \frac{1-4y_3^2}{8y_3}\right). \tag{18}$$

In this solution p_4 is on L' and L' is the bisector line of the segment with endpoints p_1 and p_2. This completes the proof.

We note that the first solution in (17) and the solution in (18) coincide when $y_5 = (1-4y_3^2)/(8y_3)$. □

4.4. **Three zeros at the vertices of an equilateral triangle.** Let p_1, p_2 and p_3 be at the vertices of an equilateral triangle. Without loss of generality we can assume that $y_3 > 0$, then the positions of the zeros are $p_1 = 0$, $p_2 = 1$, $p_3 = 1/2 + \sqrt{3}/2\, i$, $p_4 = x_4 + y_4\, i$, $p_5 = x_5 + y_5\, i$.

Since $y_3 \neq 0$ all the isochronous configurations of this type are given by Theorem 8. By substituting $(x_3, y_3) = (1/2, \sqrt{3}/2)$ into equation $\widetilde{g}_{72} = 0$ and solving the resulting equation we get the solutions

$$x_5 = 1/2, \quad y_5 = -\frac{x_5 - 1}{\sqrt{3}}, \quad y_5 = \frac{x_5}{\sqrt{3}}, \quad y_5 = \frac{1 \pm \sqrt{-12x_5^2 + 12x_5 + 1}}{2\sqrt{3}}.$$

The first three solutions provide the following valid solutions of (3) with $(x_4, y_4) = (x_{40}, y_{40})$

$$\begin{aligned}
&(x_4, y_4) = \left(\frac{1}{2}, \frac{2y_5 + \sqrt{3}}{4\sqrt{3}y_5 - 2}\right), && (x_5, y_5) = \left(\frac{1}{2}, y_5\right),\\
&(x_4, y_4) = \left(\frac{x_5}{2x_5 - 1}, \frac{\sqrt{3}(x_5 - 1)}{6x_5 - 3}\right), && (x_5, y_5) = \left(x_5, -\frac{x_5 - 1}{\sqrt{3}}\right)\\
&(x_4, y_4) = \left(\frac{x_5}{2x_5 - 1}, \frac{x_5}{\sqrt{3}(2x_5 - 1)}\right), && (x_5, y_5) = \left(x_5, \frac{x_5}{\sqrt{3}}\right).
\end{aligned} \tag{19}$$

The last solution provides solutions of (3) satisfying condition K_3, thus the solution $(x_4, y_4) = (x_{40}, y_{40})$ is not defined in this case. Finally, the solutions of (3) with $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ and $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$ is not defined when $(x_3, y_3) = (1/2, \sqrt{3}/2)$ because $d = 0$.

Analyzing the shape of the solutions of (19) we see that in the first solution of (19) the zeros p_4 and p_5 are on the median of the triangle that passes through the vertex p_3, in the second solution of (19) the zeros p_4 and p_5 are on the median passing through the vertex p_2, and in the third solution p_4 and p_5 are on the median passing through the vertex p_1. We note that when $y_5 = -\sqrt{3}/2$ the first solution of (19) is a a square of vertices p_1, p_2, p_3 and p_5 and p_4 at its center.

We have proved the following result.

Lemma 14. *Assume that the configuration $[p_1, \ldots, p_5]$ is isochronous. The following statements hold.*

(a) *If p_1, p_2 and p_3 are at the vertices of an equilateral triangle, then p_4 and p_5 are in one of the medians of the triangle and therefore p_1, p_2, p_3, p_4 and p_5 satisfy statements of Lemma 13.*

(b) *There exists three different one parameter families of configurations with three zeros at the vertices of a given equilateral triangle and two zeros* *in a median of the triangle, one for each median. If the vertices of the equilateral triangle are $p_1 = 0$, $p_2 = 1$ and $p_3 = 1/2 + i\sqrt{3}/2$, then these three one parameter families of configurations are given by $p_4 = x_4 + i\,y_4$, $p_5 = x_5 + i\,y_5$ with (x_4, y_4), (x_5, y_5) given in* (19).

We have plotted the phase portrait of the isochronous vector fields associated to the configurations given by Lemma 14(b) for many values of the variable that acts as parameter. After analyzing the obtained results we conjecture the following.

