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LIMIT CYCLES OF PLANAR DISCONTINUOUS PIECEWISE LINEAR HAMILTONIAN SYSTEMS WITHOUT EQUILIBRIUM POINTS AND SEPARATED BY IRREDUCIBLE CUBICS

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Abstract. This paper is devoted to study the limit cycles of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by irreducible cubics. We study the limit cycles that intersect the cubic in two or four points. We provide lower bounds for the maximum number of limit cycles intersecting the cubic either in two points, or in four points, or in both classes simultaneously. All the computations of this paper has been verified with the algebraic manipulator mathematica.

1. Introduction

We recall that a *limit cycle* of a differential system is an isolated periodic orbit in the set of all periodic orbits of this system. It is well-known that among the many problems of the differential systems in the plane one of the most difficult is to finding the best upper bound for the maximum number of limit cycles that a given differential system or a class of differential systems can exhibit, see for instance the 16th Hilbert problem [12, 14, 16]. Here we consider this problem for the planar discontinuous piecewise linear Hamiltonian systems without equilibrium points and separated by irreducible cubics.

Recently an increasing interest appeared for the piecewise differential systems, mainly due to its applications in engineering, mechanics, electric circuits, ... see for instance the books of [1, 3, 25] and the hundreds of references therein. A good deal of that interest is placed in studying the limit cycles of these piecewise differential systems. See for instance the papers dedicated to study the limit cycles of the piecewise linear differential systems separated by a straight line or by other kind of curves. See, without trying to be exhaustive, for instance [2, 6, 7, 8, 9, 10, 11, 17, 18, 19, 20, 24].

In particular in the papers [4, 15, 21] the authors studied the maximum number of limit cycles of piecewise linear centers separated by algebraic curves of the form $y = x^n$, or by a conic, or by a reducible or irreducible cubic curve.

In this paper we consider planar discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by an irreducible cubic. It is known

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and easy to prove that the Hamiltonian vector fields of such piecewise differential systems in each piece can be written into the form

$$X_i(x,y) = (-\lambda_i b_i x + b_i y + \gamma_i, -\lambda_i^2 b_i x + \lambda_i b_i y + \delta_i),$$

where $\delta_i \neq \lambda_i \gamma_i$ and $b_i \neq 0$ for $i = 1 \dots 4$, see for details [8]. The Hamiltonian function associated to the Hamiltonian vector field X_i is

$$H_i(x,y) = (-\lambda_i^2 b_i/2)x^2 + \lambda_i b_i xy - (b_i/2)y^2 + \delta_i x - \gamma_i y.$$

1.1. **Classification of the irreducible cubics.** An *algebraic cubic curve* or simple a *cubic* is the set of points $(x,y) \in \mathbb{R}^2$ satisfying $P(x,y) = 0$ for some polynomial $P(x,y)$ of degree three. This real cubic is *irreducible* (respectively *reducible*) if the polynomial $P(x,y)$ is irreducible (respectively reducible) in the ring of all real polynomials in the variables x and y.

A point (x_0, y_0) of a cubic $P(x,y) = 0$ is *singular* if $P_x(x_0,y_0) = P_y(x_0,y_0) = 0$. A cubic curve is *singular* if it has some singular point.

A *flex* of an algebraic curve C is a point p of C such that C is nonsingular at p and the tangent at p of the curve C intersects C at least three times.

Theorem 1. *The following statements classify all the irreducible cubic algebraic curves.*

(a) *A cubic is nonsingular and irreducible if and only if it can be transformed with an affine transformation into one of the following two curves*

$$c_1 = c_1(x,y) = y^2 - x(x^2 + bx + 1) = 0 \quad \text{with } b \in (-2,2), \text{ or}$$
$$c_2 = c_2(x,y) = y^2 - x(x-1)(x-r) = 0 \quad \text{with } r > 1.$$

(b) *A cubic is singular and irreducible if and only if it can be transformed with an affine transformation into one of the following three curves:*

$$c_3 = c_3(x,y) = y^2 - x^3 = 0, \qquad \text{or}$$
$$c_4 = c_4(x,y) = y^2 - x^2(x-1) = 0, \quad \text{or}$$
$$c_5 = c_5(x,y) = y^2 - x^2(x+1) = 0.$$

Statement (a) of Theorem 1 is proved in Theorem 8.3 of the book [5] under the additional assumption that the cubic has a flex, but in section 12 of that book it is shown that every nonsingular irreducible cubic curve has a flex. While statement (b) of Theorem 1 follows directly from Theorem 8.4 of [5].

1.2. **Statement of the main results.** We denote by C_k the class of planar discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by the irreducible cubic $c_k = 0$ for $k = 1, \dots, 5$

Our first objective is to provide the maximum number of limit cycles with two points on the cubic for the discontinuous piecewise linear Hamiltonian systems separated by a cubic $c_k = 0$, with $k = 1, \dots, 5$. We note that such limit cycles are contained only in two pieces of the discontinuous piecewise linear Hamiltonian system.

Theorem 2. *For $k = 1, \ldots, 5$ the maximum number of limit cycles of the discontinuous piecewise linear Hamiltonian systems intersecting the cubic $c_k = 0$ in two points is three.*

This maximum is reached in Figures 1, 2 and 3 for the classes C_1, C_3 and C_4 respectively; and in Figures 4, 5 and 6 for the class C_2; and in Figure 7 for the class C_5.

Theorem 2 is proved in section 2.

The second objective of this work is to give the maximum number of simultaneous limit cycles with two or four points on the cubic for the discontinuous piecewise linear Hamiltonian systems which intersect the cubics $c_2 = 0$ or $c_5 = 0$. We note that such limit cycles are contained in three pieces of the discontinuous piecewise linear Hamiltonian system.

Theorem 3. *The following statements hold.*

- (a) *The maximum number of limit cycles of the discontinuous piecewise linear Hamiltonian systems intersecting in four points the cubics c_2 or c_5 is three. See Figures 8 and 9 for the classes C_2 and C_5, respectively.*
- (b) *The maximum number of limit cycles of the discontinuous piecewise linear Hamiltonian systems intersecting simultaneously in four points and two points the cubics c_2 or c_5 is three.*

  This maximum is reached in Figures 10 and 11 for the class C_2, and in Figure 12 for the class C_5 where there are examples of systems exhibiting simultaneously one limit cycle with four intersection points and two limit cycles with two intersection points with the cubic.

  The maximum is also reached in Figures 13 and 14 for the class C_2, and in Figure 15 for the class C_5 where there are examples of systems exhibiting simultaneously two limit cycle with four intersection points and one limit cycles with two intersection points with the cubic.

Theorem 3 is proved in section 3.

