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QUADRATIC SYSTEMS WITH AN INVARIANT ALGEBRAIC CURVE OF DEGREE 3 AND A DARBOUX INVARIANT

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ABSTRACT. The planar quadratic systems having a Darboux invariant defined by invariant straight lines of total multiplicity two or by an invariant conic have been studied in [13] and [14], respectively. Here we shall present the normal forms of the planar quadratic systems having an invariant cubic. Moreover we classify the phase portraits in the Poincare disc of all planar quadratic polynomial differential systems with invariant cubic curve and having a Darboux invariant defined by it.

1. Introduction and statements of the results

Even after hundreds of studies on the topology of real planar quadratic vector fields the complete characterization of their phase portraits is a quite complex task. This family of systems depends on twelve parameters but, after affine transformations and time rescaling, we arrive at families with five parameters, which is still a big number of parameters. Many subclasses have been considered.

Denote by $\mathbb{R}[x, y]$ the ring of the real polynomials in the variables x and y. Consider the differential system in $\mathbb{R}^2$ given by

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where $P, Q \in \mathbb{R}[x, y]$. Here the dot denotes derivative with respect to the *time* t and the degree of system (1) is $m = \max\{\deg P, \deg Q\}$.

When $m = 2$ we say that system (1) is a *quadratic polynomial differential system* or simply a *quadratic system*. More than one thousand papers have been published about quadratic systems, see for instance [15] for a bibliographical survey. The quadratic systems appear in the modeling of many natural phenomena described in different branches of science, in biological and physical applications. Besides the applications the quadratic systems became a matter of interest for the mathematicians. Considering algebraic invariant curves, some authors have published on the subject, for example, [3] and [12]. In the first one the authors studied cubic systems with invariant straight lines of total multiplicity eight that have three distinct infinite singularities. The second paper is dedicated to study the normal forms and global phase portraits of quadratic and cubic integrable systems when they have two nonconcentric circles as invariant algebraic curves.

In this paper we assume that the polynomials P and Q are coprime, otherwise system (1) can be reduced to a linear or constant system doing a rescaling of the time variable.

The first objective of this paper is to characterize all quadratic systems having invariant cubics. Then using the normal forms obtained, we investigate which systems have a Darboux invariant of the form $e^{st} f_1^{\lambda_1} f_2^{\lambda_2} f_3^{\lambda_3}$ if the cubic is the product of three straight

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lines $f_i = 0$ for $i = 1, 2, 3$, of the form $e^{st} f_1^{\lambda_1} f_2^{\lambda_2}$ if the cubic is the product of one straight line $f_1 = 0$ and an irreducible conic $f_2 = 0$, and of the form $e^{st} f_1^{\lambda_1}$ if $f_1 = 0$ is an irreducible cubic.

The paper is organized as follows. In Section 2 we present our main results. They are divided in two subsections. In section 3 we present definitions and results that will be used for proving our main results. Finally in Sections 4, 5 and 6 we prove the main results.

2. Statement of the main results

The objective of this section is to present the main results of this investigation. Since the cubic curves can be classified as reducible and irreducible curves (according to the polynomial defining the curve admits fatorization or not), we split the obtained results in two subsections. In the first one we consider planar quadratic systems having irreducible cubics and in the second one, the reducible ones.

Theorem A. *Each quadratic system admitting an irreducible invariant cubic after an affine change of coordinates and a rescaling of the time variable can be written as one of the following systems.*

$$(i) \quad \begin{aligned} \dot{x} &= 2(ax + by + dxy + cx^2), \\ \dot{y} &= 3(ay + bx^2 + cxy + dy^2), \end{aligned}$$

$$(ii) \quad \begin{aligned} \dot{x} &= 2(ax + by + (3b - 2c)xy + ax^2), \\ \dot{y} &= 2bx + 2ay + 2cx^2 + 3axy + (9b - 6c)y^2, \end{aligned}$$

$$(iii) \quad \begin{aligned} \dot{x} &= 2(ax - by + (3b + 2c)xy - ax^2), \\ \dot{y} &= 2bx + 2ay + 2cx^2 - 3axy + (9b + 6c)y^2, \end{aligned}$$

$$(iv) \quad \begin{aligned} \dot{x} &= 2y(a + bx), \\ \dot{y} &= ar - 2(ar + a + br)x + (3a + br + b)x^2 + 3by^2, \end{aligned}$$

$$(v) \quad \begin{aligned} \dot{x} &= 2y(b + cx), \\ \dot{y} &= b + 2(br - c)x + (3b - cr)x^2 + 3cy^2. \end{aligned}$$

Theorem B. *Each quadratic system admitting an irreducible invariant cubic having a Darboux invariant can be written after an affine change of coordinates and a rescaling of the time variable as*

$$\dot{x} = x + y, \qquad \dot{y} = \frac{3}{2}y + x^2. \tag{2}$$

In this case $y^2 = x^3$ is the invariant algebraic curve and the Darboux invariant is given by $e^{-6t}(y^2 - x^3)$. The global phase portrait of such system is given in Figure 1.

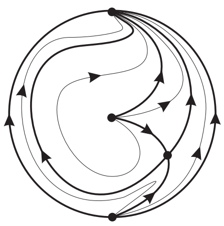


Figure 1. Phase portrait of system (2).

Theorems A and B are proved in section 4.

2.1. **Reducible invariant cubics.** Each reducible cubic can be written as the product of two polynomials one of degree two and the other of degree one (i.e, a conic and a straight line respectively). The conics can be classified in ellipses (E), complex ellipses (CE), hyperbolas (H), parabolas (P), two real straight lines intersecting in a point , two real parallel straight lines (PL), one double invariant real straight line (DL), two complex straight lines intersecting in a real point (p), and two complex parallel straight lines (CL). So the normal forms of the reducible cubics, except to an affine transformation, are

(E) $(x^2+y^2-1)(ax+by+c)=0$,
(CE) $(x^2+y^2+1)(ax+by+c)=0$,
(H) $(x^2-y^2-1)(ax+by+c)=0$,
(P) $(y-x^2)(ax+by+c)=0$,
(LV) $xy(ax+by+c)=0$,
(PL) $(x^2-1)(ax+by+c)=0$,
(DL) $x^2(ax+by+c)=0$,
(CL) $(x^2+1)(ax+by+c)=0$,
(p) $(x^2+y^2)(ax+by+c)=0$.

We shall say that a quadratic system is of type (E) if it has a real ellipse and a straight line as invariant irreducible algebraic curves; of type (CE) if it has a complex ellipse and a straight line as invariant irreducible algebraic curves, and respectively with all the nine types of conics described above.

The first result of this paper classifies the quadratic systems having a reducible invariant cubic.

Theorem C. *If a quadratic system* (1) *has a reducible invariant cubic then it can be written, after an affine change of coordinates, into one of the following forms*

$$(CE)\quad \begin{aligned}\dot x &= -(x^2+y^2+1)-2\alpha_1\, y(y+ax+c),\\ \dot y &= a(x^2+y^2+1)+2\alpha_1\, x(y+ax+c),\end{aligned}$$

$$(E.1)\quad \begin{aligned}\dot x &= -(x^2+y^2-1)-2\alpha_1 y(y+ax+c),\\ \dot y &= a(x^2+y^2-1)+2\alpha_1 x(y+ax+c),\end{aligned}$$

$$(E.2)\quad \begin{aligned}\dot x &= (\beta_1/2)(x^2+y^2-1)-y(\beta_2\, y-\alpha_2\, x+c\beta_2),\\ \dot y &= (y+c)(\alpha_2\, y+\beta_2 c\, x+\alpha_2), \qquad \text{with } \alpha_2(c+1)=0,\end{aligned}$$

$$(H.1)\quad \begin{aligned}\dot x &= (\beta_1/2)(x^2-y^2-1)+\beta_2\, y(y+c),\\ \dot y &= \beta_2\, y(y+c),\end{aligned}$$

$$(H.2)\quad \begin{aligned}\dot x &= (x+c)(\alpha_2\, x+\gamma_2 y+\alpha_2),\\ \dot y &= -(\gamma_1/2)(x^2-y^2-1)+x(\gamma_2\, x+\alpha_2\, y+c\gamma_2), \qquad \text{with } \alpha_2(c+1)=0,\end{aligned}$$

$$(H.3)\quad \begin{aligned}\dot x &= (A/2)(x^2-y^2-1)-y(\alpha-c\beta+x(\beta-c\alpha)+y(\gamma-c\alpha)),\\ \dot y &= (A/2)(x^2-y^2-1)-x(\alpha-c\beta+\beta x+y(\gamma-c\alpha))+c\alpha(y^2+1), \quad \text{with } c(\gamma+\beta)=0,\end{aligned}$$

$$(H.4)\quad \begin{aligned}\dot x &= (A/2)(x^2-y^2-1)+y(a\alpha-\beta\sqrt{d}+x(a\beta-\alpha\sqrt{d})+\beta y),\\ \dot y &= (-Aa/2)(x^2-y^2-1)+x(a\alpha-\beta\sqrt{d}+a\beta x+\beta y)-\alpha\sqrt{d}(y^2+1), \quad \text{with } d=a^2-1,\end{aligned}$$

$$(H.5) \quad \begin{aligned} \dot{x} &= -(x^2 - y^2 - 1) + 2\alpha_1 y(y + ax + c),\\ \dot{y} &= a(x^2 - y^2 - 1) + 2\alpha_1 x(y + ax + c), \qquad \textit{with } c^2 \neq a^2 - 1, \end{aligned}$$

$$(P.1) \quad \begin{aligned} \dot{x} &= x(\alpha_2 + \beta_2\, x + \gamma_2\, y),\\ \dot{y} &= \alpha_1(y - x^2) + 2\alpha_2 x^2 + 2y(\beta_2\, x + \gamma_2\, y), \end{aligned}$$

$$(P.2) \quad \begin{aligned} \dot{x} &= -\beta_1(y - x^2) + y(\beta_2 + \gamma_2 x) + (\alpha_2 + \gamma_2 c)x + c\beta_2,\\ \dot{y} &= 2(y + c)(\alpha_2 + \beta_2 x + \gamma_2 y), \qquad \textit{with } c\,\alpha_2 = 0, \end{aligned}$$

$$(P.3) \quad \begin{aligned} \dot{x} &= -(y - x^2) - \alpha(y + ax + c),\\ \dot{y} &= a(y - x^2) - 2\alpha\, x(y + ax + c), \qquad \textit{with } c \neq a^2/4, \end{aligned}$$

$$(LV.1) \quad \begin{aligned} \dot{x} &= x(\alpha + r\, y + \beta\, x),\\ \dot{y} &= y(\alpha + (r - q + \beta)\, y + q\, x), \end{aligned}$$

$$(LV.2) \quad \begin{aligned} \dot{x} &= x(p + q\, x + r\, y),\\ \dot{y} &= y(y + c), \textit{with } c(c + 1) = 0. \end{aligned}$$

$$(LV.3) \quad \begin{aligned} \dot{x} &= -x(y + \alpha(y + a\, x + c)),\\ \dot{y} &= y(a\, x + \beta(y + a\, x + c)), \qquad \textit{with } a\, c \neq 0, \end{aligned}$$

$$(RPL) \quad \begin{aligned} \dot{x} &= x^2 - 1,\\ \dot{y} &= y(\alpha + \beta\, x + \gamma\, y), \end{aligned}$$

$$(DL) \quad \begin{aligned} \dot{x} &= x^2,\\ \dot{y} &= y(\alpha + \beta\, x + \gamma\, y), \end{aligned}$$

$$(CPL) \quad \begin{aligned} \dot{x} &= x^2 + 1,\\ \dot{y} &= y(\alpha + \beta\, x + \gamma\, y), \end{aligned}$$

$$(p.1) \quad \begin{aligned} \dot{x} &= (\beta/2)(x^2 + y^2) - \beta_3\, y^2 + x(\alpha_3 + \gamma_3\, y),\\ \dot{y} &= y(\alpha_3 + \beta_3\, x + \gamma_3\, y), \end{aligned}$$

$$(p.2) \quad \begin{aligned} \dot{x} &= -(x^2 + y^2) + (\beta\, x - \alpha\, y)(y + ax + c),\\ \dot{y} &= a(x^2 + y^2) + (\beta\, y + \alpha\, x)(y + ax + c), \textit{with } c \neq 0, \end{aligned}$$

where $a, c, A, p, q, r, \alpha, \beta, \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1$ *and* γ_2 *are the parameters of the system.*

Theorem D. *The global phase portrait in the Poincaré disc of each quadratic differential system admitting a reducible invariant cubic* $f(x, y) = 0$ *and having a Darboux invariant of the form* $e^{-st} f(x, y)$ *is topologically equivalent to one of the phase portraits presented in Figures 2-7. Their normal forms according to Theorem C is labelled in the corresponding figure.*

Theorem E. *Systems of type* $(CE), (E.1), (H.1), (H.5), (P.3)$ *do not admit Darboux invariants of the form* $e^{-st} f(x, y)$.

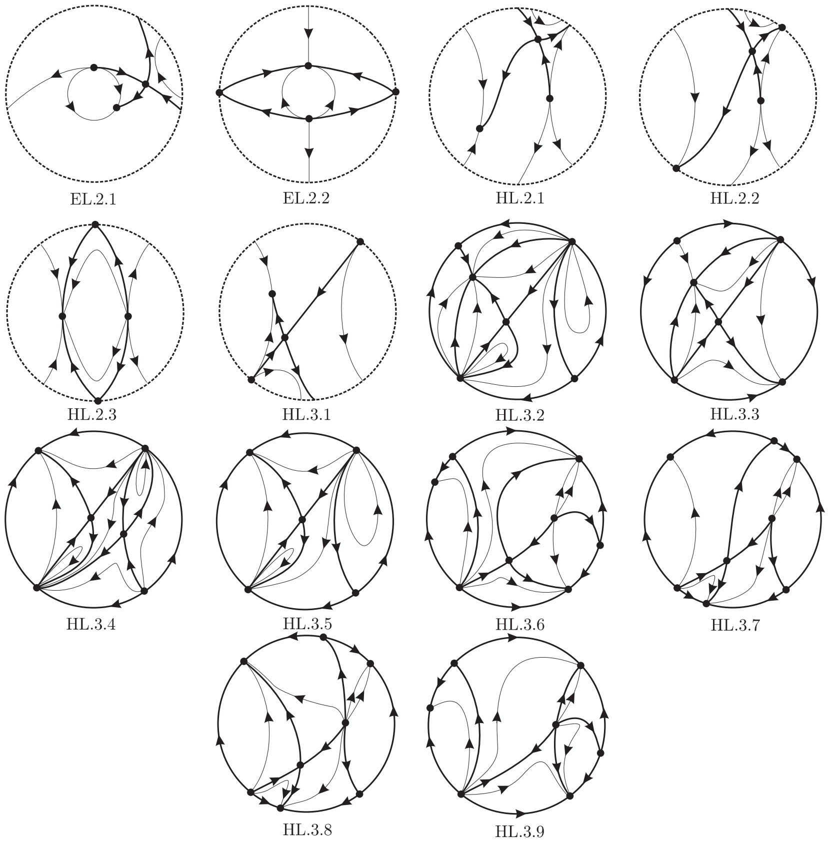


FIGURE 2. Phase portraits of systems of type (E) and (H) when they have a Darboux invariant.

3. Preliminary and basic results

The objective of this section is introduce some definitions and results which shall be used in next sections for the study of the Darboux invariants and to obtain the global phase portrait of the systems of Theorems B and C.

3.1. **Invariants.** A nonconstant C^1 function $H : U = \mathbb{R}$, defined in the open and dense set $U \subset \mathbb{R}^2$ is a *first integral* of system (1) on U if $H(x(t), y(t))$ is constant for all of the values of t for which $(x(t), y(t))$ is a solution of system (1) contained in U. In other words H is a first integral of system (1) if and only if

$$P\frac{\partial H}{\partial x} + Q\frac{\partial H}{\partial y} = 0, \tag{3}$$

for all $(x, y) \in U$.

An *invariant* of system (1) on the open subset U of $\mathbb{R}^2$ is a nonconstant C^1 function I in the variables x, y and t such that $I(x(t), y(t), t)$ is constant on all solution curves $(x(t), y(t))$ of system (1) contained in U, i.e.

$$\frac{\partial I}{\partial x}P + \frac{\partial I}{\partial y}Q + \frac{\partial I}{\partial t} = 0, \tag{4}$$

for all $(x, y) \in U$. In short, I is a first integral of system (1) depending on the time t.

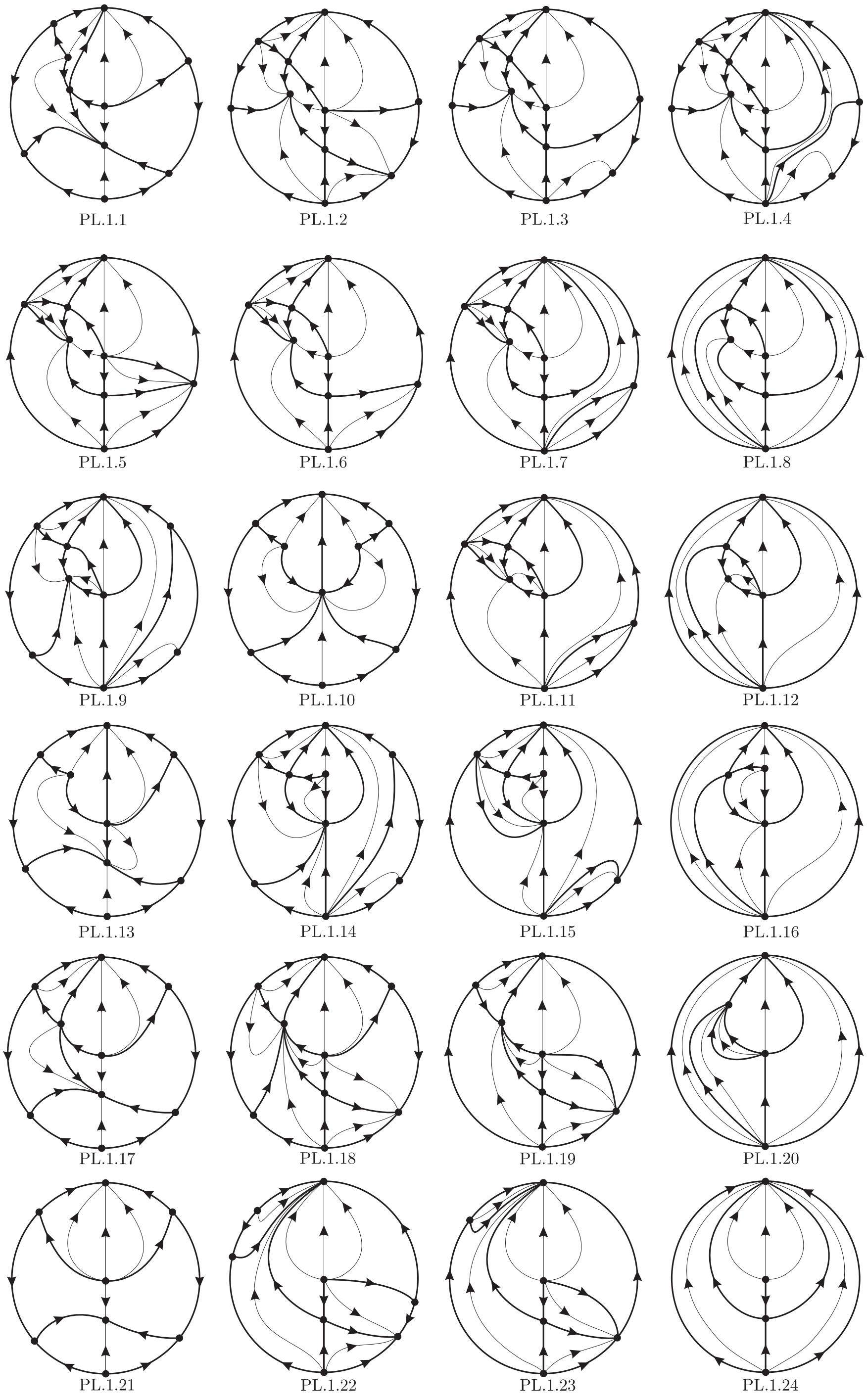


FIGURE 3. Phase portraits of systems of type (P) when they have a Darboux invariant.

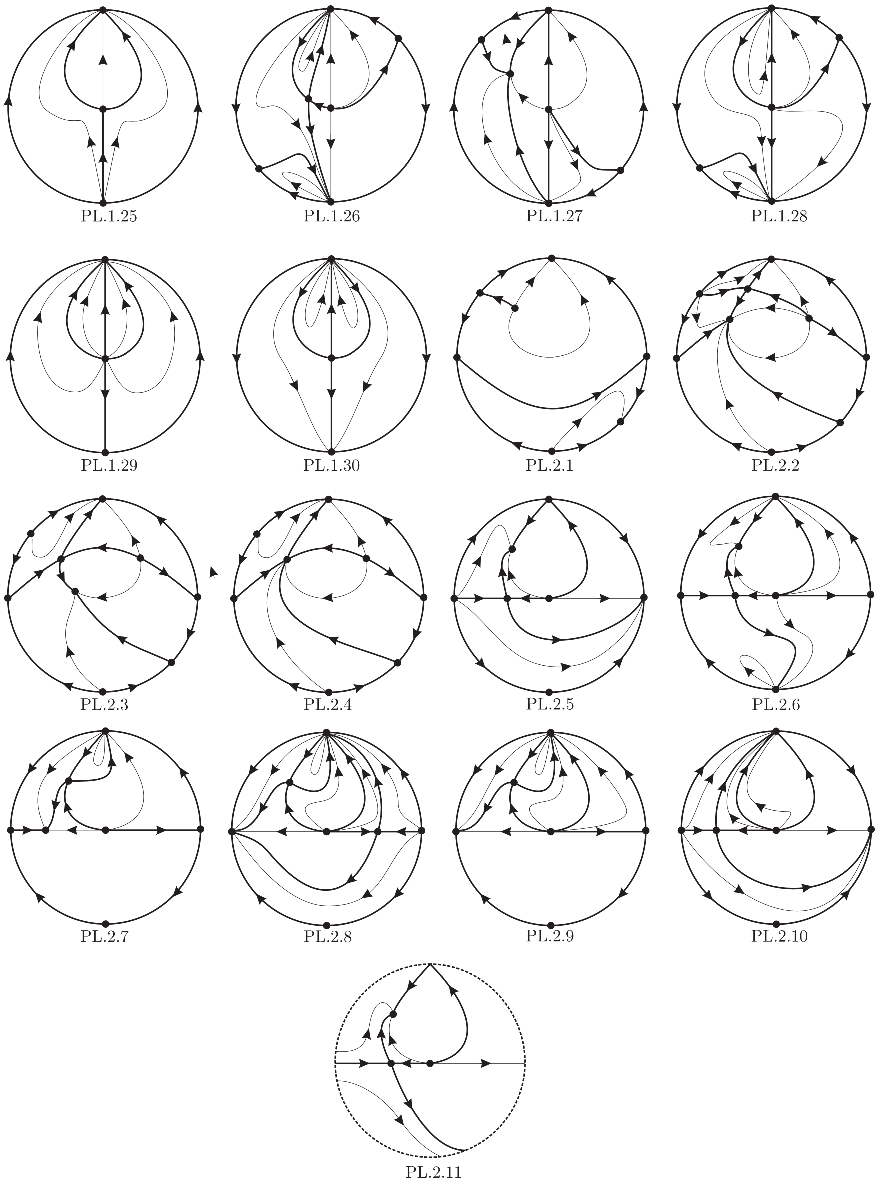


FIGURE 4. Phase portraits of systems of type (P) when they have a Darboux invariant.

