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LIMIT CYCLES OF PLANAR DISCONTINUOUS PIECEWISE LINEAR HAMILTONIAN SYSTEMS WITHOUT EQUILIBRIA SEPARATED BY REDUCIBLE CUBICS

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Abstract. Due to their applications to many physical phenomena during these last decades the interest for studying the discontinuous piecewise differential systems has increased strongly. The limit cycles play a main role in the study of any planar differential system, but to determine the maximum number of limits cycles that a class of planar differential systems can have is one of the main problems in the qualitative theory of the planar differential systems. Thus in general to provide a sharp upper bound for the number of crossing limit cycles that a given class of piecewise linear differential system can have is a very difficult problem. In this paper we characterize the existence and the number of limit cycles for the piecewise linear differential systems formed by linear Hamiltonian systems without equilibria and separated by a reducible cubic curve, formed either by an ellipse and a straight line, or by a parabola and a straight line parallel to the tangent at the vertex of the parabola. Hence we have solved the extended 16th Hilbert problem to this class of piecewise differential systems.

1. Introduction and statement of the main results

Andronov, Vitt and Khaikin [1] started around 1920's the study of the piecewise differential systems mainly motivated for their applications to some mechanical systems, and nowadays these systems still continue to receive the attention of many researchers. Thus these differential systems are widely used to model processes appearing in mechanics, electronics, economy, etc., see for instance the books [6] and [28], and the survey [25], as well as the hundreds of references cited there.

A *limit cycle* is a periodic orbit of the differential system isolated in the set of all periodic orbits of the system. Limit cycles are important in the study of the differential systems. Thus limit cycles have played and are playing a main role for explaining physical phenomena, see for instance the limit cycle of van der Pol equation [26, 27], or the one of the Belousov-Zhavotinskii model [2, 29], etc.

The *extended 16th Hilbert problem*, that is, to find an upper bound for the maximum number of limit cycles that a given class of differential systems can exhibit, is in general an unsolved problem. Only for very few classes of differential system this problem has been solved. For the class of discontinuous piecewise differential systems here studied, we can obtain its solution by using the first integrals provided by the Hamiltonians of the systems which form the discontinuous piecewise differential systems. For the statement of the classical 16th Hilbert problem see [14, 16, 20].

Of course in order that a discontinuous piecewise differential system be defined on the discontinuous line, which separates the different differential systems forming the discontinuous piecewise differential system, we follow the rules of Filippov, see [9].

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The discontinuous piecewise differential systems formed by linear differential systems can exhibit two kinds of limit cycles, the *crossing* and the *sliding limit cycles*, the first are the ones which only contain isolated points of the line of discontinuity, and the second the ones which contains arcs of the line of discontinuity. Here we only study the crossing limit cycles.

The simplest class of discontinuous piecewise differential systems are the planar ones formed by two pieces separated by a straight line having a linear differential system in each piece. Several authors have tried to determine the maximum number of crossing limit cycles for this class of discontinuous piecewise differential systems. Thus, in one of the first papers dedicated to this problem, Giannakopoulos and Pliete [12] in 2001, showed the existence of discontinuous piecewise linear differential systems with two crossing limit cycles. Then, in 2010 Han and Zhang [13] found other discontinuous piecewise linear differential systems with two crossing limit cycles and they conjectured that the maximum number of crossing limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line is two. But in 2012 Huan and Yang [15] provided numerical evidence of the existence of three crossing limit cycles in this class of discontinuous piecewise linear differential systems. In 2012, Llibre and Ponce [22] inspired by the numerical example of Huan and Yang, proved for the first time that there are discontinuous piecewise linear differential systems with two pieces separated by a straight line having three crossing limit cycles. Later on, other authors obtained also three crossing limit cycles for discontinuous piecewise linear differential systems with two pieces separated by a straight line, see Braga and Mello [7] in 2013, Buzzi, Pessoa and Torregrosa [8] in 2013, Liping Li [20] in 2014, Freire, Ponce and Torres [11] in 2014, and Llibre, Novaes and Teixeira [21] in 2015. But proving that discontinuous piecewise linear differential systems separated by a straight line have at most three crossing limit cycles is an open problem. Also some discontinuous piecewise differential systems separated by two parallel straight lines have been studied, see for instance [23, 24].

Recently, in [4, 17, 18] the authors have studied the extended 16th Hilbert problem to discontinuous piecewise linear differential centers separated by either conics, or cubics. However for the discontinuous piecewise linear Hamiltonian systems without equilibrium points, it was proven in [10] that such systems separated by two parallel straight lines can have at most one crossing limit cycle. In [5] it was proven that there is an example of two crossing limit cycles when these systems are separated by three parallel straight lines, and they can also have two crossing limit cycles if the curve of separation is a parabola, and three crossing limit cycles if the curve of separation is either an ellipse or a hyperbola. In [3] the authors provided the maximum number of crossing limit cycles when the curve of separation of these systems is an irreducible cubic.

In this paper we give the solution of the extended 16th Hilbert problem for discontinuous piecewise linear differential Hamiltonian systems without equilibrium points separated by two different reducible cubic curves, formed either by an ellipse and a straight line, or by a parabola and a straight line parallel to the tangent at the vertex of the parabola. More precisely, we provide the maximum number of crossing limit cycles for these systems, when these limit cycles intersected with the cubic of separation in four points.

Note that if a crossing limit cycle of a discontinuous piecewise linear differential Hamiltonian systems without equilibrium points intersects in two points the discontinuity line formed either by an ellipse and a straight line, or by a parabola and a straight line parallel to the tangent at the vertex of the parabola, this crossing limit cycle must intersect in two points either the straight line, or the ellipse or the parabola, and these types of crossing limit cycles already have been studied in [10, 5], as we have mention previously. For this reason in this paper we study the crossing limit cycles with intersect in four points the reducible cubic formed by either by an ellipse and a straight line, or by a parabola and a straight line parallel to the tangent at the vertex of the parabola.

Doing an affine change if the reducible cubic is formed by an ellipse and a straight line we can transform it into the reducible cubic

$$\Gamma_k = \{(x,y) \in \mathbb{R}^2 : (x-k)(x^2+y^2-1) = 0,\ k \geq 0\},$$

formed by the circle $x^2+y^2=1$ and the straight line $x=k$ with $k \geq 0$. In a similar way if the reducible cubic is formed by a parabola and a straight line parallel to the tangent at the vertex of the parabola we can transform it into the reducible cubic

$$\Sigma_k = \left\{(x,y) \in \mathbb{R}^2 : (y-k)(y-x^2) = 0,\ k \in \mathbb{R}\right\},$$

formed by the parabola $y=x^2$ and the straight line $y=k$ with $k \in \mathbb{R}$

First in Subsection 1.1 we shall consider the piecewise linear Hamiltonian systems without equilibrium points separated by the reducible cubic Γ_k, and after in Subsection 1.2 we shall consider the piecewise linear Hamiltonian systems without equilibrium points separated by the reducible cubic Σ_k.

The next result is proved in [10].

Lemma 1. *An arbitrary linear differential Hamiltonian system in $\mathbb{R}^2$ without equilibrium points can be written as*

$$\dot{x} = -\lambda b x + b y + \mu, \qquad \dot{y} = -\lambda^2 b x + \lambda b y + \sigma,$$

where $\sigma \neq \lambda\mu$ and $b \neq 0$. The Hamiltonian function of this Hamiltonian system is

$$H(x,y) = -\frac{1}{2}\lambda^2 b x^2 + \lambda b x y - \frac{b}{2} y^2 + \sigma x - \mu y. \tag{1}$$

Of course $H(x,y)$ is a first integral of the Hamiltonian system.

1.1. **The line of discontinuity is a circle and a straight line.** We denote by $\mathcal{C}_1$ the class of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by Γ_k with $k>1$. Let $\mathcal{C}_2$ be the class of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by Γ_k with $k=1$. For these two classes we get the following three zones

$$\begin{aligned} Z^1 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1\},\\ Z^2 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2>1 \text{ and } x<k\},\\ Z^3 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2>1 \text{ and } x>k\}. \end{aligned} \tag{2}$$

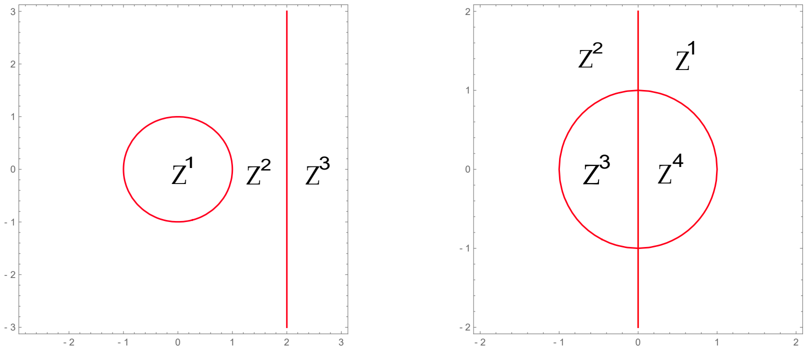
Now we denote by $\mathcal{C}_3$ the class of piecewise linear Hamiltonian systems without equilibrium points separated by Γ_k with $0 \leq k < 1$. In this case Γ_k separate the plane into four zones

$$\begin{aligned} Z^1 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2>1 \text{ and } x>k\},\\ Z^2 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2>1 \text{ and } x<k\},\\ Z^3 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1 \text{ and } x<k\},\\ Z^4 &= \{(x,y) \in \mathbb{R}^2 : x^2+y^2<1 \text{ and } x>k\}. \end{aligned} \tag{3}$$

We have three different configurations of crossing limit cycles for the class $\mathcal{C}_3$. The first one which will be denoted by **Conf 1** are the limit cycles formed by four pieces of orbits, such that in each zone of (3) we have one piece of orbit of the linear Hamiltonian system without equilibria leaving in that zone, see Figure 4a.

The second configuration of limit cycles denoted by **Conf 2** are the limit cycles formed by pieces of orbits belonging to the three zones either Z^1, Z^2 and Z^4, or Z^1, Z^2 and Z^3. We are going to consider only the three zones Z^1, Z^2 and Z^4, because by a similar analysis we obtain the crossing limit cycles intersecting the three zones Z^1, Z^2 and Z^3, for this configuration see Figure 4b.

Finally the third configuration namely **Conf 3** is formed by pieces of orbits belonging to the three zones either Z^1, Z^3 and Z^4, or Z^2, Z^3 and Z^4. For the same reason as in



(A) (B)

FIGURE 1. (A) The three zones for the class $\mathcal{C}_1$. (B) The four zones for the class $\mathcal{C}_3$.

the second configuration, we are going to consider only the three zones Z^1, Z^3 and Z^4, see Figure 4C.

We notice that we can obtain two new configurations by combining the three previous ones, such as **Conf 1** and **Conf** 2, **Conf** 1 and **Conf** 3. Note that we cannot have the configuration **Conf** 2 and **Conf** 3, and **Conf** 1, **Conf** 2 and **Conf** 3.

Our main result on the crossing limit cycles of the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity line is formed by a circle and a straight line is the following one.

Theorem 1. *The following statements hold for the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity line is formed by a circle and a straight line. The maximum number of crossing limit cycles for the class*

(i) $\mathcal{C}_1$ *or* $\mathcal{C}_2$ *is three and this maximum is reached in Example* 1 *for the class* $\mathcal{C}_1$ *and in Example* 2 *for the class* $\mathcal{C}_2$*, see Figure* 3A *and Figure* 3B*, respectively;*
(ii) $\mathcal{C}_3$ *with* ***Conf*** 1 *is three and this maximum is reached in Example* 3*, see Figure* 4A*;*
(iii) $\mathcal{C}_3$ *with* ***Conf*** 2 *is three and this maximum is reached in Example* 4*, see Figure* 4B*;*
(iv) $\mathcal{C}_3$ *with* ***Conf*** 3 *is three and this maximum is reached in Example* 5*, see Figure* 4C*;*
(v) $\mathcal{C}_3$ *with* ***Conf*** 1 *and* ***Conf*** 2 *simultaneously is six and this maximum is reached in Example* 6*, see Figure* 5*;*
(vi) $\mathcal{C}_3$ *with* ***Conf*** 1 *and* ***Conf*** 3 *simultaneously is six and this maximum is reached in Example* 7*, see Figure* 6.

Theorem 1 is proved in section 2.

1.2. **The line of discontinuity is a parabola and a straight line parallel to the tangent at the vertex of the parabola.** Let $\mathcal{C}_{\Sigma_{k^-}}$ be the class of discontinuous piecewise linear Hamiltonian systems without equilibria separated by Σ_{k^-} with $k < 0$. In this case we have following three zones in the plane

$$
\begin{aligned}
Z^1_{\Sigma_{k^-}} &= \{(x,y) \in \mathbb{R}^2 : y > x^2\},\\
Z^2_{\Sigma_{k^-}} &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y > k\},\\
Z^3_{\Sigma_{k^-}} &= \{(x,y) \in \mathbb{R}^2 : y < x^2 \text{ and } y < k\}.
\end{aligned}
$$

Let $\mathcal{C}_{\Sigma_0}$ be the class of discontinuous piecewise linear Hamiltonian systems without equilibria separated by Σ_k with $k = 0$. When the discontinuity curve is Σ_0 we have following four

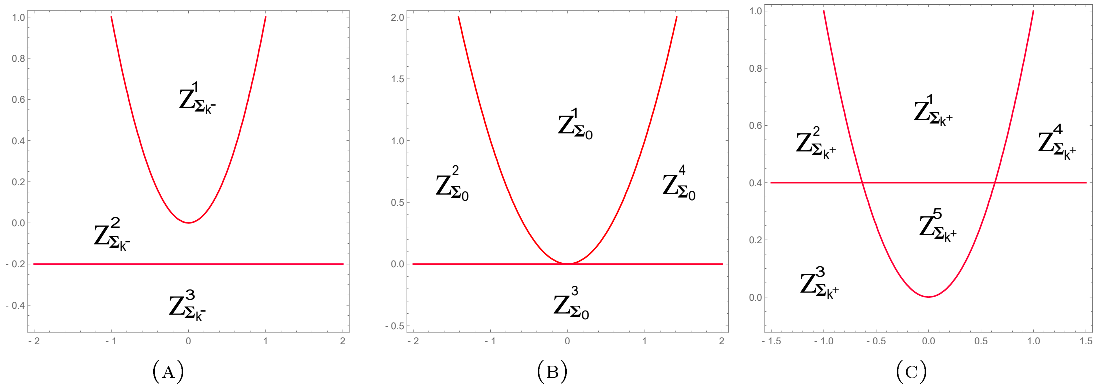


FIGURE 2. (A) Three zones for the class $\mathcal{C}_{k^-}$. (B) Four zones for the class $\mathcal{C}_0$. (C) Five zones for the class $\mathcal{C}_{k^+}$.

zones in the plane

$$\begin{aligned}
Z^1_{\Sigma_0} =&\{(x,y)\in\mathbb{R}^2 : y > x^2 \},\\
Z^2_{\Sigma_0} =&\{(x,y)\in\mathbb{R}^2 : y < x^2 \text{ and } y>0,\ x<0\},\\
Z^3_{\Sigma_0} =&\{(x,y)\in\mathbb{R}^2 : y < x^2 \text{ and } y<0\},\\
Z^4_{\Sigma_0} =&\{(x,y)\in\mathbb{R}^2 : y < x^2 \text{ and } y>0,\ x>0\}.
\end{aligned}$$

In this class we have two configurations of crossing limit cycles, first crossing limit cycles with **Conf 4** which are constituted by pieces of orbits of the four Hamiltonian systems considered, see Figure 8A. Second crossing limit cycles with **Conf 5** which intersect only three zones, in this case we have two options, first we have the case where the crossing limit cycles are formed by parts of orbits of the Hamiltonian systems in the zones $Z^1_{\Sigma_0}, Z^3_{\Sigma_0}$ and $Z^4_{\Sigma_0}$ and second the crossing limit cycles that intersect only the three zones $Z^1_{\Sigma_0}$, $Z^2_{\Sigma_0}$ and $Z^3_{\Sigma_0}$, without loss of generality we can consider the first case because the study of the second is the same, see Figure 8B. Here we observe that it is not possible to have crossing limit cycles with **Conf 5** that satisfy those two cases simultaneously, because the orbits of the Hamiltonian system in the zone $Z^3_{\Sigma_0}$ would not be nested. In statement (ii) of Theorem 2 we study the discontinuous piecewise linear Hamiltonian systems without equilibria in $\mathcal{C}_{\Sigma_0}$ which have crossing limit cycles with **Conf 4** and **Conf 5** separately, and in statement (iii) of Theorem 2 we study the case when the crossing limit cycles with **Conf 4** and **Conf 5** appear simultaneously.

