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Meromorphic integrability of the Hamiltonian systems with homogeneous potentials of degree -4 *

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Abstract We characterize the meromorphic Liouville integrability of the Hamiltonian systems with Hamiltonian $H = \left(p_1^2 + p_2^2\right)/2 + V\left(q_1, q_2\right)$, being $V\left(q_1, q_2\right)$ a homogeneous potential of degree -4, with the exception of the potential $V_8 = 1/\left(q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4\right)$ when $\mu \in \{-5/3, -2/3\}$. For this potential we only can prove that it has no polynomial first integral.

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1 Introduction and main results

Hamiltonian systems play an important role in the theory of the dynamical systems due to the fact that these systems occur frequently in mathematical physics, particularly in mechanics, engineering and other fields. In order to describe global information on the Hamiltonian systems is good to find sufficient number of first integrals. The fact that a Hamiltonian system has some additional first integral independent with its Hamiltonian is a rare phenomenon which lead to a difficult problem, how to determine whether a given Hamiltonian system has additional independent first integrals.

In this work we consider the Hamiltonian systems of two degrees of freedom

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1, 2, \tag{1.1}$$

with the Hamiltonian

$$H = \frac{1}{2}\sum_{i=1}^{2} p_i + V\left(\mathbf{q}\right), \tag{1.2}$$

where $V\left(\mathbf{q}\right) = V\left(q_1, q_2\right) \in \mathbb{C}\left[q_1, q_2\right]$ is a homogeneous polynomial potential of degree $k \in \mathbb{Z}$.

Let $A = A\left(\mathbf{q}, \mathbf{p}\right)$ and $B = B\left(\mathbf{q}, \mathbf{p}\right)$ be two functions with $\mathbf{q} = \left(q_1, q_2\right)$ and $\mathbf{p} = \left(p_1, p_2\right)$. Their *Poisson bracket* is defined as

$$\{A, B\} = \sum_{i=1}^{2}\left(\frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}\right). \tag{1.3}$$

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The functions A and B are *in involution* if $\{A,B\}=0$. A non-constant function $I=I(\mathbf{q},\mathbf{p})$ is called a *first integral* of the Hamiltonian system (1.1) if I is in involution with the Hamiltonian function H, i.e. $\{H,I\}=0$. Since the Poisson bracket is antisymmetric, it is obvious that H itself is always a first integral. The functions H and I are *functionally independent* if their gradients are linearly independent for all points of $\mathbb{C}^4$ except perhaps in a zero Lebesgue measure set. The Hamiltonian system (1.1) is *completely or Liouville integrable* if it has two functionally independent first integrals H and I.

During the past four decades, there have had an extensive study on the integrability of the Hamiltonian systems (1.1), as it is shown in the papers [1, 2, 4-6, 8, 9, 14, 15, 19]. The fundamental tools to investigate the integrability problem for system (1.1) are Painlevé test [7] and direct methods [10]. In [22, 23] Ziglin proved a relation between the integrability of Hamiltonian systems and the monodromy group of variational equations along a particular solution, and gave the necessary conditions of integrability for complex Hamiltonian systems. Yoshida proved some criteria for the nonexistence of an additional first integral in Hamiltonian systems with homogeneous potentials, see [20, 21]. Using the differential Galois group Morales and Ramis [18] obtained necessary conditions for the existence of an additional meromorphic first integral of Hamiltonian systems (1.1) with homogeneous potentials.

Recently Maciejewski and Przybylska [16] completely solved the meromorphic integrability problem of Hamiltonian systems (1.1) with homogeneous polynomial potentials of degree 3. Maciejewski et al. [17] and Llibre et al. [12] characterized all integrable Hamiltonian systems (1.1) with homogeneous polynomial potentials of degree 4. The integrability of Hamiltonian systems (1.1) with homogeneous potentials of degrees $2,1,0,-1$ or -2 were characterized in [13]. We note that Hamiltonian systems (1.1) with homogeneous potentials of degrees $2,1,0$ or -1 has always a polynomial first integral independent with the Hamiltonian. The polynomial integrability of Hamiltonian systems (1.1) with homogeneous potentials of degree -3 have been classified in [11].

The objective of this paper is to study the integrability of the Hamiltonian system (1.1) with homogeneous potentials of degree -4, i.e. with

$$V=V(q_1,q_2)=\frac{1}{aq_1^4+bq_1^3q_2+cq_1^2q_2^2+dq_1q_2^3+eq_2^4}=\frac{1}{Q(q_1,q_2)}, \tag{1.4}$$

where $Q(q_1,q_2)\not\equiv 0$.

Our main results are the following.

Proposition 1. *After a linear change of variables and a rescaling, the potential (1.4) can be written in one of the following forms:*

$$V_0=\pm\frac{1}{q_1^4};\ V_1=\frac{1}{4q_1^3q_2};\ V_2=\pm\frac{1}{6p_1^2p_2^2};\ V_3=\pm\frac{1}{(q_1^2+q_2^2)^2};\ V_4=\pm\frac{1}{q_2^2(6q_1^2-q_2^2)};\ V_5=\pm\frac{1}{q_2^2(6q_1^2+q_2^2)};$$
$$V_6=\frac{1}{q_1^4+6\mu q_1^2q_2^2-q_2^4};\ V_7=-\frac{1}{q_1^4+6\mu q_1^2q_2^2+q_2^4}\ \textit{with}\ \mu>-\tfrac{1}{3}\ \textit{and}\ \mu\neq\tfrac{1}{3};\ V_8=\frac{1}{q_1^4+6\mu q_1^2q_2^2+q_2^4}\ \textit{with}\ \mu\neq\pm\tfrac{1}{3}.$$

Proposition 1 is proved in Section 4.

Theorem 1. *Hamiltonian system (1.1) with the potential:*

(a) V_0 is completely integrable with the additional polynomial first integral p_2.