(i) If (x_4, y_4), (x_5, y_5) are given by the first solution in (19), then the phase portrait of the isochronous configuration has star topology when $y_5 < 1/(2\sqrt{3})$ and fork topology when $y_5 > 1/(2\sqrt{3})$.
(ii) If (x_4, y_4), (x_5, y_5) are given by the second solution in (19), then the phase portrait of the isochronous configuration has star topology when $x_5 < 1/2$ and fork topology when $x_5 > 1/2$.
(iii) If (x_4, y_4), (x_5, y_5) are given by the third solution in (19), then the phase portrait of the isochronous configuration has fork topology when $x_5 < 1/2$ and star topology when $x_5 > 1/2$.

4.5. **Three zeros at the vertices of an isosceles triangle.** Let p_1, p_2 and p_3 be at the vertices of an isosceles triangle. Without loss of generality we can assume that the positions of the zeros are $p_1 = 0$, $p_2 = 1$, $p_3 = 1/2 + y_3\,i$, $p_4 = x_4 + y_4\,i$, $p_5 = x_5 + y_5\,i$ with $y_3 > 0$.

Since $y_3 \neq 0$ all the isochronous configurations of this type are given by Theorem 8. From Theorem 8(d) the solutions of (3) with $(x_4, y_4) = (x_4^{(\star,1)}, y_4^{(\star,1)})$ and $(x_5, y_5) = (x_5^{(\star,1)}, y_5^{\star})$ become

$$(x_3, y_3) = \left(\frac{1}{2}, y_3\right), \quad (x_4, y_4) = \left(\frac{1}{2}, -\frac{1}{4y_3}\right), \quad (x_5, y_5) = \left(\frac{1}{2}, \frac{1-4y_3^2}{8y_3},\right). \tag{20}$$

From Theorem 8(c) we obtain the two parameter family of solutions

$$(x_3, y_3) = \left(\frac{1}{2}, y_3\right), \quad (x_4, y_4) = \left(\frac{1}{2}, \frac{y_3 + y_5}{4y_3y_5 - 1}\right), \quad (x_5, y_5) = \left(\frac{1}{2}, y_5\right). \tag{21}$$

This solution corresponds to the configuration given by Theorem 4(d). We note that the solutions (20) and (21) coincide when

$$y_5 = \frac{1-4y_3^2}{8y_3}.$$

From Theorem 8(b) we know the existence of up to six different two parameter families of solutions with $x_5 \neq 1/2$, $(x_4, y_4) = (x_{40}, y_{40})$ and $y_5 = y_5(y_3, x_5)$ satisfying $\widetilde{g}_{72} = 0$. The exact number of such families will depend on the values of y_3 and x_5. Unfortunately equation $\widetilde{g}_{72} = 0$ for the isosceles triangle configurations cannot be solved explicitly. We can solve it numerically by setting the values of the parameters y_3, x_5. Hence, we have proved the following result.

Lemma 15. *Assume that X is isochronous and that p_1, p_2 and p_3 are at the vertices of an isosceles triangle. Let $p_1 = 0$, $p_2 = 1$ and $p_3 = 1/2 + i\,y_3$ with $y_3 > 0$ be the vertices of the isosceles triangle. Then the following statements hold.*

(a) *There exist up to six different two parameter families of isochronous configurations with $x_5 \neq 1/2$, $(x_4, y_4) = (x_{40}, y_{40})$ and $y_5 = y_5(y_3, x_5)$ satisfying $\widetilde{g}_{72} = 0$.*

(b) *There exists the two parameter family given by* (21).

(c) *There exists the additional one parameter family given by* (20).

We note that configurations given in statements (b) and (c) coincide when

$$y_5 = \frac{1-4y_3^2}{8y_3}.$$

To give some examples of how the families given by Lemma 15 are, now we will analyze numerically the families of solutions of $\widetilde{g}_{72} = 0$ with $x_5 \neq 1/2$ for two particular values of y_3. We have chosen $y_3 = 1/25$ which provides a zero near $p_3 = 1/2$ (i.e. the configuration with p_3 at the midpoint of the line segment joining p_1 and p_2), and $y_3 = 9/10$ which provides a zero near $p_3 = \sqrt{3}/2$ (i.e. the configuration with p_1, p_2 and p_3 at the vertices of an equilateral triangle).