Let $\mathcal{C}_{\Sigma_{k+}}$ be the class of discontinuous piecewise linear Hamiltonian systems without equilibria separated by Σ_k with $k>0$, in this case we have the following five zones in the plane

$$\begin{aligned}
Z^1_{\Sigma_{k+}} =&\{(x,y)\in\mathbb{R}^2 : y > x^2 \text{ and } y>k\},\\
Z^2_{\Sigma_{k+}} =&\{(x,y)\in\mathbb{R}^2 : y < x^2 \text{ and } y>k,\ \ x<-\sqrt{k}\},\\
Z^3_{\Sigma_{k+}} =&\{(x,y)\in\mathbb{R}^2 : y > x^2 \text{ and } y<k\},\\
Z^4_{\Sigma_{k+}} =&\{(x,y)\in\mathbb{R}^2 : y < x^2 \text{ and } y>k,\ x>\sqrt{k}\},\\
Z^5_{\Sigma_{k+}} =&\{(x,y)\in\mathbb{R}^2 : x^2<y<k\}.
\end{aligned}$$

In this class we have six different configurations of crossing limit cycles.

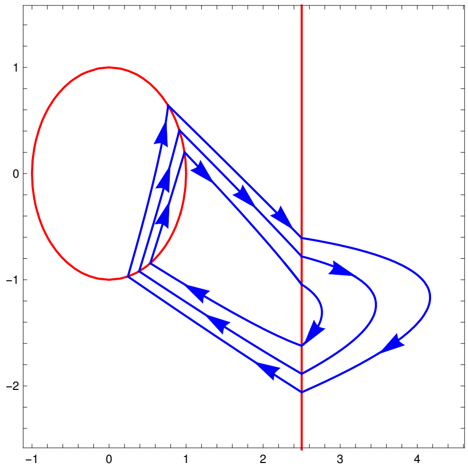
First we have crossing limit cycles such that are formed by pieces of orbits of the four Hamiltonian systems in the zones $Z^1_{\Sigma_{k+}}$, $Z^5_{\Sigma_{k+}}$, $Z^3_{\Sigma_{k+}}$ and $Z^4_{\Sigma_{k+}}$, or crossing limit cycles formed by pieces of orbits of the four Hamiltonian systems in the zones $Z^1_{\Sigma_{k+}}$, $Z^2_{\Sigma_{k+}}$, $Z^3_{\Sigma_{k+}}$ and $Z^5_{\Sigma_{k+}}$, namely crossing limit cycles with **Conf** 6^+ and crossing limit cycles with **Conf** 6^-, respectively, see Figure 11. In statement (ii) of Theorem 2 we study the crossing limit cycles with **Conf** 6^+ because the study for the case of crossing limit cycles with **Conf** 6^- is the same. Second we have crossing limit cycles with **Conf 7**, which intersect the three zones $Z^1_{\Sigma_{k+}}$, $Z^5_{\Sigma_{k+}}$ and $Z^3_{\Sigma_{k+}}$, see Figure 9B. Third we have the crossing limit cycles with **Conf 8**, which intersect the zones $Z^1_{\Sigma_{k+}}$, $Z^2_{\Sigma_{k+}}$, $Z^3_{\Sigma_{k+}}$ and $Z^4_{\Sigma_{k+}}$, see Figure 9C. And finally we have the crossing limit cycles formed by pieces of orbits of the three Hamiltonian systems in the zones $Z^1_{\Sigma_{k+}}$, $Z^3_{\Sigma_{k+}}$ and $Z^4_{\Sigma_{k+}}$, or crossing limit cycles formed by pieces of orbits of the three Hamiltonian systems in the zones $Z^1_{\Sigma_{k+}}$, $Z^2_{\Sigma_{k+}}$ and $Z^3_{\Sigma_{k+}}$, namely crossing limit cycles with **Conf** 9^+ and crossing limit cycles with **Conf** 9^-, respectively, see Figure 9D. Without loss of generality in statement (ii) of Theorem 2 we study the crossing limit cycles with **Conf** 9^+ because the study by the crossing limit cycles with **Conf** 9^- is the same. We observe that there are no crossing limit cycles that intersect the five zones $Z^i_{\Sigma_{k+}}$ for $i = 1, 2, 3, 4, 5$. Then in statement (ii) of Theorem 2 we study the crossing limit cycles with **Conf** 6^+, **Conf 7**, **Conf 8** and **Conf** 9^+ separately. In statements (iii)-(ix) of Theorem 2 we study the discontinuous piecewise linear Hamiltonian systems without equilibria in the class $\mathcal{C}_{\Sigma_{k+}}$ which have crossing limit cycles with two configurations simultaneously. Finally in statements (x)-(xii) we study the discontinuous piecewise linear Hamiltonian systems without equilibria in the class $\mathcal{C}_{\Sigma_{k+}}$ which have crossing limit cycles with three different configurations simultaneously.

Our main result on the crossing limit cycles of the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity curve is formed by a parabola and a straight line parallel to the tangent at the vertex of the parabola is the following one.

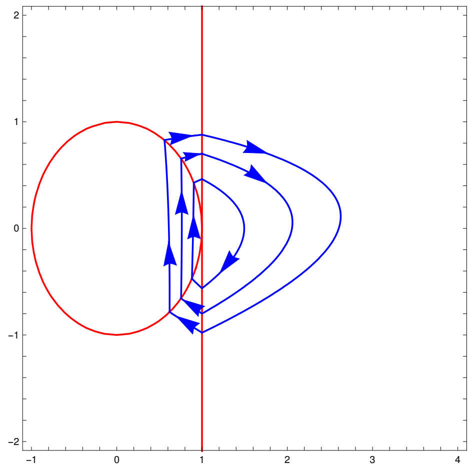
Theorem 2. *The following statements hold for the discontinuous piecewise linear Hamiltonian systems without equilibria when the discontinuity line is formed by a parabola and a straight line parallel to the tangent at the vertex of the parabola. The maximum number of crossing limit cycles for the class*

(i) $\mathcal{C}_{\Sigma_{k-}}$ *is three and this maximum is reached, see Figure* 7*;*
(ii) $\mathcal{C}_{\Sigma_0}$ *or* $\mathcal{C}_{\Sigma_{k+}}$ *with either* **Conf 4**, *or* **Conf 5**, *or* **Conf** 6^+, *or* **Conf 7**, *or* **Conf 8**, *or* **Conf** 9^+ *is three, respectively, see Figure* 8A*-Figure* 9D*;*
(iii) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf 4** *and* **Conf 5** *simultaneously is six, see Figure* 10*;*
(iv) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf** 6^+ *and* **Conf** 6^- *simultaneously is six, see Figure* 11*;*
(v) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf** 6^- *and* **Conf 7** *simultaneously is six, see Figure* 12A*;*
(vi) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf** 6^+ *and* **Conf 8** *simultaneously is six, see Figure* 12B*;*
(vii) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf** 6^+ *and* **Conf** 9^+ *simultaneously is six, see Figure* 13*;*
(viii) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf 7** *and* **Conf 8** *simultaneously is six, see Figure* 14*;*
(ix) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf 8** *and* **Conf** 9^+ *simultaneously is six, see Figure* 15*;*
(x) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf** 6^-, **Conf 7** *and* **Conf 8** *simultaneously is nine, see Figure* 16*;*
(xi) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf** 6^+, **Conf 8** *and* **Conf** 9^+ *simultaneously is nine, see Figure* 17*;*
(xii) $\mathcal{C}_{\Sigma_{k+}}$ *with* **Conf** 6^-, **Conf** 6^+ *and* **Conf** 8 *simultaneously is six with* 2 (*resp.* 3) *limit cycles with* **Conf** 6^-, 3 (*resp.* 2) *limit cycles with* **Conf** 6^+ *and* 1 *limit cycle with* **Conf** 8, *Figure* 18 (*resp.* 19).

Theorem 2 is proved in section 3.



(A)



(B)

FIGURE 3. (A) The three limit cycles of the discontinuous piecewise differential system (5). (B) The three limit cycles of the discontinuous piecewise differential system (6).

2. Proof of Theorem 1

Proof of statement (i) of Theorem 1. We have to prove that the maximum number of crossing limit cycles of the class $\mathcal{C}_1$ intersecting the curve Γ_k in four points is three. In a similar way we should prove the statement for the classes $\mathcal{C}_2$ and $\mathcal{C}_3$.

By Lemma 1 we can consider the discontinuous piecewise linear Hamiltonian systems

$$\dot{x} = -\lambda_i b_i x + b_i y + \mu_i, \quad \dot{y} = -\lambda_i^2 b_i x + \lambda_i b_i y + \sigma_i, \text{ in the zone } Z_i, \;\; with \;\; i = 1,2,3.$$

with $b_i \neq 0$ and $\sigma_i \neq \lambda_i \mu_i$, and the three zones Z_i are defined in (2). Their corresponding Hamiltonian first integrals are as (1)

$$H_i(x,y) = -(\lambda_i^2 b_i/2)x^2 + \lambda_i b_i xy - (b_i/2)y^2 + \sigma_i x - \mu_i y, \;\; with \;\; i = 1,2,3.$$

In order to have a crossing limit cycle which intersects Γ_k in the points $A_i = (x_i, y_i)$, $B_i = (z_i, w_i)$, $C_i = (k, f_i)$ and $D_i = (k, h_i)$, where A_i and B_i are points on the circle $x^2 + y^2 - 1 = 0$ and $i = 1,2,3$, these points must satisfy the following system

$$\begin{aligned}
&e_1 = H_1(x_i, y_i) - H_1(z_i, w_i) = 0,\\
&e_2 = H_2(x_i, y_i) - H_2(k, f_i) = 0,\\
&e_3 = H_2(z_i, w_i) - H_2(k, h_i) = 0,\\
&e_4 = H_3(k, f_i) - H_3(k, h_i) = 0,\\
&x_i^2 + y_i^2 - 1 = 0,\\
&z_i^2 + w_i^2 - 1 = 0.
\end{aligned} \tag{4}$$

The points A_i and B_i can take the form $A_i = (\cos r_i, \sin r_i)$, $B_i = (\cos s_i, \sin s_i)$, and due to the fact that they satisfy equation e_1, then by solving $e_1 = 0$ when $i = 1$ for the parameter σ_1, we get

$$\sigma_1 = \frac{1}{2(\cos r_1 - \cos s_1)}\Big(b_1 \sin(r_1 - s_1)\Big(-\Big(\lambda_1^2 - 1\Big)\sin(r_1 + s_1) - 2\lambda_1 \cos(r_1 + s_1)\Big) + 2\mu_1(\sin r_1 - \sin s_1)\Big).$$

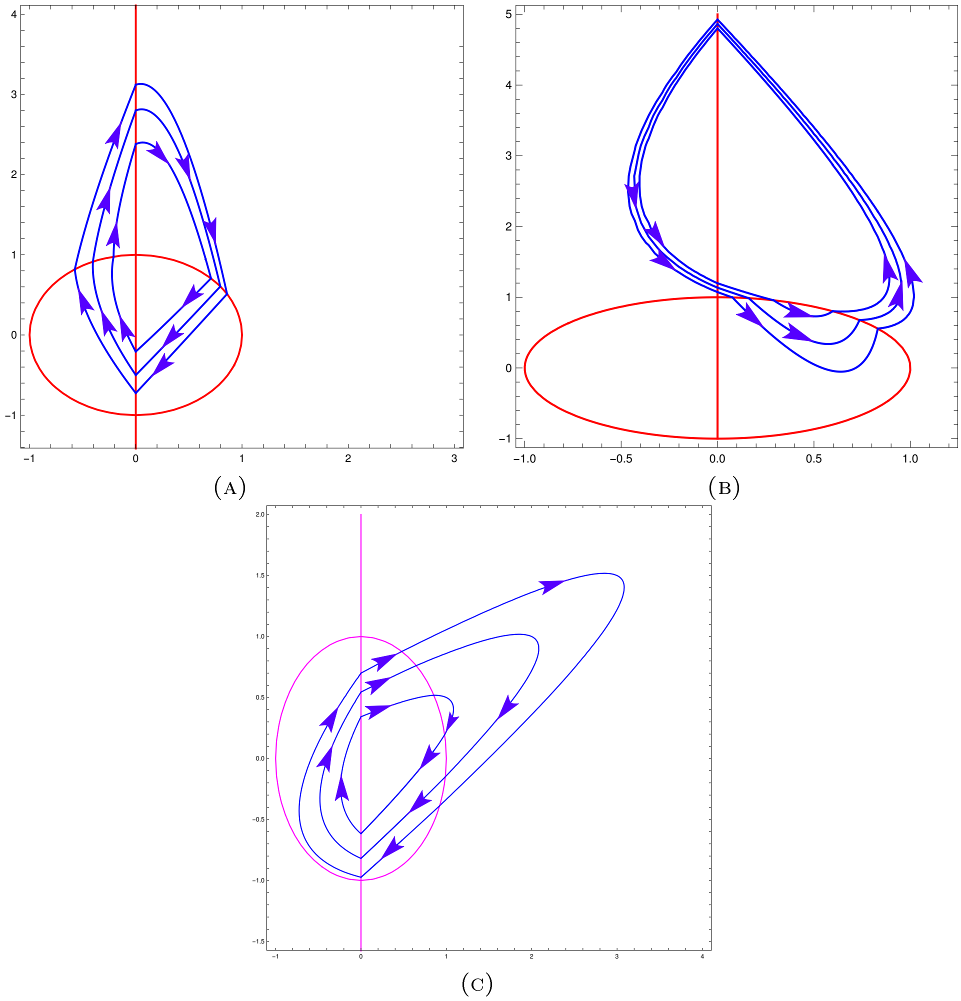


FIGURE 4. (A) The three limit cycles of **Conf 1** of the discontinuous piecewise differential system (7). (B) The three limit cycles of **Conf 2** of the discontinuous piecewise differential system (9). (C) The three limit cycles of **Conf 3** of the discontinuous piecewise differential system (11).

Since the points $A_2 = (\cos r_2, \sin r_2)$, $B_2 = (\cos s_2, \sin s_2)$ also satisfy equation $e_1 = 0$, we obtain the following expression of the parameter μ_1

$$
\begin{aligned}
\mu_1 = \; & \frac{1}{4\Big(\cos\left(\frac{1}{2}(r_1 - 2r_2 + s_1)\right) - \cos\left(\frac{1}{2}(r_1 + s_1 - 2s_2)\right)\Big)}\Big(b_1 \csc\left(\tfrac{r_1 - s_1}{2}\right) \\
& \Big(-\lambda_1 \cos r_1 \sin(2r_2) + \cos r_2 \sin(r_1 - s_1)\Big(\Big(\lambda_1^2 - 1\Big)\sin(r_1 + s_1) + 2\lambda_1 \\
& \cos(r_1 + s_1)\Big) - \Big(\lambda_1^2 - 1\Big)\cos r_1 \sin(r_2 - s_2)\sin(r_2 + s_2) + \lambda_1^2(-\cos s_2) \\
& \sin(r_1 - s_1)\sin(r_1 + s_1) + \cos s_2 \sin(r_1 - s_1)\sin(r_1 + s_1) - \lambda_1 \sin(2r_1) \\
& \cos s_2 + \lambda_1 \cos r_1 \sin(2s_2) - \lambda_1^2 \cos^2 r_2 \cos s_1 + \cos s_1 \Big(\lambda_1 \sin(2r_2) \\
& - \sin^2 r_2 + (\sin s_2 - \lambda_1 \cos s_2)^2\Big) + \lambda_1 \sin(2s_1)\cos s_2\Big)\Big).
\end{aligned}
$$

Likewise the points $A_3 = (\cos r_3, \sin r_3)$, $B_3 = (\cos s_3, \sin s_3)$ satisfy equation $e_3 = 0$, by solving it for the parameter λ_1 and we get the two values $\lambda_1^{1,2} = (A \pm 2\sqrt{2}\sin\left(\frac{1}{2}(r_1 - r_2 + s_1 - s_2)\right)\sqrt{B})/C$ given in the appendix. Now if we suppose that the points $A_4 = (\cos r_4, \sin r_4)$,

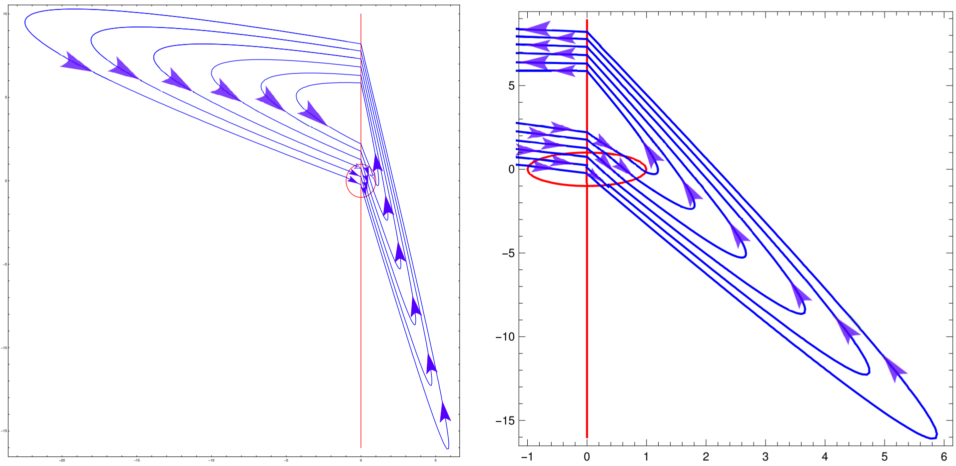


FIGURE 5. Three limit cycles of **Conf 1** and three limit cycles of **Conf 2** for the class of the discontinuous piecewise differential system (12).