(b) V_3 is completely integrable with the additional polynomial first integral $q_2p_1-q_1p_2$.

(c) V_i for $i=1,2,4,5,6,7$ does not admit an additional meromorphic first integral.

(d) V_8 does not admit an additional meromorphic first integral if $\mu\notin\{-\frac{5}{3},-\frac{2}{3}\}$. If $\mu\in\{-\frac{5}{3},-\frac{2}{3}\}$ then V_8 does not admit an additional polynomial first integral.

The proof of Theorem 1 is given in Section 5.

2 Darboux point

The point $\mathbf{d} = (d_1, d_2) \in \mathbb{C}^2 \setminus \{0\}$ is called a *Darboux point* of system (1.1) if it satisfies the gradient equation

$$V'(\mathbf{d}) = \gamma \mathbf{d}, \tag{2.1}$$

where $V'(\mathbf{d})$ is the gradient of $V(\mathbf{d})$ and $\gamma \in \mathbb{C} \setminus \{0\}$. If $\mathbf{d}$ is a Darboux point, then the point $\tilde{\mathbf{d}} = \omega \mathbf{d}$, where $\omega \in \mathbb{C} \setminus \{0\}$ satisfies $V'\left(\tilde{\mathbf{d}}\right) = \omega^{k-2}\gamma\tilde{\mathbf{d}} = \tilde{\gamma}\tilde{\mathbf{d}}$, and consequently $\tilde{\mathbf{d}}$ is also a Darboux point. Thus these Darboux points form an equivalence class. We can view a Darboux point $\mathbf{d}$ as a point in a projective space $\mathbf{d} = [d_1 : d_2] \in \mathbb{CP}^1$, for more details see [17].

Given a Darboux point $\mathbf{d}$ we consider the Hessian matrix $V''(\mathbf{q})$ evaluated at the Darboux point $\mathbf{d}$, that is,

$$V''(\mathbf{d}) = \begin{pmatrix} \frac{\partial^2 V}{\partial q_1^2}(\mathbf{d}) & \frac{\partial^2 V}{\partial q_1 \partial q_2}(\mathbf{d}) \\ \frac{\partial^2 V}{\partial q_2 \partial q_1}(\mathbf{d}) & \frac{\partial^2 V}{\partial q_2^2}(\mathbf{d}) \end{pmatrix}. \tag{2.2}$$

Since the potential V is a homogeneous function of degree k, the number $k-1$ is always an eigenvalue for any Darboux point $\mathbf{d}$. The eigenvalues of the Hessian matrix $V''(\mathbf{d})$ are denoted by $\{k-1, \lambda\}$. We say that λ is the *non-trivial eigenvalue*.

Consider the following potential

$$V(q_1, q_2) = \frac{1}{a_0 q_1^m + a_1 q_1^{m-1} q_2 + \cdots + a_{m-1} q_1 q_2^{m-1} + a_m q_2^m} \tag{2.3}$$

with $m \geq 2$. Let $z = q_2/q_1$. The potential (2.3) can be rewrite as

$$V = \frac{1}{q_1^m v(z)},$$

where $v(z) = a_0 + a_1 z + \cdots + a_m z^m$. In order to calculate the Darboux points and the non-trivial eigenvalue λ associated to potential (2.3), we define the polynomials

$$h(z) = mv(z) - zv'(z), \quad g(z) = \left(1 + z^2\right) v'(z) - mzv(z), \tag{2.4}$$

where $v'(z)$ denotes the derivative of $v(z)$ with respect to z.

The next result is due to Llibre et al., see Proposition 6 of [11].

Proposition 2. *Assume that $g \not\equiv 0$. The Darboux points $\mathbf{d} = [1 : z_*]$ associated with potential (2.3) are given by the zeros of $g(z) = 0$ for which $h(z) \neq 0$. Moreover the non-trivial eigenvalue of $V''(\mathbf{d})$ is given by $\lambda(z_*) = g'(z_*)/h(z_*) + 1$.*

3 Morales-Ramis theory

The necessary condition for the complete meromorphic integrability of Hamiltonian systems (1.1) with homogeneous potentials V was given by Morales and Ramis, see page 100 of [18].

Theorem 2 (Morales-Ramis)**.** *Assume that the Hamiltonian system defined by the Hamiltonian (1.2) with a homogeneous potential V of degree $k \in \mathbb{Z} \setminus \{0\}$ is completely integrable with meromorphic first integrals, then the non-trivial eigenvalues λ associated to its Darboux points must satisfy the conditions of Table 1, where p is an integer.*

Table 1: The Morales-Ramis table.

Degree	Eigenvalue λ	Degree	Eigenvalue λ
k	$p+p\left(p-1\right)\frac{k}{2}$	-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{12}{5}+6p\right)^2$
2	arbitrary	3	$-\frac{1}{24}+\frac{1}{24}\left(2+6p\right)^2$
-2	arbitrary	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{3}{2}+6p\right)^2$
-5	$\frac{49}{40}-\frac{1}{40}\left(\frac{10}{3}+10p\right)^2$	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{6}{5}+6p\right)^2$
-5	$\frac{49}{40}-\frac{1}{40}\left(4+10p\right)^2$	3	$-\frac{1}{24}+\frac{1}{24}\left(\frac{12}{5}+6p\right)^2$
-4	$\frac{9}{8}-\frac{1}{8}\left(\frac{4}{3}+4p\right)^2$	4	$-\frac{1}{8}+\frac{1}{8}\left(\frac{4}{3}+4p\right)^2$
-3	$\frac{25}{24}-\frac{1}{24}\left(2+6p\right)^2$	5	$-\frac{9}{40}+\frac{1}{40}\left(\frac{10}{3}+10p\right)^2$
-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{3}{2}+6p\right)^2$	5	$-\frac{9}{40}+\frac{1}{40}\left(4+10p\right)^2$
-3	$\frac{25}{24}-\frac{1}{24}\left(\frac{6}{5}+6p\right)^2$	k	$\frac{1}{2}\left(\frac{k-1}{k}+p\left(p+1\right)k\right)$

4 Homogeneous potentials of degree -4

To prove Proposition 1, we need the following result which was given in Theorem 2.6 of Cima and Llibre in [3].