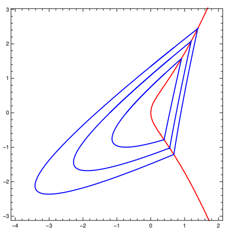

FIGURE 1. The three limit cycles of the discontinuous piecewise differential systems (4)–(5).

2. Proof of Theorem 2

We shall prove that the maximum number of limit cycles of the discontinuous piecewise linear Hamiltonian systems intersecting the cubic $c_3 = 0$ in two points is three. For the other four cubics the proof is completely similar.

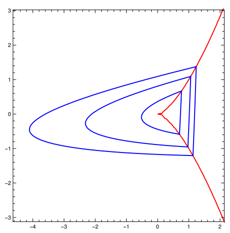

FIGURE 2. The three limit cycles of the discontinuous piecewise differential systems (7)–(8).

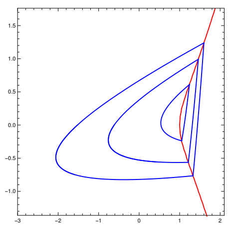

FIGURE 3. The three limit cycles of the discontinuous piecewise differential systems (9)–(10).

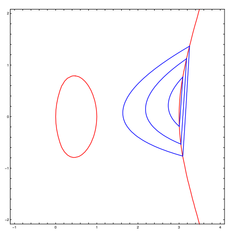

FIGURE 4. The three limit cycles of the discontinuous piecewise differential systems (12)–(13).

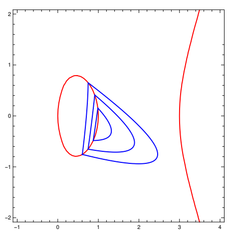

FIGURE 5. The three limit cycles of the discontinuous piecewise differential systems (14)–(15).

We consider the discontinuous piecewise linear Hamiltonian system such that in the region $R_1 = \{(x, y) : y^2 - x^3 \geq 0\}$ is defined as

$$\dot{x} = -\lambda_1 b_1 x + b_1 y + \mu_1, \quad \dot{y} = -\lambda_1^2 b_1 x + \lambda_1 b_1 y + \sigma_1, \tag{1}$$

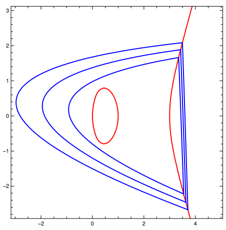

FIGURE 6. The three limit cycles of the discontinuous piecewise differential systems (16)–(17).

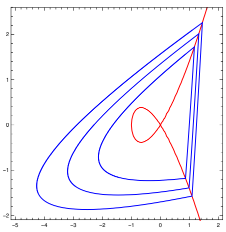

FIGURE 7. The three limit cycles of the discontinuous piecewise differential systems (19)–(20).

with $b_1 \neq 0$ and $\sigma_1 \neq \lambda_1\mu_1$. This system has the first integral

$$H_1(x,y) = -(\lambda_1^2 b_1/2)x^2 + \lambda_1 b_1 xy - (b_1/2)y^2 + \sigma_1 x - \mu_1 y.$$

In the region $R_2 = \{(x,y) : y^2 - x^3 \leq 0\}$ we consider the linear Hamiltonian system

$$(2) \qquad \dot{x} = -\lambda_2 b_2 x + b_2 y + \mu_2, \quad \dot{y} = -\lambda_2^2 b_2 x + \lambda_2 b_2 y + \sigma_2,$$

with $b_2 \neq 0$ and $\sigma_2 \neq \lambda_2\mu_2$. Its corresponding Hamiltonian first integral is

$$H_2(x,y) = -(\lambda_2^2 b_2/2)x^2 + \lambda_2 b_2 xy - (b_2/2)y^2 + \sigma_2 x - \mu_2 y.$$

In order to have a limit cycle which intersects the cubic $y^2 - x^3 = 0$ in the points (x_i, y_i) and (x_k, y_k), these points must satisfy the system

$$(3) \qquad \begin{array}{l} H_1(x_i,y_i) - H_1(x_k,y_k) = 0, \\ H_2(x_i,y_i) - H_2(x_k,y_k) = 0, \\ y_i^2 - x_i^3 = 0, \\ y_k^2 - x_k^3 = 0. \end{array}$$

Suppose that the piecewise differential system formed by the systems (1) and (2) has four limit cycles. Then system (3) must have four pairs of points of solutions of the form $p_i = (r_i^2, r_i^3)$ and $q_i = (s_i^2, s_i^3)$ for $i = 1, \ldots 4$. Due to the fact that these points must satisfy the first two equations of system (3), from these two equations and for $i = 1$ we get that

$$\sigma_1 = \frac{b_1 r_1^6 - 2b_1\lambda_1 r_1^5 + b_1\lambda_1^2 r_1^4 - b_1 s_1^6 + 2b_1\lambda_1 s_1^5 - b_1\lambda_1^2 s_1^4 + 2\mu_1 r_1^3 - 2\mu_1 s_1^3}{2(r_1^2 - s_1^2)},$$

and σ_2 has the same expression than σ_1 changing (b_1, λ_1, μ_1) by (b_2, λ_2, μ_2).

Since the points $p_2=(r_2^2,r_2^3)$ and $q_2=(s_2^2,s_2^3)$ also satisfy system (3), we obtain that parameters μ_1 and μ_2 must be $\mu_1=A/B$, where

$$\begin{aligned} A = \; & -(b_1(r_1^5(r_2+s_2)+r_1^4(r_2+s_2)(s_1-2\lambda_1)+r_1^3(r_2+s_2)(s_1-\lambda_1)^2+r_1^2s_1(r_2\\ & +s_2)(s_1-\lambda_1)^2-r_1(r_2^5+r_2^4(s_2-2\lambda_1)+r_2^3(s_2-\lambda_1)^2+r_2^2s_2(s_2-\lambda_1)^2+r_2\\ & (-s_1^4+s_2^2(s_2-\lambda_1)^2+2s_1^3\lambda_1-s_1^2\lambda_1^2)+s_2(-s_1^4+s_2^2(s_2-\lambda_1)^2+2s_1^3\lambda_1-s_1^2\lambda_1^2))\\ & +s_1(-r_2^5-r_2^4(s_2-2\lambda_1)-r_2^3(s_2-\lambda_1)^2-r_2^2s_2(s_2-\lambda_1)^2+r_2(s_1^4-s_2^2(s_2-\lambda_1)^2\\ & -2s_1^3\lambda_1+s_1^2\lambda_1^2)+s_2(s_1^4-s_2^2(s_2-\lambda_1)^2-2s_1^3\lambda_1+s_1^2\lambda_1^2))),\\ B = \; & 2(r_1^2(r_2+s_2)+s_1(-r_2^2+r_2(s_1-s_2)+(s_1-s_2)s_2)-r_1(r_2^2+r_2(-s_1+s_2)\\ & +s_2(-s_1+s_2))). \end{aligned}$$

And μ_2 has the same expression than μ_1 changing (b_1,λ_1) by (b_2,λ_2).