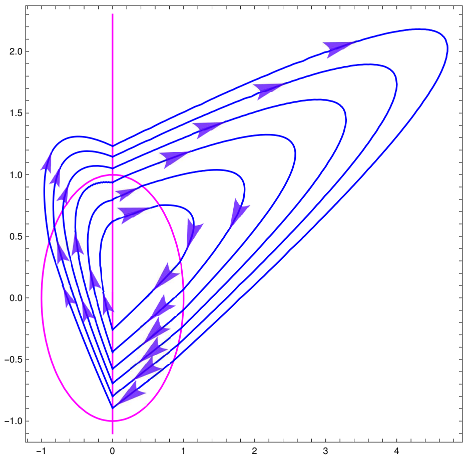


FIGURE 6. Three limit cycles of **Conf 1** and three limit cycles of **Conf 3** for the class of the discontinuous piecewise differential system (13).

$B_4 = (\cos s_4, \sin s_4)$ satisfy equation $e_3 = 0$, then we obtain $b_1 = 0$. This is a contradiction to the assumptions. Therefore we have proved that the maximum number of crossing limit cycles for the class $\mathcal{C}_1$ intersecting the curve Γ_k in four points is three.

Now we shall provide differential systems of class $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ separated by Γ_k with three limit cycles.

We will explain the method for constructing an example of three crossing limit cycles intersecting Γ_k in four points, and by a similar way we build the remaining examples.

Example 1: Three crossing limit cycles for the class $\mathcal{C}_1$. Here we consider the three zones defined in (2) for $k = 2.5$. We consider the Hamiltonian systems

$$\begin{aligned} \dot{x} =& \ -0.02..x + 0.2..y + 0.316667.., \dot{y} = 0.02..y - 0.002..x \text{ in } Z^1, \\ \dot{x} =& \ 10.8..x + 18y - 3, \dot{y} = -6.48..x - 10.8..y \text{ in } Z^2, \\ \dot{x} =& \ 5.x - 3y - 1.38889.., \dot{y} = 8.33333..x - 5y \text{ in } Z^3. \end{aligned} \tag{5}$$

The first integrals of the linear Hamiltonian systems (5) are

$$\begin{array}{rl} H_1(x,y) = & -0.001..x^2 + 0.02..xy - 0.1..y^2 - 0.316667..y, \\ H_2(x,y) = & -3.24..x^2 - 10.8..xy - 9y^2 + 3y, \\ H_3(x,y) = & 4.16667..x^2 - 5xy + 1.5y^2 + 1.38889..y, \end{array}$$

respectively.

The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (5) has exactly three crossing limit cycles, because the system of equations (4) has the three real solutions $S_i = (x_i, y_i, z_i, w_i, f_i, h_i)$ for $i = 1, 2, 3$, where

$$\begin{array}{l} S_1 = (0.244811.., -0.969571.., 0.767202.., 0.641406.., -2.05982.., -0.60685..), \\ S_2 = (0.390566.., -0.920575.., 0.912879.., 0.40823.., -1.8861.., -0.780563..), \\ S_3 = (0.535321.., -0.844649.., 0.979509.., 0.201401.., -1.62201.., 1.04466..). \end{array}$$

Then these three limit cycles are drawn in Figure 3A.

Example 2: Three crossing limit cycles for the class $\mathcal{C}_2$. We consider the three zones defined in (2) with $k = 1$. We consider the Hamiltonian systems

$$\begin{array}{rl} \dot{x} = & 15x - 3y - 11.25.., \dot{y} = 75.x - 15.y + 22.5 \text{ in } Z^1, \\ \dot{x} = & 4x + 20y - 3, \dot{y} = -0.8x - 4y + 6 \text{ in } Z^2, \\ \dot{x} = & -0.4x + 4y + 0.6, \dot{y} = -0.04x + 0.4y - 1 \text{ in } Z^3. \end{array} \tag{6}$$

The first integrals of the Hamiltonian systems (6) are

$$\begin{array}{rl} H_1(x,y) = & 37.5..x^2 - 15xy + 22.5..x + \dfrac{3y^2}{2} + 11.25..y, \\ H_2(x,y) = & -0.4x^2 - 4xy + 6x - 10y^2 + 3y, \\ H_3(x,y) = & -0.02..x^2 + 0.4..xy - x - 2y^2 - 0.6..y, \end{array}$$

respectively.

The discontinuous piecewise linear Hamiltonian system (6) has exactly three crossing limit cycles, because the system of equations (4) has the three real solutions $S_i = (x_i, y_i, z_i, w_i, f_i, h_i)$ for $i = 1, 2, 3$, where

$$\begin{array}{l} S_1 = (0.559983.., 0.828504.., 0.619895.., -0.784685.., 0.878709.., -0.978709..), \\ S_2 = (0.755607.., 0.655025.., 0.754335.., -0.65649.., 0.7, -0.8), \\ S_3 = (0.903742.., 0.881627.., -0.471947.., 0.428077.., 0.462348.., -0.562348..). \end{array}$$

These solutions provide three crossing limit cycles of the piecewise linear differential Hamiltonian system (4), which are illustrate in Figure 3B. This completes the proof of statement (i). □

To complete the proof of statements (ii)-(iv) of Theorem 1 we shall provide discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by the cubic curve Γ_k with three limit cycles for the class $\mathcal{C}_3$ of **Conf 1**; **Conf 2**; **Conf 3**.

Example 3: Three crossing limit cycles of Conf 1 for the class $\mathcal{C}_3$. For this class we consider the four zones defined in (3). We consider the Hamiltonian systems

$$\begin{array}{rl} \dot{x} = & -6.8..x + 4y - 2, \dot{y} = -11.56x + 6.8y - 2 \text{ in } Z^1, \\ \dot{x} = & 1.06216..x + 2y - 1.28925.., \dot{y} = -0.564089..x - 1.06216..y + 3.92358.. \text{ in } Z^2, \\ \dot{x} = & -4x + 2y - 2.8.., \dot{y} = -8x + 4y - 1 \text{ in } Z^3, \\ \dot{x} = & 121.33..x + 3y + 508.239.., \dot{y} = -4907.01..x - 121.33..y + 611.017.. \text{ in } Z^4. \end{array} \tag{7}$$

The linear Hamiltonian systems in (7) have the first integrals

$$
\begin{aligned}
H_1(x,y) = &\quad -5.78..x^2 + 6.8..xy - 2x - 2y^2 + 2y,\\
H_2(x,y) = &\quad -0.282045..x^2 - 1.06216..xy + 3.92358..x - y^2 + 1.28925..y,\\
H_3(x,y) = &\quad -4x^2 + 4xy - x - y^2 + 2.8..y,\\
H_4(x,y) = &\quad -2453.5..x^2 - 121.33..xy + 611.017..x - 1.5y^2 - 508.239..y,
\end{aligned}
$$

respectively.

The discontinuous piecewise linear Hamiltonian system (7) has exactly three crossing limit cycles intersecting Γ_k in the points $A_i = (x_i, y_i)$, $B_i = (z_i, w_i)$, $C_i = (k, f_i)$ and $D_4 = (k, h_i)$ for $i = 1, 2, 3$, where A_i and B_i are points on the circle $x^2 + y^2 - 1 = 0$, because the system of equations

$$
\begin{aligned}
&H_1(x_i, y_i) - H_1(k, f_i) = 0,\\
&H_2(z_i, w_i) - H_2(k, f_i) = 0,\\
&H_3(z_i, w_i) - H_3(k, h_i) = 0,\\
&H_4(x_i, y_i) - H_4(k, h_i) = 0,\\
&x_i^2 + y_i^2 - 1 = 0,\\
&z_i^2 + w_i^2 - 1 = 0,
\end{aligned} \tag{8}
$$

with $k = 0$, has only three real solutions $S_i = (x_i, y_, i, z_i, w_i, f_i, h_i)$ for $i = 1, 2, 3$, where

$$
\begin{aligned}
S_1 &= (0.859402.., 0.5113.., -0.573716.., 0.819054.., 3.12047.., -0.724745..),\\
S_2 &= (0.795991.., 0.605309.., -0.403541.., 0.914962.., 2.8.., -0.5..),\\
S_3 &= (0.708174.., 0.706038.., -0.208691.., 0.977982.., 2.3798.., -0.207107..).
\end{aligned}
$$

These three limit cycles are drawn in Figure 4A. This completes the proof of statement (ii).

Example 4: Three crossing limit cycles of Conf 2 for the class $\mathcal{C}_3$. In (3), we work only with the three zones Z^1, Z^2 and Z^4, with $k = 0$, and we consider the Hamiltonian systems

$$
\begin{aligned}
&\dot{x} = 19 - 18x - 3y, \dot{y} = -68 + 108x + 18y \text{ in } Z^1,\\
&\dot{x} = -3.88389x - 2y + 5.99641.., \dot{y} = 7.54231..x + 3.88389..y - 7.99048.. \text{ in } Z^2,\\
&\dot{x} = 6 + 2x - 2y, \dot{y} = -2 + 2x - 2y \text{ in } Z^4.
\end{aligned} \tag{9}
$$

The first integrals of the Hamiltonian systems (9) are

$$
\begin{aligned}
H_1(x,y) = &\quad 54x^2 + 18xy - 68x + \frac{3y^2}{2} - 19y,\\
H_2(x,y) = &\quad 3.77115..x^2 + 3.88389..xy - 7.99048..x + y^2 - 5.99641..y,\\
H_4(x,y) = &\quad x^2 - 2xy - 2x + y^2 - 6y,
\end{aligned}
$$

respectively

The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (9) has exactly three crossing limit cycles, because the system of equations

$$
\begin{aligned}
&H_1(x_i, y_i) - H_1(k, f_i) = 0,\\
&H_1(z_i, w_i) - H_1(k, h_i) = 0,\\
&H_2(z_i, w_i) - H_2(k, f_i) = 0,\\
&H_4(x_i, y_i) - H_4(z_i, w_i) = 0,\\
&x_i^2 + y_i^2 - 1 = 0,\\
&z_i^2 + w_i^2 - 1 = 0,
\end{aligned} \tag{10}
$$

for $k = 0$ has the three real solutions $S_i = (x_i, y_i, z_i, w_i, f_i, h_i)$ for $i = 1, 2, 3$, where

$$
\begin{aligned}
S_1 &= (0.597407.., 0.801938.., 0.29046.., 0.956887.., 4.80282.., 1.19718..),\\
S_2 &= (0.736107.., 0.676866.., 0.161682.., 0.986843.., 4.86511.., 1.13489..),\\
S_3 &= (0.831057.., 0.556188.., 0.0773343.., 0.997005.., 4.92764.., 1.07236..).
\end{aligned}
$$

These three limit cycles are drawn in Figure 4B. This completes the proof of statement (iii).

Example 5: Three crossing limit cycles of Conf 3 for the class $\mathcal{C}_3$. Here we consider the three zones Z^1, Z^3 and Z^4 defined in (3) with $k = 0$.

$$
\begin{aligned}
\dot{x} = &\ -\frac{43x}{2} + 43y + 6, \dot{y} = -\frac{43x}{4} + \frac{43y}{2} - 2, \text{ in } Z^1,\\
\dot{x} = &\ -5.01788..x + 10y + 1.37209.., \dot{y} = -2.51792..x + 5.01788..y\\
&\ -0.356396.., \text{ in } Z^4,\\
\dot{x} = &\ -5.2..x + 13y + 1.78427.., \dot{y} = -2.08..x + 5.2..y + 7, \text{ in } Z^3.
\end{aligned} \tag{11}
$$

The first integrals of the Hamiltonian systems (11) are

$$
\begin{aligned}
H_1(x,y) = &\ -\frac{43x^2}{8} + \frac{43xy}{2} - 2x - \frac{43y^2}{2} - 6y,\\
H_2(x,y) = &\ -1.25896..x^2 + 5.01788..xy - 0.356396..x - 5y^2 - 1.37209..y,\\
H_3(x,y) = &\ -1.04..x^2 + 5.2xy + 7x - \frac{13y^2}{2} - 1.78427..y,
\end{aligned}
$$

respectively.

The discontinuous piecewise linear differential system formed by the linear Hamiltonian systems (11) has exactly three crossing limit cycles, because the system of equations (4) has the solutions $S_i = (x_i, y_i, z_i, w_i, f_i, h_i)$ for $i = 1, 2, 3$, where

$$
\begin{aligned}
&S_1 = (0.92178.., -0.387712.., 0.478499.., 0.878088.., -0.974503.., 0.7),\\
&S_2 = (\{0.988715.., -0.149808.., 0.647429.., 0.762126.., -0.819428.., 0.544924..),\\
&S_3 = (0.980618.., 0.195928.., 0.855019.., 0.518597.., -0.616986.., 0.342483..).
\end{aligned}
$$

These three limit cycles are drawn in Figure 4c. This completes the proof of statement (iv).

Proof of statement (v) *of Theorem* 1. In order to have limit cycles with **Conf 1** and **Conf 2** simultaneously, the points of intersection of the limit cycles with **Conf 1** with Γ_k must satisfy system (8) with $k = 0$, and the points of intersection of the limit cycles with **Conf 2** with Γ_k must satisfy system (10). In statement (ii) and (iii) of Theorem 1 we proved that the maximum number of limit cycles with **Conf 1** and **Conf 2** is three , then we know that the upper bound of maximum number of limit cycles with both configurations is six.

Example 6: Six crossing limit cycles for the class $\mathcal{C}_3$, with three limit cycles of Conf 1 and three limit cycles of Conf 2. Here we consider the four zones defined in (3).

(12)

$$
\begin{aligned}
\dot{x} = &\ -3.37125..x - y + 3.95604.., \dot{y} = 11.3653..x + 3.37125..y - 11.7972.. \text{ in } Z^1,\\
\dot{x} = &\ -0.121473..x - \frac{1}{2}y + 2.02017.., \dot{y} = 0.0295115..x + 0.121473..y - 0.684232.. \text{ in } Z^2,\\
\dot{x} = &\ 0.328515..x + y + 3, \dot{y} = -0.107922..x - 0.328515..y - 1.29868.. \text{ in } Z^3,\\
\dot{x} = &\ -9.2x - 2.3y + 17, \dot{y} = 36.8x + 9.2y - 56 \text{ in } Z^4.
\end{aligned}
$$

The first integrals of the Hamiltonian systems (12) are

$$
\begin{aligned}
H_1(x,y) = &\ 5.68265..x^2 + 3.37125..xy - 11.7972..x + \frac{y^2}{2} - 3.95604..y,\\
H_2(x,y) = &\ 0.0147557..x^2 + 0.121473..xy - 0.684232..x + \frac{1}{4}y^2 - 2.02017..y,\\
H_3(x,y) = &\ -0.0539609x^2 - 0.328515xy - 1.29868x - \frac{y^2}{2} - 3y,\\
H_4(x,y) = &\ 18.4x^2 + 9.2xy - 56x + 1.15y^2 - 17y,
\end{aligned}
$$

respectively.

For the discontinuous piecewise differential system (13), system (8) with $k = 0$, has the three real solutions

$$S_1 = (0.224513.., -0.974471.., -0.98, 0.198997, 8.21167.., -0.231664..),$$
$$S_2 = (0.359928.., -0.93298.., -0.812094.., 0.583526.., 7.77944.., 0.239163..),$$
$$S_2 = (0.503738.., -0.863856.., -0.41, 0.912086.., 7.31697.., 0.743252..).$$

and system (10), has the three real solutions

$$S_1 = (0.65827.., -0.752782.., 0.093398.., 0.995629.., 6.82398.., 1.25669..),$$
$$S_2 = (0.825187.., -0.56486.., 0.309897.., 0.95077.., 6.31504.., 1.76563..),$$
$$S_2 = (0.986374.., -0.164516.., 0.630863.., 0.775894.., 5.87164.., 2.20904..).$$

These six limit cycles are presented in Figure 5. This completes the proof of statement (v). □

Proof of statement (vi) of Theorem 1. To get limit cycles with **Conf 1** and **Conf 3** simultaneously, the points of intersection of the limit cycles with **Conf 1** and **Conf 3** with Γ_k must satisfy system (8) and (4), respectively, with $k = 0$. In statement (ii) and (iv) of Theorem 1 we showed that the maximum number of limit cycles with **Conf 1** and **Conf 3** is three , then we know that the upper bound of maximum number of limit cycles with both configurations is six.