First, fixed the value of $y_3 = y_{30}$ we find the values of x_5 where the number of solutions $y_5 = y_5(y_{30}, x_5)$ can change by solving numerically the following system of polynomial equations

$$\widetilde{g}_{72} = 0, \qquad \frac{d\,\widetilde{g}_{72}}{d\,y_5} = 0. \tag{22}$$

Let $x_5 = \alpha_i$ for $i = 1, \dots, n$ denote the values of x_5 corresponding to the solutions of (22) and let $\alpha_0 = -\infty$ and $\alpha_{n+1} = \infty$. For all $i = 0, \dots, n$ we find the number of solutions of equation $\widetilde{g}_{72} = 0$ for a value $x_5 \in (\alpha_i, \alpha_{i+1})$ by solving numerically the polynomial equation $\widetilde{g}_{72} = 0$. We note that due to the symmetry of the configuration, it is sufficient to consider values of $x_5 < 1/2$. Finally, to understand how the families of solutions are related we compute numerically the solutions of $\widetilde{g}_{72} = 0$ for values of x_5 near the bifurcation values α_i for $i = 1, \dots, n$. Here we only give the solution at a value $x_5 \in (\alpha_i, \alpha_{i+1})$ for all $i = 0, \dots, n$ and at $x_5 = \alpha_i$ for all $i = 1, \dots, n$. These are the results that we have obtained.

Case: $y_3 = 1/25$

Equation $\widetilde{g}_{72} = 0$ with $y_3 = 1/25$ has the following families of solutions

(i) two one parameter families of solutions $y_5 = y_5(1/25, x_5)$ when $x_5 \in (\alpha_0, \alpha_1)$ with $\alpha_0 = -\infty$ and $\alpha_1 = -0.878799\dots$,

(ii) three solutions $y_5 = y_5(1/25, x_5)$ when $x_5 = \alpha_1$,

(iii) four one parameter families of solutions $y_5 = y_5(1/25, x_5)$ when $x_5 \in (\alpha_1, \alpha_2)$ with $\alpha_2 = -0.126632\dots$,

(iv) five solutions $y_5 = y_5(1/25, x_5)$ when $x_5 = \alpha_2$,

(v) six one parameter families of solutions $y_5 = y_5(1/25, x_5)$ when $x_5 \in (\alpha_2, \alpha_3)$ with $\alpha_3 = 0$,

(vi) four one parameter families of solutions $y_5 = y_5(1/25, x_5)$ when $x_5 = \alpha_3$,

(vii) six one parameter families of solutions $y_5 = y_5(1/25, x_5)$ when $x_5 \in (\alpha_3, \alpha_4)$ with $\alpha_4 = 0.0258965\dots$,

(viii) five solutions $y_5 = y_5(1/25, x_5)$ when $x_5 = \alpha_4$,

(ix) four solutions $y_5 = y_5(1/25, x_5)$ when $x_5 \in (\alpha_4, \alpha_5)$ with $\alpha_5 = 1/2$.

Now we give the isochronous configurations $p_1 = 0$, $p_2 = 1$, $p_3 = 1/2 + i\,1/25$, $p_4 = x_4 + i\,y_4$ and $p_5 = x_5 + i\,y_5$ at a given value x_5 in each interval (α_i, α_{i+1}) with $i = 0, \dots, 5$ and at $x_5 = \alpha_i$ with $i = 1, \dots, 5$. We will use the following notation: we denote the families of solutions by f_i for $i = 1, \dots, 6$, and the notation f_{ij} denotes a solution from which bifurcate the two families f_i and f_j.

(i) If $x_5 = -1$, then the two solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$\begin{aligned} f_1: \ (x_4, y_4) &= (1.841263521, 3.376637013), \quad (x_5, y_5) = (-1, -0.1950042462), \\ f_2: \ (x_4, y_4) &= (15.61743666, 13.23676750), \quad (x_5, y_5) = (-1, 7.575957509). \end{aligned}$$

The phase portrait associated to the isochronous vector field given by f_1 has the fork topology and the one given by f_2 has the star topology.