Theorem 3. *For each real fourth-order binary form $Q\left(q_1,q_2\right)$ there exists some $\sigma\in GL\left(2,\mathbb{R}\right)$ which transforms $Q\left(q_1,q_2\right)$ in one and only one of the following canonical forms:*

(I) $Q=q_1^4+6\mu q_1^2q_2^2+q_2^4$ *with* $\mu<-\frac{1}{3}$;

(II) $Q=\alpha\left(q_1^4+6\mu q_1^2q_2^2+q_2^4\right)$ *with* $\mu>-\frac{1}{3}$ *and* $\mu\neq\frac{1}{3}$;

(III) $Q=q_1^4+6\mu q_1^2q_2^2-q_2^4$;

(IV) $Q=\alpha q_2^2\left(6q_1^2+q_2^2\right)$;

(V) $Q=\alpha q_2^2\left(6q_1^2-q_2^2\right)$;

(VI) $Q=\alpha\left(q_1^2+q_2^2\right)^2$;

(VII) $Q=6\alpha p_1^2p_2^2$;

(VIII) $Q=4q_1^3q_2$;

(IX) $Q=\alpha q_1^4$;

(X) $Q=0$;

where $\alpha=\pm1$ *and* $GL\left(2,\mathbb{R}\right)$ *is the linear group.*

Proof of Proposition 1. It follows immediately from Theorem 3 and the fact that the linear changes of variables are canonical changes in the theory of Hamiltonian systems. □

5 Proof of Theorem 1

Potentials V_0 and V_3. It is immediate to check that the Hamiltonian system (1.1) with the potentials V_0 and V_3 have polynomial first integral $I=p_2$ and $I=q_2p_1-q_1p_2$, respectively. So statements (a) and (b) hold.

By Theorem 2 we know that if potential (1.4) is integrable, then its eigenvalue λ must be one of Table 1. For convenience, we define the following sets

$$\mathcal{Z}_{-4}^{1} = \left\{ \frac{9}{8} - \frac{1}{8}\left(\frac{4}{3} + 4p\right)^2 : \; p \in \mathbb{Z} \right\},$$
$$\mathcal{Z}_{-4}^{2} = \left\{ p - 2(p-1)p : \; p \in \mathbb{Z} \right\},$$
$$\mathcal{Z}_{-4}^{3} = \left\{ \frac{1}{2}\left(\frac{5}{4} - 4p(p+1)\right) : \; p \in \mathbb{Z} \right\}.$$

Potential V_1. Using Proposition 2 we obtain the Darboux points $z_* = \pm 1/\sqrt{3}$ and the corresponding non-trivial eigenvalue $\lambda = -1$. It is easy to check that $-1 \notin \left\{\mathcal{Z}_{-4}^{1} \cup \mathcal{Z}_{-4}^{2} \cup \mathcal{Z}_{-4}^{3}\right\}$. By Theorem 2 V_1 is not integrable.

Potential V_2. By Proposition 2 the potential V_2 has the Darboux points $z_* = \pm 1$. The corresponding non-trivial eigenvalue is $\lambda = -1$. Since $-1 \notin \left\{\mathcal{Z}_{-4}^{1} \cup \mathcal{Z}_{-4}^{2} \cup \mathcal{Z}_{-4}^{3}\right\}$, by Theorem 2 the Hamiltonian system (1.1) with the potential V_2 is not integrable.

Potential V_4. Analogously the potential V_4 has the Darboux points $z_* = \pm\sqrt{3}/2$ with non-trivial eigenvalue $\lambda = -5/3$. It is not difficult to check that $-5/3 \notin \left\{\mathcal{Z}_{-4}^{1} \cup \mathcal{Z}_{-4}^{2} \cup \mathcal{Z}_{-4}^{3}\right\}$. By Theorem 2 V_4 is not integrable.

Potential V_5. The potential V_5 has the Darboux points $z_* = \pm\sqrt{6}/2$. The corresponding non-trivial eigenvalue is $\lambda = -1/3$. Since $-1/3 \notin \left\{\mathcal{Z}_{-4}^{1} \cup \mathcal{Z}_{-4}^{2} \cup \mathcal{Z}_{-4}^{3}\right\}$, the Hamiltonian system (1.1) with the potential V_5 is not integrable.

Potential V_6. Using the notations of Proposition 2, we get $g(z_*) = 4\left(3\mu - 1 - (3\mu+1)z_*^2\right) z_*$ and $h(z_*) = 12 z_*^2 \mu + 4$.

If $\mu = -1/3$, then this potential has Darboux points $z_* = 0$ with non-trivial eigenvalue $\lambda = -1 \notin \left\{\mathcal{Z}_{-4}^{1} \cup \mathcal{Z}_{-4}^{2} \cup \mathcal{Z}_{-4}^{3}\right\}$. Therefore the potential V_6 for $\mu = -1/3$ is not integrable.

If $\mu \neq -1/3$, it has 3 Darboux points of the form $[1 : z_*]$

$$z_{*,1} = 0 \quad \text{and} \quad z_{*,2} = \pm\sqrt{\frac{3\mu - 1}{3\mu + 1}},$$

and 2 Darboux points of the form $[0 : z]$ $z = \pm\sqrt[3]{2}$. The corresponding non-trivial eigenvalues are $\pm 3\mu$ and $4/\left(1 + 9\mu^2\right) - 1$. If potential V_6 is integrable, then eigenvalues $\pm 3\mu \in \left\{\mathcal{Z}_{-4}^{1} \cup \mathcal{Z}_{-4}^{2} \cup \mathcal{Z}_{-4}^{3}\right\}$. So we need to analyse nine possible cases, that is, $3\mu \in \mathcal{Z}_{-4}^{j}$ and $-3\mu \in \mathcal{Z}_{-4}^{k}$ for $j, k = 1, 2, 3$. The integer of Table 1 corresponding to the case $3\mu \in \mathcal{Z}_{-4}^{j}$ (respectively $-3\mu \in \mathcal{Z}_{-4}^{k}$) is denoted by p_0 (respectively p). All the possible values of integers p_0 and p are shown in Table 2, where $\eta = \sqrt{3(3 - 8\mu)}$ and $\xi = \sqrt{3(3 + 8\mu)}$ satisfy $\eta^2 + \xi^2 = 18$.