Example 7: Six crossing limit cycles for the class $\mathcal{C}_3$, with three limit cycles of Conf 1 and three others of Conf 3. Here we consider the four zones defined in (3) with $k = 0$ with the following Hamiltonian systems
(13)
$\dot{x} = -8.8x + 22y - 3, \dot{y} = -3.52x + 8.8y - 4$ in Z^1,
$\dot{x} = 30.9637..x + 30y + 0.9.., \dot{y} = -31.9584..x - 30.9637..y + 24.1071..$ in Z^2,
$\dot{x} = 0.713131..x + 0.9y - 0.162525.., \dot{y} = -0.565063..x - 0.713131..y + 0.620587..$ in Z^3,
$\dot{x} = -8.37872..x + 22y - 3.97708.., \dot{y} = -3.19104..x + 8.37872..y - 3.05205..$ in Z^4.

The first integrals of the Hamiltonian systems (13) are

$$H_1(x, y) = -1.76x^2 + 8.8xy - 4x - 11y^2 + 3y,$$
$$H_2(x, y) = -13.0028..x^2 - 27.9315..xy + 24.0252..x - 15y^2 - 0.9y,$$
$$H_3(x, y) = -0.282531..x^2 - 0.713131..xy + 0.620587..x - 0.45..y^2 + 0.162525..y,$$
$$H_4(x, y) = -1.59552..x^2 + 8.37872..xy - 3.05205..x - 11y^2 + 3.97708..y,$$

respectively.

For the discontinuous piecewise differential system (13), system (8) with $k = 0$, has the three real solutions

$$S_1 = (0.859956.., -0.510369.., 0.89, 0.45596.., 1.232.., -0.895261..),$$
$$S_2 = (0.925727.., -0.378193.., -0.818732.., 0.574176.., 1.14562.., -0.79916..),$$
$$S_3 = (0.969836.., -0.243758.., -0.7, 0.714143.., 1.05112.., -0.694334..),$$

and system (4), has the three real solutions

$$S_1 = (0.995048.., -0.0993944.., 0.167496.., 0.985873.., 0.937707.., -0.576541..),$$
$$S_2 = (0.997733.., 0.0672986.., 0.41691.., 0.908948.., 0.799221.., -0.438055..),$$
$$S_3 = (0.954489.., 0.298247.., 0.659704.., 0.751525..0.621163.., -0.259997..).$$

These six limit cycles are drawn in Figure 6. This completes the proof of statement (vi). □

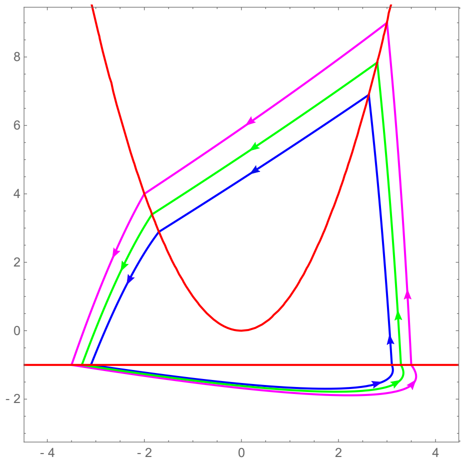
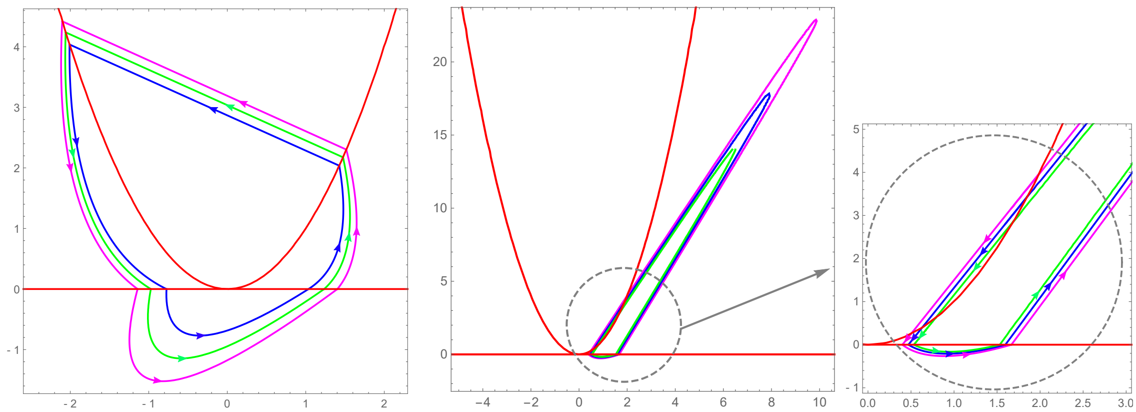


Figure 7. Three limit cycles of system (16) intersecting Σ_{-1}.



(A) (B)

Figure 8. (A) Three limit cycles with **Conf 4** of system (17). (B) Three limit cycles with **Conf 5** of system (19).

3. Proof of Theorem 2

We will prove the statement (i). For the other statements the proof is completely analogous.

Proof of statement (i) *of Theorem* 2. From Lemma 1 we can consider an arbitrary piecewise linear differential Hamiltonian system in $\mathcal{C}_{\Sigma_{k^-}}$ formed by the following three linear Hamiltonian systems without equilibrium points

$$\dot{x} = -\lambda_i b_i x + b_i y + \mu_i, \quad \dot{y} = -\lambda_i^2 b_i x + \lambda_i b_i y + \sigma_i \text{ in } Z^i_{\Sigma_{k^-}}, \tag{14}$$

for $i = 1, 2, 3$, where $\sigma_i \neq \lambda_i \mu_i$ and $b_i \neq 0$. The Hamiltonian functions associated to these systems are

$$H_i(x, y) = -\frac{1}{2}\lambda_i^2 b_i x^2 + \lambda_i b_i xy - \frac{b_i}{2}y^2 + \sigma_i x - \mu_i y, \text{ in } Z^i_{\Sigma_{k^-}} \text{ for } i = 1, 2, 3.$$

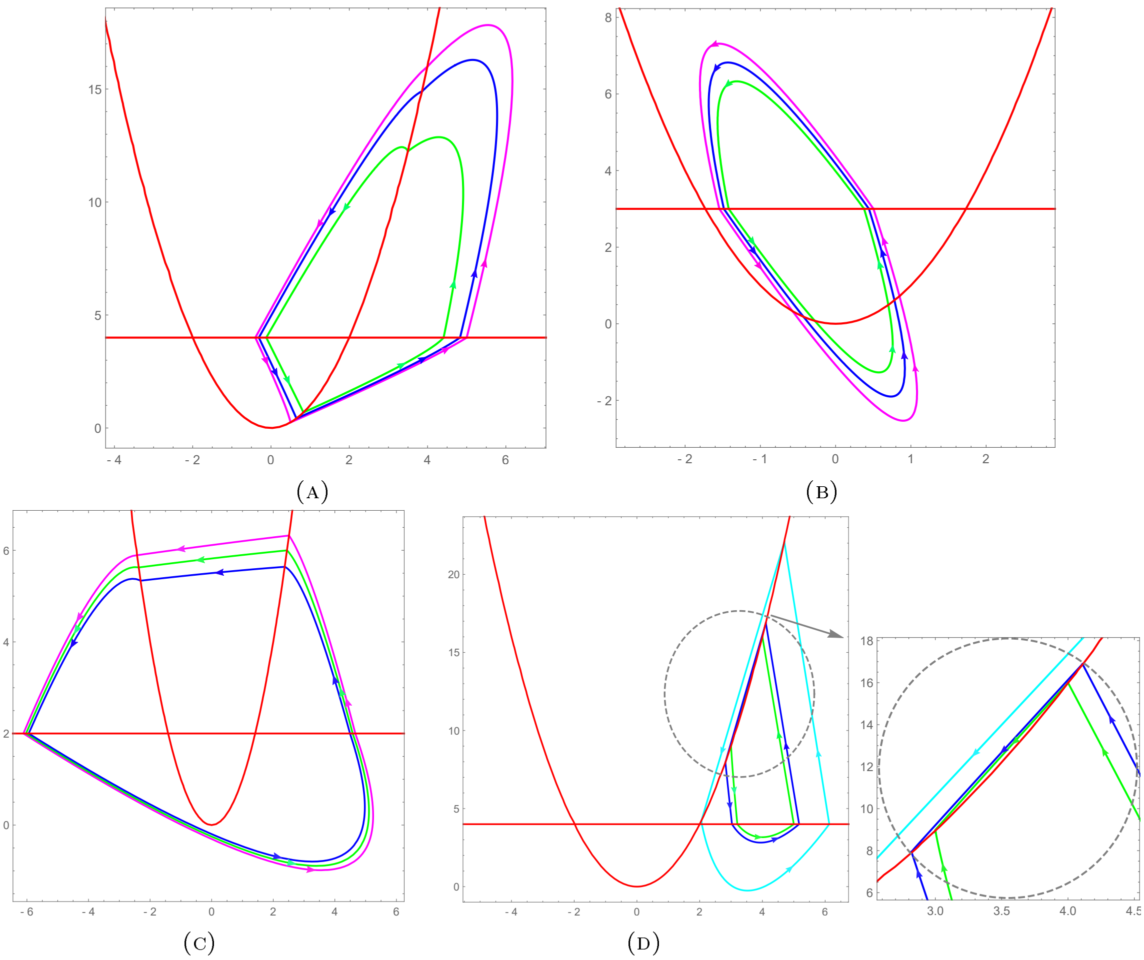


FIGURE 9. (A) Three limit cycles with **Conf** 6^+ of system (22). (B) Three limit cycles with **Conf 7** of system (24). (C) Three limit cycles with **Conf 8** of system (26). (D) Three limit cycles with **Conf** 9^+ of system (27).

In order to have a limit cycle which intersects Σ_{k^-} in four different points (x_1, x_1^2), (x_2, x_2^2), (x_3, k) and (x_4, k), these points must satisfy the system

$$
(15) \qquad \begin{aligned}
&H_1(x_1, x_1^2) - H_1(x_2, x_2^2) = 0,\\
&H_2(x_2, x_2^2) - H_2(x_3, k) = 0,\\
&H_3(x_3, k) - H_3(x_4, k) = 0,\\
&H_4(x_4, k) - H_4(x_1, x_1^2) = 0, \quad k < 0.
\end{aligned}
$$

Assume that the discontinuous piecewise linear differential system (14) has four limit cycles. For this we must suppose that system (15) has four real solutions, namely $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})$, with $i = 1, 2, 3, 4$. We consider that $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, x_4^{(1)})$ satisfies system (15), solving the first and the third equations we obtain the expressions of the parameters μ_1 and σ_3, respectively

$$
\begin{aligned}
\mu_1 =& \frac{2\sigma_1 - b_1(x_1^{(1)} + x_2^{(1)} - \lambda_1)((x_1^{(1)})^2 + (x_2^{(1)})^2 - (x_1^{(1)} + x_2^{(1)})\lambda_1)}{2(x_1^{(1)} + x_2^{(1)})},\\
\sigma_3 =& \frac{1}{2} b_3 \lambda_3(-2k + (x_3^{(1)} + x_4^{(1)})\lambda_3).
\end{aligned}
$$

When we suppose the second solution of system (15) we fixed the points $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)})$ and by third equation we get that $x_4^{(2)} = x_3^{(1)} + x_4^{(1)} - x_3^{(2)}$. Moreover from the first equation of system (15) we get

$$\sigma_1 = \frac{b_1(x_1^{(2)} - x_2^{(2)})}{2\left(x_1^{(2)} - x_2^{(2)} + \dfrac{-(x_1^{(2)})^2 + (x_2^{(2)})^2}{x_1^{(1)} + x_2^{(1)}}\right)} \Bigg((x_1^{(2)} + x_2^{(2)} - \lambda_1)((x_1^{(2)})^2 + ((x_2^{(2)})^2 - (x_1^{(2)} + x_2^{(2)})\lambda_1) - \frac{(x_1^{(2)} + x_2^{(2)})(x_1^{(1)} + x_2^{(1)} - \lambda_1)((x_1^{(1)})^2 + (x_2^{(1)})^2 - (x_1^{(1)} + x_2^{(1)})\lambda_1)}{x_1^{(1)} + x_2^{(1)}} \Bigg).$$

If we suppose a third solution, we fixed the points $(x_1^{(3)}, x_2^{(3)}, x_3^{(3)})$, and from the third equation we get that $x_4^{(3)} = x_3^{(1)} + x_4^{(1)} - x_3^{(3)}$. By first equation we obtain

$$\begin{aligned}
\lambda_1 =& \big((x_1^{(1)})^3(x_1^{(2)} + x_2^{(2)} - x_1^{(3)} - x_2^{(3)}) + (x_1^{(1)})^2 x_2^{(1)}(x_1^{(2)} + x_2^{(2)} - x_1^{(3)} - x_2^{(3)}) + (x_2^{(1)})^3(x_1^{(2)} \\
&+ x_2^{(2)} - x_1^{(3)} - x_2^{(3)}) + (x_1^{(2)} + x_2^{(2)})(x_1^{(3)} + x_2^{(3)})((x_1^{(2)})^2 + (x_2^{(2)})^2 - (x_1^{(3)})^2 - (x_2^{(3)})^2) \\
&+ x_2^{(1)}(-(x_1^{(2)})^3 - (x_1^{(2)})^2 x_2^{(2)} - x_1^{(2)}(x_2^{(2)})^2 - (x_2^{(2)})^3 + (x_1^{(3)})^3 + (x_1^{(3)})^2 x_2^{(3)} + x_1^{(3)}(x_2^{(3)})^2 \\
&+ (x_2^{(3)})^3) + x_1^{(1)}(-(x_1^{(2)})^3 - (x_1^{(2)})^2 x_2^{(2)} - x_1^{(2)}(x_2^{(2)})^2 - (x_2^{(2)})^3 + (x_1^{(3)})^3 + (x_2^{(1)})^2(x_1^{(2)} \\
&+ x_2^{(2)} - x_1^{(3)} - x_2^{(3)}) + (x_1^{(3)})^2 x_2^{(3)} + x_1^{(3)}(x_2^{(3)})^2 + (x_2^{(3)})^3)\big) / \big(2((x_1^{(2)})^2 x_1^{(3)} + x_1^{(2)} x_2^{(2)} x_1^{(3)} \\
&+ (x_2^{(2)})^2 x_1^{(3)} - x_1^{(2)}(x_1^{(3)})^2 - x_2^{(2)}(x_1^{(3)})^2 + (x_1^{(1)})^2(x_1^{(2)} + x_2^{(2)} - x_1^{(3)} - x_2^{(3)}) + (x_2^{(1)})^2 \\
&(x_1^{(2)} + x_2^{(2)} - x_1^{(3)} - x_2^{(3)}) + (x_1^{(2)})^2 x_2^{(3)} + x_1^{(2)} x_2^{(2)} x_2^{(3)} + (x_2^{(2)})^2 x_2^{(3)} - x_1^{(2)} x_1^{(3)} x_2^{(3)} \\
&- x_2^{(2)} x_1^{(3)} x_2^{(3)} - x_1^{(2)}(x_2^{(3)})^2 - x_2^{(2)}(x_2^{(3)})^2 + x_2^{(1)}(-(x_1^{(2)})^2 - x_1^{(2)} x_2^{(2)} - (x_2^{(2)})^2 + (x_1^{(3)})^2 \\
&+ x_1^{(3)} x_2^{(3)} + (x_2^{(3)})^2) + x_1^{(1)}(-(x_1^{(2)})^2 - x_1^{(2)} x_2^{(2)} - (x_2^{(2)})^2 + (x_1^{(3)})^2 + x_2^{(1)}(x_1^{(2)} + x_2^{(2)} \\
&- x_1^{(3)} - x_2^{(3)}) + x_1^{(3)} x_2^{(3)} + (x_2^{(3)})^2))\big).
\end{aligned}$$

When we suppose the fourth solution $(x_1^{(4)}, x_2^{(4)}, x_3^{(4)}, x_3^{(1)} + x_4^{(1)} - x_3^{(4)})$, from the first equation we have that $b_1 = 0$ which is a contradiction, because from Lemma 1 $b_i \neq 0$ for $i = 1, 2, 3$. Therefore the maximum number of limit cycles in this case is three. Now we prove that this upper bound is attached. We observed that the upper bound found does not depend of value of parameter $k < 0$, then we can consider without loss of generality that $k = -1$. We consider the discontinuous piecewise linear differential system defined by the following three linear Hamiltonian systems

(16)

$$\begin{aligned}
&\dot{x} = -24.293899.. - 0.692634..x + \frac{3}{2}y, \;\; \dot{y} = -19.232427.. - 0.319828..x + 0.692634..y, \\
&\dot{x} = -378.204351.. + 62.383901..x - 4y, \;\; \dot{y} = 916.621187.. + 972.937795..x - 62.383901..y, \\
&\dot{x} = \frac{9}{10} - \frac{7}{2}x - \frac{35}{4}y, \;\; \dot{y} = \frac{7}{2} + \frac{7}{5}x + \frac{7}{2}y,
\end{aligned}$$

in the zones $Z^1_{\Sigma_{-1}}$, $Z^2_{\Sigma_{-1}}$ and $Z^3_{\Sigma_{-1}}$, respectively. Then for system (16), we have that system (15) has three real solutions $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})$, $i = 1, 2, 3$, namely

$$\left(3, -2, -\tfrac{7}{2}, \tfrac{7}{2}\right), \; \left(2.625658.., -\tfrac{17}{10}.., -\tfrac{31}{10}, \tfrac{31}{10}\right), \; \left(\tfrac{14}{5}, -1.843412.., -3.287307.., 3.287307..\right).$$

These three real solutions provide the three limit cycles intersecting Σ_{-1} shown in Figure 7. This completes the proof of statement (i). $\square$

Proof of statement (ii) *of Theorem* 2. The proof in this statement is similar to the proof of statement (i). For each configuration of limit cycles that intersect Σ_k with $k \geq 0$ we have that the upper bound of limit cycles is three. In what follows we show examples of piecewise linear differential system in $\mathcal{C}_{\Sigma_0}$ with three limit cycles with **Conf 4** and **Conf 5**, respectively. And piecewise linear differential system in $\mathcal{C}_{\Sigma_{k^+}}$ with three limit cycles with **Conf** 6^+, **Conf 7**, **Conf 8** and **Conf** 9^+, respectively.