(ii) If $x_5 = \alpha_1$, then the there solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$\begin{aligned} f_1: \ (x_4, y_4) &= (1.848444615, 3.317980858), & (x_5, y_5) &= (\alpha_1, -0.1654776458), \\ f_{34}: \ (x_4, y_4) &= (1.011696918, -0.001515843709), & (x_5, y_5) &= (\alpha_1, 2.797454186), \\ f_2: \ (x_4, y_4) &= (16.58023312, 13.70216652), & (x_5, y_5) &= (\alpha_1, 7.491373514). \end{aligned}$$

The phase portraits associated to the isochronous vector fields given by f_1 and f_{34} have the fork topology and the one given by f_2 has the star topology.

(iii) If $x_5 = -1/2$, then the four solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$\begin{aligned} f_1: \ (x_4, y_4) &= (1.867414547, 3.115470806), & (x_5, y_5) &= (-1/2, -0.08264979077), \\ f_3: \ (x_4, y_4) &= (-0.1232221041, -0.05229100159), & (x_5, y_5) &= (-1/2, 1.064778767), \\ f_4: \ (x_4, y_4) &= (6.907958078, -2.048938409), & (x_5, y_5) &= (-1/2, 4.644413617), \\ f_2: \ (x_4, y_4) &= (20.94262385, 15.91785952), & (x_5, y_5) &= (-1/2, 7.206590362). \end{aligned}$$

Note that from the family f_{34} given in ii) bifurcate two families of solutions that we denote by f_3 and f_4.

The phase portraits associated to the isochronous vector fields given by f_1, f_3 and f_4 have the fork topology and the one given by f_2 has the star topology.

(iv) If $x_5 = \alpha_2$, then the five solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$\begin{aligned} f_{56}: \ (x_4, y_4) &= (-0.4495265553, 0.9614485424), & (x_5, y_5) &= (\alpha_2, -0.05979381213), \\ f_1: \ (x_4, y_4) &= (1.878034978, 2.879573610), & (x_5, y_5) &= (\alpha_2, -0.01757256381), \\ f_3: \ (x_4, y_4) &= (-0.03560001187, -0.03645028451), & (x_5, y_5) &= (\alpha_2, 0.4152272923), \\ f_4: \ (x_4, y_4) &= (16.27721421, -6.804796775), & (x_5, y_5) &= (\alpha_2, 5.385171289), \\ f_2: \ (x_4, y_4) &= (30.02386019, 20.82877636), & (x_5, y_5) &= (\alpha_2, 6.890094940). \end{aligned}$$

The phase portraits associated to the isochronous vector fields given by f_1, f_3, f_4 and f_{56} have the fork topology and the one given by f_2 has the star topology.

(v) If $x_5 = -1/20$, then the six solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$\begin{aligned}
f_5 :\ & (x_4, y_4) = (-0.1579167010, 0.4639398830), && (x_5, y_5) = (-1/20, -0.04620281861),\\
f_6 :\ & (x_4, y_4) = (-0.6577053363, 1.452342544), && (x_5, y_5) = (-1/20, -0.01438561542),\\
f_1 :\ & (x_4, y_4) = (1.878710630, 2.825402973), && (x_5, y_5) = (-1/20, -0.006635633004),\\
f_3 :\ & (x_4, y_4) = (0.001439481676, 0.002093770845), && (x_5, y_5) = (-1/20, 0.2910582405),\\
f_4 :\ & (x_4, y_4) = (19.64404887, -8.633711360), && (x_5, y_5) = (-1/20, 5.509564199),\\
f_2 :\ & (x_4, y_4) = (33.35628670, 22.67993219), && (x_5, y_5) = (-1/20, 6.819934961).
\end{aligned}$$

The phase portraits associated to the isochronous vector fields given by f_1, f_3, f_4, f_5 and f_6 have the fork topology and the one given by f_2 has the star topology.