Again the points $p_3=(r_3^2,r_3^3)$ and $q_3=(s_3^2,s_3^3)$ satisfy system (3), then we obtain two values of λ_1 we name them $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$ and two values of λ_2 we name them $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$. The first value of λ_1 is given by $\lambda_1^{(1)}=(C-(1/2)\sqrt{D})/E$ and the second one is $\lambda_1^{(2)}=(C+(1/2)\sqrt{D})/E$, where the values of C, D and E are given in the appendix. We get the expression of $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$ by changing b_1 by b_2 in the expression of $\lambda_1^{(1)}$ and $\lambda_1^{(2)}$, respectively.

We replace μ_1, $\lambda_1^{(i)}$ and σ_1 in the expression of $H_1(x,y)$, and μ_2, $\lambda_2^{(i)}$ and σ_2 in the expression of $H_2(x,y)$ and we obtain $H_1(x,y)=H_2(x,y)$, for $i=1,2$. Hence the discontinuous piecewise linear differential system becomes a linear differential system, and consequently the system has no limit cycles. So the maximum number of limit cycles in this case is two.

Now we consider the pairs either $\lambda_1^{(2)}$ and $\lambda_2^{(1)}$, or $\lambda_1^{(1)}$ and $\lambda_2^{(2)}$. By replacing the expressions of σ_1, μ_1 and $\lambda_1^{(2)}$ (resp. $\lambda_1^{(1)}$) in the expression of $H_1(x,y)$, and σ_2, μ_2 and $\lambda_2^{(1)}$ (resp. $\lambda_2^{(2)}$) in the expression of $H_2(x,y)$ and we obtain that $H_1(x,y)\neq H_2(x,y)$. Since the points $p_4=(r_4^2,r_4^3)$ and $q_4=(s_4^2,s_4^3)$ satisfy system (3), then we obtain $b_1=0$ and $b_2=0$. This is a contradiction because by the assumptions they are not zero. In summary, we proved that the maximum number of limit cycles for PHS separated by a irreducible cubic curve $c_3=0$ is at most three.

In order to complete the proof of the theorem we shall provide discontinuous piecewise linear Hamiltonian systems without equilibrium points separated by the cubic $c_k=0$ with three limit cycles for $k=1,\ldots,5$.

Example with three limit cycles when the cubic of separation is $c_1=0$. In the region $R_1=\{(x,y): y^2-x(x^2+x+1)\geq 0\}$, we consider the linear Hamiltonian system

$$(\dot{x},\dot{y})=\left(-\frac{9x}{5}+3y+\frac{1}{5},-\frac{27x}{25}+\frac{9y}{5}+1\right). \tag{4}$$

It has the Hamiltonian function $H_1(x,y)=-27x^2/50+9xy/5-3y^2/2-y/5$. Now we consider the second linear Hamiltonian system

$$(\dot{x},\dot{y})=(5.54426..x-2y-9.52503..,15.3694..x-5.54426..y-38.5097..) \tag{5}$$

in the region $R_2=\{(x,y): y^2-x(x^2+x+1)\leq 0\}$. This Hamiltonian system has the Hamiltonian function $H_2(x,y)=7.68471x^2-5.54426xy-38.5097x+y^2+9.52503y$.

The discontinuous piecewise differential system (4)–(5) has exactly three limit cycles, because the system of equations

$$(6)\qquad \begin{array}{l} H_1(\alpha,\beta)-H_1(\gamma,\delta)=0,\\ H_2(\alpha,\beta)-H_2(\gamma,\delta)=0,\\ c_i(\alpha,\beta)=0,\\ c_i(\gamma,\delta)=0, \end{array}$$

when $i=1$, has only three real solutions

$$\begin{array}{l} (\alpha_1,\beta_1,\gamma_1,\delta_1)=(0.393342..,-0.780303..,0.908209..,1.5755..),\\ (\alpha_2,\beta_2,\gamma_2,\delta_2)=(0.558506..,-1.02208..,1.19256..,2.07625..),\\ (\alpha_3,\beta_3,\gamma_3,\delta_3)=(0.680997,-1.20854..,1.3862..,2.44365..), \end{array}$$

see Figure 1.

Example with three limit cycles when the cubic of separation is $c_3=0$. In the region $R_1=\{(x,y):x^3-y^2\le 0\}$ we consider the Hamiltonian system

$$(7)\qquad (\dot x,\dot y)=\left(-\frac{x}{2}+5y+\frac{1}{5},-\frac{x}{20}+\frac{y}{2}+\frac{4}{5}\right).$$

It has the Hamiltonian function $H_1(x,y)=-x^2/40+xy/2+4x/5-5y^2/2-y/5$. Now we consider the second Hamiltonian system

$$(8)\qquad (\dot x,\dot y)=(2.8254..x+\frac{51y}{100}-9.47986..,-15.6528..x-2.8254..y-132.539..)$$

in the region $R_2=\{(x,y):x^3-y^2\ge 0\}$. This Hamiltonian system has the Hamiltonian function $H_2(x,y)=-7.82638..x^2-2.8254..xy-132.539..x-\frac{51y^2}{200}+9.47986..y$. The discontinuous piecewise differential system (7)–(8) has exactly three limit cycles, because the system of equations (6) when $i=3$ has only three real solutions

$$\begin{array}{l} (\alpha_1,\beta_1,\gamma_1,\delta_1)=(0.700707..,-0.58655..,0.765078..,0.669204..),\\ (\alpha_2,\beta_2,\gamma_2,\delta_2)=(0.969795..,-0.955036..,1.06038..,1.09192..),\\ (\alpha_3,\beta_3,\gamma_3,\delta_3)=(1.13263..,-1.2054..,1.23647..,1.37492..), \end{array}$$

see Figure 2.

Example with three limit cycles when the cubic of separation is $c_4=0$. We consider the Hamiltonian system

$$(9)\qquad (\dot x,\dot y)=\left(-\frac{4x}{5}+4y+\frac{3}{10},-\frac{4x}{25}+\frac{4y}{5}+\frac{3}{5}\right),$$

in the region $R_1=\{(x,y):y^2-x^2(x-1)\le 0\}$. This Hamiltonian system has the Hamiltonian function $H_1(x,y)=-2x^2/25+4xy/5+3x/5-2y^2-3y/10$. In the region $R_2=\{(x,y):y^2-x^2(x-1)\ge 0\}$ we consider the Hamiltonian system

$$(10)\qquad (\dot x,\dot y)=(-4.28711..x+9y/100-13.3828..,-204.215..x+4.28711..y+151.777..),$$

which has the Hamiltonian function $H_2(x,y)=-102.107..x^2+4.28711..xy+151.777..x-9y^2/200+13.3828..y$.