Crossing limit cycles with Conf 4: In order to have a limit cycle with **Conf 4** which intersects Σ_0 in four different points (x_1, x_1^2), (x_2, x_2^2), $(x_3, 0)$ and $(x_4, 0)$, these points must satisfy system (15) with $k = 0$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

(17)
$$
\begin{aligned}
&\dot{x} = \frac{17}{2} - \frac{207}{50}x - \frac{69}{10}y,\ \dot{y} = -\frac{53}{10} + \frac{621}{250}x + \frac{207}{50}y,\\
&\dot{x} = 48.069511.. + 11.825263..x - \frac{31}{5}y,\ \dot{y} = -7.155434.. + 22.554330..x - 11.825263..y,\\
&\dot{x} = \frac{9}{2} + \frac{156}{25}x - \frac{39}{10}y,\ \dot{y} = -\frac{13}{10} + 9.984000..x - \frac{156}{25}y,\\
&\dot{x} = 17.727172.. - 7.176019..x - \frac{27}{5}y,\ \dot{y} = -1.428092.. + 9.536159..x + 7.176019..y,
\end{aligned}
$$

in the zones $Z^1_{\Sigma_0}$, $Z^2_{\Sigma_0}$, $Z^3_{\Sigma_0}$ and $Z^4_{\Sigma_0}$, respectively. For the discontinuous piecewise differential system (17), system (15) has three real solutions $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})$, $i = 1, 2, 3$ given by

$$
\begin{aligned}
&\big(1.517382.., -2.102549.., -1.142394.., 1.402811..\big),\\
&\big(1.474836.., -2.058730.., -0.973819.., 1.234236..\big),\\
&\big(1.427170.., -2.00939.., -0.774355.., 1.034772..\big).
\end{aligned}
$$

These solutions provide the three limit cycles with **Conf 4** shown in Figure 8A.

Crossing limit cycles with Conf 5: In order to have a limit cycle with **Conf 5** which intersects Σ_0 in the four different points (x_1, x_1^2), (x_2, x_2^2), $(x_3, 0)$ and $(x_4, 0)$, they must satisfy

$$
\begin{aligned}
&H_1(x_1, x_1^2) - H_1(x_2, x_2^2) = 0,\\
&H_4(x_2, x_2^2) - H_4(x_3, k) = 0,\\
&H_3(x_3, k) - H_3(x_4, k) = 0,\\
&H_4(x_4, k) - H_4(x_1, x_1^2) = 0, \quad \text{with} \quad k = 0.
\end{aligned} \tag{18}
$$

We consider the discontinuous piecewise linear differential system defined by the following three linear Hamiltonian systems

(19)
$$
\begin{aligned}
&\dot{x} = -4.711119.. + 3.915394..x - \frac{3}{2}y,\ \dot{y} = -11.965988.. + 10.220210..x - 3.915394..y,\\
&\dot{x} = \frac{9}{10} + \frac{27}{10}x - \frac{3}{2}y,\ \dot{y} = -5.022000.. + \frac{243}{50}x - \frac{27}{10}y,\\
&\dot{x} = -3.005265.. + 2.848936..x - \frac{11}{10}y,\ \dot{y} = -7.616106.. + 7.378583..x - 2.848936..y,
\end{aligned}
$$

in the zones $Z^1_{\Sigma_0}$, $Z^3_{\Sigma_0}$ and $Z^4_{\Sigma_0}$, respectively. For the discontinuous piecewise differential system (19), system (18) has three real solutions $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})$, $i = 1, 2, 3$ given by

$$
\big(2, \tfrac{1}{2}, \tfrac{2}{5}, \tfrac{5}{3}\big),\ \big(\tfrac{93}{50}, 0.628914.., \tfrac{47}{100}, \tfrac{479}{300}\big),\ \big(1.696225.., 0.780317.., 0.534387.., 1.532279..\big).
$$

These solutions provide the three limit cycles with **Conf 5** shown in Figure 8B.

Crossing limit cycles with Conf 6^+: In order to have a limit cycle with **Conf** 6^+ which intersects Σ^+ in four different points $(x_1, x_1^2), (x_2, k), (x_3, x_3^2)$ and (x_4, k), these points must satisfy

$$\begin{aligned}
&H_1(x_1, x_1^2) - H_1(x_2, k) = 0,\\
&H_5(x_2, k) - H_5(x_3, x_3^2) = 0,\\
&H_3(x_3, x_3^2) - H_3(x_4, k) = 0,\\
&H_4(x_4, k) - H_4(x_1, x_1^2) = 0, \quad \text{for} \quad k > 0.
\end{aligned} \tag{20}$$

To have a limit cycle with **Conf** 6^- which intersects Σ^+ in four different points, these points must satisfy the system

$$\begin{aligned}
&H_2(x_1, x_1^2) - H_2(x_2, k) = 0,\\
&H_3(x_2, k) - H_3(x_3, x_3^2) = 0,\\
&H_5(x_3, x_3^2) - H_5(x_4, k) = 0,\\
&H_1(x_4, k) - H_1(x_1, x_1^2) = 0, \quad \text{for} \quad k > 0.
\end{aligned} \tag{21}$$

We provide an example of a piecewise linear differential system with three limit cycles with **Conf** 6^+. We observed that the upper bound found does not depend of the value of the parameter $k > 0$, then we can consider without loss of generality that $k = 4$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

(22)

$$\begin{aligned}
&\dot{x} = -4.133710.. + 6.015251..x - \frac{3}{2}y, \ \dot{y} = -5.920817.. + 24.122170..x - 6.015251..y,\\
&\dot{x} = 5.325742.. + 2.017936..x - \frac{17}{5}y, \ \dot{y} = 4.036253.. + 1.197666..x - 2.017936..y,\\
&\dot{x} = -8.981178.. + 3.946297..x - y, \ \dot{y} = -15.942643.. + 15.573265..x - 3.946297..y,\\
&\dot{x} = -2.454956.. + 4.664679..x + \frac{3}{2}y, \ \dot{y} = 6.613677.. - 14.506158..x - 4.664679..y,
\end{aligned}$$

in the zones $Z^1_{\Sigma_4}$, $Z^3_{\Sigma_4}$, $Z^4_{\Sigma_4}$ and $Z^5_{\Sigma_4}$, respectively. For the discontinuous piecewise differential system (22), system (20) has three real solutions $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})$, $i = 1, 2, 3$ given by

$$\left(4, -\tfrac{2}{5}, \tfrac{1}{2}, 5\right), \ \left(\tfrac{193}{50}, -\tfrac{31}{100}, \tfrac{13}{20}, \tfrac{483}{100}\right), \ \left(\tfrac{7}{2}, -\tfrac{3}{25}, \tfrac{83}{100}, \tfrac{441}{100}\right).$$

These solutions provide the three limit cycles with **Conf** 6^+ shown in Figure 9A.

Crossing limit cycles with Conf 7: In order to have a limit cycle with **Conf 7** which intersects Σ^+ in the four different points (x_1, k), (x_2, k), (x_3, x_3^2) and (x_4, x_4^2), they must satisfy the system

$$\begin{aligned}
&H_1(x_1, k) - H_1(x_2, k) = 0,\\
&H_5(x_2, k) - H_5(x_3, x_3^2) = 0,\\
&H_3(x_3, x_3^2) - H_3(x_4, x_4^2) = 0,\\
&H_5(x_4, x_4^2) - H_5(x_1, k) = 0, \quad \text{with } k > 0.
\end{aligned} \tag{23}$$

We can suppose without loss of generality that $k = 4$. We consider the discontinuous piecewise linear differential system defined by the following three linear Hamiltonian systems

(24)

$$\begin{aligned}
&\dot{x} = -2 - 6x - \frac{3}{2}y, \ \dot{y} = -5.491482.. + 24x + 6y,\\
&\dot{x} = 22.645454.. - 36.659999..x - \frac{47}{5}y, \ \dot{y} = -35.463636.. + 142.973999..x + 36.659999..y,\\
&\dot{x} = 5.300000.. - 8.579999..x - \frac{11}{5}y, \ \dot{y} = -8.300000.. + 33.461999..x + 8.579999..y,
\end{aligned}$$

in the zones $Z^1_{\Sigma_3}$, $Z^3_{\Sigma_3}$ and $Z^5_{\Sigma_3}$, respectively. For the discontinuous piecewise differential system (24), system (23) has three real solutions $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})$, $i = 1, 2, 3$ given by

$$\begin{aligned}&(0.502842.., -1.545218.., -0.572025.., 0.848539..),\\&(0.442709.., -1.485086.., -0.427227.., 0.781483..),\\&(0.378567.., -1.420944.., -0.276975.., 0.700080..).\end{aligned}$$

These solutions provide the three limit cycles with **Conf 7** shown in Figure 9B.

Crossing limit cycles with Conf 8: In order to have a limit cycle with **Conf 8** which intersects Σ^+ in four different points (x_1, x_1^2), (x_2, x_2^2), (x_3, k) and (x_4, k), they must satisfy

$$\begin{aligned}&H_1(x_1, x_1^2) - H_1(x_2, x_2^2) = 0,\\&H_2(x_2, x_2^2) - H_2(x_3, k) = 0,\\&H_3(x_3, k) - H_3(x_4, k) = 0,\\&H_4(x_4, k) - H_4(x_1, x_1^2) = 0, \quad \text{with } k > 0.\end{aligned} \tag{25}$$

We can consider without loss of generality that $k = 2$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

(26)

$$\begin{aligned}&\dot{x} = \frac{9}{2} + \frac{19}{50}x - \frac{19}{10}y, \ \dot{y} = \frac{17}{10} + 0.076000..x - \frac{19}{50}y,\\&\dot{x} = 10.930108.. + 7.204668..x - \frac{11}{5}y, \ \dot{y} = 99.090506.. + 23.594202.x - 7.204668..y,\\&\dot{x} = -\frac{69}{2} - 6.229999..x - \frac{89}{10}y, \ \dot{y} = -\frac{93}{10} + \frac{4361}{1000}x + 6.229999..y,\\&\dot{x} = 32.954952.. - 16.575663..x - \frac{17}{5}y, \ \dot{y} = -277.274017.. + 80.809593..x + 16.575663..y,\end{aligned}$$

in the pieces $Z^1_{\Sigma_2}$, $Z^2_{\Sigma_2}$, $Z^3_{\Sigma_2}$ and $Z^4_{\Sigma_2}$, respectively. For the discontinuous piecewise differential system (26), system (25) has three real solutions $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})$, $i = 1, 2, 3$ given by

$$\begin{aligned}&(2.514526.., -2.427396.., -6.114467.., 4.665258..),\\&(2.449236.., -2.371782.., -6.028697.., 4.579488..),\\&(2.374832.., -2.310077.., -5.941517.., 4.492308..).\end{aligned}$$

These solutions provide the three limit cycles with **Conf 8** shown in Figure 9C.

Crossing limit cycles with Conf 9^+: In order to have a limit cycle with **Conf** 9^+ which intersects Σ^+ in the four different points $(x_1, x_1^2), (x_2, x_2^2), (x_3, k)$ and (x_4, k), they must satisfy system (18) with $k > 0$. Without loss of generality we can suppose that $k = 4$. We consider the discontinuous piecewise linear differential system defined by the following three linear Hamiltonian systems

(27)

$$\begin{aligned}&\dot{x} = -170.859539.. + 99.779168..x - 15y, \dot{y} = -1139.726782.. + 663.725497..x - 99.779168..y,\\&\dot{x} = \frac{9}{10} + \frac{148}{5}x - 4y, \dot{y} = -779.664000.. + 219.040000..x - \frac{148}{5}y,\\&\dot{x} = 116.632274.. - 30.946111..x - \frac{23}{10}y, \dot{y} = -1635.644521.. + 416.374692..x + 30.946111..y,\end{aligned}$$

in the zones $Z^1_{\Sigma_4}$, $Z^3_{\Sigma_4}$, and $Z^4_{\Sigma_4}$, respectively. For the discontinuous piecewise differential system (27), system (18) with $k = 4$, has three real solutions $(x_1^{(i)}, x_2^{(i)}, x_3^{(i)}, x_4^{(i)})$, $i = 1, 2, 3$ given by

$$\left(4, 3, \tfrac{16}{5}, 5\right), \ \left(4.109491.., \tfrac{141}{50}, \tfrac{303}{100}, \tfrac{517}{100}\right), \ \left(\tfrac{47}{10}, 2.053733.., 2.068270.., 6.131729..\right).$$

These solutions provide the three limit cycles with **Conf** 9^+ shown in Figure 9D. This completes the proof of statement (ii). □

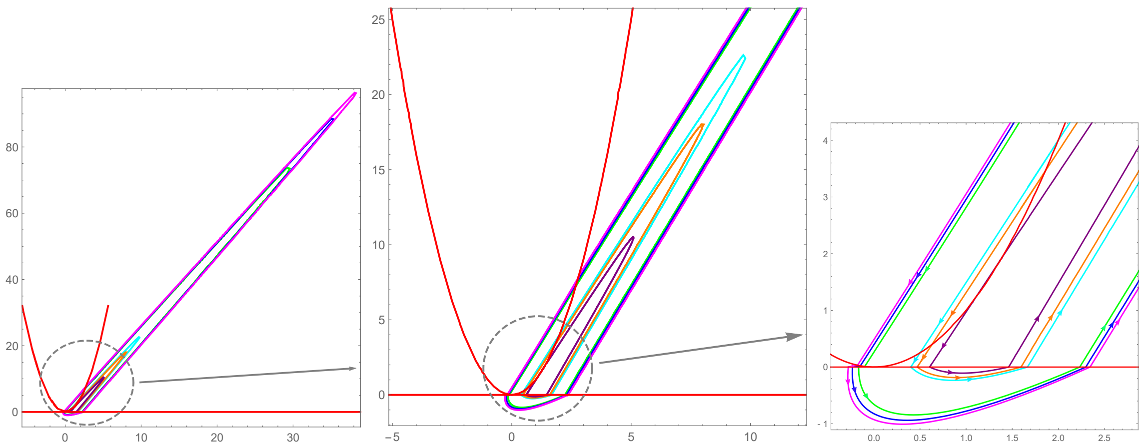


FIGURE 10. Three limit cycles with **Conf 4**, and three limit cycles with **Conf 5** of system (28) simultaneously.