Table 2: Integers p_0 and p.

Eigenvalue 3μ	Integer p_0	Eigenvalue -3μ	Integer p
$3\mu \in \mathcal{Z}_{-4}^{1}$	$\frac{1}{12}(-4 \pm 3\eta)$	$-3\mu \in \mathcal{Z}_{-4}^{1}$	$\frac{1}{12}(-4 \pm 3\xi)$
$3\mu \in \mathcal{Z}_{-4}^{2}$	$\frac{1}{4}(3 \pm \eta)$	$-3\mu \in \mathcal{Z}_{-4}^{2}$	$\frac{1}{4}(3 \pm \xi)$
$3\mu \in \mathcal{Z}_{-4}^{3}$	$\frac{1}{4}(-2 \pm \eta)$	$-3\mu \in \mathcal{Z}_{-4}^{3}$	$\frac{1}{4}(-2 \pm \xi)$

Case $3\mu \in \mathcal{Z}_{-4}^{1}$ and $-3\mu \in \mathcal{Z}_{-4}^{1}$. Since $p_0, p \in \mathbb{Z}$ we have $(3\eta, 3\xi) \in \mathbb{N} \times \mathbb{N}$. From the equation $\eta^2 + \xi^2 = 18$, it follows that $(\eta, \xi) = (3, 3)$. Thus $\mu = 0$.

Case $3\mu \in \mathcal{Z}_{-4}^{1}$ and $-3\mu \in \mathcal{Z}_{-4}^{2}$. Similarly $(3\eta, \xi) \in \mathbb{N} \times \mathbb{N}$. Using the equation $\eta^2 + \xi^2 = 18$ we obtain $(\eta, \xi) = (3, 3)$. So $\mu = 0$.

The remaining cases can be analyzed in a similar way. We summarize all the possible values for η, ξ and μ in Table 3. Consequently if the potential V_6 is completely integrable, then $\mu = 0$. For $\mu = 0$ the non-trivial eigenvalue $4/\left(1 + 9\mu^2\right) - 1$ becomes $3 \notin \left\{\mathcal{Z}_{-4}^{1} \cup \mathcal{Z}_{-4}^{2} \cup \mathcal{Z}_{-4}^{3}\right\}$. So the Hamiltonian system (1.1) with the potential V_6 is not integrable.

Table 3: The values of η, ξ and μ.

Condition	(η, ξ)	μ
$(3\eta, 3\xi) \in \mathbb{N} \times \mathbb{N}$	$(3,3)$	0
$(3\eta, \xi) \in \mathbb{N} \times \mathbb{N}$	$(3,3)$	0
$(\eta, 3\xi) \in \mathbb{N} \times \mathbb{N}$	$(3,3)$	0
$(\eta, \xi) \in \mathbb{N} \times \mathbb{N}$	$(3,3)$	0

Potential V_7. From Proposition 2 it follows that the Darboux points of the form $[1 : z_*]$ are given by the zeros of $g(z_*) = 4(3\mu - 1)z_*\left(z_*^2 - 1\right)$ for which $h(z_*) = -4\left(3\mu z_*^2 + 1\right) \neq 0$. We have that the Darboux points $[1 : z_*]$ are $z_{*,1} = 0$ and $z_{*,2} = \pm 1$, and the Darboux points of the form $[0 : z]$ are $z = \pm\sqrt[3]{2}$. The corresponding non-trivial eigenvalues are $\lambda_1 = 3\mu$ and $\lambda_2 = -1 + 4/(1 + 3\mu)$. The necessary conditions for the complete meromorphic integrability of potential V_7 are $\lambda_{1,2} \in \left\{\mathcal{Z}_{-4}^1 \cup \mathcal{Z}_{-4}^2 \cup \mathcal{Z}_{-4}^3\right\}$. We need to consider nine possible cases, that is, $\lambda_1 \in \mathcal{Z}_{-4}^j$ and $\lambda_2 \in \mathcal{Z}_{-4}^k$ for $j, k = 1, 2, 3$. For each case there exist two integers p_0 and p such that $\lambda_1 \in \mathcal{Z}_{-4}^j$ and $\lambda_2 \in \mathcal{Z}_{-4}^k$. All the possible values of the integers p_0 and p are given in Table 4, where $\eta = \sqrt{3(3 - 8\mu)}$ and $\zeta = \sqrt{17 - 32/(1 + 3\mu)}$ satisfy $\eta^2\zeta^2 - 17\zeta^2 - 17\eta^2 + 33 = 0$. Recall that $\mu > -1/3$ and $\mu \neq 1/3$. Thus $\eta \in [0, 1) \cup \left(1, \sqrt{17}\,\right]$ and $\zeta \in [0, 1) \cup \left(1, \sqrt{17}\,\right]$.

Table 4: Integers p_0 and p.