The discontinuous piecewise differential system (9)–(10) has exactly three limit cycles, because the system of equations (6) when $i = 4$ has only three real solutions

$$\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1) &= (1.05134.., -0.238211.., 1.24153.., 0.610155..),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2) &= (1.21453.., -0.562538.., 1.46092.., 0.991826..),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) &= (1.33077.., -0.765367.., 1.59962.., 1.23867..),
\end{aligned}$$

see Figure 3.

Three examples with three limit cycles when the cubic of separation is $c_2 = 0$. We define the following regions associated to the curve $c_2 = 0$

$$\begin{aligned}
R_1 &= \{(x, y) : y^2 - x(x-1)(x-3) \geq 0\},\\
R_2 &= \{(x, y) : y^2 - x(x-1)(x-3) \leq 0, x \geq 3\},\\
R_3 &= \{(x, y) : y^2 - x(x-1)(x-3) \leq 0, 0 \leq x \leq 1\}.
\end{aligned} \tag{11}$$

For the first configuration of limit cycles separated by the curve $c_2 = 0$ we consider the Hamiltonian system

$$(\dot{x}, \dot{y}) = \left(-\frac{13x}{20} + 5y + \frac{7}{10}, -\frac{169x}{2000} + \frac{13y}{20} + \frac{19}{10}\right), \tag{12}$$

in the region R_1. It has the Hamiltonian function $H_1(x, y) = -169x^2/4000 + 13xy/20 + 19x/10 - 5y^2/2 - 7y/10$. In the region R_2 we consider the Hamiltonian system

$$(\dot{x}, \dot{y}) = (44.7429..x - 3y/10 + 289.458.., 6673.09..x - 44.7429..y - 15544.3..), \tag{13}$$

which has the Hamiltonian function $H_2(x, y) = 3336.54..x^2 - 44.7429..xy - 15544.3..x + 3y^2/20 - 289.458..y$.

The discontinuous piecewise differential system (12)–(13) has exactly three limit cycles, because the system of equations (6) when $i = 2$, has only the three real solutions

$$\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1) &= (3.00602.., -0.190529.., 3.09131.., 0.768297..),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2) &= (3.04571.., -0.533662.., 3.18176.., 1.12329..),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) &= (3.09077.., -0.765857.., 3.25463.., 1.36691..),
\end{aligned}$$

see Figure 4.

For the second configuration we consider the Hamiltonian system

$$(\dot{x}, \dot{y}) = \left(-\frac{14x}{5} - 7y + \frac{163}{100}, \frac{28x}{25} + \frac{14y}{5} + \frac{2}{5}\right), \tag{14}$$

in the region R_1. It has the Hamiltonian function $H_1(x, y) = 14x^2/25 + 14xy/5 + 2x/5 + 7y^2/2 - 163y/100$. Now we consider the second Hamiltonian system

$$(\dot{x}, \dot{y}) = (-0.860462..x + 3y/10 + 0.4357.., -2.46798..x + 0.860462..y + 0.115128..), \tag{15}$$

in the region R_3. This Hamiltonian system has the Hamiltonian function $H_2(x, y) = -1.23399..x^2 + 0.860462..xy + 0.115128..x - 3y^2/20 - 0.4357..y$.

The discontinuous piecewise differential sysem (14)–(15) has exactly three limit cycles, because the system of equations (6) when $i = 2$, has only the three real solutions

$$\begin{aligned}(\alpha_1,\beta_1,\gamma_1,\delta_1) &= (0.605489..,-0.756295..,0.748414..,0.651116..),\\ (\alpha_2,\beta_2,\gamma_2,\delta_2) &= (0.747078..,-0.652453..,0.911161..,0.4112..),\\ (\alpha_3,\beta_3,\gamma_3,\delta_3) &= (0.87431..,-0.483319..,0.988069..,0.154006..),\end{aligned}$$

see Figure 5.

To obtain the third configuration we consider in the region R_1 the Hamiltonian system

$$(\dot{x},\dot{y}) = \left(\frac{23x}{100}+\frac{23y}{10}-\frac{1}{5},-\frac{23x}{1000}-\frac{23y}{100}+1\right), \tag{16}$$

which has the Hamiltonian function $H_1(x,y) = -23x^2/2000 - 23xy/100 + x - 23y^2/20 + y/5$. In the region R_2 we consider the Hamiltonian system

$$(\dot{x},\dot{y}) = (0.0984281..x + 0.000233032..y + 2, -41.574..x - 0.0984281..y + 96.8899..), \tag{17}$$

This differential system has the Hamiltonian function $H_2(x,y) = -20.787x^2 - 0.0984281xy + 96.8899x - 0.000116516y^2 - 2y$.

When $i = 2$ in the system of equations (6) the discontinuous piecewise differential system (16)–(17) has exactly three limit cycles intersecting the cubic curve $c_2 = 0$ in the points

$$\begin{aligned}(\alpha_1,\beta_1,\gamma_1,\delta_1) &= (3.54594..,-2.22005..,3.35042..,1.66118..),\\ (\alpha_2,\beta_2,\gamma_2,\delta_2) &= (3.63153..,-2.45666..,3.42694..,1.88438..),\\ (\alpha_3,\beta_3,\gamma_3,\delta_3) &= (3.70911..,-2.66935..,3.49745..,2.08449..),\end{aligned}$$

see Figure 6.

Example with three limit cycles when the cubic of separation is $c_5 = 0$. We define the following three regions associated to the curve $c_5 = 0$

$$\begin{aligned}R_1 &= \{(x,y) : y^2 - x^2(x+1) \le 0, x \ge 0\},\\ R_2 &= \{(x,y) : y^2 - x^2(x+1) \ge 0\},\\ R_3 &= \{(x,y) : y^2 - x^2(x+1) \le 0, -1 \le x \le 0\}.\end{aligned} \tag{18}$$

For the class C_5 and in the region R_1 we consider the Hamiltonian system

$$(\dot{x},\dot{y}) = \left(-\frac{9x}{10}+3y+\frac{1}{5},-\frac{27x}{100}+\frac{9y}{10}+1\right), \tag{19}$$

which has the Hamiltonian function $H_1(x,y) = -27x^2/200 + 9xy/10 + x - 3y^2/2 - y/5$. Now we consider the Hamiltonian system

$$(\dot{x},\dot{y}) = (-0.192471..x + y/1000 - 6.61081.., -37.0449..x + 0.192471..y - 26.7945..), \tag{20}$$

in the region R_2. This differential system has the Hamiltonian function $H_2(x,y) = -18.5225..x^2 + 0.192471..xy - 26.7945.. - y^2/2000 + 6.61081..y$.