Proof of statement (iii) of Theorem 2. In order to have limit cycles with **Conf 4** and **Conf 5** simultaneously, the points of intersection of the limit cycles with **Conf 4** with Σ_0 must satisfy system (15) with $k = 0$, and the points of intersection of the limit cycles with **Conf 5** with Σ_0 must satisfy system (18). In statement (ii) we proved that the maximum number of limit cycles with **Conf 4** and **Conf 5** is three, then we have that the upper bound of maximum number of limit cycles with both configurations is six. We provide an example of a piecewise linear differential system in $\mathcal{C}_{\Sigma_0}$ such that have six limit cycles with three limit cycles with **Conf 4** and **Conf 5**, respectively. This is the upper bound is reached. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

(28)

$$\dot{x} = -154.076990.. - 28.017658..x + 15y, \ \dot{y} = -387.181918.. - 52.332611..x + 28.017658..y,$$

$$\dot{x} = -0.400024.. + 0.532848..x - \frac{3}{10}y, \ \dot{y} = 0.058205.. + 0.946425..x - 0.532848..y,$$

$$\dot{x} = \frac{9}{10} + \frac{28}{5}x - 4y, \ \dot{y} = -8.101333.. + 7.839999..x - \frac{28}{5}y,$$

$$\dot{x} = -3.005265.. + 2.848936..x - \frac{11}{10}y, \ \dot{y} = -7.616106.. + 7.378583..x - 2.848936..y,$$

in the zones $Z^1_{\Sigma_0}$, $Z^2_{\Sigma_0}$, $Z^3_{\Sigma_0}$ and $Z^4_{\Sigma_0}$, respectively. For the discontinuous piecewise differential system (28), system (15) with $k = 0$, has the following three real solutions

$$\left(\tfrac{27}{10}, -\tfrac{1}{10}, -0.166825.., 2.233491..\right), \left(\tfrac{69}{25}, -0.147032.., -0.232725.., 2.299392..\right),$$
$$\left(\tfrac{14}{5}, -0.177898.., -0.278200.., 2.344866..\right),$$

and system (18) with $k = 0$, has the three real solutions

$$\left(2, \tfrac{1}{2}, \tfrac{2}{5}, \tfrac{5}{3}\right), \left(\tfrac{93}{50}, 0.628914.., \tfrac{47}{100}, \tfrac{479}{300}\right), \left(1.393438.., 1.075216.., 0.604474.., 1.462192..\right).$$

These solutions provide the three limit cycles with **Conf 4** and **Conf 5** shown in Figure 10. This completes the proof of statement (iii). □

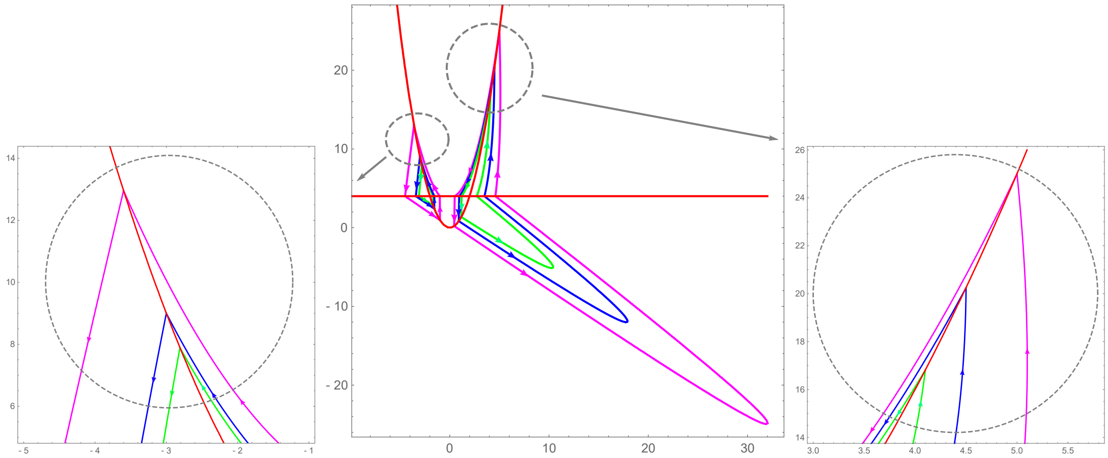


FIGURE 11. Three limit cycles with **Conf** 6^- and **Conf** 6^+ of system (29).

Proof of statement (iv) of Theorem 2. In order to have limit cycles with **Conf** 6^- and **Conf** 6^+ simultaneously, the points of intersection of the limit cycles with **Conf** 6^+ and Σ_{k^+} must satisfy system (20), and the points of intersection of the limit cycles with **Conf** 6^- and Σ_{k^+} must satisfy system (21). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the upper bound of maximum number of limit cycles with both configurations is six. We provide an example of a piecewise linear differential system in $\mathcal{C}_{\Sigma_{k^+}}$ such that have six limit cycles with three limit cycles with each configuration. This is the upper bound is reached. Without loss of generality we can suppose that $k = 4$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

(29)
$$\begin{aligned}
&\dot{x} = -23138.489410.. + 403.676452..x + \frac{9}{2}y,\ \dot{y} = 2942.120325.. - 36212.150741..x \\
&\qquad\qquad -403.676452..y,\\
&\dot{x} = 4.276633.. + 1.873985..x - \frac{3}{10}y,\ \dot{y} = 4.991226.. + 11.706072..x - 1.873985..y,\\
&\dot{x} = 15.472057.. - 3.117904..x - \frac{17}{5}y,\ \dot{y} = -13.354567.. + 2.859213..x + 3.117904..y,\\
&\dot{x} = 48.158492.. - 6.082779..x - y,\ \dot{y} = -31.590984.. + 37.000210..x + 6.082779..y,\\
&\dot{x} = -151.854124.. - 136.354901..x + \frac{3}{2}y,\ \dot{y} = -10611.949690.. - 12395.106180..x \\
&\qquad\qquad +136.354901..y,
\end{aligned}$$

in the pieces $Z^1_{\Sigma_4}$, $Z^2_{\Sigma_4}$, $Z^3_{\Sigma_4}$, $Z^4_{\Sigma_4}$ and $Z^5_{\Sigma_4}$, respectively. For the discontinuous piecewise differential system (29), (20), has the three real solutions

$$\big(5, \tfrac{1}{2}, \tfrac{9}{20}, \tfrac{23}{5}\big), \big(\tfrac{9}{2}, \tfrac{19}{20}, \tfrac{91}{100}, \tfrac{7}{2}\big), \big(\tfrac{41}{10}, 1.196150.., 1.163297.., 2.719447..\big),$$

and system (21), has the following three real solutions

$$\big(-\tfrac{18}{5}, -\tfrac{9}{2}, -\tfrac{49}{50}, -1\big), \big(-3, -3.411586.., -1.557354.., -1.546135..\big),$$
$$\big(-2.809209.., -\tfrac{31}{10}, -1.671884.., -1.662653..\big).$$

These solutions provide the three limit cycles with **Conf** 6^- and **Conf** 6^+ shown in Figure 11. This completes the proof of statement (iv). $\square$

Proof of statement (v) of Theorem 2. In order to have limit cycles with **Conf** 6^- and **Conf 7** simultaneously, the points of intersection of the limit cycles with **Conf** 6^- and Σ_{k+} must satisfy system (21), and the points of intersection of the limit cycles with **Conf 7** and Σ_{k+} must satisfy system (23). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. Without loss of generality we can suppose that $k = 3$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

(30)

$$\dot{x} = -0.567977.. - 5.151614..x - \frac{3}{2}y,\ \ \dot{y} = -6.233588.. + 17.692757..x + 5.151614..y,$$

$$\dot{x} = 11.250254.. + 0.637407..x - \frac{2}{5}y,\ \ \dot{y} = -35.085985.. + 1.015720..x - 0.637407..y,$$

$$\dot{x} = 22.645454.. - 36.659999..x - \frac{47}{5}y,\ \ \dot{y} = -35.463636.. + 142.973999..x + 36.659999..y,$$

$$\dot{x} = 5.300000.. - 8.579999..x - \frac{11}{5}y,\ \ \dot{y} = -8.300000.. + 33.461999..x + 8.579999..y,$$

in the pieces $Z^1_{\Sigma_3}$, $Z^2_{\Sigma_3}$, $Z^3_{\Sigma_3}$ and $Z^5_{\Sigma_3}$, respectively. For the discontinuous piecewise differ-

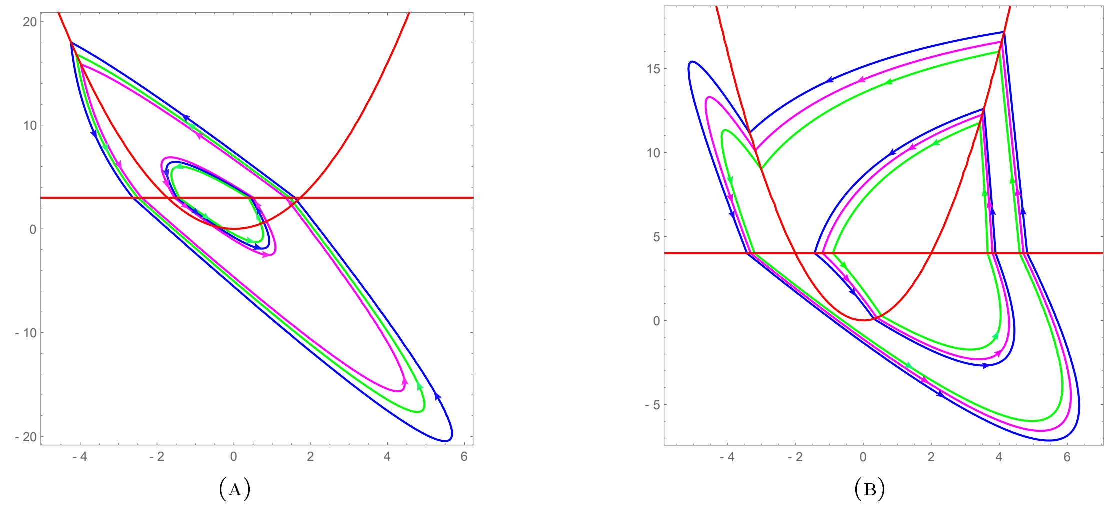


FIGURE 12. (A) Three limit cycles with **Conf** 6^- and **Conf 7** of system (30). (B) Three limit cycles with **Conf** 6^+ and **Conf 8** of system (31).

ential system (30), system (21), has the following three real solutions

$$\big(-\tfrac{41}{10}, -\tfrac{5}{2}, 1.569412.., 1.457623..\big), \big(-\tfrac{106}{25}, -\tfrac{263}{100}, 1.647799.., 1.587623..\big),$$
$$\big(-\tfrac{199}{50}, -2.401954.., 1.508473.., 1.359577..\big),$$

and (23), has the three real solutions

$$\big(0.502842.., -1.545218.., -0.572025.., 0.848539..\big),$$
$$\big(0.442709.., -1.485086.., -0.427227.., 0.781483..\big),$$
$$\big(0.378567.., -1.420944.., -0.276975.., 0.700080..\big).$$

These solutions provide the three limit cycles with **Conf** 6^- and **Conf 7** shown in Figure 12A. This completes the proof of statement (v). □

Proof of statement (vi) of Theorem 2. In order to have limit cycles with **Conf** 6^+ and **Conf 8** simultaneously, the points of intersection of the limit cycles with **Conf** 6^+ and

Σ_{k+} must satisfy system (21), and the points of intersection of the limit cycles with **Conf 8** and Σ_{k+} must satisfy system (25). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. Without loss of generality we can consider that $k = 4$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

(31)

$$\dot{x} = -0.325270.. - 1.247316..x - \frac{3}{2}y,\ \dot{y} = -35.808990.. + 1.037199..x + 1.247316..y,$$
$$\dot{x} = -13.295856.. - 8.394370..x - 2y,\ \dot{y} = 47.825295.. + 35.232724..x + 8.394370..y,$$
$$\dot{x} = 33.366737.. - 7.961636..x - \frac{17}{5}y,\ \dot{y} = -44.896945.. + 18.643428..x + 7.961636..y,$$
$$\dot{x} = 65.056521.. - 17.010074..x - y,\ \dot{y} = -1052.5380642.. + 289.342621..x + 17.010074..y,$$
$$\dot{x} = 74.167422.. + 12.003227..x + \frac{3}{2}y,\ \dot{y} = -187.420662.. - 96.051650..x - 12.003227..y,$$

in the pieces $Z^1_{\Sigma_4}$, $Z^2_{\Sigma_4}$, $Z^3_{\Sigma_4}$, $Z^4_{\Sigma_4}$ and $Z^5_{\Sigma_4}$, respectively. For the discontinuous piecewise differential system (31), system (20), has the following three real solutions

$$\left(\tfrac{7}{2}, -\tfrac{6}{5}, \tfrac{2}{5}, \tfrac{19}{5}\right), \left(\tfrac{71}{20}, -\tfrac{143}{100}, \tfrac{31}{100}, \tfrac{389}{100}\right), \left(\tfrac{343}{100}, -0.893313.., 0.533583.., 3.654678..\right),$$

and (25), has the three real solutions

$$\left(4, -3, -\tfrac{16}{5}, \tfrac{23}{5}\right), \left(4.073407.., -3.179377.., -3.311999.., \tfrac{589}{125}\right),$$
$$\left(4.144187.., -3.341881.., -3.420000.., \tfrac{241}{50}\right).$$

These solutions provide the three limit cycles with **Conf** 6^+ and **Conf 8** shown in Figure 12B. This completes the proof of statement (vi). □

Proof of statement (vii) of Theorem 2. In order to have limit cycles with **Conf** 6^+ and **Conf** 9^+ simultaneously, the points of intersection of the limit cycles with **Conf** $6^+$and Σ_{k+} must satisfy system (20), and the points of intersection of the limit cycles with **Conf** 9^+ and Σ_{k+} must satisfy system (18) with $k > 0$. In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. We can suppose without loss of generality that $k = 4$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

(32)

$$\dot{x} = -17.085953.. + 9.977916..x - \frac{3}{2}y,\ \dot{y} = -113.972678.. + 66.372549..x - 9.977916..y,$$
$$\dot{x} = 34.897550.. - 4.677048..x - \frac{7}{2}y,\ \dot{y} = -44.332934.. + 6.249936..x + 4.677048..y,$$
$$\dot{x} = 65.922589.. - 17.491280..x - \frac{13}{10}y,\ \dot{y} = -924.494729.. + 235.342217..x + 17.491280..y,$$
$$\dot{x} = 13.883036.. + 3.280745..x - \frac{9}{2}y,\ \dot{y} = -44.382913.. + 2.391842..x - 3.280745..y,$$

in the pieces $Z^1_{\Sigma_4}$, $Z^3_{\Sigma_4}$, $Z^4_{\Sigma_4}$ and $Z^5_{\Sigma_4}$, respectively. For the discontinuous piecewise differential system (32), system (20), has the following three real solutions

$$\left(6, 1.209968.., \tfrac{7}{5}, 8.457532..\right), \left(\tfrac{156}{25}, 1.006799.., \tfrac{5}{4}, 8.915579..\right),$$
$$\left(\tfrac{117}{20}, 1.328327.., 1.486618.., 8.175706..\right),$$

and (18), has the three real solutions

$$\left(4, 3, \tfrac{16}{5}, 5\right), \left(4.109491.., \tfrac{141}{50}, \tfrac{303}{100}, \tfrac{517}{100}\right), \left(\tfrac{47}{10}, 2.053733.., 2.068270.., 6.131729..\right).$$

These solutions provide the three limit cycles with **Conf** 6^+ and **Conf** 9^+ shown in Figure 13. This completes the proof of statement (vii). □

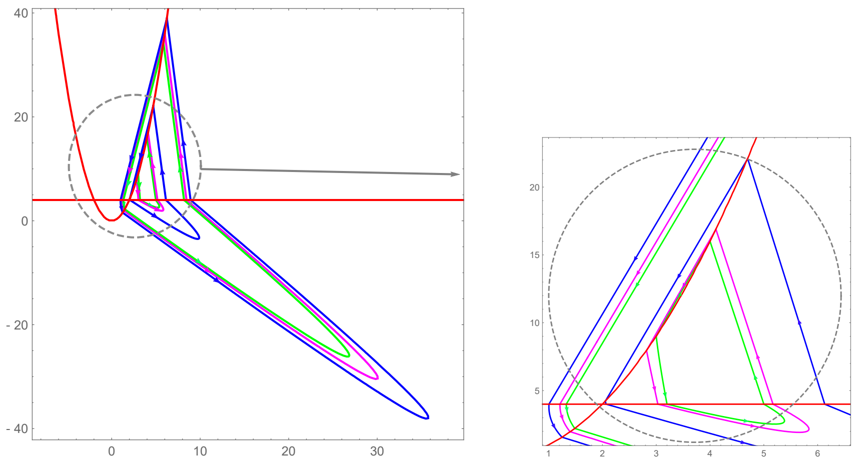


FIGURE 13. Three limit cycles with **Conf** 6^+ and **Conf** 9^+ of system (32).