(vi) If $x_5 = 0$, then the four solutions of $\widetilde{g}_{72} = 0$ provide the three isochronous configurations with

$$\begin{aligned}
f_3 :\ & (x_4, y_4) = (0.02289908105, 0.05342007123), && (x_5, y_5) = (0, 0.1930349677),\\
f_4 :\ & (x_4, y_4) = (22.37646822, -10.13881729), && (x_5, y_5) = (0, 5.587207040),\\
f_2 :\ & (x_4, y_4) = (36.06938270, 24.19789721), && (x_5, y_5) = (0, 6.773091325).
\end{aligned}$$

The fourth solution of $\widetilde{g}_{72} = 0$ is $(x_5, y_5) = (0, 0)$ and it does not provide an isochronous configuration. At this solution the three families of solutions $y_5 = y_5(1/25, x_5)$ of $\widetilde{g}_{72} = 0$ corresponding to f_5, f_6 and f_1 coincide.

The phase portraits associated to the isochronous vector fields given by f_3 and f_4 have the fork topology and the one given by f_2 has the star topology.

(vii) If $x_5 = 1/100$, then the six solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$\begin{aligned}
f_5 :\ & (x_4, y_4) = (-0.7105285631, 1.616886803), && (x_5, y_5) = (1/100, 0.002419518544),\\
f_6 :\ & (x_4, y_4) = (-0.03179981882, 0.2585482764), && (x_5, y_5) = (1/100, 0.01652281623),\\
f_1 :\ & (x_4, y_4) = (1.878769832, 2.781350016), && (x_5, y_5) = (1/100, 0.001277977173),\\
f_3 :\ & (x_4, y_4) = (0.02541034717, 0.07171752269), && (x_5, y_5) = (1/100, 0.1670678630),\\
f_4 :\ & (x_4, y_4) = (22.98795700, -10.47759276), && (x_5, y_5) = (1/100, 5.602426971),\\
f_2 :\ & (x_4, y_4) = (36.67728304, 24.53909014), && (x_5, y_5) = (1/100, 6.763618187).
\end{aligned}$$

The phase portraits associated to the isochronous vector fields given by f_1, f_3, f_4, f_5 and f_6 have the fork topology and the one given by f_2 has the star topology.

(viii) If $x_5 = \alpha_4$, then the five solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$\begin{aligned}
f_5 :\ & (x_4, y_4) = (-0.7213143913, 1.653877238), && (x_5, y_5) = (\alpha_4, 0.006016399475),\\
f_{36} :\ & (x_4, y_4) = (0.01527596548, 0.1507327875), && (x_5, y_5) = (\alpha_4, 0.08456876555),\\
f_1 :\ & (x_4, y_4) = (1.878709929, 2.769422027), && (x_5, y_5) = (\alpha_4, 0.003275093401),\\
f_4 :\ & (x_4, y_4) = (24.01178345, -11.04616649), && (x_5, y_5) = (\alpha_4, 5.626418095),\\
f_2 :\ & (x_4, y_4) = (37.69558803, 25.11140165), && (x_5, y_5) = (\alpha_4, 6.748486215).
\end{aligned}$$

The phase portraits associated to the isochronous vector fields given by f_1, f_{36}, f_4 and f_5 have the fork topology and the one given by f_2 has the star topology.

(ix) If $x_5 = 3/10$, then the four solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$
\begin{aligned}
f_5 &: \ (x_4, y_4) = (-0.8239444869, 2.112991297), && (x_5, y_5) = (3/10, 0.03808849818),\\
f_1 &: \ (x_4, y_4) = (1.871423616, 2.543910571), && (x_5, y_5) = (3/10, 0.03043761676),\\
f_4 &: \ (x_4, y_4) = (66.70297000, -35.33061762), && (x_5, y_5) = (3/10, 6.006030328),\\
f_2 &: \ (x_4, y_4) = (80.32320661, 49.44048347), && (x_5, y_5) = (3/10, 6.472332652).
\end{aligned}
$$

The phase portraits associated to the isochronous vector fields given by f_1, f_4 and f_5 have the fork topology and the one given by f_2 has the star topology.