The discontinuous piecewise differential system (19)–(20) has exactly three limit cycles, because the system of equations (6) when $i = 5$, has the three real solutions

$$\begin{aligned}(\alpha_1,\beta_1,\gamma_1,\delta_1) &= (0.863262..,-1.17836..,1.16993..,1.72339..),\\ (\alpha_2,\beta_2,\gamma_2,\delta_2) &= (0.986378..,-1.39019..,1.32103..,2.01258..),\\ (\alpha_3,\beta_3,\gamma_3,\delta_3) &= (1.0876..,-1.57142..,1.44408..,2.25761..),\end{aligned}$$

see Figure 7.

This completes the proof of Theorem 2.

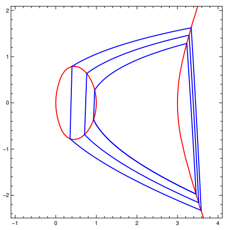

Figure 8. The three limit cycles of the discontinuous piecewise differential system (24).

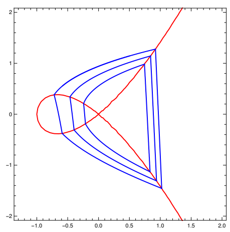

Figure 9. The three limit cycles of the discontinuous piecewise differential systems (26).

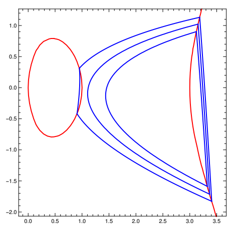

Figure 10. The three limit cycles of the discontinuous piecewise differential system (27).

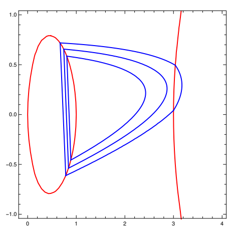

Figure 11. The three limit cycles of the discontinuous piecewise differential systems (28).

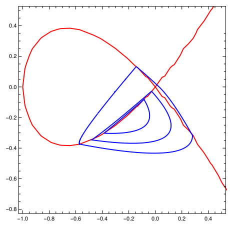

Figure 12. The three limit cycles of the discontinuous piecewise differential systems (29).

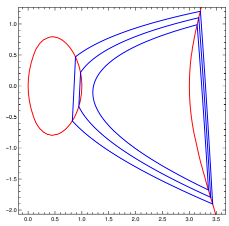

Figure 13. The three limit cycles of the discontinuous piecewise differential systems (30).

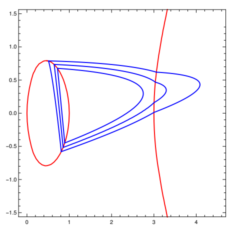

Figure 14. The three limit cycles of the discontinuous piecewise differential systems (31).

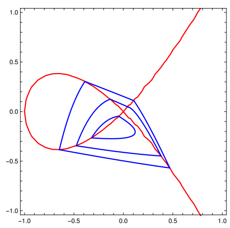

Figure 15. The three limit cycles of the discontinuous piecewise differential systems (32).

3. Proof of Theorem 3

We give the proof for the maximum number of limit cycles for the statements (a) and (b) of the class C_5, and the proof for the class C_2 is similar.

We consider the Hamiltonian systems

$$
\begin{array}{lll}
\dot{x} = -\lambda_1 b_1 x + b_1 y + \mu_1, & \dot{y} = -\lambda_1^2 b_1 x + \lambda_1 b_1 y + \sigma_1, & \text{in the region } R_1,\\
\dot{x} = -\lambda_2 b_2 x + b_2 y + \mu_2, & \dot{y} = -\lambda_2^2 b_2 x + \lambda_2 b_2 y + \sigma_2, & \text{in the region } R_2,\\
\dot{x} = -\lambda_3 b_3 x + b_3 y + \mu_3, & \dot{y} = -\lambda_3^2 b_3 x + \lambda_3 b_3 y + \sigma_3, & \text{in the region } R_3,
\end{array} \tag{21}
$$

with $b_i \neq 0$ and $\sigma_i \neq \lambda_i \mu_i$, when $i = 1, 2, 3$. The regions R_i for $i = 1, 2, 3$ are defined in (18). Their corresponding Hamiltonian first integrals are

$$
\begin{array}{l}
H_1(x,y) = -(\lambda_1^2 b_1/2)x^2 + \lambda_1 b_1 xy - (b_1/2)y^2 + \sigma_1 x - \mu_1 y,\\
H_2(x,y) = -(\lambda_2^2 b_2/2)x^2 + \lambda_2 b_2 xy - (b_2/2)y^2 + \sigma_2 x - \mu_2 y,\\
H_3(x,y) = -(\lambda_3^2 b_3/2)x^2 + \lambda_3 b_3 xy - (b_3/2)y^2 + \sigma_3 x - \mu_3 y.
\end{array} \tag{22}
$$

In order that the discontinuous piecewise differential system (21) has limit cycles which intersect the cubic $c_5 = 0$ in the points $p_1^{(i)} = (r_i^2 - 1, r_i(r_i^2 - 1))$, $p_2^{(i)} = (s_i^2 - 1, s_i(s_i^2 - 1))$, $p_3^{(i)} = (f_i^2 - 1, f_i(f_i^2 - 1))$ and $p_4^{(i)} = (h_i^2 - 1, h_i(h_i^2 - 1))$ they must satisfy the following system

$$
\begin{array}{l}
H_1(r_i^2 - 1, r_i(r_i^2 - 1)) - H_1(s_i^2 - 1, s_i(s_i^2 - 1)) = 0,\\
H_2(s_i^2 - 1, s_i(s_i^2 - 1)) - H_2(h_i^2 - 1, h_i(h_i^2 - 1)) = 0,\\
H_2(r_i^2 - 1, r_i(r_i^2 - 1)) - H_2(f_i^2 - 1, f_i(f_i^2 - 1)) = 0,\\
H_3(f_i^2 - 1, f_i(f_i^2 - 1)) - H_3(h_i^2 - 1, h_i(h_i^2 - 1)) = 0.
\end{array} \tag{23}
$$

Now we consider the first and the last equations of system (23), by solving the first equation for $i = 1, 2, 3$ we get the expressions of λ_1, μ_1 and σ_1, and we get λ_3, μ_3 and σ_3 by solving the last equation. If we suppose that these two equations have a fourth solution, then from the first we get $b_1 = 0$ and from the last one we get $b_3 = 0$. This is a contradiction because by the assumptions they are not zero. Then we proved that the maximum number of limit cycles intersecting the cubic $c_5 = 0$ in four points is at most three.