Proof of statement (viii) of Theorem 2. In order to have limit cycles with **Conf 7** and **Conf 8** simultaneously, the points of intersection of the limit cycles with **Conf 7** and Σ_{k+} must satisfy system (23), and the points of intersection of the limit cycles with **Conf 8** and Σ_{k+} must satisfy system (25). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. Without loss of generality we can suppose that $k = 3$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems
(33)

$$\dot{x} = -453.807220.. - 20.414445..x - \frac{3}{2}y,\ \ \dot{y} = 83.559977.. + 277.833055..x + 20.414445..y,$$

$$\dot{x} = -29.218386.. - 5.465711..x + \frac{2}{5}y,\ \ \dot{y} = -414.702614.. - 74.685013..x + 5.465711..y,$$

$$\dot{x} = 22.645454.. - 36.659999..x - \frac{47}{5}y,\ \ \dot{y} = -35.463636.. + 142.973999..x + 36.659999..y,$$

$$\dot{x} = 3.918325.. - 3.744301..x + \frac{6}{5}y,\ \ \dot{y} = 33.556264.. - 11.683162..x + 3.744301..y,$$

$$\dot{x} = 5.300000.. - 8.579999..x - \frac{11}{5}y,\ \ \dot{y} = -8.300000.. + 33.461999..x + 8.579999..y,$$

in the zones $Z^1_{\Sigma_3}$, $Z^2_{\Sigma_3}$, $Z^3_{\Sigma_3}$, $Z^4_{\Sigma_3}$ and $Z^5_{\Sigma_3}$, respectively. For the discontinuous piecewise differential system (33), system (23), has the following three real solutions

$$\left(0.502842.., -1.545218.., -0.572025.., 0.848539..\right),$$
$$\left(0.442709.., -1.485086.., -0.427227.., 0.781483..\right),$$
$$\left(0.378567.., -1.420944.., -0.276975.., 0.700080..\right),$$

and (25), has the three real solutions

$$\left(\tfrac{84}{25}, -\tfrac{79}{20}, -4.562376.., \tfrac{88}{25}\right), \left(\tfrac{3297}{1000}, -\tfrac{387}{100}, -4.492376.., \tfrac{69}{20}\right), \left(\tfrac{34301}{10000}, -4.039424.., -4.622376.., \tfrac{179}{50}\right).$$

These solutions provide the three limit cycles with **Conf 7** and **Conf 8** shown in Figure 14. This completes the proof of statement (viii). □

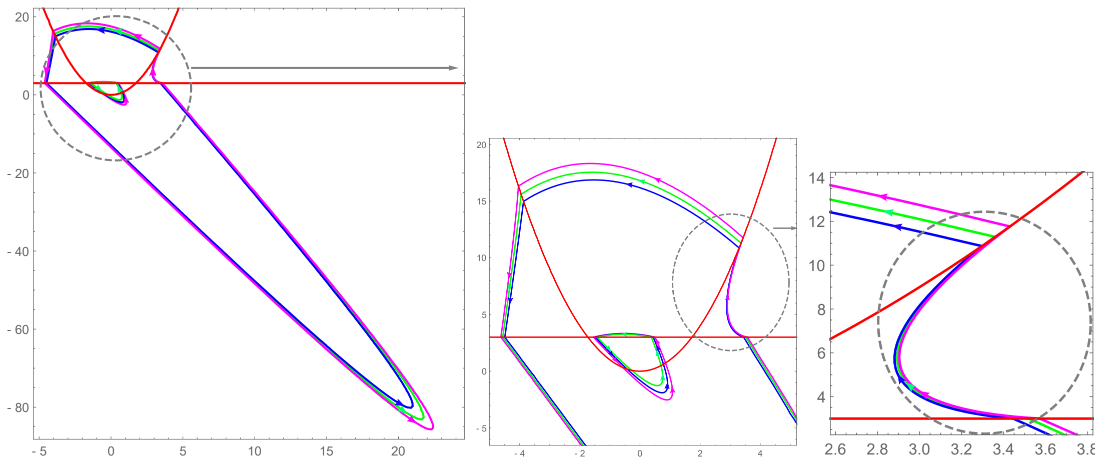


FIGURE 14. Three limit cycles with **Conf 7** and **Conf 8** of system (33).

Proof of statement (ix) of Theorem 2. In order to have limit cycles with **Conf 8** and **Conf** 9^+ simultaneously, the points of intersection of the limit cycles with **Conf 8** and Σ_{k^+} must satisfy system (25), and the points of intersection of the limit cycles with **Conf** 9^+ and Σ_{k^+} must satisfy system (18) with $k > 0$. In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with both configurations is six. Moreover this upper bound is reached. Without loss of generality we can suppose that $k = 4$. We consider the discontinuous piecewise linear differential system defined by the following four linear Hamiltonian systems

(34)

$$\dot{x} = -17.085953.. + 9.977916..x - \frac{3}{2}y,\ \dot{y} = -113.972678.. + 66.372549..x - 9.977916..y,$$

$$\dot{x} = -23.136372.. + 2.354826..x - \frac{3}{10}y,\ \dot{y} = 81.642102.. + 18.484031..x - 2.354826..y,$$

$$\dot{x} = \frac{431}{10} + 14.700000..x - \frac{7}{2}y,\ \dot{y} = -194.334000.. + \frac{3087}{50}x - 14.700000..y,$$

$$\dot{x} = 65.922589.. - 17.491280..x - \frac{13}{10}y,\ \dot{y} = -924.494729.. + 235.342217..x + 17.491280..y,$$

in the zones $Z^1_{\Sigma_4}$, $Z^2_{\Sigma_4}$, $Z^3_{\Sigma_4}$ and $Z^4_{\Sigma_4}$, respectively. For the discontinuous piecewise differential system (34), system (25), has the following three real solutions

$$\left(9, -2.341532.., -6.604799.., 14.804799..\right), \left(\tfrac{457}{50}, -2.481840.., -6.933823.., 15.133823..\right), \left(\tfrac{187}{20}, -2.692262.., -7.432829.., 15.632829..\right),$$

and (18), has the three real solutions

$$\left(4, 3, \tfrac{16}{5}, 5\right), \left(4.109491.., \tfrac{141}{50}, \tfrac{303}{100}, \tfrac{517}{100}\right), \left(\tfrac{47}{10}, 2.053733.., 2.068270.., 6.131729..\right).$$

These solutions provide the three limit cycles with **Conf 8** and **Conf** 9^+ shown in Figure 15. This completes the proof of statement (ix). □

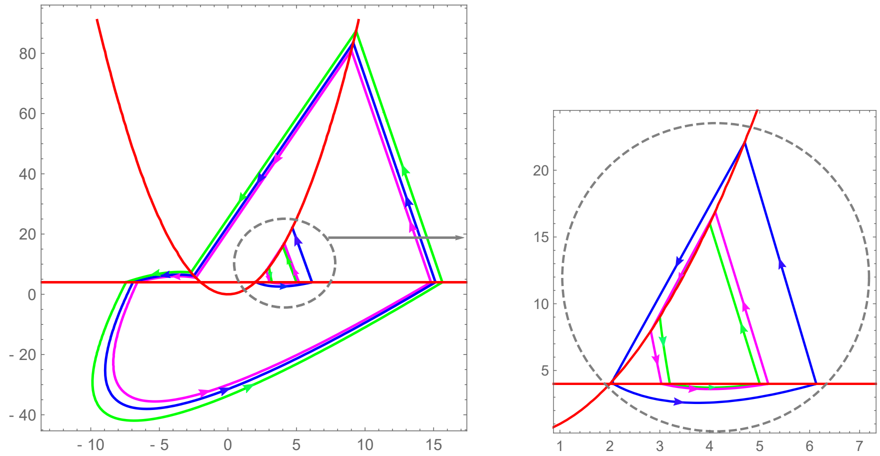


FIGURE 15. Three limit cycles with **Conf 8** and **Conf** 9^{+} of system (34).

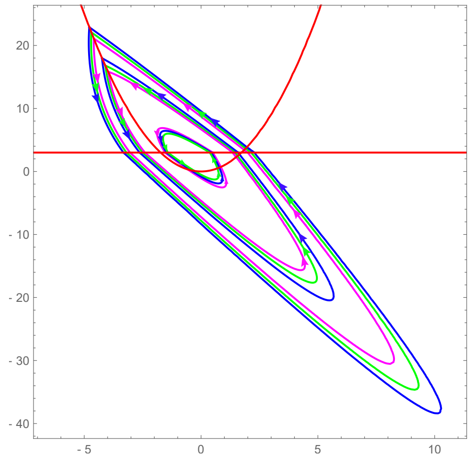


FIGURE 16. Three limit cycles with **Conf** 6^{-}, **Conf 7** and **Conf 8** of system (35).

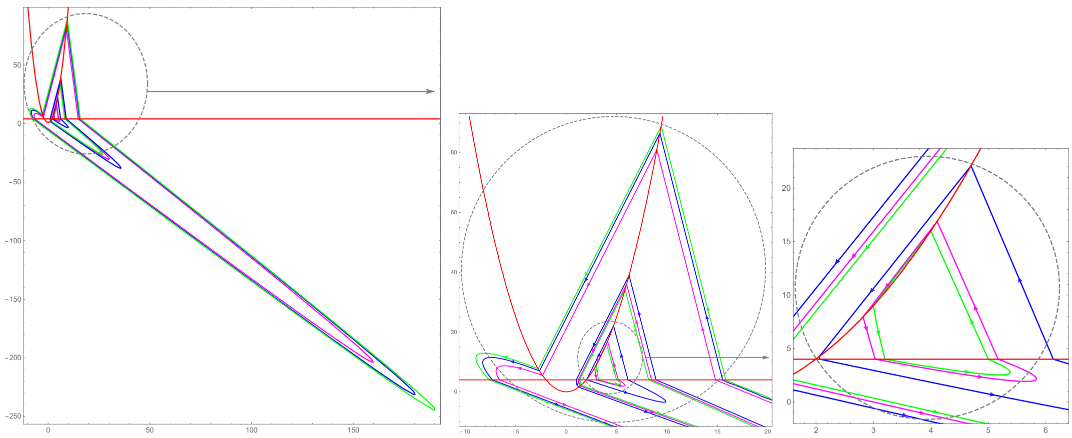


FIGURE 17. Three limit cycles with **Conf** 6^{+}, **Conf 8** and **Conf** 9^{+} of system (36).

Proof of statement (x) of Theorem 2. In order to have limit cycles with **Conf** 6^-, **Conf 7** and **Conf 8** simultaneously, the points of intersection of the limit cycles with **Conf** 6^- and Σ_{k+} must satisfy system (21), the points of intersection of the limit cycles with **Conf 7** and Σ_{k+} must satisfy system (23) and the points of intersection of the limit cycles with **Conf 8** and Σ_{k+} must satisfy system (25). In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with the three configurations simultaneously, is nine. Moreover this upper bound is reached. Without loss of generality we can suppose that $k = 3$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

(35)

$$\dot{x} = -0.567977.. - 5.151614..x - \frac{3}{2}y,\ \dot{y} = -6.233588.. + 17.692757..x + 5.151614..y,$$
$$\dot{x} = 11.250254.. + 0.637407..x - \frac{2}{5}y,\ \dot{y} = -35.085985.. + 1.015720..x - 0.637407..y,$$
$$\dot{x} = 22.645454.. - 36.659999..x - \frac{47}{5}y,\ \dot{y} = -35.463636.. + 142.973999..x + 36.659999..y,$$
$$\dot{x} = -13.170507.. + 3.348185..x + \frac{6}{5}y,\ \dot{y} = 36.547853.. - 9.341954..x - 3.348185..y,$$
$$\dot{x} = 5.300000.. - 8.579999..x - \frac{11}{5}y,\ \dot{y} = -8.300000.. + 33.461999..x + 8.579999..y,$$

in the zones $Z^1_{\Sigma_3}$, $Z^2_{\Sigma_3}$, $Z^3_{\Sigma_3}$, $Z^4_{\Sigma_3}$ and $Z^5_{\Sigma_3}$, respectively. For the discontinuous piecewise differential system (35), system (21), has the following three real solutions

$$\big(-\tfrac{41}{10}, -\tfrac{5}{2}, 1.569412.., 1.457623..\big), \big(-\tfrac{106}{25}, -\tfrac{263}{100}, 1.647799.., 1.587623..\big),$$
$$\big(-\tfrac{199}{50}, -2.401954.., 1.508473.., 1.359577..\big),$$

and (23), has the three real solutions

$$\big(0.502842.., -1.545218.., -0.572025.., 0.848539..\big),$$
$$\big(0.442709.., -1.485086.., -0.427227.., 0.781483..\big),$$
$$\big(0.378567.., -1.420944.., -0.276975.., 0.700080..\big),$$

and system (25), has the three real solutions

$$\big(1.847758.., -4.593279.., -3.042376.., 2\big), \big(1.910216.., -4.699349.., -3.192376.., \tfrac{43}{20}\big),$$
$$\big(1.962805.., -4.784530.., -3.322376.., \tfrac{57}{25}\big).$$

These solutions provide the three limit cycles with **Conf** 6^-, **Conf 7** and **Conf 8** shown in Figure 16. This completes the proof of statement (x). □

Proof of statement (xi) of Theorem 2. In order to have limit cycles with **Conf** 6^+,**Conf 8** and **Conf** 9^+ simultaneously, the points of intersection of the limit cycles with **Conf** 6^+ and Σ_{k+} must satisfy system (20), the points of intersection of the limit cycles with **Conf 8** and Σ_{k+} must satisfy system (25) and the points of intersection of the limit cycles with **Conf** 9^+ and Σ_{k+} must satisfy system (18) with $k > 0$. In statement (ii) we proved that the maximum number of limit cycles with each configuration is three, then we have that the maximum number of limit cycles with the three configurations, is nine. Moreover this upper bound is reached. Without loss of generality we can suppose that $k = 3$. We consider the discontinuous piecewise linear differential system defined by the following five linear

Hamiltonian systems
(36)

$$\dot{x} = -17.085953.. + 9.977916..x - \frac{3}{2}y,\ \ \dot{y} = -113.972678.. + 66.372549..x - 9.977916..y,$$

$$\dot{x} = -2.306102.. - 0.633078..x - \frac{3}{10}y,\ \ \dot{y} = 2.662449.. + 1.335961..x + 0.633078..y,$$

$$\dot{x} = 34.897550.. - 4.677048..x - \frac{7}{2}y,\ \ \dot{y} = -44.332934.. + 6.249936..x + 4.677048..y,$$

$$\dot{x} = 65.922589.. - 17.491280..x - \frac{13}{10}y,\ \ \dot{y} = -924.494729.. + 235.342217..x + 17.491280..y,$$

$$\dot{x} = 13.883036.. + 3.280745..x - \frac{9}{2}y,\ \ \dot{y} = -44.382913.. + 2.391842..x - 3.280745..y,$$

in the zones $Z^1_{\Sigma_4}$, $Z^2_{\Sigma_4}$, $Z^3_{\Sigma_4}$, $Z^4_{\Sigma_4}$ and $Z^5_{\Sigma_4}$, respectively. For the discontinuous piecewise differential system (36), system (20), has the following three real solutions

$$\big(6, 1.209968.., \tfrac{7}{5}, 8.457532..\big), \big(\tfrac{156}{25}, 1.006799.., \tfrac{5}{4}, 8.915579..\big),$$
$$\big(\tfrac{117}{20}, 1.328327.., 1.486618.., 8.175706..\big),$$

and (25), has the three real solutions

$$\big(9, -2.341532.., -6.604799.., 14.804799..\big), \big(\tfrac{93}{10}, -2.642166.., -7.313423.., 15.513423..\big),$$
$$\big(\tfrac{943}{100}, -2.772412.., -7.624652.., 15.824652..\big),$$

and system (18), has the three real solutions

$$\big(4, 3, \tfrac{16}{5}, 5\big), \big(4.109491.., \tfrac{141}{50}, \tfrac{303}{100}, \tfrac{517}{100}\big), \big(\tfrac{47}{10}, 2.053733.., 2.068270.., 6.131729..\big).$$

These solutions provide the three limit cycles with **Conf** 6^+, **Conf 8** and **Conf** 9^+ shown in Figure 17. This completes the proof of statement (xi). □

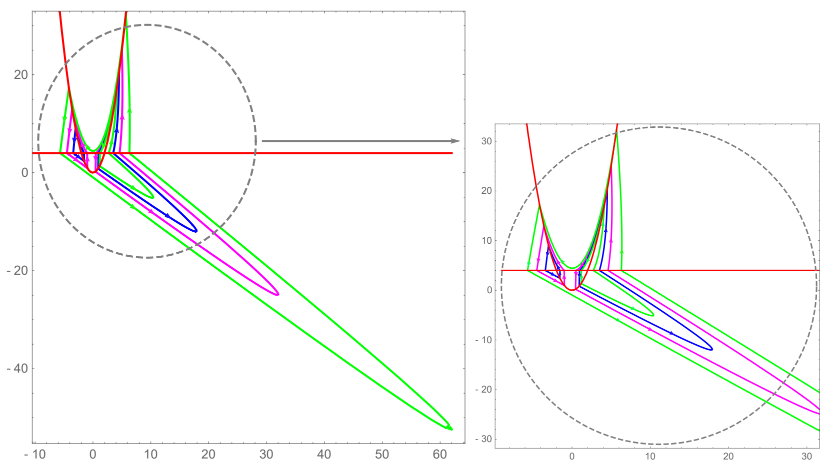


FIGURE 18. Three limit cycles with **Conf** 6^+, two limit cycles with **Conf** 6^- and one limit cycle with **Conf** 8 of system (37).