On the other hand given $f \in \mathbb{C}[x, y]$ we say that the curve $f(x, y) = 0$ is an *invariant algebraic curve* of system (1) if there exists $K \in \mathbb{C}[x, y]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = Kf. \tag{5}$$

The polynomial K is called the *cofactor* of the invariant algebraic curve $f = 0$. When $K = 0$, f is a polynomial first integral. Note that if a real polynomial differential system has a complex invariant algebraic curve then it has also its conjugate. It is important to consider the complex invariant algebraic curves of the real systems because sometimes

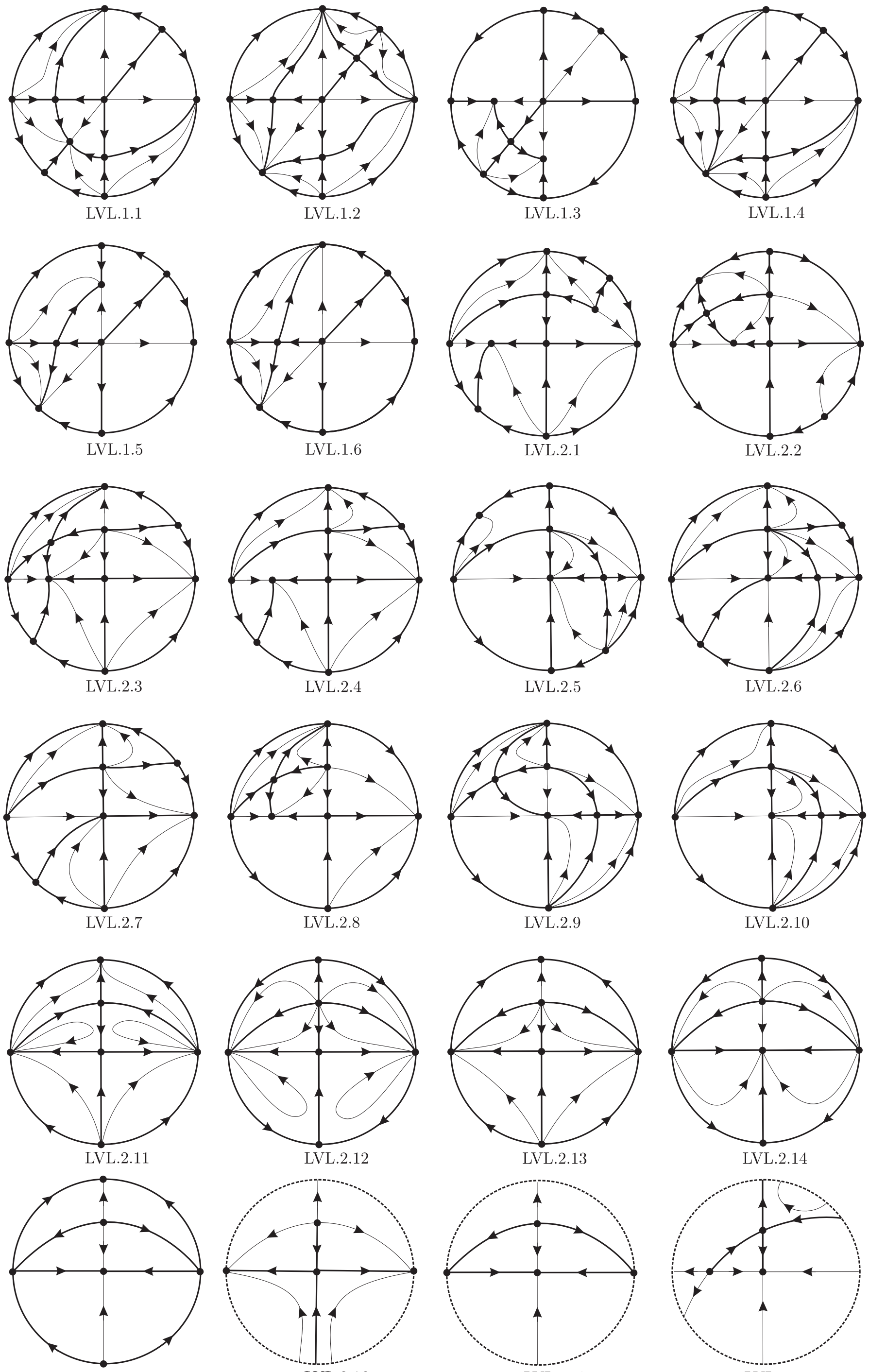


FIGURE 5. Phase portraits of systems of type (LV) when they have a Darboux invariant.

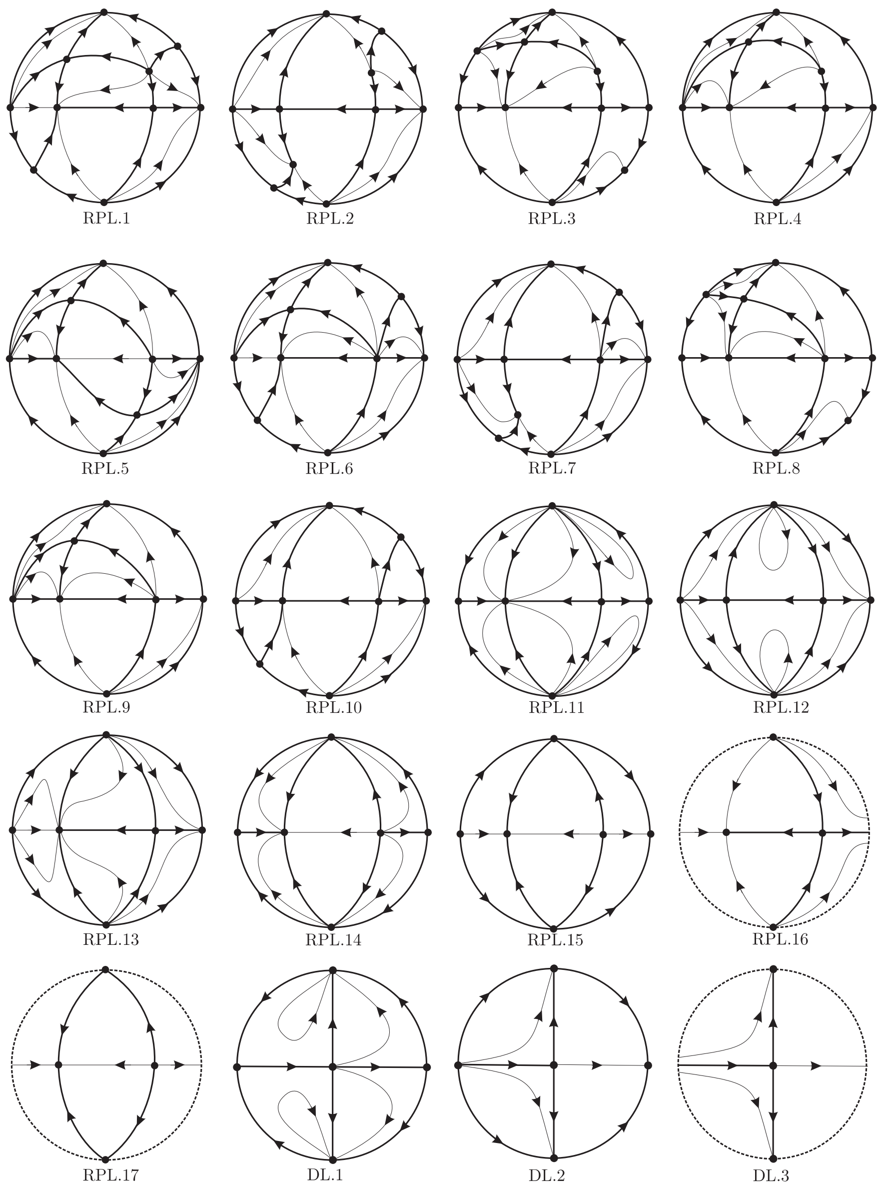


FIGURE 6. Phase portraits of systems of type (RPL) and (DL) when they have a Darboux invariant.

these force the real integrability of the system, for more details see Chapter 8 of [9], or the subsection 3.2.

Let $f, g \in \mathbb{C}[x, y]$ and assume that f and g are relatively prime in the ring $\mathbb{C}[x, y]$, or that $g = 1$. Then the function $\exp(f/g)$ is called a *exponential factor* of system (1) if for some polynomial $L \in \mathbb{C}[x, y]$ of degree at most $m - 1$ we have

$$P\frac{\partial \exp(f/g)}{\partial x} + Q\frac{\partial \exp(f/g)}{\partial y} = L \exp(f/g). \tag{6}$$

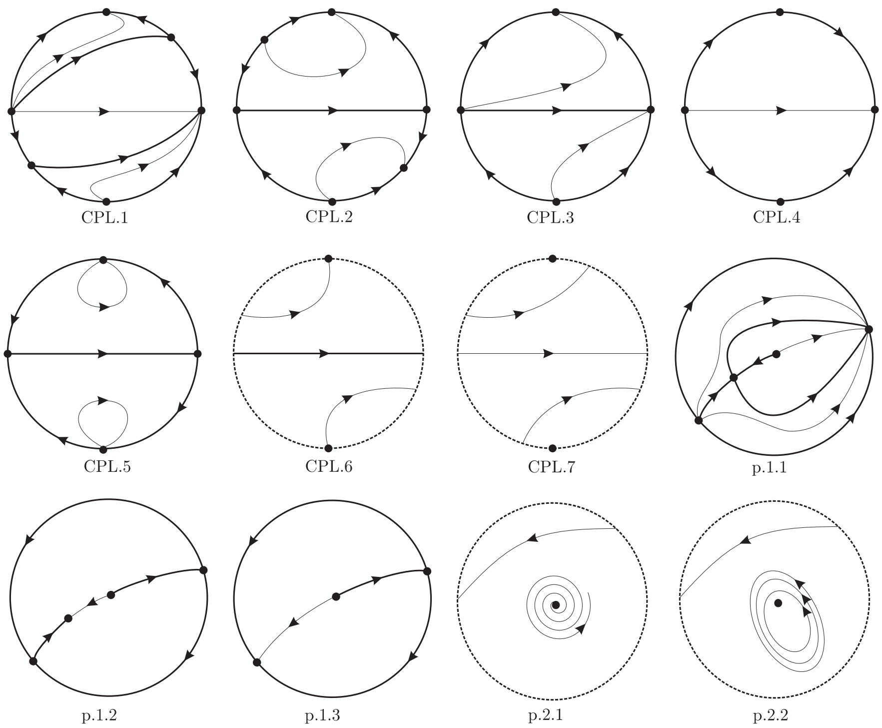


FIGURE 7. Phase portraits of systems of type (CPL) and (p) when they have a Darboux invariant.

As previously we say that L is the *cofactor* of the exponential factor $\exp(f/g)$. We observe that in the definition of exponential factor $\exp(f/g)$ if $f, g \in \mathbb{C}[x, y]$ then the exponential factor is a complex function. Again when we look for a complex exponential factor of a real polynomial system we are thinking the real polynomial system as a complex polynomial system.

3.2. **Darboux invariants.** An invariant I is called a *Darboux invariant* if it can be written into the form

$$I(x, y, t) = f_1^{\lambda_1} \cdots f_p^{\lambda_p} F_1^{\mu_1} \cdots F_q^{\mu_q} e^{s\,t}, \tag{7}$$

where $f_i = 0$ are invariant algebraic curves of system (1) for $i = 1, \ldots p$, and F_j are exponential factors of system (1) for $j = 1, \ldots, q$, $\lambda_i, \mu_j \in \mathbb{C}$ and $s \in \mathbb{R} \setminus \{0\}$.

Observe that if among the invariant algebraic curves a complex conjugate pair $f = \mathrm{Re}(f) + \mathrm{Im}(f)i = 0$ and $\bar{f} = \mathrm{Re}(f) - \mathrm{Im}(f)i = 0$ occurs, then the Darboux invariant has a factor of the form $f^{\lambda}\bar{f}^{\bar{\lambda}}$, which is the real multi-valued function

$$\Big((\mathrm{Re}(f))^2 + (\mathrm{Im}(f))^2\Big)^{\mathrm{Re}(\lambda)} e^{-2\mathrm{Im}(\lambda)\arctan(\mathrm{Im}(f)/\mathrm{Re}(f))}.$$

So if system (1) is real then the Darboux invariant is also real, independently of the fact of having complex invariant curves or complex exponential factors.

The next result is proved in Proposition 8.4 of [9].

Proposition 1. *Suppose that* $f \in \mathbb{C}[x, y]$ *and let* $f = f_1^{n_1} \ldots f_r^{n_r}$ *be its factorization into irreducible factors over* $\mathbb{C}[x, y]$*. Then for a polynomial differential system* (1)*,* $f = 0$ *is an* *invariant algebraic curve with cofactor k_f if and only if $f_i = 0$ is an invariant algebraic curve for each $i = 1, \ldots, r$ with cofactor k_{f_i}. Moreover $k_f = n_1 k_{f_1} + \ldots + n_r k_{f_r}$.*

The next result, proved in [6], explain how to obtain a Darboux invariant using the algebraic invariant curves of a polynomial differential system.

Proposition 2. *Suppose that a polynomial system* (1) *of degree m admits p invariant algebraic curves $f_i = 0$ with cofactors k_i for $i = 1, ..., p$, q exponential factors* $\exp(g_j/h_j)$ *with cofactors L_j for $j = 1, ..., q$, then, if there exist λ_i and $\mu_j \in \mathbb{C}$ not all zero such that*

$$\sum_{i=1}^{p} \lambda_i k_i + \sum_{j=1}^{q} \mu_j L_j = -s, \tag{8}$$

for some $s \in \mathbb{R} \backslash \{0\}$, then substituting $f_i^{\lambda_i}$ by $|f_i|^{\lambda_i}$ if $\lambda_i \in \mathbb{R}$, the real (multi-valued) function

$$f_1^{\lambda_1} \ldots f_p^{\lambda_p} \left(\exp\left(\frac{g_1}{h_1}\right)\right)^{\mu_1} \ldots \left(\exp\left(\frac{g_q}{h_q}\right)\right)^{\mu_q} e^{st}$$

is a Darboux invariant of system (1).

The search of first integrals is a classic tool in order to describe the phase portraits of a 2–dimensional differential system. As usual the *phase portrait* of a system is the decomposition of the domain of definition of this system as union of all its orbits.

It is well known that the existence of a first integral or an a invariant for a planar differential system allow to draw its phase portrait. Here we investigate the existence of invariants of the form $f(x, y)e^{st}$, called Darboux invariants, see section 3.2 for details. Such invariants describe the asymptotic behavior of the solutions of the system.

Indeed let $\phi_p(t)$ be the solution of system (1) passing through the point $p \in \mathbb{R}^2$, defined on its maximal interval (α_p, ω_p) such that $\phi_p(0) = p$. If $\omega_p = \infty$ we define the *ω-limit set* of p as

$$\omega(p) = \{q \in \mathbb{R}^2 : \exists \{t_n\} \text{ with } t_n = \infty \text{ and } \phi_p(t_n) = q \text{ when } n = \infty\}.$$

In the same way, if $\alpha_p = -\infty$ we define the *α-limit set* of p as

$$\alpha(p) = \{q \in \mathbb{R}^2 : \exists \{t_n\} \text{ with } t_n = -\infty \text{ and } \phi_p(t_n) = q \text{ when } n = \infty\}.$$

For more details on the ω– and α–limit sets see for instance section 1.4 of [9].

The existence of a Darboux invariant of system (1) provides information about the ω– and α–limit sets of all orbits of system (1). More precisely, we have the following result, where the definition of Poincaré compactification and Poincaré disc is given in subsection 3.3. Its proof can be found in [13].

Proposition 3. *Let $I(x, y, t) = f(x, y)e^{st}$ be a Darboux invariant of system* (1). *Let $p \in \mathbb{R}^2$ and $\phi_p(t)$ the solution of system* (1) *with maximal interval (α_p, ω_p) such that $\phi_p(0) = p$.*

(1) *If $\omega_p = \infty$ then $\omega(p) \subset \{f(x, y) = 0\} \cup \mathbb{S}^1$,*
(2) *If $\alpha_p = -\infty$ then $\alpha(p) \subset \{f(x, y) = 0\} \cup \mathbb{S}^1$.*

Here $\mathbb{S}^1$ denotes the infinity of the Poincaré disc.

3.3. **Poincaré compactification.** Let$\mathcal{X} = P(x,y)\frac{\partial}{\partial x} + Q(x,y)\frac{\partial}{\partial y}$ be the planar polynomial vector field of degree m associated to the polynomial differential system (1). The *Poincaré compactified vector field* $\pi(\mathcal{X})$ *corresponding to* $\mathcal{X}$ is an analytic vector field induced on $\mathbb{S}^2$ as follows (for more details, see [9]).

Let $\mathbb{S}^2 = \{y = (y_1, y_2, y_3) \in \mathbb{R}^3;\ y_1^2 + y_2^2 + y_3^2 = 1\}$ and $T_y\mathbb{S}^2$ be the tangent plane to $\mathbb{S}^2$ at point y. We identify $\mathbb{R}^2$ with $T_{(0,0,1)}\mathbb{S}^2$ and we consider the central projection $f : T_{(0,0,1)}\mathbb{S}^2 = \mathbb{S}^2$. The map f defines two copies of $\mathcal{X}$ on $\mathbb{S}^2$, one in the southern hemisphere and the other in the northern hemisphere. Denote by $\mathcal{X}'$ the vector field $D(f \circ \mathcal{X})$ defined on $\mathbb{S}^2 \setminus \mathbb{S}^1$, where $\mathbb{S}^1 = \{y \in \mathbb{S}^2;\ y_3 = 0\}$ is identified with the infinity of $\mathbb{R}^2$.

For extending $\mathcal{X}'$ to a vector field on $\mathbb{S}^2$, including $\mathbb{S}^1$, $\mathcal{X}$ must satisfy convenient conditions. Since the degree of $\mathcal{X}$ is m, $\pi(\mathcal{X})$ is the unique analytic extension of $y_3^{m-1}\mathcal{X}'$ to $\mathbb{S}^2$. On $\mathbb{S}^2 \setminus \mathbb{S}^1$ there is two symmetric copies of $\mathcal{X}$, and once we know the behavior of $\pi(\mathcal{X})$ near $\mathbb{S}^1$, we know the behavior of $\mathcal{X}$ in a neighborhood of the infinity. The Poincaré compactification has the property that $\mathbb{S}^1$ is invariant under the flow of $\pi(\mathcal{X})$. The projection of the closed northern hemisphere of $\mathbb{S}^2$ on $y_3 = 0$ under $(y_1, y_2, y_3) \mapsto (y_1, y_2)$ is called the *Poincaré disc*, and its boundary is $\mathbb{S}^1$.

Two polynomial vector fields $\mathcal{X}$ and $\mathcal{Y}$ on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on $\mathbb{S}^2$ preserving the infinity $\mathbb{S}^1$ carrying orbits of the flow induced by $\pi(\mathcal{X})$ into orbits of the flow induced by $\pi(\mathcal{Y})$ preserving or not the orientation of all the orbits.

As $\mathbb{S}^2$ is a differentiable manifold, in order to compute the explicit expression of $\pi(\mathcal{X})$, we consider six local charts $U_i = \{y \in \mathbb{S}^2;\ y_i > 0\}$ and $V_i = \{y \in \mathbb{S}^2;\ y_i < 0\}$, where $i = 1, 2, 3$, and the diffeomorphisms $F_i : U_i = \mathbb{R}^2$ and $G_i : V_i = \mathbb{R}^2$, for $i = 1, 2, 3$, which are the inverses of the central projections from the tangent planes at the points $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$ and $(0, 0, -1)$, respectively. We denote by $z = (u, v)$ the value of $F_i(y)$ and $G_i(y)$, for any $i = 1, 2, 3$, therefore z means different things depending on the local charts where we are working. So after some computations $\pi(\mathcal{X})$ is given by:

$$v^m \Delta(z) \left(Q\left(\frac{1}{v}, \frac{u}{v}\right) - uP\left(\frac{1}{v}, \frac{u}{v}\right), -vP\left(\frac{1}{v}, \frac{u}{v}\right) \right) \text{ in } U_1, \tag{9}$$

$$v^m \Delta(z) \left(P\left(\frac{u}{v}, \frac{1}{v}\right) - uQ\left(\frac{u}{v}, \frac{1}{v}\right), -vQ\left(\frac{u}{v}, \frac{1}{v}\right) \right) \text{ in } U_2, \tag{10}$$

$$\Delta(z)(P(u,v), Q(u,v)) \text{ in } U_3, \tag{11}$$

where $\Delta(z) = (u^2 + v^2 + 1)^{-(m-1)/2}$. The expressions for V_i's are the same as that for U_i's but multiplied by the factor $(-1)^{m-1}$. In these coordinates $v = 0$ always denotes the points of the infinity $\mathbb{S}^1$.

3.4. **Irreducible invariant cubics.** The next results characterize all irreducible cubics, their proofs can be found in [2].

Proposition 4 (Theorem 8.3 in [2])**.** *A cubic is non–singular and irreducible and has a flex (a generalized inflection point) if and only if it can be transformed with affine transformations into either*

$$y^2 = x(x-1)(x-r) \qquad \textit{with } r > 1,$$

or

$$y^2 = x(x^2+sx+1) \qquad \text{with } -2<s<2.$$

Proposition 5 (Theorem 8.4 in [2]). *A cubic is singular and irreducible if and only if it can be transformed with affine transformations into one of the forms*

$$y^2 = x^3, \qquad y^2 = x^2(x+1), \qquad y^2 = x^2(x-1).$$

Moreover in [2] it is proved that every non-singular and irreducible curve has a flex. So we have the complete characterization of the irreducible cubics.

3.5. **Reducible invariant cubics.**

Proposition 6. *A real quadratic system having an invariant conic after an affine change of coordinates can be written in one of the following forms*

$$\begin{array}{ll}
\textit{(real ellipse)} & \begin{aligned}\dot x &= (A/2)(x^2+y^2-1)+2y(p+q\,x+r\,y),\\ \dot y &= (B/2)(x^2+y^2-1)-2x(p+q\,x+r\,y),\end{aligned}\\[2ex]
\textit{(complex ellipse)} & \begin{aligned}\dot x &= (A/2)(x^2+y^2+1)+2y(p+q\,x+r\,y),\\ \dot y &= (B/2)(x^2+y^2+1)-2x(p+q\,x+r\,y),\end{aligned}\\[2ex]
\textit{(hyperbola)} & \begin{aligned}\dot x &= (A/2)(x^2-y^2-1)-2y(p+q\,x+r\,y),\\ \dot y &= -(B/2)(x^2-y^2-1)-2x(p+q\,x+r\,y),\end{aligned}\\[2ex]
\textit{(parabola)} & \begin{aligned}\dot x &= A(y-x^2)-(p+q\,x+r\,y),\\ \dot y &= B(y-x^2)-2x(p+q\,x+r\,y),\end{aligned}\\[2ex]
\textit{(Lotka-Volterra)} & \begin{aligned}\dot x &= x(p_1+q_1\,x+r_1\,y)\\ \dot y &= y(p_2+q_2\,x+r_2\,y),\end{aligned}\\[2ex]
\textit{(two parallel real lines)} & \begin{aligned}\dot x &= x^2-1\\ \dot y &= Q(x,y),\end{aligned}\\[2ex]
\textit{(double line)} & \begin{aligned}\dot x &= x^2\\ \dot y &= Q(x,y),\end{aligned}\\[2ex]
\textit{(two parallel complex lines)} & \begin{aligned}\dot x &= x^2+1\\ \dot y &= Q(x,y),\end{aligned}\\[2ex]
\textit{(two non-parallel complex lines)} & \begin{aligned}\dot x &= (A/2)(x^2+y^2)+(C/2)x+2y(p+q\,x+r\,y),\\ \dot y &= (B/2)(x^2+y^2)+(C/2)y-2x(p+q\,x+r\,y).\end{aligned}
\end{array}$$

Here $Q(x,y)$ denotes an arbitrary polynomial of degree 2.