Example with three limit cycles intersecting the curve $c_2 = 0$ in four points. We consider the Hamiltonian systems

$$
\begin{array}{ll}
\dot{x} = & -9x/25 - 18y/5 + 1/10, \dot{y} = 9x/250 + 9y/25 - 19/10, \text{ in } R_1,\\
\dot{x} = & -409.623..x - 3y/25 - 91053.6.., \dot{y} = 1.39826 \times 10^6 x + 409.623..y\\
 & -3.35682 \times 10^6, \text{ in } R_2,\\
\dot{x} = & -23.0405..x - y - 49.2425.., \dot{y} = 530.865..x + 23.0405..y\\
 & -2056.81.., \text{ in } R_3,
\end{array} \tag{24}
$$

where the regions R_i for $i = 1, 2, 3$ are defined in (11). The Hamiltonian first integrals of the Hamiltonian systems (24) are

$$
\begin{array}{l}
H_1(x,y) = 9x^2/500 + 9xy/25 - 19x/10 + 9y^2/5 - y/10\\
H_2(x,y) = 699130..x^2 + 409.623..xy - 3.35682 \times 10^6 x + 3y^2/50 + 91053.6..y,\\
H_3(x,y) = 265.432..x^2 + 23.0405..xy - 2056.81..x + y^2/2 + 49.2425..y,
\end{array}
$$

In order that discontinuous piecewise differential system (24) has limit cycles which intersect the cubic $c_2 = 0$ in the points $p_1^{(i)} = (\alpha_i, \beta_i)$, $p_2^{(i)} = (\gamma_i, \delta_i)$, $p_3^{(i)} = (f_i, g_i)$

and $p_4^{(i)} = (h_i, k_i)$ these points must satisfy the system

$$
(25)\qquad
\begin{array}{l}
H_1(\alpha_i, \beta_i) - H_1(\gamma_i, \delta_i) = 0,\\
H_2(\alpha_i, \beta_i)) - H_2(a_i, b_i) = 0,\\
H_2(\gamma_i, \delta_i) - H_2(c_i, d_i) = 0,\\
H_3(a_i, b_i) - H_3(c_i, d_i) = 0,\\
c_2(a_i, b_i) = c_2(c_i, d_i) = 0,\\
c_2(\alpha_i, \beta_i) = c_2(\gamma_i, \delta_i) = 0.
\end{array}
$$

For the discontinuos piecewise differential system (24) all the real solutions of the system of equations (25) are

$$
\begin{array}{rl}
(\alpha_1, \beta_1, \gamma_1, \delta_1, a_1, b_1, c_1, d_1) = & (3.22996.., 1.28698.., 3.45846.., -1.97435..,\\
& 0.960385.., 0.278564.., 0.931044.., -0.364457..),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2, a_2, b_2, c_2, d_2) = & (3.28571.., 1.46483.., 3.52436.., -2.15987..,\\
& 0.763003.., 0.636015.., 0.710476.., -0.686261..),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3, a_3, b_3, c_3, d_3) = & (3.33822.., 1.62479.., 3.58465.., -2.3274..,\\
& 0.400766.., 0.790071.., 0.351882.., -0.777131..),
\end{array}
$$

Then the discontinuous piecewise linear differential system (24) has exactly three limit cycles, see Figure 8.

Example with three limit cycles intersecting the curve $c_5 = 0$ in four points. We consider the following Hamiltonian systems

$$
(26)\qquad
\begin{array}{rl}
\dot{x} = & -82.0596..x - y - 176.297.., \dot{y} = 6733.77x + 82.0596y\\
& +333.127, \text{ in } R_1,\\
\dot{x} = & -9x/25 - 18y/5 + 1/10, \dot{y} = 9x/250 + 9y/25 - 19/10, \text{ in } R_2,\\
\dot{x} = & -138.156..x + 2y + 550.05.., \dot{y} = -9543.6..x + 138.156..y\\
& -9983.81.., \text{ in } R_3.
\end{array}
$$

The regions R_i for $i = 1, 2, 3$ are defined in (18). The Hamiltonian first integrals of the Hamiltonian systems (26) are

$$
\begin{array}{l}
H_1(x, y) = 3366.89..x^2 + 82.0596..xy + 333.127..x + \; y^2/2 + 176.297..y,\\
H_2(x, y) = 9x^2/500 + 9xy/25 - 19x/10 + 9y^2/5 - y/10,\\
H_3(x, y) = -4771.8..x^2 + 138.156..xy - 9983.81..x - y^2 - 550.05..y,
\end{array}
$$

respectively. For the discontinuos piecewise linear differential system (26) the real solutions of the system of equations (23) are

$$
\begin{array}{rl}
(\alpha_1, \beta_1, \gamma_1, \delta_1, a_1, b_1, c_1, d_1) = & (0.744242.., 0.982918.., 0.834499.., -1.13028..,\\
& -0.241407.., 0.210259.., -0.211795.., -0.188034..),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2, a_2, b_2, c_2, d_2) = & (0.842745.., 1.14401.., 0.939388.., -1.30821..,\\
& -0.462869.., 0.339233.., -0.395867.., -0.307691..),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3, a_3, b_3, c_3, d_3) = & (0.92187.., 1.278.., 1.02368.., -1.45624..,\\
& -0.718929.., 0.381148.., -0.589074.., -0.377617..),
\end{array}
$$

where $\alpha_i = r_i^2 - 1$, $\beta_i = r_i(r_i^2 - 1)$, $\gamma_i = s_i^2 - 1$, $\delta_i = s_i(s_i^2 - 1)$, $a_i = f_i^2 - 1$, $b_i = f_i(f_i^2 - 1)$, $c_i = h_i^2 - 1$, $d_i = h_i(h_i^2 - 1)$. Then the discontinuous piecewise linear differential system (26) has exactly three limit cycles, see Figure 9. This completes the proof of statement (a) of Theorem 3.

Now we start the proof of statement (b) of Theorem 3.

We consider the discontinuous piecewise differential system (21) with their corresponding first integrals (22). Assume that there are piecewise differential systems (21) having two limit cycles intersecting the cubic $c_2 = 0$ in four points and two limit cycles intersecingt $c_2 = 0$ in two points. Then such systems must have two solutions in system (25), and two solutions in system (6) with $i = 2$ where eventually H_1 can be permuted with H_2. From the fourth first equations of these two systems related with the first integral H_1 and the fourth mentioned solutions, we obtain the expressions of the parameters λ_1, μ_1, σ_1 and b_1, and it results that $b_1 = 0$, which is a contradiction.