Proof of statement (xii) of Theorem 2. In order to have limit cycles with **Conf** 6^+,**Conf** 6^- and **Conf** 8 simultaneously, the points of intersection of the limit cycles with **Conf** 6^+ and Σ_{k^+} must satisfy system (20), the points of intersection of the limit cycles with **Conf** 6^- and Σ_{k^+} must satisfy system (21) and the points of intersection of the limit cycles with **Conf 8** and Σ_{k^+} must satisfy system (25). If we suppose that there is one solution for each

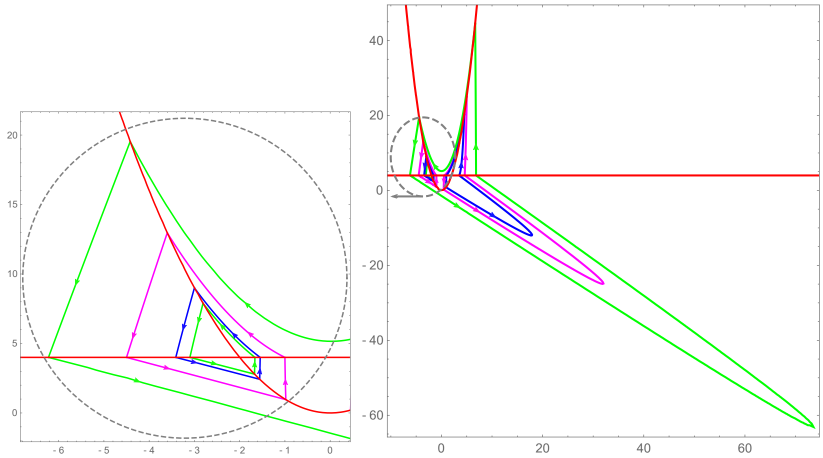


FIGURE 19. Two limit cycles with **Conf** 6^+, three limit cycles with **Conf** 6^- and one limit cycle with **Conf** 8 of system (38).

system (20) and (21), then similar to statement (i) of Theorem 2, we obtain the value of the parameters $\gamma_1, \delta_1, \gamma_2, \gamma_3, \delta_3, \gamma_4, \gamma_5, \delta_5$.

Now we have two options, first we suppose that there is a solution of system (25), then we obtain the value of the parameters $\lambda_1, \delta_2, \lambda_3$ and δ_4, therefore we have two options, first we can suppose that there is a second solution of system (25) then we obtain the value of the parameters λ_2 and λ_4, hence in the zones $Z^1_{\Sigma+}, Z^2_{\Sigma+}, Z^3_{\Sigma+}, Z^4_{\Sigma+}$ we only have the parameters b_1, b_2, b_3, b_4 as unknowns and in the zone $Z^5_{\Sigma+}$ we have λ_5, b_5 as unknowns. Therefore we can obtain at most one solution either of system (20) or of system (21) and we cannot obtain more solutions of systems (20), (21) and (25), because we would have that $b_i = 0$ for some $i = 1, 2, 3, 4, 5$. Hence we would have five limit cycles with two (resp. one) limit cycles with **Conf** 6^+, one (resp. two) limit cycle(s) with **Conf** 6^- and two limit cycles with **Conf** 8. Second we can suppose that there is a second solution for each system (20) and (21), then we obtain the values of the parameters $\lambda_2, \lambda_4, \lambda_5$ hence we cannot obtain more solutions of systems (20), (21) and (25), because we would have that $b_i = 0$ for some $i = 1, 2, 3, 4, 5$. Therefore in this case we would obtain five limit cycles with two limit cycles with **Conf** 6^+, two limit cycles with **Conf** 6^- and one limit cycle with **Conf** 8.

Second, after considering the first solution of each system (20) and (21), we can suppose that there is a second solution for each system (20) and (21), then we obtain the values of $\lambda_1, \delta_2, \lambda_3$ and δ_4. Then in the zones $Z^1_{\Sigma+}, Z^3_{\Sigma+}$ and $Z^5_{\Sigma+}$ we only have the parameters b_1, b_3 and b_5 as unknowns and in the zones $Z^2_{\Sigma+}$ and $Z^4_{\Sigma+}$ we have the parameters $\lambda_2, b_2, \lambda_4, b_4$ unknowns. Hence can have two cases, first we can suppose that there is a solution of system (25), then we determine the value of parameters λ_2 and λ_4, hence we cannot to have more limit cycles because we would have that $b_i = 0$ for some $i = 1, 2, 3, 4, 5$. Therefore we would have five limit cycles with two limit cycles with **Conf** 6^+, two limit cycles with **Conf** 6^- and one limit cycle with **Conf** 8. Second we can suppose that there is a third solution of system (20) (resp. (21)) and we obtain the value of parameter λ_4 (res. λ_2), then in the zone $Z^4_{\Sigma+}$ (resp. $Z^2_{\Sigma+}$) we only have the parameter b_4 (resp. b_2) as unknown and in the zone $Z^2_{\Sigma+}$ (resp. $Z^4_{\Sigma+}$) we have that the parameters λ_2, b_2 (res. λ_4, b_4) as unknowns. Now we suppose that there is one solution of system (25) and we obtain the parameter λ_2 (res. λ_4). We observe that we cannot obtain more solutions of systems (20), (21) and (25), because we would have that $b_i = 0$ for some $i = 1, 2, 3, 4, 5$. Therefore we have at most six limit cycles with three (resp. two) limit cycles with **Conf** 6^+, two (resp. three) limit cycles with **Conf** 6^- and one limit cycle with **Conf** 8. We observe that these six limit cycles can be either three limit cycles with **Conf** 6^+, two limit cycles with **Conf** 6^- and one limit cycle with **Conf** 8 or two limit cycles with **Conf** 6^+, three limit cycles with **Conf** 6^- and one limit cycle with **Conf** 8. We shall give an example of each case.

We observe that without loss of generality we can suppose that $k = 4$. We consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

(37)

$$\dot{x} = -23138.489410.. + 403.676452..x + \frac{9}{2}y,\ \dot{y} = 2942.120325.. - 36212.150741..x - 403.676452..y,$$

$$\dot{x} = 1.812606.. + 1.308936..x - \frac{3}{10}y,\ \dot{y} = -25.828218.. + 5.711045..x - 1.308936..y,$$

$$\dot{x} = 15.472057.. - 3.117904..x - \frac{17}{5}y,\ \dot{y} = -13.354567.. + 2.859213..x + 3.117904..y,$$

$$\dot{x} = 48.158492.. - 6.082779..x - y,\ \dot{y} = -31.590984.. + 37.000210..x + 6.082779..y,$$

$$\dot{x} = -151.854124.. - 136.354901..x + \frac{3}{2}y,\ \dot{y} = -10611.949690.. - 12395.106180..x + 136.354901..y,$$

in the zones $Z^1_{\Sigma_3}$, $Z^2_{\Sigma_3}$, $Z^3_{\Sigma_3}$, $Z^4_{\Sigma_3}$ and $Z^5_{\Sigma_3}$, respectively. For the discontinuous piecewise differential system (37), system (20) has the following three real solutions

$$\left(5, \tfrac{1}{2}, \tfrac{9}{20}, \tfrac{23}{5}\right), \left(\tfrac{9}{2}, \tfrac{19}{20}, \tfrac{91}{100}, \tfrac{7}{2}\right), \left(\tfrac{41}{10}, 1.196150.., 1.163297.., 2.719447..\right);$$

and (21) has the two real solutions

$$\left(-\tfrac{18}{5}, -\tfrac{9}{2}, -\tfrac{49}{50}, -1\right), \left(-3, -3.411586.., -1.557354.., -1.546135..\right);$$

and system (25) has the real solution

$$\left(5.688640.., -4.154651.., -5.682382.., \tfrac{63}{10}\right).$$

These solutions provide the three limit cycles with **Conf** 6^+, the two limits cycles with **Conf** 6^- and the limit cycle with **Conf** 8 shown in Figure 18.

Now we consider the discontinuous piecewise linear differential system defined by the following five linear Hamiltonian systems

(38)

$$\dot{x} = -23138.489410.. + 403.676452..x + \frac{9}{2}y,\ \dot{y} = 2942.120325.. - 36212.150741..x - 403.676452..y,$$

$$\dot{x} = 4.276633.. + 1.873985..x - \frac{3}{10}y,\ \dot{y} = 4.991226.. + 11.706072..x - 1.873985..y,$$

$$\dot{x} = 15.472057.. - 3.117904..x - \frac{17}{5}y,\ \dot{y} = -13.354567.. + 2.859213..x + 3.117904..y,$$

$$\dot{x} = 293.931246.. - 44.804431..x - y,\ \dot{y} = -6905.938713.. + 2007.437106..x + 44.804431..y,$$

$$\dot{x} = -151.854124.. - 136.354901..x + \frac{3}{2}y,\ \dot{y} = -10611.949690.. - 12395.106180..x + 136.354901..y,$$

in the zones $Z^1_{\Sigma_3}$, $Z^2_{\Sigma_3}$, $Z^3_{\Sigma_3}$, $Z^4_{\Sigma_3}$ and $Z^5_{\Sigma_3}$, respectively. For the discontinuous piecewise differential system (38), system (20) has the following two real solutions

$$\left(5, \tfrac{1}{2}, \tfrac{9}{20}, \tfrac{23}{5}\right)\left(\tfrac{9}{2}, \tfrac{19}{20}, \tfrac{91}{100}, \tfrac{7}{2}\right);$$

and (21) has the three real solutions

$$\big(-\tfrac{18}{5},-\tfrac{9}{2},-\tfrac{49}{50},-1\big),\big(-3,-3.411586..,-1.557354..,-1.546135..\big),$$
$$\big(-2.809209..,-\tfrac{31}{10},-1.671884..,-1.662653..\big);$$

and system (25) has the real solution

$$\big(\tfrac{667}{100},-4.419374..,-6.225080..,6.842698..\big).$$

These solutions provide the two limit cycles with **Conf** 6^+, the three limits cycles with **Conf** 6^- and the limit cycle with **Conf** 8 shown in Figure 19. This completes the proof of statement (xii). □

4. The Appendix

Here we provide the values A, B and C

$$\begin{aligned}A = {} & \csc\left(\tfrac{r_3-s_3}{2}\right)\Big(-\cos\left(\tfrac{1}{2}(3r_1-r_2+2r_3+s_1-s_2)\right)+\cos\Big(\tfrac{1}{2}(r_1-r_2+4r_3\\
&+s_1-s_2)\Big)+\cos\left(\tfrac{1}{2}(r_1-3r_2-2r_3+s_1-s_2)\right)-\cos\Big(\tfrac{1}{2}(r_1-r_2-4r_3\\
&+s_1-s_2)\Big)-\cos\left(\tfrac{1}{2}(r_1-r_2+2r_3+3s_1-s_2)\right)-\cos\Big(\tfrac{1}{2}(3r_1+r_2-2r_3\\
&+s_1+s_2)\Big)+\cos\left(\tfrac{1}{2}(r_1+3r_2-2r_3+s_1+s_2)\right)-\cos\Big(\tfrac{1}{2}(r_1+r_2-2r_3\\
&+3s_1+s_2)\Big)+\cos\left(\tfrac{1}{2}(r_1+r_2-2r_3+s_1+3s_2)\right)+\cos\Big(\tfrac{1}{2}(r_1-r_2-2r_3\\
&+s_1-3s_2)\Big)+\cos\left(\tfrac{1}{2}(3r_1-r_2+s_1-s_2+2s_3)\right)+\cos\Big(\tfrac{1}{2}(r_1-r_2+3s_1\\
&-s_2+2s_3)\Big)-\cos\left(\tfrac{1}{2}(r_1-r_2+s_1-s_2+4s_3)\right)-\cos\Big(\tfrac{1}{2}(r_1-3r_2+s_1\\
&-s_2-2s_3)\Big)+\cos\left(\tfrac{1}{2}(3r_1+r_2+s_1+s_2-2s_3)\right)-\cos\Big(\tfrac{1}{2}(r_1+3r_2+s_1\\
&+s_2-2s_3)\Big)+\cos\left(\tfrac{1}{2}(r_1+r_2+3s_1+s_2-2s_3)\right)-\cos\Big(\tfrac{1}{2}(r_1+r_2+s_1\\
&+3s_2-2s_3)\Big)-\cos\left(\tfrac{1}{2}(r_1-r_2+s_1-3s_2-2s_3)\right)+\cos\Big(\tfrac{1}{2}(r_1-r_2+s_1\\
&-s_2-4s_3)\Big)\Big),\end{aligned}$$

$$\begin{aligned}B = {} & -(\cos(r_1-r_2)+\cos(r_1-r_3)+\cos(r_2-r_3)-2\cos(r_1-s_1)+\cos(r_2-s_1)\\
&+\cos(r_3-s_1)+\cos(r_1-s_2)-2\cos(r_2-s_2)+\cos(r_3-s_2)+\cos(s_1-s_2)\\
&+2\cos(r_1-r_2+s_1-s_2)+\cos(2r_1-2r_2+s_1-s_2)-\cos(2r_1-r_2-r_3\\
&+s_1-s_2)-\cos(r_1-2r_2+r_3+s_1-s_2)+\cos(r_1-2r_2+2s_1-s_2)-\cos(r_1\\
&-r_2-r_3+2s_1-s_2)+\cos(2r_1-r_2+s_1-2s_2)-\cos(r_1-r_2+r_3+s_1-2s_2)\\
&+\cos(r_1-r_2+2s_1-2s_2)+\cos(r_1-s_3)+\cos(r_2-s_3)-2\cos(r_3-s_3)\\
&+\cos(s_1-s_3)+2\cos(r_1-r_3+s_1-s_3)-\cos(2r_1-r_2-r_3+s_1-s_3)\\
&+\cos(2r_1-2r_3+s_1-s_3)-\cos(r_1+r_2-2r_3+s_1-s_3)-\cos(r_1-r_2-r_3\\
&+2s_1-s_3)+\cos(r_1-2r_3+2s_1-s_3)-\cos(2r_1-r_2+s_1-s_2-s_3)-\cos(2r_1\\
&-r_3+s_1-s_2-s_3)+\cos(r_1+r_2-r_3+s_1-s_2-s_3)+\cos(r_1-r_2+r_3+s_1\\
&-s_2-s_3)-\cos(r_1-r_2+2s_1-s_2-s_3)-\cos(r_1-r_3+2s_1-s_2-s_3)+\cos(s_2\\
&-s_3)+2\cos(r_2-r_3+s_2-s_3)-\cos(r_1+r_2-2r_3+s_2-s_3)+\cos(2r_2-2r_3\\
&+s_2-s_3)+\cos(r_1+r_2-r_3-s_1+s_2-s_3)-\cos(2r_2-r_3-s_1+s_2-s_3)\\
&+\cos(r_1-r_2-r_3+s_1+s_2-s_3)-\cos(r_1-2r_3+s_1+s_2-s_3)-\cos(r_2-2r_3\\
&+s_1+s_2-s_3)+\cos(r_2-2r_3+2s_2-s_3)-\cos(r_2-r_3-s_1+2s_2-s_3)-\cos(r_1\\
&-2r_2+r_3-s_2+s_3)+\cos(r_1-r_2+r_3-s_1-s_2+s_3)-\cos(r_1-2r_2+s_1-s_2\\
&+s_3)+\cos(r_1-r_2-r_3+s_1-s_2+s_3)-\cos(r_1-r_2+r_3-2s_2+s_3)-\cos(r_1\\
&-r_2+s_1-2s_2+s_3)+\cos(2r_1-r_3+s_1-2s_3)-\cos(r_1+r_2-r_3+s_1-2s_3)\\
&+\cos(r_1-r_3+2s_1-2s_3)-\cos(r_1+r_2-r_3+s_2-2s_3)+\cos(2r_2-r_3+s_2\\
&-2s_3)-\cos(r_1-r_3+s_1+s_2-2s_3)-\cos(r_2-r_3+s_1+s_2-2s_3)+\cos(r_2-r_3\\
&+2s_2-2s_3)-6)\csc^2\left(\tfrac{1}{2}(r_1-r_2+s_1-s_2)\right)\sin^2\left(\tfrac{r_3-s_3}{2}\right),\end{aligned}$$

$$
\begin{aligned}
C = \quad & 2\Big(-\cos\Big(\tfrac{1}{2}(r_1 - 3r_2 - r_3 + s_1 - s_2 - s_3)\Big) + \cos\Big(\tfrac{1}{2}(r_1 - r_2 - 3r_3 + s_1 - s_2 - s_3)\Big) \\
& -\cos\Big(\tfrac{1}{2}(3r_1 + r_2 - r_3 + s_1 + s_2 - s_3)\Big) + \cos\Big(\tfrac{1}{2}(r_1 + 3r_2 - r_3 + s_1 + s_2 - s_3)\Big) \\
& -\cos\Big(\tfrac{1}{2}(r_1 + r_2 - r_3 + 3s_1 + s_2 - s_3)\Big) + \cos\Big(\tfrac{1}{2}(r_1 + r_2 - r_3 + s_1 + 3s_2 - s_3)\Big) \\
& -\cos\Big(\tfrac{1}{2}(r_1 - r_2 - r_3 + s_1 - 3s_2 - s_3)\Big) + \cos\Big(\tfrac{1}{2}(3r_1 - r_2 + r_3 + s_1 - s_2 + s_3)\Big) \\
& -\cos\Big(\tfrac{1}{2}(r_1 - r_2 + 3r_3 + s_1 - s_2 + s_3)\Big) + \cos\Big(\tfrac{1}{2}(r_1 - r_2 + r_3 + 3s_1 - s_2 + s_3)\Big) \\
& -\cos\Big(\tfrac{1}{2}(r_1 - r_2 + r_3 + s_1 - s_2 + 3s_3)\Big) + \cos\Big(\tfrac{1}{2}(r_1 - r_2 - r_3 + s_1 - s_2 - 3s_3)\Big)\Big).
\end{aligned}
$$

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