The proof of the previous result can be found in [4], except to the normal form of the system with a parabola that is proved in [11]. The next result is due to Christopher, Llibre, Pantazi, Zhang and Zholadek, see [5, 7, 17]. An algebraic proof of it also can be found in [7].

Theorem 7. *Let $f_i = 0$ for $i = 1, \ldots, q$ be q irreducible algebraic curves in $\mathbb{C}^2$, and let $k = \sum_{i=1}^{q} deg f_i$. We assume*

(i) *there are no points at which f_i and its first derivatives all vanish,*
(ii) *the highest order terms of f_i have no repeated factors,*
(iii) *no more than two curves meet at any point in the finite plane and are not tangent at these points,*

(iv) *no two curves have a common factor in their highest order terms, then any polynomial vector field X of degree m tangent to all $f_i = 0$ is of the form describe bellow.*

(a) *If $m > k-1$ then*

$$X = Y\left(\prod_{i=1}^{q} f_i\right) + \sum_{i=1}^{q}\left(\prod_{j=1, j\neq i}^{q} f_j\right) X_{f_i}, \tag{12}$$

where $X_{f_i} = (-\partial f_i/\partial y, \partial f_i/\partial x)$ is a Hamiltonian vector field, the h_i are polynomials of degree $\leq m-k+1$ and Y is a polynomial vector field of degree $\leq m-k$.

(b) *If $m = k-1$ then*

$$X = \sum_{i=1}^{q} \alpha_i \left(\prod_{j=1, j\neq i}^{q} f_j\right) X_{f_i}, \tag{13}$$

where $\alpha_i \in \mathbb{C}$. In this case a Darboux first integral exists.

(c) *If $m < k-1$ then $X \equiv 0$.*

Theorem 8 (Lemma 7 of [7])**.** *Assume that $f = 0$ and $g = 0$ are different irreducible invariant algebraic curves of system* (1) *of degree m, and that they satisfy conditions* (*i*) *and* (*iii*) *of Theorem 7. If $gcd(f_x, f_y) = 1$ and $gcd(g_x, g_y) = 1$, then system* (1) *has the normal form*

$$\dot{x} = Afg - h_1 f_y g - h_2 f g_y \qquad \dot{y} = Bfg + h_1 f_x g + h_2 f g_x, \tag{14}$$

where A, B and h_j are polynomials, for $i = 1, 2$.

4. Proof of Theorems A and B

Here we denote

$$P(x,y) = a_{00} + a_{01}y + a_{02}y^2 + a_{10}x + a_{11}xy + a_{20}x^2,$$
$$Q(x,y) = b_{00} + b_{01}y + b_{02}y^2 + b_{10}x + b_{11}xy + b_{20}x^2.$$

4.1. **Proof of Theorem A.** If a quadratic system (1) has a singular irreducible invariant cubic $f(x,y) = 0$ by Proposition 5 the function f can be written as $f(x,y) = y^2 - x^3$ or $f(x,y) = y^2 - x^2(x+1)$ or $f(x,y) = y^2 - x^2(x-1)$. The curve $f(x,y) = y^2 - x^3 = 0$ is an invariant cubic for system (1) if and only if equation (5) is satisfied. The solution of this equation in terms of the parameters of the system is

$$a_{00} = a_{02} = b_{00} = b_{10} = 0, \quad b_{01} = 3a_{10}/2, \quad b_{02} = 3a_{11}/2, \quad b_{11} = 3a_{20}/2, \quad b_{20} = 3a_{01}/2.$$

So the cofactor of f is $K = 3(a_{10} + a_{20}x + a_{11}y)$. Doing $a_{10} = a$, $a_{20} = b$, $a_{01} = c$, $a_{11} = d$ and a rescaling of the time we obtain system (*i*) of Theorem A.

When $f(x,y) = y^2 - x^2(x \pm 1)$ we obtain the normal forms given in (*ii*) and (*iii*) of the theorem following similar steps.

Now if a quadratic system (1) has an invariant non–singular irreducible cubic $f(x,y) = 0$ then by Proposition 4 we can write $f(x,y) = y^2 - x(x-1)(x-r)$ with $r > 1$ or $f(x,y) = y^2 - x(x^2 + sx + 1)$ with $-2 < s < 2$. In the first case solving equation (5) we obtain three solution but fixing $r > 1$ only one solution can hold $a_{00} = a_{02} = a_{10} = a_{20} = b_{01} = b_{11} = 0$, $b_{00} = a_{01}r/2$, $b_{02} = 3a_{11}/2$, $b_{10} = -a_{01}(r+1) - a_{11}r$, $b_{20} = (3a_{01} + a_{11}r + a_{11})/2$. It corresponds to system (*iv*) of Theorem A.

For $f(x,y) = y^2 - x(x^2 + sx + 1)$ we obtain only one solution corresponding to system (v) of the theorem. ∎

Using the normal forms described in Theorem A we investigate when these systems admit a Darboux invariant of the form $e^{st}f(x,y)$.

4.2. **Proof of Theorem B.** First of all is easy to see that the cofactor K of f in systems $(ii) - (v)$ of Theorem A has no constant terms. Then equation (8) becomes $\lambda K + s = 0$ which never holds if $s \in \mathbb{R} \setminus \{0\}$ and $\lambda \in \mathbb{C} \setminus \{0\}$. Therefore we conclude that systems $(ii) - (v)$ do not admit a Darboux invariant of such form.

Now considering system (i) of Theorem A we have $f(x,y) = y^2 - x^3 = 0$ as invariant curve with cofactor $K = 6(a + cx + dy)$. In this case the solution of equation (8) is given by $\{c = 0,\ d = 0,\ s = -6a\lambda\}$. Taking $\lambda = -s/(6a)$ we obtain the system

$$\dot{x} = 2(ax + by), \qquad \dot{y} = 3(ay + bx^2),$$

with Darboux invariant $e^{-6at}(y^2 - x^3)$.

The normal form described in Theorem B is obtained doing the following change of coordinates and rescaling of the time $x = \dfrac{2a^2}{3b^2}X$, $y = \dfrac{2a^3}{3b^3}Y$, $t = \dfrac{1}{2a}T$.

Now it remains to study the phase portrait of system (2). This system has two singular points, namely $z_1 = (0,0)$ yperbolic unstable node, and $z_2 = (3/2, -3/2)$ a hyperbolic saddle. Applying the Poincaré compactification in the local chart U_1 and on the line $v = 0$ the compactified system has no singular points. However in the local chart U_2 the origin $(0,0)$ is a nilpotent singularity. With the notation of Theorem 3.5 of [9] the compactified system has $F(u) = -u^5 - (3/2)u^6$ and $G(u) = -4u^2 - (7/2)u^3$. Hence the origin of U_2 is a nilpotent stable node. By the previous statements it follows that the phase portrait of system (2) is the one described in Figure 1. ∎

5. Proof of Theorem C

The proof is done according to the conic that appears in the expression of the reducible cubic.

5.1. **Systems of type** (E)**.** If system (1) has an invariant cubic of the form $f(x,y) = f_1(x,y)f_2(x,y)$ with $f_1 = x^2 + y^2 - 1$ and $f_2 = ax + by + c$, then applying a rotation we can assume $b = 1$. Therefore it follows from Proposition 1 that f_j is an invariant curve with cofactor $k_j = \alpha_j + \beta_j x + \gamma_j y$, $j = 1, 2$. Consider two cases: $a = 0$ and $a \neq 0$.

If $a = 0$ then using equation (5) we have $Q = k_2 f_2$ and $P = (k_1 f_1 - 2yk_2 f_2)/(2x)$. As P is a polynomial the parameters of the system must satisfy on the of following conditions

$$\begin{aligned} s_1 &= \{c = -1,\ \alpha_1 = 0,\ \gamma_1 = 2\alpha_2,\ \gamma_2 = \alpha_2\}, \\ s_2 &= \{c = 1,\ \alpha_1 = 0,\ \gamma_1 = -2\alpha_2,\ \gamma_2 = -\alpha_2\}, \\ s_3 &= \{\alpha_1 = 0,\ \gamma_1 = 0,\ \gamma_2 = 0\}. \end{aligned}$$

Moreover the solutions s_1 and s_2 provide equivalent systems, and we can summarize the solutions s_1 and s_3 writing the system

$$\begin{aligned} \dot{x} &= (\beta_1/2)(x^2 + y^2 - 1) - y(\beta_2\, y - \alpha_2\, x + c\beta_2), \\ \dot{y} &= (y + c)(\alpha_2\, y + \beta_2 c\, x + \alpha_2), \end{aligned} \tag{15}$$

with $\alpha_2(c+1) = 0$. This is exactly system $(E.1)$ of Theorem C.

When $a \neq 0$ we check when the hypotheses of Theorem 7 are satisfied. Clearly f_1 and f_2 satisfies $(i), (ii)$ and (iv). Condition (iii) is not satisfied when $c^2 = a^2 + 1$ because the line

$f_2 = 0$ is tangent to the real ellipse $f_1 = 0$. Indeed if the straight line $f_2 = y + ax + c = 0$ is tangent to the real ellipse $f_1 = x^2 + y^2 - 1 = 0$ at the point (x_0, y_0), then their gradients are parallel in such point, what means that $x_0 - ay_0 = 0$. Replacing $y_0 = x_0/a$ in the ellipse we conclude that $x_0 = \pm a/\sqrt{a^2+1}$. From $f_2 = 0$ we get $c = \mp\sqrt{a^2+1}$. Therefore the condition for the tangency is $c^2 = a^2 + 1$. In this case applying a rotation we can get $f_2 = y - 1$. Again we are in system (15) with $c = -1$.

Now assuming $c^2 \neq a^2 + 1$ it follows from Theorem 7 that our system is given by

$$\dot{x} = -\alpha_2(x^2+y^2-1) - 2\alpha_1 y(y+ax+c), \quad \dot{y} = a\alpha_2(x^2+y^2-1) + 2\alpha_1 x(y+ax+c), \tag{16}$$

where α_1, $\alpha_2 \in \mathbb{C}$ and a, $c \in \mathbb{R}$. As we are looking for a real system, then α_1, $\alpha_2 \in \mathbb{R}$, and doing a rescaling of the time we can assume $\alpha_2 = 1$. Note that system (16) is exactly system $(E.2)$ of Theorem C.

5.2. **Systems of type** (CE)**.** In this case we can follow the same steps applied previously. If system (1) has an invariant cubic of the form $f = f_1 f_2$ with $f_1 = x^2 + y^2 + 1$ and $f_2 = ax + by + c$ we suppose, without loss of generality, $b = 1$. Since the coefficients a, b and c are real numbers the straight line $f_2 = 0$ cannot be tangent to the complex ellipse $f_1 = 0$. So we get

$$\dot{x} = -\alpha_2(x^2+y^2+1) - 2\alpha_1\, y(y+ax+c), \quad \dot{y} = a\,\alpha_2(x^2+y^2+1) + 2\alpha_1\, x(y+ax+c), \tag{17}$$

where α_1, $\alpha_2 \in \mathbb{C}$ and a, $c \in \mathbb{R}$. Applying a rescaling we have $\alpha_2 = 1$ in (17), and we get the normal form for the systems of type (CE).

5.3. **Systems of type** (H)**.** Let $f_1 = x^2 - y^2 - 1$ and $f_2 = ax + by + c$ be two real algebraic invariant curves of system (1), so $a^2 + b^2 \neq 0$. Proceeding as before if $a = 0$ then we can assume $b = 1$ and the system can be written in the form

$$\dot{x} = (\beta_1/2)(x^2 - y^2 - 1) + \beta_2\, y(y + c), \quad \dot{y} = \beta_2\, y(y + c), \tag{18}$$

with $\beta_1\beta_2 \neq 0$. This is system $(H.1)$ of Theorem C.

If $a \neq 0$ and $b = 0$ we take $a = 1$ and system (1) satisfies $P = k_2 f_2$ and $2y\,Q = 2xP - k_1 f_1$, where $k_j = \alpha_j + \beta_j\, x + \gamma_j\, y$, for $j = 1, 2$. Since Q is a polynomial in the parameters of the system it must satisfy one of the following conditions

$$s_1 = \{c = -1,\ \alpha_1 = 0,\ \beta_1 = 2\alpha_2,\ \beta_2 = \alpha_2\},$$
$$s_2 = \{c = 1,\ \alpha_1 = 0,\ \beta_1 = -2\alpha_2,\ \beta_2 = -\alpha_2\},$$
$$s_3 = \{\alpha_1 = 0,\ \alpha_2 = 0,\ \beta_1 = 0,\ \beta_2 = 0\}.$$

Applying the change of coordinates $x = -X, y = Y$ we conclude that case s_1 and s_2 provide equivalent systems. Moreover we can summarize solutions s_1 and s_3 in the unique system

$$\dot{x} = (x + c)(\alpha_2\, x + \gamma_2 y + \alpha_2), \quad \dot{y} = -(\gamma_1/2)(x^2 - y^2 - 1) + x(\gamma_2\, x + \alpha_2\, y + c\gamma_2), \tag{19}$$

with $\alpha_2(c + 1) = 0$. System (19) corresponds to system $(H.2)$ of Theorem C.

If $ab \neq 0$ we assume $b = 1$ and consider three cases, according to the conditions of Theorem 7. Note that condition (i) of Theorem 7 holds because $\nabla f_1(x, y) = (2x, -2y)$ and $\nabla f_2(x, y) = (a, 1)$, where ∇ indicates the gradient. Condition (ii) also holds. However condition (iv) is not verified when $a^2 - 1 = 0$. Indeed in this case $f_1 = (x + y)(x - y) - 1$ and $f_2 = (y \pm x) + c$. Condition (iii) does not hold when $c^2 = a^2 - 1$ since the straight line $f_2 = y + ax + c = 0$ is tangent to the hyperbola. The proof of this last statement can be done analogously as for the systems of type (E). Hence when $a^2 - 1 = 0$ or $c^2 = a^2 - 1$

Theorem 7 does not hold and we split the study of systems of type (H) for $ab \neq 0$ in three cases: $a^2 - 1 = 0$, $c^2 = a^2 - 1$ and $(a^2 - 1)(c^2 - a^2 + 1) \neq 0$.

For the first two cases we apply Propositions 1 and 6 to conclude that f_1 is an algebraic invariant curve of a quadratic system (1) and it can be written as

$$(20)\quad \dot{x} = (A/2)(x^2-y^2-1)-2y(p+q\,x+r\,y), \qquad \dot{y} = -(B/2)(x^2-y^2-1)-2x(p+q\,x+r\,y),$$

where $A, B, p, q, r \in \mathbb{R}$. Fixing the cofactor of $f_2 = 0$ as $k_2 = \alpha + \beta\,x + \gamma\,y$, where $\alpha, \beta, \gamma \in \mathbb{R}$ and using system (20) we solve (5). First considering $a = -1$ (the case $a = 1$ is analogous except by a reflection) equation (5) has two possible solutions

$$s_1 = \{B = -A,\ c = 0, p = \alpha/2,\ q = \beta/2,\ r = \gamma/2\},$$
$$s_2 = \{B = -A + 2c\alpha,\ p = (\alpha c - \beta)/2,\ q = (\beta - c\alpha)/2, r = -(\beta + c\alpha)/2,\ \gamma = -\beta\}.$$

Using the two above solutions we get the system

$$\begin{aligned}\dot{x} =&\ \ (A/2)(x^2 - y^2 - 1) - y(\alpha - c\beta + x(\beta - c\alpha) + y(\gamma - c\alpha)),\\ \dot{y} =&\ \ (A/2)(x^2 - y^2 - 1) - x(\alpha - c\beta + \beta x + y(\gamma - c\alpha)) + c\alpha(y^2 + 1),\end{aligned}$$

with $c(\gamma + \beta) = 0$. This is system $(H.3)$ of Theorem C.

Now considering $c^2 = a^2 - 1$ we investigate the conditions that must be satisfied by the parameters of system (20) in order that $f_2 = y + ax \pm \sqrt{a^2 - 1}$ be an invariant curve. Without loss of generality we can assume $c = \sqrt{a^2 - 1}$. Equation (5) has one solution, namely

$$B = aA - 2\alpha\sqrt{d}, p = (\beta\sqrt{d} - a\alpha)/2, r = -\beta/2, q = (\alpha\sqrt{d} - a\beta)/2, \gamma = a\beta - \alpha\sqrt{d},$$

where $d = a^2 - 1$. Replacing it in (20) we get

$$(21)\qquad \begin{aligned}\dot{x} =&\ \ (A/2)(x^2 - y^2 - 1) + y(a\alpha - \beta\sqrt{d} + x(a\beta - \alpha\sqrt{d}) + \beta y),\\ \dot{y} =&\ \ (-Aa/2)(x^2 - y^2 - 1) + x(a\alpha - \beta\sqrt{d} + a\beta x + \beta y) - \alpha\sqrt{d}(y^2 + 1),\end{aligned}$$

where $d = a^2 - 1$, and this systems corresponds to system $(H.4)$ of Theorem C.

Finally if $(a^2 - 1)(c^2 - a^2 + 1) \neq 0$ applying Theorem 7 we obtain the system

$$(22)\qquad \begin{aligned}\dot{x} =&\ \ -\alpha_2(x^2 - y^2 - 1) + 2\alpha_1 y(y + ax + c),\\ \dot{y} =&\ \ a\alpha_2(x^2 - y^2 - 1) + 2\alpha_1 x(y + ax + c),\end{aligned}$$

which is system $(H.5)$ of Theorem C.

5.4. **Systems of type** (P)**.** Let $f = (y - x^2)(ax + by + c) = 0$ be an invariant cubic of system (1). When $b = 0$ we can assume $f = x(y - x^2)$. Indeed if $b = 0$ we take $a = 1$ and do the change of coordinates $x = X - c$, $y = Y - 2cX + c^2$. Using that $f_2 = x = 0$ is an invariant straight line we have $P = k_2 f_2$ with $k_2 = \alpha_2 + \beta_2\,x + \gamma_2\,y$, and a quadratic system (1) can be written as

$$(23)\qquad \dot{x} = x(\alpha_2 + \beta_2\,x + \gamma_2\,y), \quad \dot{y} = \alpha_1(y - x^2) + 2\alpha_2 x^2 + 2y(\beta_2\,x + \gamma_2\,y).$$

If $b \neq 0$ and $a = 0$ we can take $b = 1$ and proceed as in systems of type (H) and (E), then we get the system

$$(24)\quad \dot{x} = -\beta_1(y - x^2) + y(\beta_2 + \gamma_2 x) + (\alpha_2 + \gamma_2 c)x + c\beta_2, \quad \dot{y} = 2(y + c)(\alpha_2 + \beta_2 x + \gamma_2 y),$$

with $c\,\alpha_2 = 0$. Observe that when $c = 0$ the invariant line is $y = 0$ and when $\alpha_2 = 0$ it is $y + c = 0$.

If $ab \neq 0$ and $f_2 = y \pm ax + a^2/4$, $f_2 = 0$ is tangent to the parabola. In this case we can assume $f_2 = y + ax + a^2/4$ (the other case is a reflection). Applying the change of coordinates $x = -X - a/2$ and $y = Y + aX + a^2/4$ the cubic $f = (y - x^2)(y + ax + a^2/4)$

becomes $f = (Y - X^2)Y$, which already has been studied above. Indeed it corresponds to system (23) with $c = 0$.

Otherwise there is no tangency between the straight line and the parabola, and we apply Theorem 7 to get the differential system

$$\dot{x} = -(y - x^2) - \alpha(y + ax + c), \quad \dot{y} = a(y - x^2) - 2\alpha\, x(y + ax + c). \tag{25}$$

Systems (23), (24) and (25) correspond to systems $(P.1), (P.2)$ and $(P.3)$ of Theorem C, respectively.

5.5. **Systems of type** (LV)**.** In this case $f = x\,y\,(a\,x + b\,y + c) = 0$ is the invariant curve and except by a rotation we can assume $b = 1$. We consider different cases according to $ac = 0$ or $ac \neq 0$. Note that if $c = 0$ hypothesis (iii) of Theorem 7 is not valid, whereas $a = 0$ breaks the hypothesis (iv).

When $c = 0$ and $a \neq 0$, doing the change of coordinates $x = -\frac{Y}{\sqrt[3]{a^2}}$, $y = \sqrt[3]{a}X$ the cubic becomes $F = X\,Y\,(Y - X)$. So using Proposition 6 the differential system can be written as

$$\dot{x} = x(p_1 + q_1\,x + r_1\,y) \quad \dot{y} = y(p_2 + q_2\,x + r_2\,y). \tag{26}$$

If (26) has $f_3 = y - x$ as an invariant curve with cofactor $k = \alpha + \beta\,x + \gamma\,y$, then equation (5) must be satisfied. Solving it we get

$$s_1 = \{p_2 = \alpha,\ r_2 = \beta - q_2 + r_1,\ q_1 = \beta,\ p_1 = \alpha,\ \gamma = \beta - q_2 + r_1\}.$$

Replacing in (26) and writing $q = q_2$, $r = r_1$ we obtain system $(LV.1)$ of Theorem C.

Now if $c = a = 0$ then $f_2 = y = 0$ is a double line, and it is not difficult to see that we can write the system as

$$\dot{x} = x(p + q\,x + r\,y), \quad \dot{y} = y^2. \tag{27}$$

Finally, when $a = 0$ and $c \neq 0$, doing the change of coordinates $x = X/c^2$, $y = cY - c$ the cubic $f = 0$ becomes $F = X\,Y\,(Y - 1)$. So without loss of generality we can work with $f_3 = y - 1$. Again the idea is to write the system as in (26), and see what are the conditions in order that $f_3 = 0$ to be an invariant curve for such system. Solving equation (5) and replacing the solutions in (26) we get

$$\dot{x} = x(p + q\,x + r\,y), \quad \dot{y} = y(y - 1). \tag{28}$$

Systems (27) and (28) can be summarized as

$$\dot{x} = x(p + q\,x + r\,y), \quad \dot{y} = y(y + c),$$

with $c = 0$ or $c = -1$. This is exactly system $(LV.2)$ of Theorem C.

In the last case, $a\,c \neq 0$ the invariant cubic is $f = x\,y\,(y + a\,x + c) = 0$ and by the geometry to the curves we can assume $a < 0$ and $c < 0$. Applying Theorem 7 we get the system

$$\dot{x} = -\alpha_2\,x(y + a\,x + c) - \alpha_3\,x\,y, \quad \dot{y} = \alpha_1\,y(y + a\,x + c) + a\,\alpha_3\,x\,y.$$

Note that we can take $\alpha_3 = 1$. Doing $\alpha = \alpha_2$, $\beta = \alpha_1$ we obtain system $(LV.3)$ of Theorem C.

5.6. **Systems of type** (RPL)**.** Here the invariant cubic is $f = f_1 f_2 f_3 = 0$ where $f_1 = x+1$, $f_2 = x-1$ and $f_3 = a\,x + b\,y + c$. When $b = 0$ we apply Proposition 6 (case *(RPL)*), then it is easy to see that the corresponding normal form has one additional invariant curve $f_3 = 0$ as invariant straight line if and only if it is a multiple of f_1 or f_2. However we cannot consider any of these cases because if the system has f_2 as an invariant double straight line for example, then there would be a change of coordinates so that the system would be written as

$$\dot{x} = (x-1)(x+1)^2, \quad \dot{y} = Q(x,y),$$

then having degree 3 instead of 2.