Now assume that the discontinuous piecewise differential system (21) has one (resp. three) limit cycle intersecting the cubic $c_2 = 0$ in four points and three (resp. one) limit cycles intersecingt $c_2 = 0$ in two points, we get in the region R_3, defined in (11), four equations on H_3 from which we obtain the expressions of the parameters λ_3, μ_3, σ_3 and a zero value for b_3, which is again a contradiction.

In summary, we conclude that the maximum number of simultaneous limit cycles intersecting the cubic $c_2 = 0$ in four points and two points is three.

Examples with one limit cycle with four points on $c_2 = 0$ and two limit cycles with two points on $c_2 = 0$. As usual we consider the regions defined in (11). For the first possible configuration we consider the following Hamiltonian systems separated by the cubic $c_2 = 0$

$$
\begin{aligned}
\dot{x} = {} & 308.837x - y/10 - 67017.8, \dot{y} = 953806.x - 308.837y \\
& -2.2674 \times 10^6, \text{ in } R_1, \\
\dot{x} = {} & -7x/10 - 7y + 1/10, \dot{y} = 7x/100 + 7y/10 - 3, \text{ in } R_2, \\
\dot{x} = {} & -4x + 2y + 3.55037, \dot{y} = -8x + 4y + 3, \text{ in } R_3.
\end{aligned} \tag{27}
$$

The Hamiltonian systems in (27) have the Hamiltonian first integrals

$$
\begin{aligned}
&H_1(x, y) = 476903..x^2 - 308.837..xy - 2.2674 \times 10^6 x + y^2/20 + 67017.8..y, \\
&H_2(x, y) = 7x^2/200 + 7xy/10 - 3x + 7y^2/2 - y/10, \\
&H_3(x, y) = -4x^2 + 4xy + 3x - y^2 - 3.55037..y.
\end{aligned}
$$

The discontinuos piecewise differential system (27) has one limit cycle intersecting the cubic $c_2 = 0$ in four points satisfying system (25) and two limit cycles intersecting the cubic $c_2 = 0$ in two points satisfying system (6) with $i = 2$, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1, a_1, b_1, c_1, d_1) = {} & (3.12267.., 0.901721.., 3.3261.., -1.58839.., \\
& 0.949883.., 0.312404.., 0.907114.., -0.419931..), \\
(\alpha_2, \beta_2, \gamma_2, \delta_2) = {} & (3.15388.., 1.0224.., 3.36809.., -1.71344..), \\
(\alpha_3, \beta_3, \gamma_3, \delta_3) = {} & (3.18433.., 1.1323.., 3.40728.., -1.82772..).
\end{aligned}
$$

Then the discontinuous piecewise differential system (27) has exactly three limit cycles, see Figure 10.

For the second possible configuration we consider the following Hamiltonian systems separated by the cubic $c_2 = 0$

(28)

$$
\begin{aligned}
\dot{x} = &\ -1.02031..x + 12y - 3/10, \dot{y} = -0.0867522..x + 1.02031..y - 2, \text{ in } R_1,\\
\dot{x} = &\ -13x/5 + 26y + 7/10, \dot{y} = -13x/50 + 13y/5 - 11/5, \text{ in } R_2,\\
\dot{x} = &\ -2.71966..x - y/5 + 1.64749.., \dot{y} = 36.9826..x + 2.71966..y\\
&\ -23.1126.., \text{ in } R_3.
\end{aligned}
$$

The Hamiltonian first integrals of the Hamltonian systemsn (28) are

$$
\begin{aligned}
&H_1(x,y) = -0.0433761..x^2 + 1.02031..xy - 2x - 6y^2 + 3y/10,\\
&H_2(x,y) = -13x^2/100 + 13xy/5 - 11x/5 - 13y^2 - 7y/10,\\
&H_3(x,y) = 18.4913..x^2 + 2.71966..xy - 23.1126..x + y^2/10 - 1.64749..y,
\end{aligned}
$$

respectively. The discontinuos piecewise differential system (28) has one limit cycle intersecting the cubic $c_2 = 0$ in four points satisfying system (25) and two limit cycles intersecting the cubic $c_2 = 0$ in two points satisfying system (6) with $i = 2$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1,\beta_1,\gamma_1,\delta_1,a_1,b_1,c_1,d_1) = &\ (3.03932.., 0.493657.., 3.00028.., 0.04126..,\\
&\ 0.806003.., 0.585712.., 0.889093.., -0.456234..),\\
(\alpha_2,\beta_2,\gamma_2,\delta_2) = &\ (0.743027.., 0.656462., 0.840689.., -0.537772...),\\
(\alpha_3,\beta_3,\gamma_3,\delta_3) = &\ (0.668452.., 0.718837.., 0.78419.., -0.612369..).
\end{aligned}
$$

Then the discontinuous piecewise differential system (28) has exactly three limit cycles, see Figure 11.

Example with one limit cycle with four points on $c_5 = 0$ and two limit cycles with two points on $c_5 = 0$. As usual we consider the regions defined in (18). We consider the following Hamiltonian systems

$$
\begin{aligned}
\dot{x} = &\ -12x - 4y - 3.26985.., \dot{y} = 36x + 12y + 2, \text{ in } R_1,\\
\dot{x} = &\ -16x - 40y - 41/5, \dot{y} = 32x/5 + 16y + 699/100, \text{ in } R_2,\\
\dot{x} = &\ -17.5227..x + 20y - 2.73401.., \dot{y} = -15.3522..x\\
&\ +17.5227..y - 2.04797.., \text{ in } R_3.
\end{aligned}
\tag{29}
$$