Eigenvalue λ_1	Integer p_0	Eigenvalue λ_2	Integer p
$\lambda_1 \in \mathcal{Z}_{-4}^1$	$\frac{1}{12}(-4 \pm 3\eta)$	$\lambda_2 \in \mathcal{Z}_{-4}^1$	$\frac{1}{12}(-4 \pm 3\zeta)$
$\lambda_1 \in \mathcal{Z}_{-4}^2$	$\frac{1}{4}(3 \pm \eta)$	$\lambda_2 \in \mathcal{Z}_{-4}^2$	$\frac{1}{4}(3 \pm \zeta)$
$\lambda_1 \in \mathcal{Z}_{-4}^3$	$\frac{1}{4}(-2 \pm \eta)$	$\lambda_2 \in \mathcal{Z}_{-4}^3$	$\frac{1}{4}(-2 \pm \zeta)$

For $\lambda_{1,2} \in \mathcal{Z}_{-4}^1$, we have $(3\eta, 3\zeta) \in \mathbb{N} \times \mathbb{N}$ due to the fact that p_0 and p are integers. Combining with the conditions $\eta \in [0, 1) \cup \left(1, \sqrt{17}\,\right]$, $\zeta \in [0, 1) \cup \left(1, \sqrt{17}\,\right]$ and $\eta^2\zeta^2 - 17\zeta^2 - 17\eta^2 + 33 = 0$, we conclude that η and ζ do not exist. The analysis of other cases are similar. We list all the possible values of η, ξ and ζ in Table 5. So $\lambda_1 \notin \left\{\mathcal{Z}_{-4}^1 \cup \mathcal{Z}_{-4}^2 \cup \mathcal{Z}_{-4}^3\right\}$ or $\lambda_2 \notin \left\{\mathcal{Z}_{-4}^1 \cup \mathcal{Z}_{-4}^2 \cup \mathcal{Z}_{-4}^3\right\}$. Therefore the Hamiltonian system (1.1) with the potential V_7 is not integrable.

In summary statement (c) of Theorem 1 is proved.

Table 5: The values of η, ζ and μ.

Condition	(η, ζ)	μ
$(3\eta, 3\zeta) \in \mathbb{N} \times \mathbb{N}$	$\varnothing$	$\varnothing$
$(3\eta, \zeta) \in \mathbb{N} \times \mathbb{N}$	$\varnothing$	$\varnothing$
$(\eta, 3\zeta) \in \mathbb{N} \times \mathbb{N}$	$\varnothing$	$\varnothing$
$(\eta, \xi) \in \mathbb{N} \times \mathbb{N}$	$\varnothing$	$\varnothing$

Potential V_8. Applying Proposition 2 the Darboux points of the form $[1 : z_*]$ are given by the zeros of $g(z_*) = 4(1 - 3\mu)z_*\left(z_*^2 - 1\right)$ for which $h(z_*) = 4\left(3\mu z_*^2 + 1\right) \neq 0$. The Darboux points of the form $[1 : z_*]$ and $[0 : z]$ are respectively $z_{*,1} = 0$ and $z_{*,2} = \pm 1$, and $z = \pm\mathbf{i}\sqrt[3]{2}$ with $\mathbf{i} = \sqrt{-1}$. The corresponding non-trivial eigenvalues are $\lambda_1 = 3\mu$ and $\lambda_2 = -1 + 4/(1 + 3\mu)$. By the same reasons as above one can get Table 4. Next we will divide the proof into two cases:

Case (i) $\mu > -\frac{1}{3}$ and $\mu \neq \frac{1}{3}$.

Case (ii) $\mu < -\frac{1}{3}$.

The **Case (i)** is the same as the potential V_7. Thus statement (d) holds for **Case (i)**.

Now we consider **Case (ii)**. Since $\mu < -\frac{1}{3}$, $\eta = \sqrt{3(3 - 8\mu)}$ and $\zeta = \sqrt{17 - 32/(1 + 3\mu)}$, we get that $\eta > \sqrt{17}$, $\zeta > \sqrt{17}$ and

$$\zeta = \sqrt{\frac{256}{\eta^2 - 17} + 17}. \tag{5.1}$$

Since p must be an integer we have Table 6. Let $\tilde{\eta} = 3\eta$ and $\tilde{\zeta} = 3\zeta$. Then equation (5.1) becomes

$$\tilde{\zeta} = 3\sqrt{\frac{2304}{\tilde{\eta}^2 - 153} + 17}, \tag{5.2}$$

with $\tilde{\eta} > 3\sqrt{17}$ and $\tilde{\zeta} > 3\sqrt{17}$. If $(3\eta, 3\zeta) \in \mathbb{N} \times \mathbb{N}$, then $\left(\tilde{\eta}, \tilde{\zeta}\right) \in \mathbb{N} \times \mathbb{N}$ with $\tilde{\eta} \geq 13$ and $\tilde{\zeta} \geq 13$. Furthermore, $13 \leq \tilde{\zeta} \leq 38$ due to the fact that equation (5.2) is decreasing with $\tilde{\eta}$. For each value of $\tilde{\zeta} = 13, 14, \ldots, 38$, equation (5.2) provides a value of $\tilde{\eta}$. We get the first row of Table 6. For the second row of Table 6, equation (5.1) can be rewrite as

$$\zeta = \sqrt{\frac{2304}{\tilde{\eta}^2 - 153} + 17}, \tag{5.3}$$

with $\tilde{\eta} > 3\sqrt{17}$ and $\zeta > \sqrt{17}$. If $(\tilde{\eta}, \zeta) \in \mathbb{N} \times \mathbb{N}$, then $\tilde{\eta} \geq 13$ and $\zeta \geq 5$. Since equation (5.3) is decreasing with $\tilde{\eta}$, we have $5 \leq \zeta \leq 12$. The second row of Table 6 holds. The remaining rows of Table 6 are obtained in a similar way.

Table 6: The values of η, ζ and μ.