When $b \neq 0$ we can fix $b = 1$. In this case the cubic $f = (x^2-1)(y + ax + c) = 0$ can be reduced to $F = y(x^2-1)$ by change of coordinates $x = X, y = Y - a\,X - c$. If the quadratic differential system (1) has the invariant curve $f = y(x^2-1) = 0$, then $f_1 = 0$ and $f_2 = 0$ are invariant curves and by Proposition 6 such system can be written as

$$\dot{x} = x^2 - 1, \quad \dot{y} = Q(x,y), \tag{29}$$

where $Q(x,y)$ is an arbitrary polynomial of degree 2. Imposing that $f_3 = y = 0$ is an additional invariant curve with cofactor $k_3 = \alpha + \beta\,x + \gamma\,y$, the above system must satisfy $Q(x,y) = y(\alpha + \beta\,x + \gamma\,y)$. This expression justify the normal form given in (RPL) of Theorem C.

5.7. **Systems of type** (DL)**.** These systems have a double straight line as invariant curve which can be taken as $f_1 = x$. We write $f_2 = a\,x + b\,y + c$ and use the normal form of a system having $f = f_1 f_2 = 0$ as an invariant cubic. For such normal form, if $b = 0$ then $f_2 = 0$ is an invariant straight line if and only if $c = 0$ but in this case the system cannot have a triple invariant straight line.

If $b \neq 0$ we can take $b = 1$ and $f = x^2(y + a\,x + c)$. Doing the change $x = X, y = Y - a\,X - c$ the function f can be written as $F = X^2 Y$. Hence it is enough to consider $f_2 = y$. By Proposition 6 a quadratic system (1) can be written as

$$\dot{x} = x^2, \quad \dot{y} = Q(x,y),$$

where $Q(x,y)$ is an arbitrary polynomial of degree 2. Imposing that $f_2 = 0$ is an additional invariant curve with cofactor $k_2 = \alpha + \beta\,x + \gamma\,y$, we conclude that $Q(x,y) = y(\alpha + \beta\,x + \gamma\,y)$. This expression justify the normal form given in (DL) of Theorem C.

5.8. **Systems of type** (CPL)**.** The proof for this case is analogous to the case (DL) so we will omit some details. In short the cubic is given by $f = f_1 f_2 f_3 = 0$ where $f_1 = x + i$, $f_2 = x - i$ and $f_3 = ax + by + c$. In order to $f_3 = 0$ to be an invariant curve with $b = 0$ it is necessary that $c = \pm i$. So $b \neq 0$ and we assume $b = 1$. This reduce f to the cubic $F = y(x^2+1)$ and then we get the normal form (CPL) described in Theorem C.

5.9. **Systems of type** (p)**.** In this case the cubic is given by $f = (x^2+y^2)(a\,x + b\,y + c) = 0$ and except by a rotation we can assume $b = 1$. When $c = 0$ the three curves intersect at the same point and the conditions of Theorem 7 are not satisfied. But if $c = 0$ doing the change of coordinates

$$x = -\frac{X}{\sqrt[3]{(a^2+1)^2}} + \frac{aY}{\sqrt[3]{(a^2+1)^2}}, \quad y = \frac{aX}{\sqrt[3]{(a^2+1)^2}} + \frac{Y}{\sqrt[3]{(a^2+1)^2}},$$

the cubic $f = (x^2+y^2)(y + a\,x) = 0$ is reduced to $f = Y(X^2+Y^2)$. Now using that system (1) has $f_3 = y = 0$ as a third invariant curve it follows that $Q(x,y) = k_3 f_3$ where

$k_3 = \alpha_3 + \beta_3\, x + \gamma_3\, y$ is the cofactor of f_3. Moreover $f_1 f_2 = 0$ is also an invariant curve then we must have

$$2xP(x,y) + 2yQ(x,y) = k(x,y)(x^2+y^2),$$

with $k(x,y) = \alpha + \beta\, x + \gamma\, y$ being the sum of the cofactors of f_1 and f_2. So a quadratic system (1) can be written as

$$\dot{x} = (\beta/2)(x^2+y^2) - \beta_3\, y^2 + x(\alpha_3 + \gamma_3\, y), \quad \dot{y} = y(\alpha_3 + \beta_3\, x + \gamma_3\, y),$$

which is exactly system $(p.1)$ of Theorem C.

When $c \neq 0$ we apply Theorem 7 and conclude that a quadratic system (1) can be written as

$$\begin{aligned} \dot{x} &= -\alpha_3(x^2+y^2) - ((\alpha_2+\alpha_1)y - i(\alpha_2-\alpha_1)x)(y+ax+c), \\ \dot{y} &= a\alpha_3(x^2+y^2) + ((\alpha_2+\alpha_1)x - i(\alpha_2-\alpha_1)y)(y+ax+c), \end{aligned} \tag{30}$$

with α_1, α_2 and $\alpha_3 \in \mathbb{C}$. Writing $\alpha_j = m_j + i\, n_j$ with $m_j, n_j \in \mathbb{R}$ and using that such system have real parameters we conclude that $m_2 = m_1$, $n_2 = -n_1$ and $n_3 = 0$. Replacing this conditions in (30) we get the system

(31)

$$\dot{x} = -m_3(x^2+y^2) + 2(n_1\, x - m_1\, y)(y + a\, x + c), \quad \dot{y} = am_3(x^2+y^2) + 2(m_1\, x + n_1\, y)(y + a\, x + c).$$

Note that if $m_3 = 0$ then the system has a common factor, so we can take $m_3 = 2$. Applying a rescaling of the time and writing $\alpha = m_1$, $\beta = n_1$ we obtain system $(p.2)$ of Theorem C.

It follows from the previous study the proof of Theorem C.

6. Proof of Theorems D and E

In this section we investigate the conditions in order that a given quadratic system with an algebraic invariant cubic has a Darboux invariant. Moreover, using the obtained normal forms in Theorem C we study the phase portrait in the Poincaré disc of such systems.

Proposition 9 (E)**.** *Each real planar quadratic differential system with a real ellipse and a straight line having a Darboux invariant can be written, after an affine change of coordinates, as system* $(E.2)$ *with* $c = -1$, $\alpha_2 \neq 0$. *Moreover, such system has the Darboux invariant*

$$I_1(t,x,y) = e^{-t}(y-1)^{1/\alpha_2}(x^2+y^2-1)^{-\frac{1}{2\alpha_2}}.$$

and, these systems have only two non equivalent phase portraits, see phase portraits EL.2.1 *and* EL.2.2 *of Figure 2.*

Proof. If follows from the reducible cubic classification that we can fix $f_1 = x^2+y^2-1 = 0$ as the real ellipse and by Theorem C there are only two families of systems having $f_1 = 0$ and a straight line as invariant curves $(E.1)$ and $(E.2)$. We shall prove later that $(E.1)$ does not admit a Darboux invariant. Now we study system $(E.2)$. By Proposition 2 system $(E.2)$ has a Darboux invariant if there exist $\lambda_1, \lambda_2 \in \mathbb{R}$ not both equal to zero such that (8) holds with $s \in \mathbb{R} \setminus \{0\}$ and k_1, k_2 being the cofactors of $f_1 = 0$ and $f_2 = y + c = 0$, respectively. But for system $(E.2)$ we must have $\alpha_2 = 0$ or $c = -1$. If $\alpha_2 = 0$ the cofactors are $k_1 = \beta_1\, x$ and $k_2 = \beta_2\, x$ and the equation $\lambda_1 k_1 + \lambda_2 k_2 + s = 0$ has no solution for $s \neq 0$. Hence if $\alpha_2 = 0$ system $(E.2)$ has no Darboux invariant.

If $\alpha_2 \neq 0$ and $c = -1$ then the cofactors are $k_1 = \beta_1\, x + 2\alpha_2\, y$ and $k_2 = \alpha_2 + \beta_2\, x + \alpha_2\, y$ and the unique solution of (8), with $s \neq 0$ is

$$\beta_1 = 2\beta_2,\ s = -\alpha_2\lambda_2,\ \lambda_1 = -\lambda_2/2. \tag{32}$$

Taking $\lambda_1 = 1/\alpha_2$ and replacing (32) in system $(E.2)$ we obtain the system

$$(33) \qquad \dot{x} = \beta_2(y-1) + x(\beta_2\, x + \alpha_2\, y), \quad \dot{y} = (y-1)(\alpha_2 + \beta_2\, x + \alpha_2\, y),$$

which has the Darboux invariant

$$I_1(t,x,y) = e^{-t}(y-1)^{1/\alpha_2}(x^2+y^2-1)^{-\frac{1}{2\alpha_2}}.$$

In order to study the global phase portrait of system $(E.2)$ we start considering its finite singularities. Note that (33) has at most three finite singularities, namely $z_1 = (0,\, 1)$, $z_2 = (-1/\beta_2,\, 1)$ and $z_3 = \left(-\frac{2\beta_2}{\beta_2^2+1},\, \frac{\beta_2^2-1}{\beta_2^2+1}\right)$. The eigenvalues associated to z_1 are 2 and 1, if $\beta_2 \neq 0$, the eigenvalues associated to z_2 are -1 and 1 and the eigenvalues of z_3 are -1 and -2. So for $\beta_2 \neq 0$ z_1, z_2 and z_3 are an unstable node, a saddle and a stable node, respectively. When $\beta_2 = 0$ we have only z_1 and z_3 as finite singularities.

In the local chart U_1 the compactified system is

$$(34) \qquad \dot{u} = -v(\beta_2 + \beta_2 u^2 - \beta_2 uv + v), \quad \dot{v} = -v(\beta_2 + \beta_2 uv + u - \beta_2 v^2),$$

so $v = 0$ is a common factor, this means that $v = 0$ is a line of singular points. Eliminating the common factor v, system (34) has no singular points if $\beta_2 \neq 0$. Otherwise $u_1 = (0,0)$ is a singular point with eigenvalues -1 and 1 which implies that the origin is a hyperbolic saddle besides the line of singular points.

In the local chart U_2 the compactified system is written as

$$\dot{u} = v(\beta_2 + \beta_2 u^2 + uv - \beta_2 v), \quad \dot{v} = v(v-1)(\beta_2 u + v + 1).$$

Eliminating the common factor v the origin is not a singular point of the compactified system.

Note that if $\beta_2 = 0$ there are an additional invariant straight line given by $y + 1 = 0$. From the study of the finite and infinite behavior of system $(E.2)$ we conclude that such system has two non–equivalent phase portraits when $c = -1$: phase portrait EL.2.1, if $\beta_2 \neq 0$ and phase portrait EL.2.2, if $\beta_2 = 0$. See Figure 2. □

Proposition 10 (H)**.** *Each real planar quadratic differential system with a hyperbola and a straight line having a Darboux invariant can be written, after an affine change of coordinates, as*

(i) *system $(H.2)$ with $\alpha_2 \neq 0$ and $c = -1$. Its Darboux invariant is*

$$I_2(t,x,y) = e^{-\alpha_2 t}(x^2-y^2-1)^{-1/2}(x-1).$$

ii) *system $(H.3)$ with $A\alpha \neq 0$, $c = 0$ and $\beta = -\gamma$. Its Darboux invariant is*

$$I_3(t,x,y) = e^{-A\alpha t}(x^2-y^2-1)^{\gamma}(y-x)^{A}.$$

(iii) *system $(H.3)$ with $\alpha \neq 0$ and $\beta = \gamma = 0$. Its Darboux invariant is*

$$I_4(t,x,y) = e^{\alpha t}(y-x+c)^{-1}.$$

(iv) *system $(H.4)$ with $\alpha \neq 0$ and $A = 2\beta$. Its Darboux invariant is*

$$I_5(t,x,y) = e^{-\alpha t}(x^2-y^2-1)^{-1/2}(y+ax-\sqrt{a^2-1}).$$

Moreover the are 13 non-equivalent phase portrait in the Poincaré disc of these systems. They are in Figure 2 HL.2.1–HL.2.3, HL.3.1–HL.3.13.

Proof. Fixing $f_1 = x^2 - y^2 - 1 = 0$, Proposition 2 says that system $(H.2)$ has a Darboux invariant if equation (8) holds for λ_1, λ_2 not both zero, where $s \in \mathbb{R} \setminus \{0\}$, and k_1, k_2 are cofactors of $f_1 = 0$ and $f_2 = x + c = 0$, respectively. Moreover $c = -1$ or $\alpha_2 = 0$ in system $(H.2)$. For $\alpha_2 = 0$ we have $k_1 = \gamma_1 y$ and $k_2 = \gamma_2 y$ and the equation $\lambda_1 k_1 + \lambda_2 k_2 + s = 0$ has no solution with $s \neq 0$. So in this case system $(H.2)$ has no Darboux invariant. If $\alpha \neq 0$ and $c = -1$ then $k_1 = 2\alpha_2 x + \gamma_1 y$ and $k_2 = \alpha_2 + \alpha_2 x + \gamma_2 y$ and (8) has a unique solution

$$s = -\alpha_2\lambda_2,\ \gamma_1 = 2\gamma_2,\ \lambda_1 = -\lambda_2/2.$$

The proof of (i) follows taking $\lambda_2 = 1$ and replacing $\gamma_1 = 2\gamma_2$ in system $(H.2)$, from that we have the system

$$\dot{x} = (x-1)(\alpha_2 + \alpha_2 x + \gamma_2 y), \quad \dot{y} = -\gamma_2(x^2 - y^2 - 1) + x(-\gamma_2 + \gamma_2 x + \alpha_2 y), \tag{35}$$

having the Darboux invariant

$$I_1(t,x,y) = e^{-\alpha_2 t}(x^2 - y^2 - 1)^{-1/2}(x-1).$$

To prove (ii) and (iii) we study system $(H.3)$ where we consider two cases: $c = 0$ and $\beta = -\gamma$. It is easy to see that if $c = 0$ $(H.3)$ has a Darboux invariant when $\alpha \neq 0$ and $\beta = -\gamma$. In this case we have the differential system

$$\dot{x} = (A/2)(x^2 - y^2 - 1) - y(\alpha - \gamma x + \gamma y), \quad \dot{y} = (A/2)(x^2 - y^2 - 1) - x(\alpha - \gamma x + \gamma y), \tag{36}$$

having the Darboux invariant

$$I_1(t,x,y) = e^{-A\alpha t}(x^2 - y^2 - 1)^{\gamma}(y - x)^A.$$

If $\beta = -\gamma$ system $(H.3)$ has a Darboux invariant only when $\gamma = 0$ and $\alpha \neq 0$. In this case the system is

$$\dot{x} = (A/2)(x^2 - y^2 - 1) - \alpha\, y(1 - cx - cy), \quad \dot{y} = (A/2)(x^2 - y^2 - 1) - \alpha\, x(1 - c\, y) + c\,\alpha(y^2 + 1), \tag{37}$$

and it has the Darboux invariant

$$I_1(t,x,y) = e^{\alpha t}(y - x + c)^{-1}.$$

The study of (iv) follows from system $(H.4)$ where the invariant line is $f_2 = y + ax - \sqrt{a^2 - 1} = 0$. In this case the unique solution of equation (8) is

$$s = -\alpha\lambda_2,\ A = 2\beta,\ \lambda_1 = -\lambda_2/2. \tag{38}$$

So taking $\lambda_2 = 1$ we obtain the Darboux invariant

$$I_2(t,x,y) = e^{-\alpha t}(x^2 - y^2 - 1)^{-1/2}(y + ax - \sqrt{a^2 - 1}).$$

We start the study of the phase portraits of system (35). Since $\alpha_2 \neq 0$ we can take $\alpha_2 = 1$ and the transformation $x = X, y = -Y$ takes the system with parameter γ_2 to the system with parameter $-\gamma_2$. So we may also assume $\gamma_2 \geq 0$.

Considering the finite singularities, if $\gamma_2 \notin \{0, 1\}$ system (35) has three finite singularities, namely $z_1 = (0, 1)$, $z_2 = (1, -1/\gamma_2)$ and $z_3 = \left(\frac{\gamma_2^2+1}{\gamma_2^2-1}, -\frac{2\gamma_2}{\gamma_2^2-1}\right)$. The eigenvalues associated to z_1 are 2 and 1, if $\beta_2 \neq 0$, the eigenvalues associated to z_2 are -1 and 1 and the eigenvalues of z_3 are -1 and -2. So for $\gamma_2 \notin \{0, 1\}$ z_1, z_2 and z_3 are respectively, an unstable node, a saddle and a stable node. When $\beta_2 = 0$ we have only z_1 and z_3 as finite singularities.

In the local chart U_1 the compactified system is

$$\dot{u} = v(-\gamma_2 + \gamma_2 u^2 + uv + \gamma_2 v), \quad \dot{v} = v(v-1)(\gamma_2 u + v + 1), \tag{39}$$

so v is a common factor, this means that $v=0$ is a line of singular points. Eliminating the common factor v, system (39) has no singular points if $\gamma_2 \neq 1$. Otherwise $u_1=(-1,0)$ is a singular point with eigenvalues -2 and -1, which implies that u_1 is a hyperbolic stable node. Moreover if $\gamma_2=0$ there an additional invariant straight line given by $x+1=0$.

In the local chart U_2 the compactified system is written as

$$\dot{u}=-v(\gamma_2-\gamma_2u^2+\gamma_2uv+v), \quad \dot{v}=-v(\gamma_2+\gamma_2v^2-\gamma_2uv+u).$$

So after eliminating the common factor v the origin is a singular point of the compactified system if and only if $\gamma_2=0$. In this case $(0,0)$ is a hyperbolic saddle.

It is easy to see that if $\gamma_2 \in (0,1)$ the singulatities z_1 and z_3 are in distinct branches of the hyperbola, and if $\gamma_2 \in (1,+\infty)$ they are in the same branch as shows Figure 8.

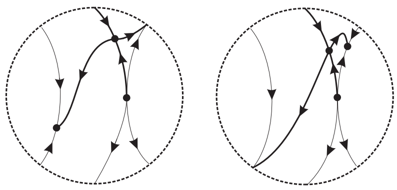


FIGURE 8. Possible phase portraits of sytem (35) when $\gamma_2 \notin \{0,1\}$.

From Theorem 1.43 of [9] (Markus-Neumann-Peixoto Theorem) we conclude that these two phase portraits are topologically equivalent. By continuity and the study done previously we conclude that system of type $(H.2)$ having a Darboux invariant can have three non-equivalent phase portrait. The case $\gamma_2 \neq 0,1$ corresponds to HL.2.1 in Figure 2 and when $\gamma_2=1$ or $\gamma_2=0$ we have the phase portraits HL.2.2 and HL.2.3 of Figure 2, respectivelly.

Now we study the global phase portrait of system $(H.3)$. Remember that the parameters of $(H.3)$ must satisfies $c(\gamma+\beta)=0$. We start considering $c=0$, then the differential system is

$$(40)\quad \dot{x}=(A/2)(x^2-y^2-1)-y(\alpha-\gamma x+\gamma y), \quad \dot{y}=(A/2)(x^2-y^2-1)-x(\alpha-\gamma x+\gamma y),$$

that has $f_1=x^2-y^2-1=0$ and $f_2=y-x=0$ as invariant algebraic curves. Since $\alpha \neq 0$ we can take $\alpha=1$ and the transformation $x=-X, y=-Y$ allows to assume $A>0$.

If $\gamma \neq 0$ then $z_1=(-A/2,-A/2)$ and $z_2=\big((\gamma^2+1)/(2\gamma),(\gamma^2-1)/(2\gamma)\big)$ are the two finite singular points. If $\gamma=0$ exists only one finite singular point.

The eigenvalues associated to z_1 are -1 and 1 so z_1 is a saddle. The eigenvalues associated to z_2 are A/γ and -1, so z_2 is a stable node if $\gamma<0$, and a saddle if $\gamma>0$. Moreover z_1 is on the straight line and z_2 is on the hyperbola.

In the local chart U_2 we have the system

$$\begin{aligned}\dot{u}&=(1/2)(u-1)(Av^2-(A+2\gamma)u^2+2uv+2v+A+2\gamma),\\ \dot{v}&=(1/2)v(Av^2-(A+2\gamma)u^2+2\gamma u+2uv+A),\end{aligned}$$

and the origin is a singular point only when $A+2\gamma=0$ but in this case the line $v=0$ is filled up of singular points.

In the local chart U_1 we have system

$$\begin{aligned}\dot{u}&=(1/2)(u-1)((A+2\gamma)u^2+Av^2+2uv+2v-A-2\gamma),\\ \dot{v}&=(1/2)v((A+2\gamma)u^2+Av^2+2uv-2\gamma u-A),\end{aligned}$$

which has the infinity filled up by singularities when $A+2\gamma=0$, otherwise, there are two singularities $u_1=(-1,0)$ and $u_2=(1,0)$.

Assuming $A+2\gamma\neq 0$. The point u_1 has eigenvalues 2γ and $2(A+2\gamma)$, and u_2 is linearly zero because the Jacobian matrix of the linear part of the system evaluated in u_2 is null. To decide the local behavior of u_2 we must do *blow up*. From now on we fix $l_1=\gamma,\ l_2=A+2\gamma$.

After translate the singular point u_2 to the origin, making the change of coordinates $u=U, v=UW$ and rescaling the common factor U we get the differential system

$$\dot U=(1/2)U(AUW^2+(A+2\gamma)U+2UW+4W+2A+4\gamma),\quad \dot W=-W(W+\gamma).$$

Note that such system have two singularities when $l_1l_2\neq 0$, namely, $\overline{U_1}=(0,0)$ and $\overline{U_2}=(0,-\gamma)$; one singular point when $l_1=0$ and $l_2\neq 0$, namely $\overline{U_1}=\overline{U_2}$. The eigenvalues of $\overline{U_1}$ are $-\gamma$ and $A+2\gamma$, whereas the eigenvalues of $\overline{U_2}$ are A and γ. From the combination of the signs of l_1 and l_2, as described in Figure 9, we get the possible local behavior of $\overline{U_1}$ and $\overline{U_2}$.

$$(1)\ l_1>0 \begin{cases} \xrightarrow{(1.1)} l_2>0 \\ \xrightarrow{(1.2)} l_2<0 \end{cases} \qquad (2)\ l_1<0 \xrightarrow{(2.1)} l_2>0 \qquad (3)\ l_1=0 \xrightarrow{(3.1)} l_2>0$$

FIGURE 9. The possible combination of signs of l_1 and l_2 describe the cases to be considered for system $(H.3)$ when $c=0$.

Applying the *blow down* we get all possible phase portraits for system $(H.3)$ when $c=0$. Note that each one is realizable, indeed, the phase portrait HL.3.2 corresponds to subcase (1.1) which is realizable with $A=4$ and $\gamma=-1$; HL.3.3 corresponds to subcase (1.2) which is realizable with $A=1$ and $\gamma=-1$. Notice that if $\gamma\neq 0$ there is a third invariant straight line, given by $f_3=\gamma(x-y)-1=0$ so HL.3.3 is the only possible phase portrait for subcase (1.2). The phase portraits HL.3.4 and HL.3.5 correspond, respectively, to subcases (2.1) and (3.1). The phase portrait HL.3.4 is realizable with $A=1$ and $\gamma=1$, and HL.3.5 is realizable with $A=1$ and $\gamma=0$.

It remains to consider the case $l_2=0$. With this condition the infinity is filled up of singular points. After eliminating the common factor v we have only one singular point at the local chart U_1. The eigenvalues associated to this point are 2 and 1, so this is a unstable node. By continuity the only possible phase portrait in this case is HL.3.1 of Figure 2, which is realizable with $A=2$ and $\gamma=-1$.