We note that when $x_5 \to 1/2$ the solution families f_2 and f_4 tend to a solution of $\widetilde{g}_{72} = 0$ with $(x_5, y_5) = (1/2, 25/4)$. This solution does not provide a solution of (3) because $D_1^2 + D_2^2 = 0$ and therefore (x_{40}, y_{40}) is not defined. Moreover $(x_4, y_4) \to (+\infty, +\infty)$ along the solutions families f_2 and f_4 when $x_5 \to 1/2$. On the other hand, the solution families f_1 and f_5 tend to a solution of $\widetilde{g}_{72} = 0$ with $(x_5, y_5) = (1/2, 1/25)$. This solution does not provide a valid solutions because $p_3 = p_5$. In this case $(x_4, y_4) \to (1.855487879, 2.348621176)$ and $(x_4, y_4) \to (-0.8554878791, 2.348621176)$ along the solutions families f_1 and f_5 respectively when $x_5 \to 1/2$.

Case: $y_3 = 9/10$
Equation $\widetilde{g}_{72} = 0$ with $y_3 = 9/10$ has the following families of solutions

(i) two one parameter families of solutions $y_5 = y_5(9/10, x_5)$ when $x_5 \in (\alpha_0, \alpha_1)$ with $\alpha_0 = -\infty$ and $\alpha_1 = -0.0137965\ldots$,
(ii) three solutions $y_5 = y_5(9/10, x_5)$ when $x_5 = \alpha_1$,
(iii) four one parameter families of solutions $y_5 = y_5(9/10, x_5)$ when $x_5 \in (\alpha_1, \alpha_2)$ with $\alpha_2 = 0$,
(iv) two solutions $y_5 = y_5(9/10, x_5)$ when $x_5 = \alpha_2$,
(v) four one parameter families of solutions $y_5 = y_5(9/10, x_5)$ when $x_5 \in (\alpha_2, \alpha_3)$ with $\alpha_3 = 1/2$.

Now we give the isochronous configurations $p_1 = 0$, $p_2 = 1$, $p_3 = 1/2 + i\,9/10$, $p_4 = x_4 + i\,y_4$ and $p_5 = x_5 + i\,y_5$ at a given value x_5 in each interval (α_i, α_{i+1}) with $i = 0, \ldots, 3$ and at $x_5 = \alpha_i$ with $i = 1, 2, 3$. As above f_i with $i = 1, \ldots, 4$ denote the families of solutions, and f_{ij} denotes a solution from which bifurcate the two families f_i and f_j.

(i) If $x_5 = -1/2$, then two solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$
\begin{aligned}
f_1 &: \ (x_4, y_4) = (0.2273153488, 0.1290590866), && (x_5, y_5) = (-1/2, -0.2873400074),\\
f_2 &: \ (x_4, y_4) = (0.2050985302, 0.3874083254), && (x_5, y_5) = (-1/2, 0.8291856821).
\end{aligned}
$$

The phase portrait associated to the isochronous vector field given by f_1 has the fork topology and the one given by f_2 has the star topology.

(ii) If $x_5 = \alpha_1$, then there solutions of $\widetilde{g}_{72} = 0$ provide the isochronous configurations with

$$\begin{aligned} f_1: \ (x_4, y_4) &= (-0.03026264711, -0.01730520403), \quad (x_5, y_5) = (\alpha_1, -0.007886903324), \\ f_2: \ (x_4, y_4) &= (0.06989971864, -0.1553809110), \qquad (x_5, y_5) = (\alpha_1, 0.01504460688), \\ f_{34}: \ (x_4, y_4) &= (-0.1760156610, 0.2985515049), \qquad (x_5, y_5) = (\alpha_1, 0.1098723790). \end{aligned}$$

The phase portrait associated to the isochronous vector field given by f_1 has the fork topology and the ones given by f_2 and f_{34} have the star topology.