Condition	(η, ζ)	μ
$(3\eta, 3\zeta) \in \mathbb{N} \times \mathbb{N}$	$(5, 7)$ or $(7, 5)$	$-\frac{2}{3}$ or $-\frac{5}{3}$
$(3\eta, \zeta) \in \mathbb{N} \times \mathbb{N}$	$(5, 7)$ or $(7, 5)$	$-\frac{2}{3}$ or $-\frac{5}{3}$
$(\eta, 3\zeta) \in \mathbb{N} \times \mathbb{N}$	$(5, 7)$ or $(7, 5)$	$-\frac{2}{3}$ or $-\frac{5}{3}$
$(\eta, \xi) \in \mathbb{N} \times \mathbb{N}$	$(5, 7)$ or $(7, 5)$	$-\frac{2}{3}$ or $-\frac{5}{3}$

For $\mu = -2/3$ the non-trivial eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -5$. For $\mu = -5/3$, the non-trivial eigenvalues are $\lambda_1 = -5$ and $\lambda_2 = -2$. Obviously $-2 \in \mathcal{Z}^2_{-4}$ $(p = 2)$ and $-5 \in \mathcal{Z}^2_{-4}$ $(p = -1)$. For $\mu = -2/3$ or $-5/3$, we cannot use Theorem 2 to decide whether or not the Hamiltonian system is meromorphically integrable. Next we show that V_8 does not admit an additional polynomial first integral when $\mu \in \{-5/3, -2/3\}$.

Consider the following potential

$$V = \frac{1}{q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4}, \tag{5.4}$$

with $\mu \in \{-5/3, -2/3\}$. The corresponding Hamiltonian system (1.1) is

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = \frac{4q_1\left(q_1^2 + 3\mu q_2^2\right)}{\left(q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4\right)^2}, \quad \dot{p}_2 = \frac{4q_2\left(3\mu q_1^2 + q_2^2\right)}{\left(q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4\right)^2}. \tag{5.5}$$

After a rescaling of the time variable $dt = \left(q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4\right)^2 ds$, system (5.5) becomes

$$\begin{aligned} \dot{q}_1 &= p_1\left(q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4\right)^2, \quad & \dot{q}_2 &= p_2\left(q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4\right)^2, \\ \dot{p}_1 &= 4q_1\left(q_1^2 + 3\mu q_2^2\right), & \dot{p}_2 &= 4q_2\left(3\mu q_1^2 + q_2^2\right). \end{aligned} \tag{5.6}$$

Doing the transformation $(q_1, q_2, p_1, p_2) \mapsto (q_1, q_2, p_1, P)$ with $P = q_2 p_1 - q_1 p_2$, system (5.6) writes

$$\begin{aligned} &\dot{q}_1 = p_1\left(q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4\right)^2, \quad \dot{q}_2 = \frac{q_2 p_1 - P}{q_1}\left(q_1^4 + 6\mu q_1^2 q_2^2 + q_2^4\right)^2, \\ &\dot{p}_1 = 4q_1\left(q_1^2 + 3\mu q_2^2\right), \quad \dot{P} = 4\left(1 - 3\mu\right) q_1 q_2 \left(q_1 + q_2\right)\left(q_1 - q_2\right). \end{aligned} \tag{5.7}$$

Suppose that system (5.6) has a polynomial first integral $F(q_1, q_2, p_1, p_2) \in \mathbb{C}[q_1, q_2, p_1, p_2]$. In the new variables (q_1, q_2, p_1, P), it can be written as

$$\bar{F}(q_1, q_2, p_1, P) = F\left(q_1, q_2, p_1, \frac{q_2 p_1 - P}{q_1}\right) = \sum_{j=-n}^{n} f_j(q_2, p_1, P)\, q_1^j, \tag{5.8}$$

where $f_j(q_2, p_1, P) \in \mathbb{C}[q_2, p_1, P]$. Since $\bar{F}(q_1, q_2, p_1, P)$ is a first integral of system (5.7), we have

$$\dot{q}_1 \frac{\partial \bar{F}}{\partial q_1} + \dot{q}_2 \frac{\partial \bar{F}}{\partial q_2} + \dot{p}_1 \frac{\partial \bar{F}}{\partial p_1} + \dot{P} \frac{\partial \bar{F}}{\partial P} = 0. \tag{5.9}$$

For clarity we introduce the following differential operators acting on $f_j(q_2, p_1, P) \in \mathbb{C}[q_2, p_1, P]$:

$$\begin{aligned}
\mathcal{A}[f_j] &:= j p_1 f_j + (q_2 p_1 - P) \frac{\partial f_j}{\partial q_2}, \\
\mathcal{B}[f_j] &:= 3\mu q_2^4 \mathcal{A}[f_j] - (1 - 3\mu) q_2 \frac{\partial f_j}{\partial P} + 3\mu \frac{\partial f_j}{\partial p_1}, \\
\mathcal{C}[f_j] &:= \left(18\mu^2 + 1\right) q_1^4 \mathcal{A}[f_j] + 2\left((1 - 3\mu) q_2 \frac{\partial f_j}{\partial P} + \frac{\partial f_j}{\partial p_1}\right).
\end{aligned}$$

Using the above notions equation (5.9) can be written as

$$\sum_{j=-n}^{n} \left(q_1^{j+7}\mathcal{A} + 12\mu q_2^2 q_1^{j+5}\mathcal{A} + 2q_1^{j+3}\mathcal{C} + 4q_2^2 q_1^{j+1}\mathcal{B} + q_2^8 q_1^{j-1}\mathcal{A}\right)[f_j] = 0, \tag{5.10}$$

with $f_j \in \mathbb{C}[q_2, p_1, P]$. Moreover $\bar{F}(q_1, q_2, p_1, P)$ is a first integral of system (5.7) if and only if the coefficients of q_1^{j-1} in equation (5.10) are

$$\mathcal{A}[f_{j-8}] + 12\mu q_2^2 \mathcal{A}[f_{j-6}] + 2\mathcal{C}[f_{j-4}] + 4q_2^2 \mathcal{B}[f_{j-2}] + q_2^8 \mathcal{A}[f_j] = 0, \tag{5.11}$$

where $j = -n, \ldots, n+8$ and $f_j = 0$ if $j < -n$ or $j > n$. Therefore system (5.6) has a polynomial first integral if and only if there exist $2n+1$ polynomials f_j such that equations (5.11) hold. The existence of such polynomials are given by the following two lemmas.