Now considering system $(H.3)$ with $\beta+\gamma=0$ we have seen above that the system has a Darboux invariant when $\beta=\gamma=0$ and $\alpha\neq 0$. Under these conditions the differential system is

$$\begin{aligned}\dot x&=(A/2)(x^2-y^2-1)-\alpha\, y(1-cx-cy),\\ \dot y&=(A/2)(x^2-y^2-1)+c\alpha(y^2+1)-\alpha\, x(1-cy).\end{aligned}\tag{41}$$

Such system has $f_1=x^2-y^2-1=0$ and $f_2=y-x+c=0$ as algebraic invariant curves. If $c=0$ then we get system (40) when $\gamma=0$, so we can take $c\neq 0$ here. Moreover, doing the transformation $x=-X$, $y=-Y$ in the algebraic cubic we can assume $c>0$. Finally, since α is different from zero we can take $\alpha=1$ in (41).

System (41) has two finite singular points, namely $z_1=((2c-A)/2,-A/2)$ and $z_2=((c^2+1)/(2c),(1-c^2)/(2c))$. Defining $l_1=c^2-Ac-1$, $l_2=A-c$ and $l_3=A-2c$, we

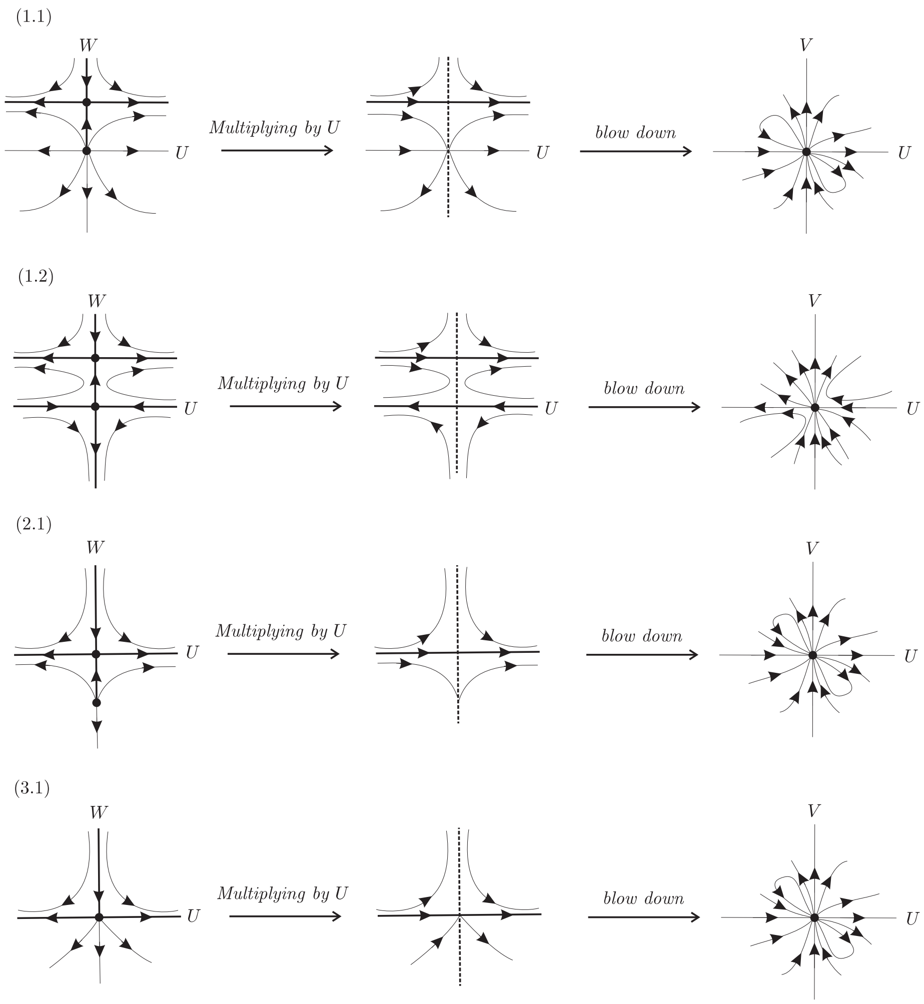


FIGURE 10. On the left the local phase portrait after blow up. Here they are indexed according to the signs of l_1 and l_2. On the right the local behavior at origin after the *Blow down* for system $(H.3)$.

have z_1 coalesces with z_2 if and only if $l_1 = 0$. Moreover the eigenvalues associated to z_1 are l_1 and 1, and the eigenvalues associated to z_2 are $-l_1$ and 1. So we conclude that z_1 is a unstable node and z_2 is a saddle if $l_1 > 0$; z_1 is a saddle and z_2, an unstable node, if $l_1 < 0$ and, if $l_1 = 0$, $z_1 = z_2$ is a saddle-node.

In the local chart U_1 the system becomes

(42)
$$\begin{aligned}\dot{u} &= (1/2)((A-2c)u^3 - Au^2 + Auv^2 - Au - Av^2 + 2cu + 2cv^2 + 2u^2v - 2v + A),\\ \dot{v} &= (1/2)v((A-2c)u^2 + Av^2 - 2cu + 2uv - A),\end{aligned}$$

which has three singularities $u_1 = (-1, 0)$ and $u_2 = (1, 0)$ and $u_3 = (\frac{A}{A-2c}, 0)$, if $A \neq 2c$. Note that when $l_3 = 0$ the point u_3 does not exist and $u_1 = u_3$ when $l_2 = 0$. The eigenvalues associated to u_1 are $2l_2$ and 0, the point u_2 has both eigenvalues equal to $-2c$, and u_3 has eigenvalues 0 and $2cl_2/l_3$. It is not difficult to see that when $l_2 \neq 0$, u_1 and u_3

are saddle–nodes. In the local chart U_2 the origin $(0,0)$ is a singular point if and only if $l_3 = 0$.

Assuming $l_1 l_2 \neq 0$ and considering all possible combinations of the sign of l_1, l_2 and l_3 we observe that there are some impossible combinations, for instance when $l_2 < 0$ we have $l_3 < 0$. In Figure 11 we describe the possible combinations and introduce a label for each one.

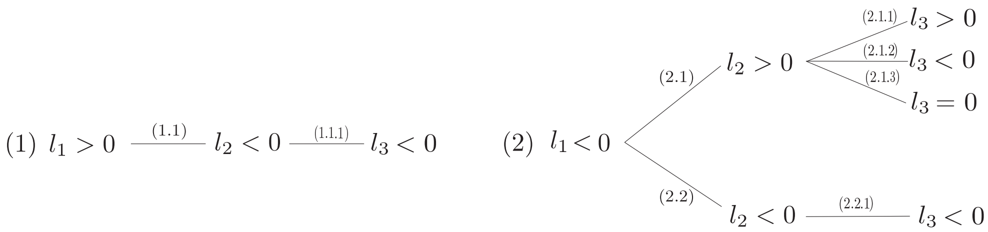


FIGURE 11. The possible combinations of signs of l_1, l_2 and l_3 for system $(H.3)$ when $c \neq 0$.

The case (2.2.1) presents a unique phase portrait, HL.3.6 of Figure 2 and it is realizable with $A = 1/2$ and $c = 1$.

In case (2.1.1) we have three possibilities for the finite saddle separatrix ω-limit set: we can have a connection of separatrix as in HL.3.7; the separatrix can go to the stable node, generating a phase portrait equivalent to HL.3.6, or the separatrix can go to the parabolic sector of the saddle node u_3 which corresponds to HL.3.8. Moreover HL.3.8 is realizable with $A = 2$ and $c = 1/2$, and as we see above, HL.3.6 is realizable with $A = 1/2$ and $c = 1$. Since HL.3.6 and HL.3.8 are realizable then by continuity of the parameters we conclude that HL.3.7 is also realizable.

The analysis of case (2.1.2) can be done as the case (2.1.1) and it has the phase portraits equivalent to them.

The possible phase portraits of (2.1.3) are also equivalent to the phase portraits of (2.1.1). Also the case (1.1.1) has a phase portrait equivalent to (2.2.1).

When $l_2 = 0$ it follows that $l_1, l_3 < 0$ and in the local chart U_1 the singular point $u_1 = u_3$ is non–elementary. After translate this singular point to the origin, making the change of coordinates $u = U$, $v = UW$ and rescaling the common factor U we get

$$\dot{U} = (U/2)(AUW^2 - AU + 2UW - 4W + 2A), \quad \dot{W} = W(W - A).$$

This system has two singularities $\overline{U_1} = (0,0)$ and $\overline{U_2} = (0, A)$ being both saddles. Figure 12 shows the *blow down*.

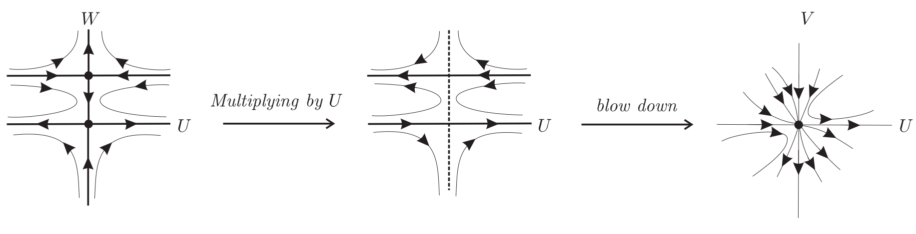


FIGURE 12. The local phase portrait of system (42) when $l_2 = 0$. On the left the local phase portrait after blow up. On the right the local behavior at the origin after blow down.

To obtain the phase portrait for system (41) with $l_2 = 0$ we note that there is more two invariant straight lines, given by $f_3 = x + y = 0$ and $f_4 = A\,x + A\,y - 1$. The finite saddle z_1 is on $f_3 = 0$ and the finite node is on the intersection of $f_2 = 0$ and $f_4 = 0$ so by continuity there is only one phase portrait, which is topologically equivalent to HL.3.3.

Finally it remains to study the case $l_1 = 0$. Here $l_2 < 0$ and $l_3 < 0$ so the only possibility is the phase portrait HL.3.9 of Figure 2, which is realizable with $A = 0$ and $c = 1$.

To conclude the proof of Proposition 10 it remains to study the global phase portrait of system $(H.4)$ when $A = 2\beta$ and $\alpha \neq 0$. In this case we assume $\alpha = 1$, so $(H.4)$ is written as

$$\begin{aligned}\dot{x} &= \beta x^2 + (a\beta - \sqrt{a^2-1})xy + (a - \sqrt{a^2-1}\beta)y - \beta,\\ \dot{y} &= (a\beta - \sqrt{a^2-1})y^2 + \beta xy + (a - \sqrt{a^2-1}\beta)x + (a\beta - \sqrt{a^2-1}).\end{aligned}$$

Denoting $\delta = a\beta - \sqrt{a^2-1}$ and $\eta = a - \sqrt{a^2-1}\beta$ there are at most three finite singularities $z_1 = (-\delta/\eta, \beta/\eta)$, $z_2 = ((\delta\eta - \beta)/(\beta^2 - \delta^2), (\delta\eta - \beta)/(\beta^2 - \delta^2))$ and $z_3 = ((\beta + \delta\eta)/(\beta^2 - \delta^2), -(\beta + \delta\eta)/(\beta^2 - \delta^2))$. We observe that such singular points never coalesce but if $\eta = 0$, z_1 does not exist and if $\beta^2 - \delta^2 = 0$ the same happens with z_2 or z_3. With respect to the localization of these points, z_3 is the intersection of the hyperbola and the straight line, z_1 is on the straight line and z_2 is on the hyperbola. Moreover is not difficult to check that z_1, z_2 and z_3 are hyperbolic points, being z_1 a saddle, z_2 a stable node and z_3 an unstable node.

Concerning to the behavior at infinity, in the local chart U_1 the compactified system is given by

$$\dot{u} = v(\eta - \eta u^2 + \beta uv + \delta v), \quad \dot{v} = -v(\beta - \beta v^2 + \eta uv + \delta u),$$

so v is a common factor what means that $v = 0$ is a line of singular points. Eliminating this common line it remains singularities if and only if $\eta = 0$ or $\beta^2 - \delta^2 = 0$. When $\eta = 0$ the point $u_1 = (-a, 0)$ is a saddle. When $\delta = \beta$ the point $u_2 = (-1, 0)$ is a node with eigenvalues η and 2η. Finally if $\delta = -\beta$ then the point $u_3 = (1, 0)$ has eigenvalues $-\eta$ and -2η so it is a node.

In the local chart U_2 the system becomes

$$\dot{u} = -v(-\eta + \beta v + \eta u^2 + \delta uv), \quad \dot{v} = v(\delta + \beta u + \eta uv + \delta v^2).$$

So eliminating the common factor v the origin is not a singular point.

By the previous study and continuity of the solutions we conclude that there exist three possible phase portraits and they are topologically equivalent to the ones obtained from system $(H.2)$ and described in Figure 2. Indeed when $\eta, \beta^2 - \delta^2 \neq 0$ we have the phase portrait HL.2.1, when $\beta^2 - \delta^2 = 0$ we have HL.2.2, and the case $\eta = 0$ corresponds to phase portrait HL.2.3. □

Before to study the systems of type (P), we present two lemmas that will help to show the realization or not of the phase portraits that follow.

Lemma 11. *On any straight line which is not composed of orbits the total number of contact points is at most two for any quadratic system. If there are two such points p_1 and p_2, then the orbits intersecting the segment ∞p_1 cross in the same sense as the orbits intersecting $p_2\infty$, and the opposite sense to the path intersecting p_1p_2.*

Lemma 12. *The straight line connecting one finite singular point and a pair of infinite singular points in a quadratic system is either formed by trajectories or a line with exactly one contact point. If this contact point is the finite singular point, the flow goes in different directions on each half straight line.*

The proof of Lemma 11 can be founded in [8]. *Lemma 12 in the case that the pair of infinite singular points are saddles was proved in* [16]. *When such a pair are saddle-nodes, the proof appeared in* [1].

Proposition 13 (P)**.** *Each real planar quadratic differential system with a parabola and a straight line having a Darboux invariant can be written, after an affine change of coordinates, as*

(i) $(P.1)$ *with* $\alpha_1 - 2\alpha_2 \neq 0$ *and Darboux invariant*
$$I_6(t,x,y) = e^{(\alpha_1-2\alpha_2)t}(y-x^2)^{-1}x^2.$$
(ii) $(P.2)$ *with* $\alpha_2(\beta_1-\beta_2)\neq 0$, $c=0$ *and Darboux invariant*
$$I_7(t,x,y) = e^{2\alpha_2(\beta_1-\beta_2)t}(y-x^2)^{\beta_2}y^{-\beta_1},$$
(iii) $(P.2)$ *with* $c\gamma_2\neq 0$, $\beta_1=\beta_2$, $\alpha_2=0$ *and Darboux invariant*
$$I_8(t,x,y) = e^{-2c\gamma_2 t}(y-x^2)(y+c)^{-1},$$

Moreover there are 41 non-equivalent phase portrait in the Poincaré disc for these systems. They are in Figures 3 and 4.

Proof. We fix the invariant parabola as $f_1 = y - x^2 = 0$. Here we describe in details the proof of the existence of a Darboux invariant for system $(P.2)$, the other cases are analogous. System $(P.2)$ is given by
$$\dot{x} = -\beta_1(y-x^2) + y(\beta_2+\gamma_2 x) + (\alpha_2+\gamma_2 c)x + c\beta_2, \quad \dot{y} = 2(y+c)(\alpha_2+\beta_2 x+\gamma_2 y),$$
where $c\,\alpha_2 = 0$. If $c=0$ then the additional invariant line is written as $f_2 = y = 0$ and if $\alpha_2 = 0$, such line is $f_2 = y + c = 0$.

System $(P.2)$ has a Darboux invariant if there exist λ_1,λ_2 not all zero satisfying equation (8) with $s\in\mathbb{R}\setminus\{0\}$, and k_1,k_2 being the cofactors of $f_1=0$ and $f_2=0$, respectively. For $c=0$, $k_1 = 2(\alpha_2+\beta_1 x+\gamma_2\,y)$ and $k_2 = 2(\alpha_2+\beta_2\,x+\gamma_2\,y)$. Equation (8), with $s\neq 0$ has the solution
$$s = -2\alpha_2(\lambda_1+\lambda_2),\ \beta_2 = -\beta_1\lambda_1/\lambda_2,\ \gamma_2 = 0, \tag{43}$$
Taking $\lambda_1=\beta_2$ and $\lambda_2=-\beta_1$ the solution can be rewritten as
$$s = -2\alpha_2(\beta_2-\beta_1),\ \lambda_1=\beta_2,\ \lambda_2=-\beta_1,\ \gamma_2=0, \tag{44}$$
and the Darboux invariant is
$$I_1(t,x,y) = e^{2\alpha_2(\beta_1-\beta_2)t}(y-x^2)^{\beta_2}y^{-\beta_1}.$$
In this case we assume $\beta_2-\beta_1\neq 0$ otherwise system $(P.2)$ has a common factor. Moreover if $\alpha_2=c=0$ $(P.2)$ does not admit a Darboux invariant.

When $\alpha_2=0$ then $f_2=y+c$ and the cofactors of $f_1=0$ and $f_2=0$ are, respectively, $k_1 = 2(c\gamma_2+\beta_1 x+\gamma_2 y)$ and $k_2 = 2(\beta_2 x+\gamma_2 y)$. In this case equation (8) has only one solution
$$s = -2c\gamma_2\lambda_1,\ \beta_2=\beta_1,\ \lambda_2=-\lambda_1.$$
So taking $\lambda_1=1$ we get the Darboux invariant
$$I_1(t,x,y) = e^{-2c\gamma_2 t}(y-x^2)(y+c)^{-1}.$$
∎

From now on we study the possible global phase portraits for systems (P) when they have a Darboux invariant. We start studying system $(P.1)$. Remember that such system is given by

$$\dot{x} = x(\alpha_2 + \beta_2\, x + \gamma_2\, y), \qquad \dot{y} = \alpha_1(y - x^2) + 2\alpha_2 x^2 + 2y(\beta_2\, x + \gamma_2\, y).$$

We consider two cases: $\gamma_2 \neq 0$ and $\gamma_2 = 0$. If $\gamma_2 \neq 0$ we assume $\gamma_2 = 1$. In this last case system $(P.1)$ have at most four singular points, given by

$$\begin{aligned} z_1 &= (0,0), \qquad z_2 = (0, -\alpha_1/2),\\ z_3 &= \left(-(\beta_2 + \sqrt{\beta_2^2 - 4\alpha_2})/2,\, (\beta_2^2 - 2\alpha_2 + \beta_2\sqrt{\beta_2^2 - 4\alpha_2})/2\right),\\ z_4 &= \left(-(\beta_2 - \sqrt{\beta_2^2 - 4\alpha_2})/2,\, (\beta_2^2 - 2\alpha_2 - \beta_2\sqrt{\beta_2^2 - 4\alpha_2})/2\right). \end{aligned}$$

Observe that unless of the change $x = -X$, $y = Y$ we can assume $\beta_2 \geq 0$. Let $l_1 = \alpha_1$, $l_2 = \alpha_2$, $l_3 = \beta_2^2 - 4\alpha_2 - \beta_2\sqrt{\beta_2^2 - 4\alpha_2}$ and $l_4 = \alpha_1 - 2\alpha_2$ be. It follows from Proposition 13 (i) $l_4 \neq 0$. Moreover

- z_1 has eigenvalues l_1 and l_2;
- z_2 has eigenvalues $-l_1$ and $-l_4$;
- z_3 has eigenvalues l_4 and $(\beta_2^2 - 4\alpha_2 + \beta_2\sqrt{\beta_2^2 - 4\alpha_2})/2$;
- z_4 has eigenvalues l_3 and l_4,

so $l_1^2 + l_2^2 \neq 0$ and the topological type of the finite singular points can be studied using the Hartman-Grobman Theorem and Theorem 2.19 of [9].

With respect to the position of the finite singularities, z_1 is on the intersection of the parabola and the straight line, z_2 is on the straight line, and z_3, z_4 are on the parabola.

In the local chart U_1 system $(P.1)$ is written as

$$\dot{u} = u^2 + \beta_2 u + (\alpha_1 - \alpha_2)uv + 2\alpha_2 - \alpha_1, \qquad \dot{v} = -v(\alpha_2 v + u + \beta_2),$$

which has at most two singular points when $v = 0$, namely

$$u_1 = (-\beta_2 - \sqrt{\beta_2^2 + 4(\alpha_1 - 2\alpha_2)}/2,\, 0), \quad u_2 = (-\beta_2 + \sqrt{\beta_2^2 + 4(\alpha_1 - 2\alpha_2)}/2,\, 0).$$

The eigenvalues associated to u_1 are $-\sqrt{\beta_2^2 + 4l_4}$ and $-(\beta_2 - \sqrt{\beta_2^2 + 4l_4})/2$ while the eigenvalues associated to u_2 are $\sqrt{\beta_2^2 + 4l_4}$ and $-(\beta_2 + \sqrt{\beta_2^2 + 4l_4})/2$.

Since we are assuming $\beta_2 \geq 0$ it follows that when $\beta_2^2 + 4l_4 > 0$ the point u_2 is a saddle and it is not difficult to see that if $l_4 > 0$, then u_1 is a saddle, and if $l_4 < 0$, u_1 is a stable node. When $\beta_2^2 + 4l_4 = 0$ u_1 and u_2 coalesce and we conclude that this point is a saddle-node, using Theorem 2.19 [9]. When $\beta_2^2 + 4l_4 < 0$ there is no infinite points in the local chart U_1.

In the local chart U_2 the origin $(0,0)$ is a stable node.

Observe that $l_1, l_2, l_3, l_4, \beta_2^2 - 4\alpha_2$ and $\beta_2^2 + 4l_4$ are bifurcation surfaces, i.e. where topological changes in the global phase portrait of (P.1) can happen. To draw all non-equivalent phase portraits of system (P.1) we split the study in three cases: $\beta_2^2 - 4\alpha_2 > 0$, $\beta_2^2 - 4\alpha_2 = 0$ and $\beta_2^2 - 4\alpha_2 < 0$.

Choosing a representative of each region defined by such surfaces we have a configuration of finite and infinite points. Considering the behavior of the separatrices of these systems we obtain all possible phase portraits when $\beta_2^2 - 4\alpha_2 > 0$, thus we obtain the 40 phase portraits described in Figures 13 and 14 and the phase portraits $41 - 50$ of Figure 18. We study all these cases bellow.