(iii) If $x_5 = -1/100$, then the four isochronous configurations are given by

$$\begin{aligned} f_1: \ (x_4, y_4) &= (-0.03421033191, -0.01956401179), \quad (x_5, y_5) = (-1/100, -0.005716181684), \\ f_2: \ (x_4, y_4) &= (0.07267869959, -0.1594371587), \qquad (x_5, y_5) = (-1/100, 0.01081966297), \\ f_3: \ (x_4, y_4) &= (-0.1515620634, 0.2353327280), \qquad (x_5, y_5) = (-1/100, 0.05238535873), \\ f_4: \ (x_4, y_4) &= (-0.1965428001, 0.3621824381), \qquad (x_5, y_5) = (-1/100, 0.1660213488). \end{aligned}$$

Note that from the family f_{34} given in ii) bifurcate two families of solutions that we denote by f_3 and f_4.

The phase portrait associated to the isochronous vector field given by f_1 has the fork topology and the ones given by f_2, f_3 and f_4 have the star topology.

(iv) If $x_5 = 0$, then the two solutions of $\widetilde{g}_{72} = 0$ provide a unique isochronous configuration with

$$f_4: \ (x_4, y_4) = (-0.2129653240, 0.4198478686), \quad (x_5, y_5) = (0, 0.2147580631).$$

The other solution of $\widetilde{g}_{72} = 0$ is $(x_5, y_5) = (0, 0)$ and it does not provide isochronous configurations. At this solution the three families of solutions $y_5 = y_5(9/10, x_5)$ of $\widetilde{g}_{72} = 0$ corresponding to $f1$, $f2$ and $f3$ coincide.

The phase portrait associated to the isochronous vector field given by f_4 has the star topology.

(v) If $x_5 = 3/10$, then the four solutions of $\widetilde{g}_{72} = 0$ provide a unique isochronous configuration with

$$\begin{aligned} f_3: \ (x_4, y_4) &= (0.2692037018, -0.1777437971), \qquad (x_5, y_5) = (3/10, -0.3420260454), \\ f_2: \ (x_4, y_4) &= (0.3264300730, -0.3521082462), \qquad (x_5, y_5) = (3/10, -0.1880843834), \\ f_1: \ (x_4, y_4) &= (-0.8646324249, -0.4976775204), \quad (x_5, y_5) = (3/10, 0.1697279313), \\ f_4: \ (x_4, y_4) &= (-0.9156482786, 1.092880633), \qquad (x_5, y_5) = (3/10, 0.3673099827). \end{aligned}$$

The phase portrait associated to the isochronous vector field given by f_1 has the fork topology and the ones given by f_2, f_3 and f_4 have the star topology.

We note that when $x_5 \to 1/2$ the solution families f_1 and f_4 tend to a solution of $\widetilde{g}_{72} = 0$ with $(x_5, y_5) = (1/2, 5/18)$. This solution does not provide a solution of (3) because $D_1^2 + D_2^2 = 0$ and therefore (x_{40}, y_{40}) is not defined. Moreover $(x_4, y_4) \to (0, 0)$ along the solutions families f_1 and f_4 when $x_5 \to 1/2$. The solution families f_2 and f_3 tend to the following solutions, which provide isochronous configurations,

$$\begin{aligned} f_3: \ (x_4, y_4) &= (1/2, -0.2183503419), \qquad (x_5, y_5) = (1/2, -0.3816496581), \\ f_2: \ (x_4, y_4) &= (1/2, -0.3816496581), \qquad (x_5, y_5) = (1/2, -0.2183503419). \end{aligned}$$

The phase portraits associated to the isochronous vector fields given by f_2, f_3 have the star topology.

4.6. **More examples.** In the previous sections we have studied particular cases where the configuration of some of the zeros is symmetric. By proceeding as in Section 4.5 we could analyze the isochronous configurations when p_1, p_2 and p_3 are in an arbitrary triangle obtaining in this way isochronous configurations having no symmetries. For example if $x_3 = -8/10$, $y_3 = 1$ and $x_5 = -2/10$ then we obtain five isochronous configurations that are given by

$$
\begin{aligned}
(x_4, y_4) &= (-4.312509774, -1.551253710) & (x_5, y_5) &= (-2/10, -2.856970625),\\
(x_4, y_4) &= (-1.164499645, 1.452969805) & (x_5, y_5) &= (-2/10, -0.9524134598),\\
(x_4, y_4) &= (1.199397231, 2.531219547) & (x_5, y_5) &= (-2/10, -0.3220182178),\\
(x_4, y_4) &= (-0.2605712264, -0.06904887457) & (x_5, y_5) &= (-2/10, -0.04520614246),\\
(x_4, y_4) &= (0.03635581518, -0.5567650011) & (x_5, y_5) &= (-2/10, 0.2820385942).
\end{aligned}
$$