Lemma 1. *If $\bar{F}(q_1, q_2, p_1, P)$ is a first integral of system (5.7), then $f_j(q_2, p_1, P) = 0$ for $j = 1, \ldots, n$.*

Proof. For $j = n+8$ equation (5.11) becomes $\mathcal{A}[f_n] = 0$. The solution of $\mathcal{A}[f_n] = 0$ is $f_n = \alpha(p_1, P) / (P - q_2 p_1)^n$, where $\alpha(p_1, P)$ is a function in the variables p_1 and P. So $f_n = 0$ due to the fact that $f_n \in \mathbb{C}[q_2, p_1, P]$. If $n = 1$ we are done.

When $n \geq 2$, taking $j = n+8, n+7$, we have $\mathcal{A}[f_n] = 0$ and $\mathcal{A}[f_{n-1}] = 0$. By the same reason as above, $f_n = f_{n-1} = 0$. If $n = 2$, then the lemma holds.

When $n \geq 3$, taking $j = n+8, n+7, n+6$, we get, respectively, $\mathcal{A}[f_n] = 0$, $\mathcal{A}[f_{n-1}] = 0$ and $\mathcal{A}[f_{n-2}] + 12\mu q_2^2 \mathcal{A}[f_n] = 0$. This implies that $\mathcal{A}[f_{n-2}] = 0$. Using similar arguments we obtain $f_n = f_{n-1} = f_{n-2} = 0$. If $n = 3$, the proof is finished.

Consider $n \geq 4$. Substituting $j = n+8, n+7, n+6, n+5$ into equation (5.11), we get, respectively, $\mathcal{A}[f_n] = 0$, $\mathcal{A}[f_{n-1}] = 0$, $\mathcal{A}[f_{n-2}] + 12\mu q_2^2 \mathcal{A}[f_n] = 0$ and $\mathcal{A}[f_{n-3}] + 12\mu q_2^2 \mathcal{A}[f_{n-1}] = 0$. Using similar arguments to the case $n \geq 3$, it is easy to prove that $f_n = f_{n-1} = f_{n-2} = f_{n-3} = 0$. If $n = 4$, the lemma is proved.

When $n \geq 5$, we analyze respectively $j = n+8, n+7, n+6, n+5, n+4$ in equation (5.11). Using similar arguments as in the previous cases, one can get that $f_n = f_{n-1} = f_{n-2} = f_{n-3} = f_{n-4} = 0$. If $n = 5$ the lemma is confirmed.

Next we prove this lemma by induction when $n \geq 6$. Assume that $f_n = f_{n-1} = \cdots = f_{i+1} = 0$, where $i \geq 1$. Now we consider equation (5.11) for $j = i+8$, that is,

$$\mathcal{A}[f_i] + 12\mu q_2^2 \mathcal{A}[f_{i+2}] + 2\mathcal{C}[f_{i+4}] + 4q_2^2 \mathcal{B}[f_{i+6}] + q_2^8 \mathcal{A}[f_{i+8}] = 0. \tag{5.12}$$

By the induction hypothesis we have $\mathcal{A}[f_{i+2}] = \mathcal{A}[f_{i+8}] = \mathcal{C}[f_{i+4}] = \mathcal{B}[f_{i+6}] = 0$. Thus equation (5.12) reduces to $\mathcal{A}[f_i] = 0$. From the above analysis we know that the only polynomial solution of differential equation $\mathcal{A}[f_i] = 0$ is $f_i = 0$. This ends the proof. □

Lemma 2. *If $\bar{F}(q_1, q_2, p_1, P)$ is a first integral of system (5.7), then $f_j(q_2, p_1, P) = 0$ for $j = -n, -n+1, \ldots, -1$, and $f_0(q_2, p_1, P)$ is a constant.*

Proof. We consider equation (5.11) for $j = -n, -n+1, -n+2$, that is, $q_2^8 \mathcal{A}[f_{-n}] = 0$, $q_2^8 \mathcal{A}[f_{-n+1}] = 0$ and $4q_2^2 \mathcal{B}[f_{-n}] + q_2^8 \mathcal{A}[f_{-n+2}] = 0$. So $\mathcal{A}[f_{-n}] = \mathcal{A}[f_{-n+1}] = 0$. The polynomial solutions of the differential equations $\mathcal{A}[f_{-n}] = 0$ and $\mathcal{A}[f_{-n+1}] = 0$ are respectively

$$f_{-n} = \alpha_{-n}(P - q_2 p_1)^n \quad \text{and} \quad f_{-n+1} = \alpha_{-n+1}(P - q_2 p_1)^{n-1}, \tag{5.13}$$

where $\alpha_{-n} = \alpha_{-n}(p_1, P)$ and $\alpha_{-n+1} = \alpha_{-n+1}(p_1, P)$ are polynomials in the variables p_1 and P.

From equations $4q_2^2 \mathcal{B}[f_{-n}] + q_2^8 \mathcal{A}[f_{-n+2}] = 0$ and (5.13), we obtain

$$f_{-n+2} = (\alpha_{-n+2} + \beta_{-n+2})(P - q_2 p_1)^{n-2}, \tag{5.14}$$

where $\alpha_{-n+2} = \alpha_{-n+2}(p_1, P)$ is an integration constant, $\beta_{-n+2} = \frac{1}{15q_2^5}\tilde{\beta}_{-n+2}$, and

$$\tilde{\beta}_{-n+2} = 15nq_2\alpha_{-n} + 5(1-3\mu)(3P - 4p_1q_2)\, q_2 \frac{\partial \alpha_{-n}}{\partial P} + 9\mu(5p_1q_2 - 4P)\frac{\partial \alpha_{-n}}{\partial p_1}. \tag{5.15}$$