Among the phase portraits $1 - 18$ of Figure 13, we claim that 1 and 3, as well as 7 to 18, are not realizable. Indeed these 18 phase portraits, $1 - 3$ present the possible combinations

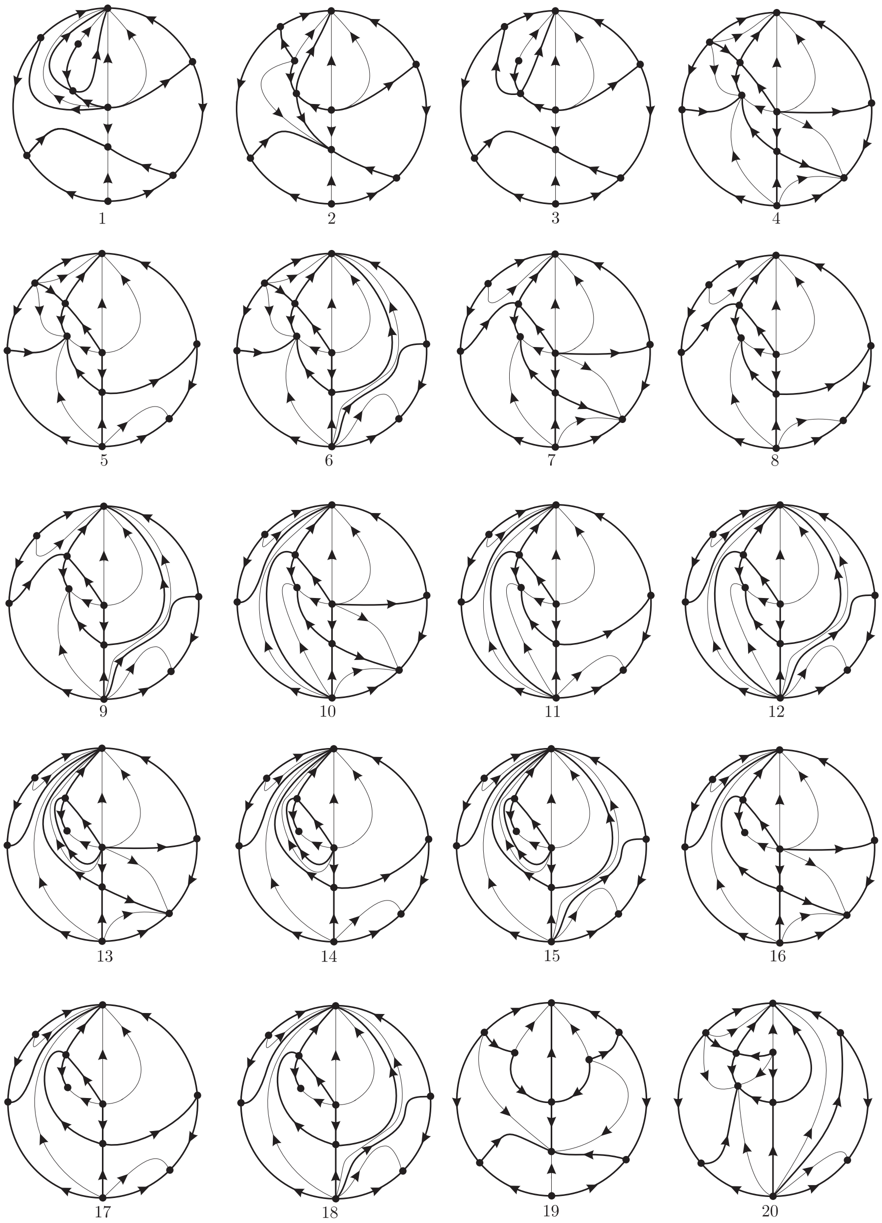


FIGURE 13. Phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 > 0$.

when the singular points in the local chart U_1 are both saddles. In the finite part we have z_1 and z_3 unstable nodes, z_2 is a stable node and z_4 is a saddle. So we have $l_1, l_2, l_4 > 0$ and $l_3 < 0$. In phase portrait 1 of Figure 13, consider the straight line joining the finite singular point z_3 to the infinity singular point u_1 as shows Figure 15. We can see that near the singular point z_3 but on opposite sides, the vector field has the same direction, which contradicts Lemma 12. So the phase portrait 1 of Figure 13 is not realizable. With the same argument the portrait 3 of Figure 13 is also not realizable. So phase portrait 2 of Figure 13 is the only realizable and corresponds to phase portrait PL.1.1 of Figure 3.

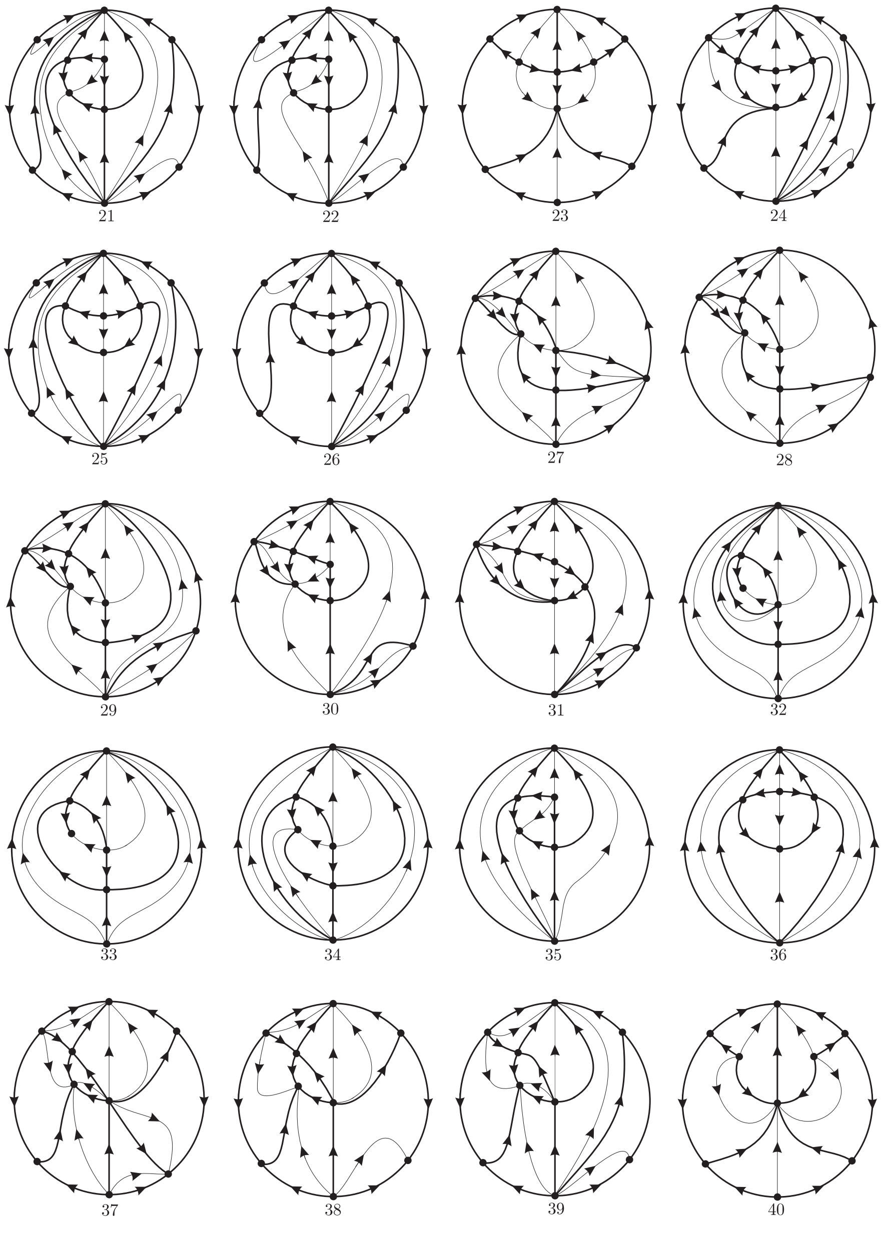


FIGURE 14. Phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 > 0$.

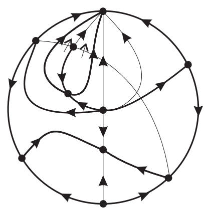


FIGURE 15. The straight line joining the finite singular point z_3 to the infinity singular point u_1 in phase portrait 1 of Figure 13.

Considering the phase portraits $4-18$ of Figure 13 we shall prove that $7-18$ are not realizable. First consider the phase portrait 7 and the straight line joining the middle point between the infinity singular points u_1 and u_2 and the middle point between the finite singular points z_3 and z_4 as shows Figure 16. By Lemma 11 this line should have at most two points of contact with the vector field, which does not occur. In Figure 16 we can see at least four contact points, represented by the smaller points that are not singularities of the system. This fact guarantees that the $\omega-$limit set of u_2 is the finite point z_4 on the parabola. So phase portraits $7-18$ are not realizable using similar arguments. So among the phase portraits $4-18$ only $4, 5$ and 6 are realizable, which correspond, respectively to phase portraits PL.1.2, PL.1.3 and PL.1.4 of Figure 3. The values of the parameters that realize these systems can be found in Table 2.

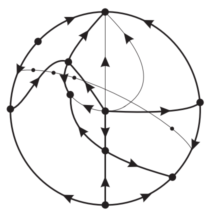


FIGURE 16. The straight line joining the middle point between the infinity singular points u_1 and u_2 and the middle point between the finite singular points z_3 and z_4 in phase portrait 7 of Figure 13.

The phase portraits $19-20$ in Figure 13 and $21-26$ in Figure 14 are topologically equivalent to one of the phase portraits $1-18$ in Figure 13 so they can be realizable or not, depends on their configuration. In Table 1 we present the relation among the equivalent phase portraits of system $(P.1)$ when $c \neq 0$. In the case where they are topologically equivalent to a realizable phase portrait, we need not consider the study again. However if they are topologically equivalent to a phase portrait which was not realizable, we need to study it.

Considering the same straight line used to prove the non-realization of phase portraits $7-18$ of Figure 13 we apply Lemma 11 to conclude that $21, 22, 25$ and 26 of Figure 14 are not realizable.

The phase portraits $27-31$ in Figure 14 present all the possibilities when there are four finite singular points and one infinite singular point on the local chart U_1. Phase portraits 27, 28 and 29 are realizable and correspond to phase portraits PL.1.5, PL.1.6 and PL.1.7 of Figure 3. The values of the parameters that realize these systems can be found in Table 2. Moreover 30 and 31 are topologically equivalent to one of these three phase portraits.

Finally if there are four finite singular points and the local chart U_1 has no singular point we get the phase portraits $32-36$ in Figure 14. For phase portraits 32 and 33 of Figure 14 we consider the straight line $x = z_4^1$ where the finite singularity z_4 is $z_4 = (z_4^1, z_4^2)$, and apply Lemma 12 to see that they are not realizable (see Figure 17).

Moreover the phase portraits 35 and 36 are topologically equivalent to the phase portrait 34 which is the only realizable phase portrait for this case and it is represented by PL.1.8 in Figure 3. The values of the parameters that realize this system can be found in Table 2.

For $\beta_2^2 - 4\alpha_2 > 0$ we consider the cases with three finite singular points. When $z_1 = z_2$ the origin is a saddle-node and there are ten possible phase portraits, namely $37-40$ in Figure 14 and $41-46$ in Figure 18. But since the nodal sector of the saddle node

Phase portrait	Topologicaly equivalent to
19	2
20	6
21	12
22	9
23	2
24	6
25	12
26	9
30	29
31	29
35	34
36	34
60	50

TABLE 1. Table of relations among all the possible phase portraits of system $(P.1)$ when $c \neq 0$.

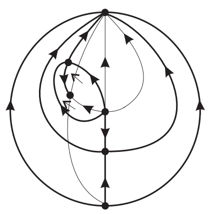


FIGURE 17. The straight line $x = z_4^1 = -(\beta_2 - \sqrt{\beta_2^2 - 4\alpha_2})/2$ in phase portrait 32 of Figure 14.

must have its orbits tangent to its separatrix, the phase portraits 37 and 38 in Figure 14 are not realizable. In other words the separatrices of the saddle-node z_1 must be on the invariant parabola. With the same argument the phase portraits $41, 42, 45$ and 46 of Figure 18 also are not realizable. So when $z_1 = z_2$ the realizable phase portraits are $39, 40,$ 43 and 44 of Figure 18, corresponding to PL.1.9, PL.1.10, PL.1.11 and PL.1.12, in Figure 3, respectively. The values of the parameters that realize these systems can be found in Table 2.

When there are three finite singularities with $z_1 = z_4$ then by continuity we have the phase portraits $47, 48, 49$ and 50 of Figure 18. All these for phase portraits are realizable and correspond, to PL.1.13, PL.1.14, PL.1.15 and PL.1.16 in Figure 3, respectively. The values of the parameters that realize these systems can be found in Table 2

For $\beta_2^2 - 4\alpha_2 = 0$ there is another case with three finite singularities that correspond to the case $z_3 = z_4$. Here we can have ten phase portraits, given by $51 - 60$ in Figure 18. The phase portraits $51, 52$ and 55 are realizable and corresponds, respectively, to PL.1.17, PL.1.18 and PL.1.19 in Figure 3. The values of the parameters that realize these systems can be found in Table 2. The phase portraits 53 and 54 are not realizable. The ideia again is to use Lemma 12 with the straight line joining the origin of the local chart U_3 to the singular point u_2 of the local chart U_1. By Figure 19 and this lemma the phase portraits 53 and 54 are not realizable.

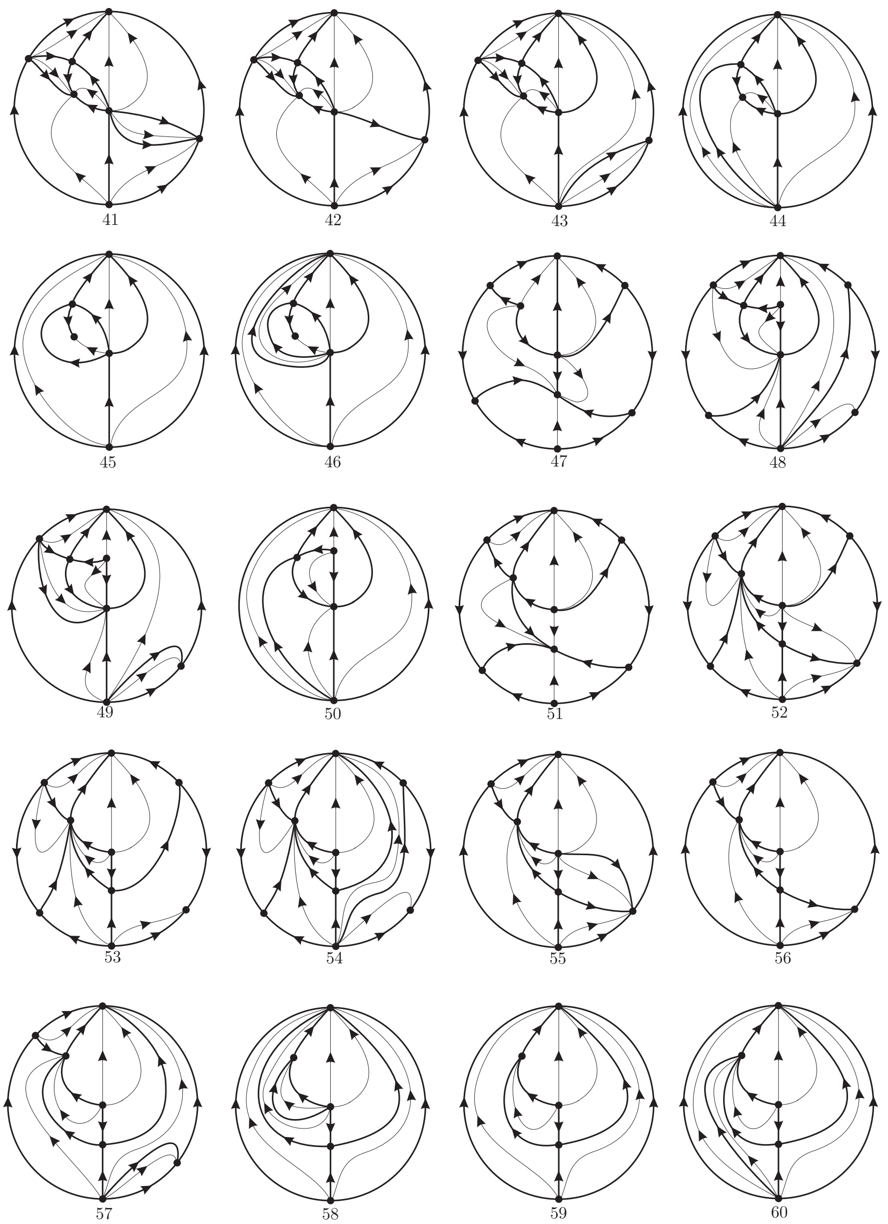


FIGURE 18. Phase portraits $41-50$ corresponds to phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 > 0$;
Phase portraits $51-60$ corresponds to phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 = 0$.

Considering the phase portraits 56 and 57 we will show that they are not realizable. Take the straight line passing through the origin of the local chart U_1 and the infinite singular point $u_1 = u_2$ (see Figure 20). The contact points on this straight line contradicts Lemma 12 so the phase portraits 56 and 57 are not realizable. About the phase portraits 58 and 59, considering the straight line passing through the points z_1 and z_3 we have Figure 21

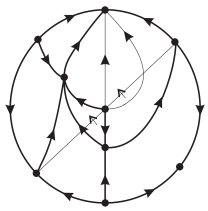


FIGURE 19. The straight line connecting the origin of the local chart U_3 with the singular poin u_2 of the local chart U_1 in phase portrait 53 of Figure 18.

that is a contradiction with Lemma 11. So they are not realizable. The phase portrait 60 is topologically equivalent to 50 of Figure 18.

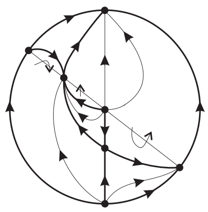


FIGURE 20. The straight line connecting the origin of the local chart U_3 with the singular poin $u_1 = u_2$ of the local chart U_1 in phase portrait 56 of Figure 18.

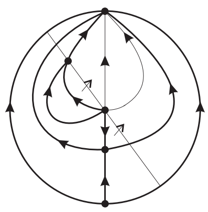


FIGURE 21. The straight line passing through the points z_1 and z_3 in phase portrait 58 of Figure 18.

If $z_3 = z_4$ and $z_1 = z_2$ we have the phase portraits $61, 62$ and 63 of Figure 22. But using the straight line joining z_1 and z_3 as done in Figure 21 and applying Lemma 11 we see that 61 and 62 are not realizable. The phase portrait 63 is realizable and corresponds to PL.1.20 in Figure 3. The values of the parameters that realize this system can be found in Table 2.

For $\beta_2^2 - 4\alpha_2 < 0$ the points z_3 and z_4 are complex. The possible phase portraits are described by $64 - 72$ of Figure 22. The phase portraits $64, 65, 68$ and 71 are realizable and corresponds, respectively, to PL.1.21, PL.1.22, PL.1.23 and PL.1.24 of Figure 3. The values of the parameters that realize these systems can be found in Table 2. To prove that the phase portraits $66, 67, 69$ and 70 are not realizable, it is enough to consider the straight line passing through the origin of the local chart U_3 and the infinity singularity $u_1 = u_2$ of the local chart U_1 (see Figure 23). This straight line generates a contradition with Lemma 12 so the phase portraits $66, 67, 69$ and 70 are not realizable.

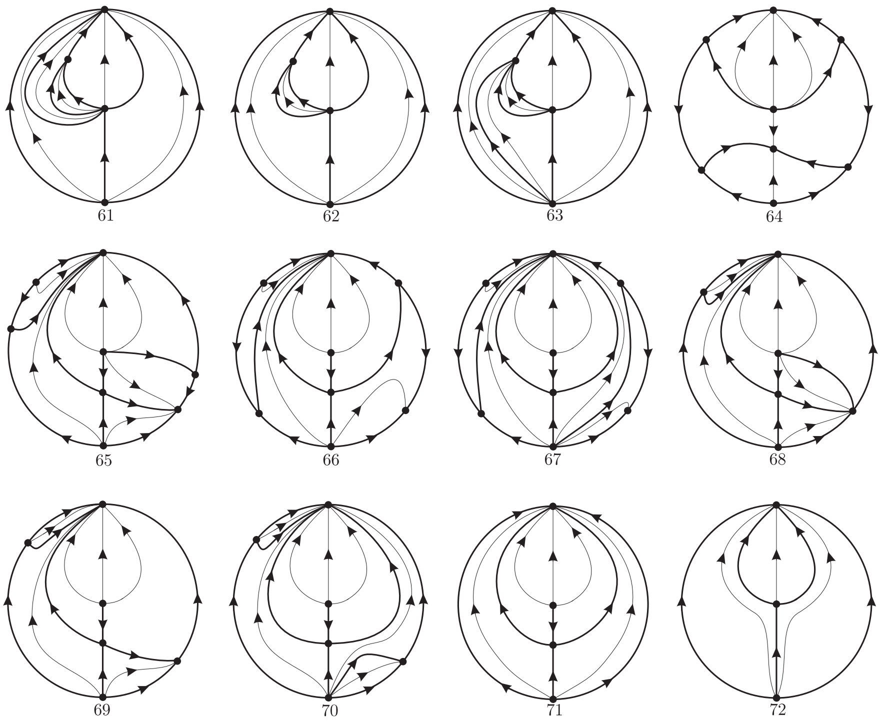


FIGURE 22. Phase portraits $61-63$ corresponds to phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 = 0$;
Phase portraits $64-72$ corresponds to phase portraits of system $(P.1)$ when $\gamma_2 = 1$ and $\beta_2^2 - 4\alpha_2 < 0$.

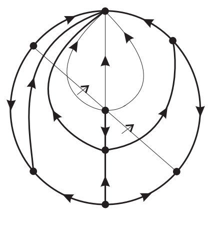


FIGURE 23. The straight line connecting the origin of the local chart U_3 with the singular poin $u_1 = u_2$ in the local chart U_1 in phase portrait 66 of Figure 22.

To end the case $\gamma_2 = 1$ we consider the case where there is only one finite singular point. Using Theorem 2.19 of [9] we can see that the point is a saddle, which generates phase portrait 72 of Figure 22 which corresponds to phase portrait PL.1.25 of Figure 4. The values of the parameters that realize this system can be found in Table 2.

Now we consider the case $\gamma_2 = 0$. The system is

$$\dot{x} = x(\alpha_2 + \beta_2\, x), \qquad \dot{y} = \alpha_1(y - x^2) + 2x(\alpha_2 x + \beta_2\, y). \tag{45}$$

When $\alpha_1 = 0$ such system has a common factor so assume $\alpha_1 = 1$. By the change $x = -X, y = Y$ it is enough to consider the case $\beta_2 \geq 0$.

Assuming $\beta_2 > 0$. In the finite part the points $z_1 = (0,0)$ and $z_2 = \big(-\alpha_2/\beta_2, (\alpha_2/\beta_2)^2\big)$ are the singular points and the system has an additional invariant straight line, given by $f_3 = x + \alpha_2/\beta_2 = 0$. Defining $l_1 = \alpha_2$ and $l_2 = 1 - 2\alpha_2$ the eigenvalues associated to z_1 are 1 and l_1, while the eigenvalues associated to z_2 are $-l_1$ and l_2. We assume $l_2 \neq 0$ (otherwise such system has a common factor and it is equivalent to a linear system).

In the local chart U_1 the unique singular point is $u_1 = (l_2/\beta_2,\, 0)$ and it is a saddle. In the local chart U_2 the compactified system is

$$\dot{u} = u((1-2\alpha_2)u^2 + (\alpha_2 - 1)v - \beta_2 u), \qquad \dot{v} = v((1-2\alpha_2)u^2 - 2\beta_2 u - v).$$

The origin $(0,0)$ is a linearly zero singularity. Doing the blow up $u = UV$, $v = V$ and rescaling by V we get the system

$$\dot{U} = U(\alpha_2 + \beta_2 U), \qquad \dot{V} = V((1 - 2\alpha_2 U^2 V) - 2\beta_2 U - 1).$$

When $V = 0$ the singularities are $\overline{u}_1 = (0,0)$ and $\overline{u}_2 = (-\alpha_2/\beta_2, 0)$. The eigenvalues associated to $\overline{u}_1$ are -1 e l_1 while the eigenvalues of $\overline{u}_2$ are $-l_1$ e $-l_2$. The blowing down process is described in Figure 24 (1)-(4) according to the signs of l_1 and l_2.

When $\beta_2 = 0$ the point z_1 is the unique finite singular point, being a saddle or an unstable node depending on the sign of l_1. In the local chart U_1 there is no singular point and the origin $(0,0)$ of U_2 is linearly zero. To study such point we apply *blow ups*, in Figure 24 is described the blowing down (5) and (6).