In Figure 2 we show the phase portraits of the isochronous vector fields associated to these configurations with their corresponding 5–trees. We observe that the phase portrait of the first and the third configurations have the star topology while the other three have the fork topology. Note that in the second configuration three zeros are in the bisector line of the other two, this does not happen in the fourth and the fifth.

For the moment we have not been able to find examples of isochronous vector fields having a phase portrait with line topology whose zeros are not aligned. We conjecture that there are no isochronous vector fields having having a phase portrait with line topology whose zeros are not aligned.

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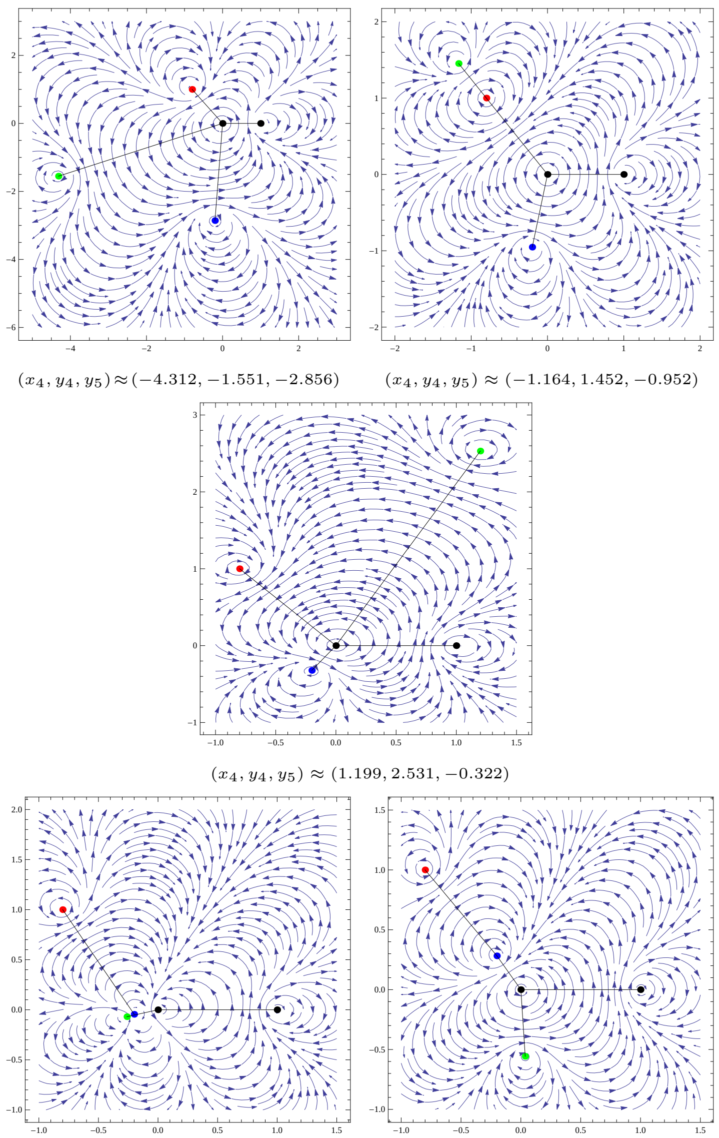

$(x_4, y_4, y_5) \approx (-4.312, -1.551, -2.856)$ $(x_4, y_4, y_5) \approx (-1.164, 1.452, -0.952)$

$(x_4, y_4, y_5) \approx (1.199, 2.531, -0.322)$

$(x_4, y_4, y_5) \approx (-0.260, -0.069, -0.045)$ $(x_4, y_4, y_5) \approx (0.036, -0.556, 0.282)$

FIGURE 2. The isochronous configurations with $(x_3, y_3, x_5) = (-8/10, 1, -2/10)$.