Since α_{-n} is a polynomial, we claim that β_{-n+2} is also a polynomial. Assume that β_{-n+2} is not a polynomial. Then $\beta_{-n+2}(P - q_2p_1)^{n-2}$ is a rational function with denominator $15q_2^5$. Since f_{-n+2} must be a polynomial, we have that $\alpha_{-n+2}(P - q_2p_1)^{n-2}$ is also a rational function with denominator $15q_2^5$. This contradicts the fact that α_{-n+2} is independent of the variable q_2. This means that β_{-n+2} is a polynomial if and only if $\tilde{\beta}_{-n+2}$ is divisible by q_2^5. Thus $\tilde{\beta}_{-n+2} = 0$. Evaluating (5.15) on $q_2 = 0$, we have $\partial\alpha_{-n}/\partial p_1 = 0$ due to the fact that $9\mu \neq 0$. After division by $5q_2$ in equation (5.15), we obtain

$$3n\alpha_{-n} + (1-3\mu)(3P - 4p_1q_2)\frac{\partial \alpha_{-n}}{\partial P} = 0. \tag{5.16}$$

Substituting $q_2 = 3P/4p_1$ into (5.16), we get $\alpha_{-n} = 0$. Consequently $f_{-n} = 0$ and $f_{-n+2} = \alpha_{-n+2}(P - q_2p_1)^{n-2}$.

For $j = -n+3$ equation (5.11) becomes $4q_2^2 \mathcal{B}[f_{-n+1}] + q_2^8 \mathcal{A}[f_{-n+3}] = 0$. In the same way as f_{-n+2}, we get

$$f_{-n+3} = (\alpha_{-n+3} + \beta_{-n+3})(P - q_2p_1)^{n-3}, \tag{5.17}$$

where $\alpha_{-n+3} = \alpha_{-n+3}(p_1, P)$ is an integration constant, $\beta_{-n+3} = \frac{1}{15q_2^5}\tilde{\beta}_{-n+3}$, and

$$\tilde{\beta}_{-n+3} = 15q_2\alpha_{-n+1}(n-1) + 5(1-3\mu)(3P - 4p_1q_2)\, q_2 \frac{\partial \alpha_{-n+1}}{\partial P} + 9\mu(5p_1q_2 - 4P)\frac{\partial \alpha_{-n+1}}{\partial p_1}. \tag{5.18}$$

By the same arguments as above one can get that $\alpha_{-n+1} = 0$ and $\beta_{-n+3} = 0$. Therefore $f_{-n+1} = 0$ and $f_{-n+3} = \alpha_{-n+3}(P - q_2p_1)^{n-3}$.

Using exactly the same steps as in the previous cases, we can prove that $f_{-n+3} = \cdots = f_{-3} = 0$, $f_{-2} = \alpha_{-2}(P - q_2p_1)^2$ and $f_{-1} = \alpha_{-1}(P - q_2p_1)$, where $\alpha_{-2} = \alpha_{-2}(p_1, P)$ and $\alpha_{-1} = \alpha_{-1}(p_1, P)$ are polynomials in the variables p_1 and P.

From Lemma 1 we know that $f_j = 0$ for $j \geq 1$. For $j = 8$ equation (5.11) becomes $\mathcal{A}[f_0] = 0$, that is $\partial f_0(q_2, p_1, P)/\partial q_2 = 0$. This implies that f_0 does not depend on q_2.

Finally we prove that $f_{-1} = f_{-2} = 0$ and $f_0 =$ constant.

Consider equation (5.11) for $j = 0$, that is

$$\mathcal{B}[f_{-2}] = \mathcal{B}\left[\alpha_{-2}(P - q_2p_1)^2\right] = 0. \tag{5.19}$$

The general solution of the linear partial differential equation (5.19) is

$$\alpha_{-2}\left(p_1,P\right)=\frac{1}{9\mu^2\left(P-p_1q_2\right)^2}\varphi\left(P-p_1q_2+\frac{p_1q_2}{3\mu}\right),$$

where φ is a function in the variable $P-p_1q_2+\frac{p_1q_2}{3\mu}$. Since α_{-2} does not depend on q_2 and $\frac{1}{3\mu}\neq 0$, we have $\alpha_{-2}=0$. Consequently $f_{-2}=0$.

Substituting $j=1$ into (5.11) we have $\mathcal{B}\left[f_{-1}\right]=\mathcal{B}\left[\alpha_{-1}\left(P-q_2p_1\right)\right]=0$. The general solution of this linear partial differential equation is

$$\alpha_{-1}\left(p_1,P\right)=\frac{1}{3\mu\left(P-p_1q_2\right)}\psi\left(P-p_1q_2+\frac{p_1q_2}{3\mu}\right).$$

By the same reasons as above we get $\alpha_{-1}=0$.

From Lemma 1 and $f_{-6}=f_{-4}=f_{-2}=0$, we obtain that

$$\begin{aligned}&\left(3\mu-1\right)q_2\frac{\partial f_0}{\partial P}+3\mu\frac{\partial f_0}{\partial p_1}=0\quad\text{for } j=2 \text{ in equation (5.11)};\\&\left(1-3\mu\right)q_2\frac{\partial f_0}{\partial P}+\frac{\partial f_0}{\partial p_1}=0\quad\text{for } j=4 \text{ in equation (5.11)}.\end{aligned}\tag{5.20}$$

System (5.20) is linear in the unknowns $q_2\partial f_0/\partial P$ and $\partial f_0/\partial p_1$. Its coefficient matrix is $9\mu^2-1\neq 0$ due to the fact that $\mu\in\{-5/3,-2/3\}$. Thus $\partial f_0/\partial P=\partial f_0/\partial p_1=0$ that is $f_0=\text{constant}$.

This completes the proof of the lemma. □

Proposition 3. *The Hamiltonian system* (1.1) *with the potential* (5.4) *does not admit an additional polynomial first integral.*

Proof. From Lemmas 1 and 2 the proposition follows. □

This completes the proof of Theorem 1.

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