Summarizing the study done previously we get the local behaviour at origin of U_2:

(1) $\beta_2 > 0$, $l_1 > 0$ and $l_2 > 0$: the origin of U_2 has two elliptic sectors;
(2) $\beta_2 > 0, l_1 > 0$ and $l_2 < 0$: the origin of U_2 has two hyperbolic sectors;
(3) $\beta_2 > 0, l_1 < 0$ and $l_2 > 0$: the origin of U_2 has two elliptic sectors;
(4) $\beta_2 > 0, l_1 = 0$ and $l_2 > 0$: the origin of U_2 has two elliptic sectors.
(5) $\beta_2 = 0, l_1 > 0$: the origin of U_2 has two hyperbolic sectors;
(6) $\beta_2 = 0, l_1 < 0$: the origin of U_2 has two elliptic sectors;

By continuity and the above analysis we conclude that the case (3) is topologically equivalent to case (1) and the cases $(1), (2), (4), (5)$ and (6) correspond, respectively, to the phase portraits PL.1.26, PL.1.27, PL.1.28, PL.1.29 e PL.1.30 of Figure 4. Table 4 has the values of the parameters that realizes the phase portraits of system $(P.1)$

System $(P.2)$ with $c \neq 0$ has a Darboux invariant if $\gamma_2 \neq 0$, and it can be written as

$$\dot{x} = \beta_1\,(x^2 + c) + \gamma_2\, x(y + c), \qquad \dot{y} = 2(y + c)(\beta_1 x + \gamma_2\, y).$$

Note that if $\beta_1 = 0$ such system has a common factor so we can assume $\beta_1 = 1$. Applying the change of coordinates $x = -X, y = Y$ and rescaling the time we can assume $\gamma_2 > 0$.

If $c < 0$ the system has three finite singular points $z_1 = (-1/\gamma_2, 1/\gamma_2^2)$, $z_2 = (-\sqrt{-c}, -c)$ and $z_3 = (\sqrt{-c}, -c)$. Otherwise, only z_1.

Defining $l_1 = c \neq 0$ and $l_2 = 1 + c\,\gamma_2^2$ the eigenvalues associated to z_1 are $2\gamma_2 l_1$ and l_2/γ_2, the eigenvalues associated to z_2 are $-2\sqrt{-c}$ and $-2(\gamma_2\, c + \sqrt{-c})$; the eigenvalues associated to z_3 are $2\sqrt{-c}$ and $-2(\gamma_2\, c - \sqrt{-c})$. So when $c < 0$ the point z_3 exists and it is an unstable node.

In the local chart U_1 we have two singular points $u_1 = (0,0)$ being a hyperbolic saddle and $u_2 = (-1/\gamma_2, 0)$ being a saddle-node. In the local chart U_2 the origin is a stable node.

When $l_2 = 0$ then $z_1 = z_3$ is a semi-hyperbolic node and the infinity part does not change. Note that z_1 is a saddle-node in this case. So by continuity and the reasoning above, if $c > 0$ we have phase portrait PL.2.1 of Figure 4 which is realizable with $c = \gamma_2 = 1$. When $c < 0$ and $l_2 \neq 0$ the system has two possible phase portraits, also described

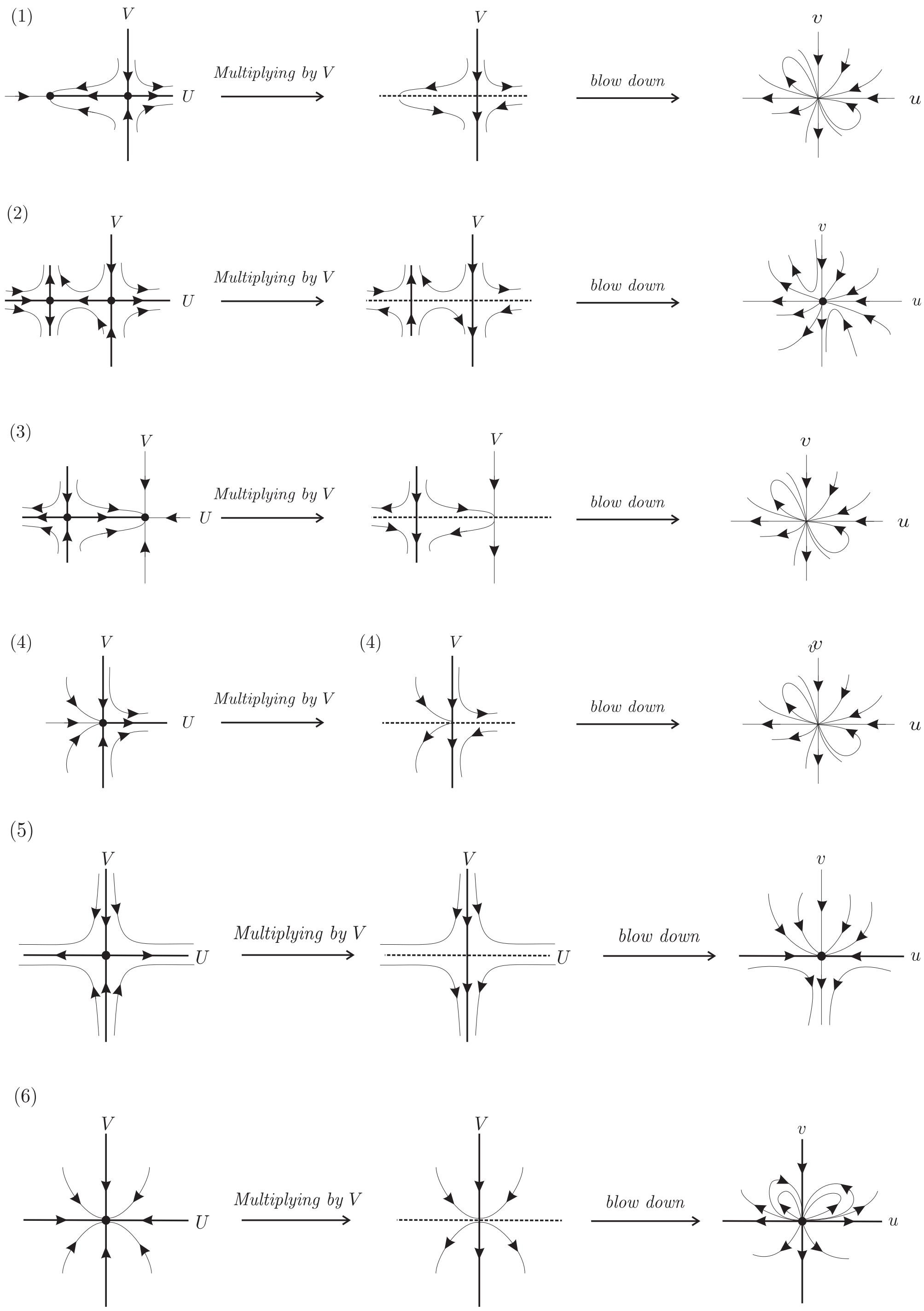
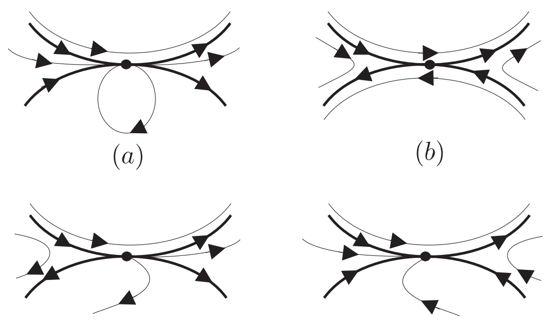


FIGURE 24. *Blow down* of system $(P.1)$ when $\gamma_2 = 0$.

	γ_2	β_2	α_2	α_1
PL.1.1	1	1	1/8	1
PL.1.2	1	1	1/16	1/16
PL.1.3	by continuity			
PL.1.4	1	1	1/16	1/150
PL.1.5	1	1/2	3/64	1/32
PL.1.6	by continuity			
PL.1.7	1	1	-3/8	-1
PL.1.8	1	1	3/16	1/16
PL.1.9	1	1	1/16	0
PL.1.10	1	1	-1	0
PL.1.11	1	1	1/18	0
PL.1.12	1	1	3/16	0
PL.1.13	1	1	0	1
PL.1.14	1	1	0	-1/8
PL.1.15	1	1	0	-1/4
PL.1.16	1	1	0	-1
PL.1.17	1	1	1/4	1
PL.1.18	1	1	1/4	3/8
PL.1.19	1	1	1/4	1/4
PL.1.20	1	1	1/4	0
PL.1.21	1	1	2	5
PL.1.22	1	3	4	6
PL.1.23	1	1	9/8	2
PL.1.24	1	1	2	13/4
PL.1.25	1	1	2	0
PL.1.26	0	1	1/4	1
PL.1.27	0	1	3/2	1
PL.1.28	0	1	0	1
PL.1.29	0	0	1	1
PL.1.30	0	0	-1	1

TABLE 2. Table of values for the parameters of system ($P.1$).



(a) (b) (c) (d)

FIGURE 25

in Figure 4: PL.2.2 (realizable with $c = -1/2$ and $\gamma_2 = 1$) and PL.2.3 (realizable with $c = -2$ and $\gamma_2 = 1$).

Finally if $c < 0$ and $l_2 = 0$, we see that the line $y + c = 0$ is one of the separatix of the saddle-node. So the only possible phase picture is PL.2.4 (realizable with $c = -1$ and $\gamma_2 = 1$).

Phase portraits of systems $(P.2)$ **when** $c = 0$ **and they have a Darboux invariant.** The differential system is

$$\dot{x} = -\beta_1(y - x^2) + \beta_2\, y + \alpha_2\, x, \qquad \dot{y} = 2y(\beta_2\, x + \alpha_2).$$

Since $\alpha_2 \neq 0$ we take $\alpha_2 = 1$. Moreover doing the change of coordinates $x = -X, y = Y$ we can assume $\beta_2 \geq 0$. The system has at most three finite singular points, namely, $z_1 = (0, 0)$ and $z_2 = (-1/\beta_1, 0)$ and $z_3 = (-1/\beta_2, 1/\beta_2^2)$. The point z_1 has eigenvalues 2 and 1, so it is an unstable node. On the other hand the topological type of z_2 and z_3 depends on the numbers $l_1 \doteq \beta_1$ and $l_2 \doteq \beta_1 - \beta_2 \neq 0$. Indeed the point z_2 has eigenvalues -1 and $2l_2/l_1$ and z_3 has the eigenvalues -1 and $-2l_2/l_1$.

In the local chart U_1 the system has $u_1 = (0, 0)$ as a singularity with eigenvalues $-l_1$ and $-l_3$, where $l_3 \doteq \beta_1 - 2\beta_2$.

In the local chart U_2 the compactified system has the origin as a *nilpotent* singularity. This mean that the linear part of the system, evaluated in $(0, 0)$, is not null but their eigenvalues are both equal to zero. To classify this type of singular point we use Theorem 3.5 of [9]. This result use two functions, $F(u) = a_M u^M + o(u^M)$ and $G(u) = b_N u^N + o(u^N)$, defined from the differential system. In short the caracterization is done using a_M, b_N and the natural numbers M, N.

For the compactified system in the local chart U_2 these functions are

$$G(u) = -\frac{2(\beta_2 - 3\beta_2)}{l_2}\, u + \frac{5l_3}{l_2^2}\, u^2, \qquad F(u) = \frac{2\beta_2 l_3}{l_2^2}\, u^3 + \frac{2l_3^2}{l_2^3}\, u^4.$$

So when $l_3 > 0$ the origin $(0, 0)$ is a saddle as in (b) of Figure 25. If $l_3 < 0$ the origins consists of one hyperbolic and one elliptic sector as in (a) of Figure 25. By continuity, when $l_1 > 0$ and $l_3 > 0$ we have the phase portrait PL.2.5 of Figure 4 (realizable with $\beta_1 = 4$ and $\beta_2 = 1$). If $l_3 < 0$ we have the phase portraits PL.2.6 (realizable with $\beta_1 = 3/2$ and $\beta_2 = 1$) and PL.2.7 (realizable with $\beta_1 = 1/2$ and $\beta_2 = 1$) of Figure 4. Now if $l_1 < 0$ the only possibility is $l_3 < 0$ and we have the phase portrait PL.2.8 (realizable with $\beta_1 = -1$ and $\beta_2 = 1$) of Figure 4.

If $l_1 = 0$ the point z_2 goes to the infinity and collide with u_1 becoming a saddle-node. Moreover $l_1 = 0$ implies $l_3 < 0$, so the origin of U_2 has a hyperbolic and one elliptic sector. This case corresponds to phase portrait PL.2.9 of Figure 4, realizable with $\beta_1 = 0$ and $\beta_2 = 1$.

If $\beta_2 = 0$ the point z_3 goes to the infinity and collide with the origin of U_2 becoming $(0, 0)$ a nilpotente saddle-node as (c) or (d) in Figure 25. Moreover the only possible phase portrait is given by PL.2.10 of Figure 4, realizable with $\beta_1 = 1$ and $\beta_2 = 0$).

Finally when $l_3 = 0$ then the infinity if filled of singular points, without special singularities and the corresponding phase portrait is PL.2.11 of Figure 4 (realizable with $\beta_1 = 2$ and $\beta_2 = 1$).

Proposition 14 (LV)**.** *Each real polynomial differential system having two real lines that intersect at a single point and a third straight line having a Darboux invariant can be written, after an affine change of coordinates, as*

(i) $(LV.1)$ *with* $\alpha(q-\beta)\neq$ *and Darboux invariant*

$$I_9(t,x,y)=e^{\alpha(q-\beta)\,t}\,y^{\beta}\,x^{\beta-q+r}(y-x)^{-(\beta+r)},$$

(ii) $(LV.2)$ *with* $c=q=0$, $p\neq 0$ *and Darboux invariant*

$$I_{10}(t,x,y)=e^{-pt}x\,y^{-r},$$

(iii) $(LV.2)$ *with* $c=-1$ *and Darboux invariant*

$$I_{11}(t,x,y)=e^{t}\,y\,(y-1)^{-1},$$

(iv) $(LV.3)$ *with* $\alpha=-(\beta+1)$, $c\,\beta\neq 0$ *and Darboux invariant*

$$I_{12}(t,x,y)=e^{-c\,\beta\,t}\,y\,(y+a\,x+c)^{-1}.$$

Moreover there are 27 non-equivalent phase portraits in the Poincaré disc. They are in Figure 5.

Proof of Proposition 14 (LV). Let $f_1=x=0$, $f_2=y=0$ be the two real straight lines intersecting in a point. Considering system $(LV.1)$ the third line is $f_3=y-x$ and the cofactors associated to f_1,f_2 and f_3 are, respectivelly, $k_1=\alpha+ry+\beta x$, $k_2=\alpha+y(\beta-q+r)+qx$ and $k_3=\alpha+y(\beta-q+r)+\beta x$. One solution for equation $\lambda_1k_1+\lambda_2k_2+s=0$ is

$$\lambda_2=\frac{\beta\lambda_1}{\beta-q+r},\ \lambda_3=-\frac{(\beta+r)\lambda_1}{\beta-q+r},\ s=\frac{\alpha(q-\beta)\lambda_1}{\beta-q+r},$$

Taking $\lambda_1=\beta-q+r$ we obtain the Darboux invariant

$$I_1(t,x,y)=e^{\alpha(q-\beta)\,t}\,y^{\beta}\,x^{\beta-q+r}(y-x)^{-(\beta+r)}.$$

Now we analize system $(LV.2)$ that has $f_3=y+c$ as the third invariant straigh line(remember that $c=0$ or $c=-1$). Here the cofactors are $k_1=p+qx+ry$, $k_2=y+c$ and $k_3=y$. If $c=0$ then equation (8) has only one the solution

$$q=0,\ \lambda_3=-r\lambda_1-\lambda_2,\ s=-p\lambda_1.$$

Taking $\lambda_1=1$ we get the Darboux invariant

$$I_1(t,x,y)=e^{-pt}xy^{-r}.$$

Otherwise if $c=-1$ then the more general solution is

$$\lambda_1=0,\ \lambda_3=-\lambda_2,\ s=\lambda_2.$$

Taking $\lambda_2=1$ we obtain the Darboux invariant

$$I_1(t,x,y)=e^{t}y(y-1)^{-1}.$$

The last case to be considered is system $(LV.3)$ that has $f_3=y+a\,x+c=0$ as the third straight line. The cofactors are $k_1=-\alpha(y+a\,x+c)-y$, $k_2=\beta(y+a\,x+c)+a\,x$ and $k_3=\beta\,y-a\,\alpha\,x$. Solving equation (8) we get the solution

$$\alpha=-(\beta+1),\ \lambda_2=-\lambda_1-\lambda_2,\ s=-c(\lambda_1+\beta(\lambda_1+\lambda_2)).$$

Taking $\lambda_1=0$ and $\lambda_2=1$ then we obtain the Darboux invariant

$$I_1(t,x,y)=e^{-c\,\beta\,t}\,y\,(y+a\,x+c)^{-1}.$$

$\square$

Phase portrait of systems $(LV.1)$ when they have a Darboux invariant. Remember that if system $(LV.1)$ has a Darboux invariant then $\beta - q \neq 0$ and $\alpha \neq 0$ so we can take $\alpha = 1$ getting

$$\dot{x} = x(1 + \beta\, x + r\, y), \qquad \dot{y} = y(1 + q\, x + (\beta - q + r)\, y). \tag{46}$$

Define $l_1 = (\beta - q)/(\beta - q + r)$, $l_2 = (\beta - q)/\beta$ and $l_3 = (\beta - q)/(\beta + r)$. The finite part presents at most four singularities

- $z_1 = (0, 0)$ with eigenvalues both equal to 1;
- $z_2 = (0, -1/(\beta - q + r))$ with eigenvalues -1 and l_1;
- $z_3 = (-1/\beta, 0)$ with eigenvalues -1 and l_2;
- $z_4 = (-1/(\beta + r),\ -1/(\beta + r))$ with eigenvalues -1 and $-l_3$.

.

In the local chart U_1 the compactified system has two singular points, being $u_1 = (0, 0)$ with eigenvalues $-\beta$ and $-(\beta - q)$ and $u_2 = (1, 0)$ with eigenvalues $\beta - q$ and $-(\beta + r)$. Moreover in the local chart U_2 the origin $(0, 0)$ is a singular point with eigenvalues $-(\beta - q)$ and $-(\beta - q + r)$. Thus when one of the finite singularities goes to infinity, it collides with u_1, u_2, or the origin of the local chart U_2.

When l_1, l_2 and l_3 are non-zero, the combinations between their signs generate the possible phase portraits of system (46). There are exactally three possible phase portraits, all of them described in Figure 5: LVL.1.1, realizable for $\beta = 1, q = r = 0$; LVL.1.2, realizable for $\beta = 1, q = r = -2$; LVL.1.3, realizable for $\beta = 1, q = -r = 3/4$.

Now we consider the case $\beta = -r \neq 0$. Here only the point z_4 goes to the infinity and collides with u_2 making it a semi hyperbolic saddle-node. There are two possible phase portraits, given by LVL.1.4 of Figure 5 (realizable with $\beta = 1, q = r = -1$) and LVL.1.5 of Figure 5 (realizable with $\beta = 2, q = 1, r = -2$). The cases where z_2 or z_3 goes to the infinity generate phase portraits equivalent to the previous ones.

Finally when two finite singular points go to the infinity (for example when $\beta = -r$ and $q = 0$), then there is only one phase portrait, given by LVL.1.6 of Figure 5. This last phase portrait is realizable for $\beta = 1, q = 0$ and $r = -1$.

Phase portraits of systems $(LV.2)$ when they have a Darboux invariant. First we consider the case $c = -1$, when the system is given by

$$\dot{x} = x(p + q\, x + r\, y), \qquad \dot{y} = y(y - 1).$$

If $q \neq 0$ unless of the change $x = X/q$ we can assume $q = 1$. Considering $q = 1$ and defining $l_1 = p$, $l_2 = -(p + r)$ and $l_3 = r - 1$ the system has at most four finite singular points, namely

- $z_1 = (0, 0)$ with eigenvalues -1 and l_1;
- $z_2 = (0, 1)$ with eigenvalues 1 and $-l_2$;
- $z_3 = (-p, 0)$ with eigenvalues -1 and $-l_1$;
- $z_4 = (-p - r, 1)$ with eigenvalues 1 and l_2.

In the local chart U_2 the origin $(0, 0)$ is a singularity with eigenvalues -1 and l_3. In the local chart U_1 the sytem has two singularities if $l_3 \neq 0$: $u_1 = (0, 0)$ being a hyperbolic unstable node and $u_2 = (1/l_3, 0)$ with eigenvalues 1 and $1/l_3$. Hence if $l_3 = 0$ the point u_2 collides with the origin of U_2 making it a semi-hyperbolic singularity of type saddle node. By continuity and using all the possible combinations of the signs of l_1, l_2 and l_3 when $q = 1$ and $l_3 \neq 0$ we obtain the phase portraits LVL.2.1- LVL.2.7 of Figure 5. When $l_3 = 0$, i.e., $r = 1$ has three possible phase portraits: LVL.2.8, LVL.2.9 and LVL.2.10 of Figure 5. The values of the parameters that realize these systems can be found in Table 3.

Now it remains to study the case $q = 0$. Note that since the system cannot have commom factors it follows that l_1 and l_2 are different from zero. When $q = 0$ both the finite part and the analyzes in the local chart U_2 remain almost the same. The only difference in the finite part is that the singularities z_3 and z_4 go to infinity. However in the local chart U_1 the compactified system is

$$\dot{u} = -u((r-1)u + (p+1)v), \qquad \dot{v} = -v(pv + ru).$$

So the origin is a linearly zero singular point if $l_3 \neq 0$ and we apply the *blow up* doing the change of coordinates $u = U, v = UW$. The new system is

$$\dot{U} = -U^2((p+1)W + r - 1), \qquad \dot{W} = UW(W-1).$$

After eliminating the common factor U it remains two singular points on $U = 0$: $\overline{u_1} = (0,0)$ with eigenvalues -1 and $-l_3$, and $\overline{u_2} = (0,1)$ with eigenvalues 1 and l_2. Hence they are hyperbolic points and doing the *blow down* the origin of U_2 has (for $l_3 \neq 0$)

- two elliptic sectors if $\overline{u_1}$ is a saddle and $\overline{u_2}$ is a unstable node. This case corresponds to phase portrait LVL.2.11 of Figure 5;
- two elliptic sectors if $\overline{u_1}$ is a stable node and $\overline{u_2}$ is a saddle. This case corresponds to phase portrait LVL.2.12 of Figure 5;
- two parabolic sectors if $\overline{u_1}$ and $\overline{u_2}$ are both saddles and there is a saddle and a node as singular finite points. This case corresponds to phase portrait LVL.2.13 of Figure 5;
- two parabolic sectors if $\overline{u_1}$ and $\overline{u_2}$ are both saddles and there are two nodes as singular finite points. This case corresponds to phase portrait LVL.2.14 of Figure 5;
- six parabolic sectors if $\overline{u_1}$ and $\overline{u_2}$ are both saddles and there are two nodes as singular finite points. This case corresponds to phase portrait LVL.2.15 of Figure 5.

The last possibility when $c = -1$ is $q = 0$ and $l_3 = 0$. But when this happens the system has the infinity line $v = 0$ filled up of singular points. After eliminating the common factor v, in the local chart U_1 the point $u_1 = (0,0)$ is a singular point, with eigenvalues $-l_1$ and l_2. In the local chart U_2, After eliminating the common factor v, the origin is a singularity. By continuity the possible phase portraits are LVL.16 and LVL.2.17 of Figure 5. In Table 3 we put the values of the parameters that realizes each one of the phase portraits described in Figure 5.

Finally when $c = 0$ we get the differential system

$$\dot{x} = x^2, \qquad \dot{y} = y(p + r\,x), \tag{47}$$

with $p \neq 0$. So we can take $p = 1$ and the system becomes a particular case of system (DL) of Theorem C. The global phase portraits of this system will be done in the proof of Proposition 16 and the correponding phase portraits of system (47) are described by DL.1, DL.2 and DL.3 of Figure 6.

Phase portraits of systems $(LV.3)$ when they have a Darboux invariant. When $(LV.3)$ has a Darboux invariant the parameter α must be equal to $-(\beta+1)$ so the differential system is

$$\dot{x} = x(a\,x + \beta(y + a\,x + c) + c), \qquad \dot{y} = y(a\,x + \beta(y + a\,x + c)).$$

In the finite part there are three singular points, namely $z_1 = (0,0)$, $z_2 = (0,-c)$ and $z_3 = (-c/a, 0)$ (remember that $a\,c \neq 0$). Defining $l_1 = c\,\beta \neq 0$ and $l_2 = c\,(\beta+1) \neq 0$,

	q	r	p
LVL.2.1	1	-1	1/2
LVL.2.2	1	2	1
LVL.2.3	1	-1	2
LVL.2.4	1	-1	1
LVL.2.5	1	2	-2
LVL.2.6	1	1/2	-1/2
LVL.2.7	1	0	0
LVL.2.8	1	1	1
LVL.2.9	1	1	-1/2
LVL.2.10	1	1	-1
LVL.2.11	0	-2	1
LVL.2.12	0	2	1
LVL.2.13	0	0	1
LVL.2.14	0	2	-1
LVL.2.15	0	3/4	-1/4
LVL.2.16	0	1	1
LVL.2.17	0	1	-1/2

TABLE 3. Table of values for the parameters of system $(LV.2)$ when $c = -1$.

then the eigenvalues of the z_1 are l_1 and l_2; the eigenvalues of z_2 are c and $-l_1$, and the eigenvalues associated to z_3 are $-c$ and $-l_2$.

In the local chart U_1 the compactified system becomes

$$\dot{u} = -c\,u\,v, \qquad \dot{v} = -v(c\,v + \beta(u + c\,v + a) + a).$$

Hence the line $v = 0$ is filled of singular points after eliminating the common factor v there are no singular points. The same happens in the local chart U_2. So by continuity the only possible phase portrait is LVL.3.1 of Figure 5, which is realizable for $\beta = 1$ and $a = c = -1$.

Proposition 15 (RPL). *Each real planar quadratic differential system with two parallel real straight lines and a third straight line having a Darboux invariant can be written, after an affine change of coordinates, as system (RPL) and it has the Darboux invariant*

$$I_{13}(t, x, y) = e^{2t}(x + 1)(x - 1)^{-1}.$$

Moreover there are 17 non-equivalent phase portraits in the Poincaré disc for this system. They are described in Figure 6.

Proof. Let $f_1 = x + 1 = 0$, $f_2 = x - 1 = 0$ and $f_3 = y = 0$ be the three invariant straight lines. The cofactors of f_1, f_2 and f_3 are, respectivelly, $k_1 = x - 1, k_2 = x + 1, k_3 = \alpha + \beta\,x + \gamma\,y$. With these cofactors equation (8) with $s \in \mathbb{R} \setminus \{0\}$ has two solutions, namely

$$\begin{aligned} s_1 &= \{\gamma = 0,\ s = 2\lambda_1 + (\beta - \alpha)\lambda_3,\ \lambda_2 = -(\lambda_1 + \beta\lambda_3)\} \\ s_2 &= \{s = 2\lambda_1,\ \lambda_2 = -\lambda_1,\ \lambda_3 = 0\}. \end{aligned}$$

Since the second solution s_2 is more general we conclude that every quadratic system that has two real parallel straight lines and a third real straight line as invariant straight lines also has a Darboux invariant. Taking $\lambda_1 = 1$ we get the invariant

$$I_{16}(t, x, y) = e^{2t}(x + 1)(x - 1)^{-1}.$$

□

Phase portraits of systems (RPL)**.** Remember that the system is

$$\dot{x} = x^2 - 1, \qquad \dot{y} = y(\alpha + \beta\, x + \gamma\, y).$$

When $\gamma \neq 0$ we can take $\gamma = 1$ (indeed, just do the change $x = X, y = Y/\gamma$). So the system can present at most four finite singularities, namely, $z_1 = (-1, 0)$, $z_2 = (-1, \beta - \alpha)$, $z_3 = (1, 0)$ and $z_4 = (1, -\beta - \alpha)$. Define $l_1 = \alpha - \beta$ and $l_2 = \alpha + \beta$. The eigenvalues associated to z_1 are -2 and l_1 while the eigenvalues associated to z_2 are -2 and $-l_1$. Moreover $z_1 = z_2$ when $l_1 = 0$. Analogously the eigenvalues of z_3 are 2 and l_2, while the eigenvalues associated to z_4 are 2 and $-l_2$, with $z_3 = z_4$ when $l_2 = 0$. So in the finite part the system can have two, three or four singulaties, depending on the values of l_1 and l_2.

In the local chart U_1 the compactified system has at most two singularities on the infinity line: $u_1 = (0, 0)$ and $u_2 = (1 - \beta, 0)$. Defining $l_3 = \beta - 1$ we see that $u_1 = u_2$ when $l_3 = 0$ and the topological type of these singularities depends on the sign of l_3. Indeed the eigenvalues associated to u_1 are -1 and l_3 while the associated to u_2 are -1 and $-l_3$.

In the local chart U_2 we just need to check if the origin $(0, 0)$ is a singularity, which is true. It is a node, with the two eigenvalues equal to -1.

So considering $\gamma \neq 0$ and combining all the possibilities of the signs of l_1, l_2 and l_3 we obtain the phase portraits RPL.1–RPL.10 of Figure 6. In Table 3 we put the values of the parameters that realizes each one of the phase portraits described in Figure 6.

If $\gamma = 0$ then z_2 and z_4 goes to the infinity and the compactified system in the local chart U_2 becomes

$$\dot{u} = (1 - \beta)u^2 - \alpha uv - v^2, \qquad \dot{v} = -v(\beta u + \alpha v).$$

Note that when $l_3 = 0 (\beta = 1)$ the line $v = 0$ is filled up of singular points, and when $l_3 \neq 0$ the origin $(0, 0)$ is a linearly zero singularity. Considering this case first and applying the *blow up* $u = U, v = UW$ and dividing by U we get the system

$$\dot{U} = -U(\beta + W^2 + \alpha W - 1), \qquad \dot{W} = W(W - 1)(W + 1). \tag{48}$$

When $U = 0$ the singularities of (48) are $\overline{u_1} = (0, -1)$ with eigenvalues 2 and l_1, $\overline{u_2} = (0, 0)$ with eigenvalues -1 and $-l_3$, and $\overline{u_3} = (0, 1)$ with eigenvalues 2 and $-l_2$.

After blow-down we get the local phase portraits of the origin of U_2 which depend on the signs of l_1, l_2 and l_3. Doing all the combinations the origin of U_2 consists of:

- two elliptic sectors and parabolic sectors, see phase portraits RPL.11 and RPL.12 of Figure 6;
- two hyperbolic sectors and parabolic sectors, see phase portraits RPL.13 and RPL.14 of Figure 6;
- six hyperbolic sectors, see phase portrait RPL.15 of Figure 6.

Finally if we consider $\beta = 1$ and after eliminating the common factor v the origin of the local chart U_2 is either a hyperbolic node or a hyperbolic saddle, described respectively by the phase portraits RPL.16 and RPL.17 of Figure 6. The Table 4 has the values of the parameters that realizes the phase portraits of Figure 6.

Proposition 16 (DL)**.** *Each real planar quadratic differential system with a double real straight line and a third straight line having a Darboux invariant can be written, after an affine change of coordinates, as system (DL), with* $\gamma = 0$ *and* $\alpha \neq 0$*, and the Darboux invariant is*

$$I_{14}(t, x, y) = e^{-\alpha\, t} y\, x^{-\beta}.$$

	α	β	γ
RPL.1	-5/4	1/4	1
RPL.2	0	-1	1
RPL.3	-3	2	1
RPL.4	-2	1	1
RPL.5	0	1	1
RPL.6	-1/2	1/2	1
RPL.7	1/2	-1/2	1
RPL.8	-2	2	1
RPL.9	-1	1	1
RPL.10	0	0	1
RPL.11	-3	2	0
RPL.12	0	-1	0
RPL.13	-1	0	0
RPL.14	-1	2	0
RPL.15	-1/4	3/4	0
RPL.16	-2	1	0
RPL.17	0	1	0

TABLE 4. Table of values for the parameters of system (RPL).

Moreover there are 3 non-equivalent phase portraits in the Poincaré disc for this systems. They are described in Figure 6.

Proof. Let $f_1 = x = 0$ be the double real invariant straight line. By the proof of Proposition 6 we know that the second invariant straight line is $f_2 = y = 0$. The cofactors of f_1 and f_2 are, respectivelly, $k_1 = x, k_2 = \alpha + \beta\, x + \gamma\, y$. Equation (8) with $s \in \mathbb{R} \setminus \{0\}$ has only one solution $\gamma = 0$, $s = -\alpha\lambda_2$, $\lambda_1 = -\beta\lambda_2$.

Taking $\lambda_2 = 1$ and using this solution we get

$$\dot{x} = x^2, \qquad \dot{y} = y(\alpha + \beta\, x),$$

with Darboux invariant $I_2(t, x, y) = e^{-\alpha\, t} y\, x^{-\beta}$. □

Phase portraits of systems (DL) when $\gamma = 0$ and $\alpha \neq 0$. Since $\alpha \neq 0$ we can take $\alpha = 1$. The origin of the system is the only finite singularity, which is a saddle-node. For the infinity singularities we assume first that $\beta - 1 \neq 0$. In the local chart U_1 the origin is a saddle if $\beta - 1 > 0$, and a stable node if $\beta - 1 < 0$. In the chart U_2 the system becomes

$$\dot{u} = -u((\beta - 1)u + v), \qquad \dot{v} - v(\beta u + v),$$

and the origin is a linearly zero singularity. Applying the *blow up* $u = U$, $v = UW$ we get the system

$$\dot{U} = -U^2(\beta - 1 + W), \qquad \dot{W} = -UW,$$

which after eliminating the common factor U has the origin as only singular point. If $\beta - 1 > 0$ the origin is a hyperbolic stable node and if $\beta - 1 < 0$ the origin is a saddle.

After *blow down* we get the local phase portraits of the origin of U_2 which depend on β. When $\beta - 1 > 0$ the origin has two elliptic sectors and parabolic sectors, see phase portrait DL.1 of Figure 6. If $\beta - 1 < 0$ then there are two hyperbolic sectors and parabolic ones, see phase portrait DL.2 of Figure 6.

When $\beta = 1$ the infinity is filled up of singular points and the origin in the local chart U_2 is a hyperbolic stable node. The phase portrait of this case can be found and it is described by DL.3 of Figure 6.

Proposition 17 (CPL)**.** *Each real planar quadratic differential system with two parallel complex straight line and a third straight line having a Darboux invariant can be written, after an affine change of coordinates, as system* (CPL)*. A Darboux invariant is given by*

$$I_{15}(t, x, y) = e^t e^{arctan(1/x)}$$

Moreover there are 6 non-equivalent phase portraits in the Poincaré disc for this system. They are described in Figure 7.

Proof. Let $f_1 = x + i = 0$, $f_2 = x - i = 0$ be the two complex parallel straight lines. By the proof of Proposition 6 we know that the third invariant straight line is $f_3 = y = 0$. The cofactors of f_1, f_2 and f_3 are, respectivelly, $k_1 = x - i, k_2 = x + i, k_3 = \alpha + \beta\, x + \gamma\, y$. The equation (8) with $s \in \mathbb{R} \setminus \{0\}$ has two solutions, namely

$$\begin{aligned} s_1 &= \{\gamma = 0,\ s = i(2\lambda_1 + (\beta + i\alpha)\lambda_3),\ \lambda_2 = -\beta\lambda_3 - \lambda_1\} \\ s_2 &= \{s = 2i\lambda_1,\ \lambda_2 = -\lambda_1,\ \lambda_3 = 0\}. \end{aligned}$$

Using s_2 (which is more general) we conclude that all systems with two parallel complex straight lines and a real straight line as invariants curves have a Darboux invariant. Moreover taking $\lambda_1 = -i/2$ we get

$$I_2(t, x, y) = e^t (x - i)^{i/2} (x + i)^{-i/2}.$$

Using the polar form of the complex numbers it follows that $(x - i)^{i/2}(x + i)^{-i/2} = e^{\arctan(1/x)}$ so the Darboux invariant is $I_2(t, x, y) = e^{\arctan(1/x)+t}$. □

Phase portraits of systems (CPL)**.** In [10] the authors already study the quadratic systems with $f = x^2 + 1 = 0$ as invariant curve, given by

$$\dot{x} = x^2 + 1, \qquad \dot{y} = Q(x, y),$$

with Q an arbitrary polynomial of degree 2. In our case we have $Q(x, y) = y(\alpha + \beta\, x + \gamma\, y)$. So the system studied here is a subcase of systems (VI) of the article [10]. In that article the study of those systems is divided in six cases and since we have the invariant straigh line $y = 0$ there are seven possible phase portraits. The case $(VI.1)$ provides the phase portraits 1 and 2 of [10](Fig. 1), i.e. the phase portraits CPL.1 and CPL.2 of Figure 7; the case $(VI.2)$ gives the phase portrait 6 of [10](Fig. 1), i.e. the phase portrait CPL.3 of Figure 7; the case $(VI.4)$ generates the phase portraits 16 and 17 of [10](Fig. 1), i.e. the phase portraits CPL.4 and CPL.5 of Figure 7; the case $(VI.5)$ gives the phase portrait 20 of [10](Fig. 1), i.e. the phase portrait CPL.6 of Figure 7. Finally the case $(VI.6)$ provides the phase portrait 21 of [10](Fig. 1), i.e. the phase portrait CPL.7 of Figure 7.

Proposition 18 (p)**.** *Each real planar quadratic differential system with two complex straight lines that intersects in a real point and a third straight line having a Darboux can be written, after an affine change of coordinates, as*

(i) $(p.1)$ *with* $\alpha_3(\beta - 2\beta_3) \neq 0$ *and Darboux invariant*

$$I_{16}(t, x, y) = e^{\alpha_3(\beta - 2\beta_3)\, t}\, e^{-2\gamma_3 \arctan(y/x)}\, (x^2 + y^2)^{\beta_3}\, y^{-\beta}.$$

(ii) $(p.2)$ *with* $c \neq 0$, $\alpha = -1$ *and Darboux invariant*

$$I_{17}(t, x, y) = e^{-\arctan(y/x) - c\, t}$$

Moreover there are 5 non-equivalent phase portraits in the Poincaré disc for this system. They are described in Figure 7.

Proof. Let $f_1 = x + iy = 0$ and $f_2 = x - iy = 0$ be the two complex straight lines that intersect at a real point. We have two systems, $(p.1)$, with $f_3 = y$, and $(p.2)$ with $f_3 = y + a\,x + c$. We shall do the calculations for $(p.1)$, and for system $(p.2)$ the computations are analogous.

Consider system $(p.1)$ the cofactors of f_1, f_2 and f_3 are, respecively,

$$\begin{aligned} k_1 &= (1/2)(\beta\, x + 2\gamma_3\, y + 2\alpha_3 - i(\beta - 2\beta_3)y),\\ k_2 &= (1/2)(\beta\, x + 2\gamma_3\, y + 2\alpha_3 + i(\beta - 2\beta_3)y),\\ k_3 &= \alpha_3 + \beta_3\, x + \gamma_3\, y. \end{aligned} \tag{49}$$

Solving equation (8) the most general solution is

$$\lambda_1 = \beta_3 + i\gamma_3, \quad \lambda_2 = \beta_3 - i\gamma_3, \quad \lambda_3 = -\beta, \quad s = \alpha_3(\beta - 2\beta_3).$$

Hence assuming $\alpha_3(\beta - 2\beta_3) \neq 0$ system $(p.1)$ of Theorem C has the Darboux invariant

$$I_2(t, x, y) = e^{\alpha_3(\beta - 2\beta_3)\,t} y^{-\beta}\,(x - iy)^{\beta_3 - i\gamma_3}(x + iy)^{\beta_3 + i\gamma_3}. \tag{50}$$

Using the polar form of the complex numbers it follows that $(x-i)^{i/2}(x - iy)^{\beta_3 - i\gamma_3}(x + iy)^{\beta_3 + i\gamma_3} = e^{-2\gamma_3 \arctan(y/x)}\,(x^2 + y^2)^{\beta_3}$ and we get the Darboux invariant

$$I_{16}(t, x, y) = e^{\alpha_3(\beta - 2\beta_3)\,t}\, e^{-2\gamma_3 \arctan(y/x)}\,(x^2 + y^2)^{\beta_3}\, y^{-\beta}$$

For system $(p.2)$ the third invariant straight line is $f_3 = y + a\,x + c$ with $c \neq 0$. In this case the system has a Darboux invariant if and only if $\alpha = -1$, and with the same reasoning applied above we get the invariant

$$I_{17}(t, x, y) = e^{-\arctan(y/x) - c\,t}.$$

Phase portraits of systems $(p.1)$ when they have a Darboux invariant. Since $\alpha_3 \neq 0$ we can take $\alpha_3 = 1$. Systems $(p.1)$ have at most two finite singularities, namely $z_1 = (0, 0)$ and $z_2 = (-2/\beta, 0)$. When $\beta = 0$ the point z_2 goes to infinity. The point z_1 is an unstable node and the eigenvalues associated to z_2 are -1 and $(\beta - 2\beta_3)/\beta$. So the point z_2 is either a stable node or a saddle.

In the local chart U_2 the origin is not a singularity for the compactfied system. In the local chart U_1 the system compactified has only one infinity singularity $u_1 = (0, 0)$ with eigenvalues $-\beta/2$ and $-(\beta - 2\beta_3)/2$.

Then if $\beta(\beta - 2\beta_3) > 0$, z_2 is a saddle and u_1 is a stable node and the only phase portrait is p.1.1 of Figure 7, realizable for $\beta = 1$, $\gamma_3 = 1$ and $\beta_3 = -1/2$. If $\beta(\beta - 2\beta_3) < 0$, z_2 is a stable node and u_1 is a saddle and the corresponding phase portrait of this case is p.1.2 of Figure 7, realizable for $\beta = 1$, $\gamma_3 = 1$ and $\beta_3 = 3/2$. Finally if $\beta = 0$ then z_2 goes to the infinity and u_2 becomes a semi hyperbolic saddle-node generating the phase portrait p.1.3 of Figure 7, which is realizable for $\beta = 0$, $\gamma_3 = 1$ and $\beta_3 = 2$.

Phase portraits of systems $(p.2)$ when they have a Darboux invariant. In the local chart U_1 system $(p.2)$ becomes

$$\dot{u} = -cv(u^2 + 1), \qquad \dot{v} = -v(a\beta + au + cuv + \beta cv + \beta u - 1).$$

So the line $v = 0$ is filled up of singular points. The same happens in the local chart U_2. In the finite part the point $(0, 0)$ is the only singularity, with complex eigenvalues. So the origin can be a node or a center. Both cases are described, respectively, by the phase portraits p.2.1 ,realizable with $a = \beta = 1$ and $c = 2$, and p.2.2, realizable with $a = 1$, $\beta = 0$ and $c = 2$, of Figure 7. $\square$

By the end we prove Theorem E. This result is about the differential systems having an invariant cubic but that do not have a Darboux invariant.

Proof of Theorem E. First we consider systems of type (CE), i.e, the ones which has an invariant cubic of the form $f = f_1 f_2 = 0$ where $f_1 = x^2 + y^2 + 1$ and $f_2 = a\,x + b\,y + c$. By Theorem C these systems can be written as

$$\dot{x} = -(x^2 + y^2 + 1) - 2\alpha_1\, y(y + ax + c), \qquad \dot{y} = a(x^2 + y^2 + 1) + 2\alpha_1\, x(y + ax + c),$$

with $f_1 = x^2+y^2+1$ and $f_2 = y+ax+c$. The cofactors of f_1 and f_2 are $k_1(x, y) = 2(a\,y+x)$ and $k_2(x, y) = -2\alpha_1(a\,y - x)$, respectively. So the cofactors have no constant terms, i.e, $k_1(0, 0) = k_2(0, 0) = 0$. The consequence of this is that equation (8) has no solution considering $s \neq 0$. Hence these systems do not have a Darboux invariant of the form $e^{s\,t} f_1^{\lambda_1} f_2^{\lambda_2}$.

The proofs for other systems are very similar. In fact it suffices to observe that the cofactors of the invariant curves never have a constant term. $\square$

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References

[1] J.C. Artés, PhD thesis, Universitat Autónoma de Barcelona, Barcelona, 1990. Advisor J. Llibre.

[2] R. Bix, *Conics and Cubics. A concrete introduction of algebraic curves.* UTM, Springer, 2006.

[3] C. Bujac and N. Vulpe, *Cubic systems with invariant straight lines of total multiplicity eight and with three distinct infinite singularities*, Qual. Theory Dyn. Syst. **14** (2015), 109–137.

[4] L. Cairó and J. Llibre, *Darbouxian first integrals and invariants for real quadratic systems having an invariant conic*, J. Phys. A: Math. Gen. **35** (2002), 589– 608.

[5] C. J. Christopher, *Invariant algebraic curves and conditions for a center*, Proc. R. Soc. Edin. **124 A** (1994), 1209–1229.

[6] C. J. Christopher and J. Llibre, *Integrability via invariant algebraic curves for planar polynomial differential systems,* Ann. Differ. Equ. **16** (2000), 5–19.

[7] C. J. Christopher, J. Llibre, C. Pantazi, X. Zhang., *Darboux integrability and invariant algebraic curves for planar polynomial systems.* J. Phys. A: Math Gen. **35** (2012) , 2457–2476.

[8] W.A. Coppel, *A survey of quadratic systems*, J. Differential Equations, **2** (1966), 293–304.

[9] F. Dumortier, J. Llibre and J. C. Artés, *Qualitative theory of planar differential systems*, Universitext, Springer, New York, 2006.

[10] A. Gasull, S. Li-Ren and J. Llibre, *Chordal quadratic system*, Rocky Mountain J. Math.**16** (1986), 751–781.

[11] J. Llibre, M. Messias and A. C. Reinol, *Darboux invariants for planar polynomial differential systems having an invariant conic.* Z. Angew. Math. Phys. **65** (2014), 1127–1136.

[12] J. Llibre, M. Messias and A. C. Reinol, *Normal forms and global phase portraits of quadratic and cubic integrable vector fields having two nonconcentric circles as invariant algebraic curves.*, Dyn. Syst. **32** (2017), 374–390.

[13] J. Llibre and R. D. S. Oliveira, *Quadratic systems with invariant straight lines of total multiplicity two having Darboux invariants*, Commun. Contemp. Math. **17** (2015), 1450018, 17 pp.

[14] J. Llibre and R. D. S. Oliveira, *Quadratic systems with invariant conics having Darboux invariants*, Commun. Contemp. Math. **20** (2018), 1750033, 15 pp.

[15] J. W. Reyn, *A bibliography of the qualitative theory of quadratic systems of differential equations in the plane*, Delf University of Technology, http://ta.twi.tudelf.nl/DV/Staff/ J.W.Reyn.html, 1997.

[16] J. Sotomayor and R. Paterlini, Quadratic Vector Fields with Finitely Many Periodic Orbits. IMPA. International Symposium on Dynamical Systems, 1983.

[17] H. Zholadek *On algebraic solutions of algebraic Pfaff equations.* Studia Math. **114** (1995), 117–126.

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