The Hamiltonian first integrals of these Hamiltonian systems are

$$
\begin{aligned}
&H_1(x,y) = 18x^2 + 12xy + 2x + 2y^2 + 3.26985..y,\\
&H_2(x,y) = 16x^2/5 + 16xy + 699x/100 + 20y^2 + 41y/5,\\
&H_3(x,y) = -7.67611..x^2 + 17.5227..xy - 2.04797..x - 10y^2 + 2.73401..y,
\end{aligned}
$$

respectively. The discontinuos piecewise differential system (29) has one limit cycle intersecting the cubic $c_5 = 0$ in four points satisfying system (23) and two limit cycles intersecting the cubic $c_5 = 0$ in two points satisfying system (6) with $i = 5$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1,\beta_1,\gamma_1,\delta_1,a_1,b_1,c_1,d_1) = &\ (0.0127029.., 0.0127834.., 0.280458.., -0.317359..,\\
&\ -0.14376.., 0.133026.., -0.571076.., -0.374011..),\\
(\alpha_2,\beta_2,\gamma_2,\delta_2) = &\ (-0.028627.., -0.028214.., -0.480257.., -0.346233..),\\
(\alpha_3,\beta_3,\gamma_3,\delta_3) = &\ (-0.085601.., -0.081855.., -0.386674.., -0.302824..),
\end{aligned}
$$

where $\alpha_1 = r_1^2 - 1$, $\beta_1 = r_1(r_1^2 - 1)$, $\gamma_1 = s_1^2 - 1$, $\delta_1 = s_1(s_1^2 - 1)$, $a_1 = f_1^2 - 1$, $b_1 = f_1(f_1^2 - 1)$, $c_1 = h_1^2 - 1$, $d_1 = h_1(h_1^2 - 1)$. Then the discontinuous piecewise differential system (29) has exactly three limit cycles, see Figure 12.

Examples with two limit cycles with four points on $c_2 = 0$ and one limit cycle with two points on $c_2 = 0$. We consider the regions defined in (11). For the first configuration of the class C_2 we consider the following Hamiltonian systems

$$
(30) \quad
\begin{aligned}
\dot{x} = &\ 1937.84..x - 7y/10 - 377775.., \dot{y} = 5.36462 \times 10^6 x - 1937.84..y \\
&-1.27506 \times 10^7, \text{ in } R_1, \\
\dot{x} = &\ -7x/10 - 7y + 1/10, \dot{y} = 7x/100 + 7y/10 - 3, \text{ in } R_2, \\
\dot{x} = &\ -6x + 3y - 11.0527.., \dot{y} = -12x + 6y - 275.389.., \text{ in } R_3,
\end{aligned}
$$

with the Hamiltonian first integrals

$$
\begin{aligned}
&H_1(x,y) = 2.68231 \times 10^6 x^2 - 1937.84..xy - 1.27506 \times 10^7 x + 7y^2/20 + 377775..y, \\
&H_2(x,y) = 7x^2/200 + 7xy/10 - 3x + 7y^2/2 - y/10, \\
&H_3(x,y) = -6x^2 + 6xy - 275.389..x - 3y^2/2 + 11.0527..y,
\end{aligned}
$$

respectively. The discontinuos piecewise differential system (30) has two limit cycles intersecting the cubic $c_2 = 0$ in four points satisfying system (25) and one limit cycle intersecting the cubic $c_2 = 0$ in two points satisfying system (6) with $i = 2$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1, a_1, b_1, c_1, d_1) = &\ (3.14459.., 0.987464.., 3.35582.., -1.67719.., \\
&\ 0.975858.., 0.218376.., 0.943351.., -0.331522..), \\
(\alpha_2, \beta_2, \gamma_2, \delta_2, a_2, b_2, c_2, d_2) = &\ (3.17528.., 1.1003.., 3.39578.., -1.79441.., \\
&\ 0.881547.., 0.470332.., 0.822573.., -0.563727..), \\
(\alpha_3, \beta_3, \gamma_3, \delta_3) = &\ (3.20514.., 1.20413.., 3.4333.., -1.9026..).
\end{aligned}
$$

Then the discontinuous piecewise differential system (30) has exactly three limit cycles, see Figure 13.

For the second possible configuration of the class C_2 we consider the following Hamiltonian systems

$$
(31) \quad
\begin{aligned}
\dot{x} = &\ 10y - 1.05447..x, \dot{y} = -0.11119..x + 1.05447..y - 0.426696.., \text{ in } R_1 \\
\dot{x} = &\ -29x/10 + 29y - 3/5, \dot{y} = -29x/100 + 29y/10 - 5/2, \text{ in } R_2 \\
\dot{x} = &\ -1.11707..x - y/10 + 0.370838.., \dot{y} = 12.4785..x + 1.11707..y \\
&-6.64471.., \text{ in } R_3,
\end{aligned}
$$

which have the Hamiltonian first integrals

$$
\begin{aligned}
&H_1(x,y) = -0.0555949..x^2 + 1.05447..xy - 0.426696..x - 5y^2, \\
&H_2(x,y) = -29x^2/200 + 29xy/10 - 5x/2 - 29y^2/2 + 3y/5, \\
&H_3(x,y) = 6.23927..x^2 + 1.11707..xy - 6.64471..x + y^2/20 - 0.370838..y,
\end{aligned}
$$

respectively. The discontinuos piecewise differential system (31) has two limit cycles intersecting the cubic $c_2 = 0$ in four points satisfying system (25) and one limit cycle intersecting the cubic $c_2 = 0$ in two points satisfying system (6) with $i = 2$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1, a_1, b_1, c_1, d_1) = &\ (3.03545.., 0.468004.., 3.00426.., 0.160123.., \\
&\ 0.72123.., 0.676877.., 0.892948.., -0.448796..), \\
(\alpha_2, \beta_2, \gamma_2, \delta_2, a_2, b_2, c_2, d_2) = &\ (3.06074.., 0.618961.., 3.00002.., 0.0115046.., \\
&\ 3.00002.., 0.0115046.., 0.854489.., -0.516495..), \\
(\alpha_3, \beta_3, \gamma_3, \delta_3) = &\ (0.508698.., 0.789074.., 0.810902.., -0.579376..).
\end{aligned}
$$

Then the discontinuous piecewise differential system (31) has exactly three limit cycles, see Figure 14.

Example with two limit cycles with four points on $c_5 = 0$ and one limit cycle with two points on $c_5 = 0$. Here we consider the regions defined in (18). We consider the Hamiltonian systems

(32)
$$
\begin{aligned}
\dot{x} = &\ -3x/10 - y/10 - 0.0118645.., \dot{y} = 9x/10 + 3y/10 - 0.049529.., \text{ in } R_1,\\
\dot{x} = &\ -203x/25 - 29y - 5, \dot{y} = 1421x/625 + 203y/25 + 5/2, \text{ in } R_2,\\
\dot{x} = &\ -13.0887x - 2y - 23.1224.., \dot{y} = 85.6566..x + 13.0887..y\\
&\ +1.62877.., \text{ in } R_3.
\end{aligned}
$$

The Hamiltonian systems in (24) have the Hamiltonian first integrals

$$
\begin{aligned}
&H_1(x,y) = 9x^2/20 + 3xy/10 - 0.049529..x + y^2/20 + 0.0118645..y,\\
&H_2(x,y) = 1421x^2/1250 + 203xy/25 + 5x/2 + 29y^2/2 + 5y,\\
&H_3(x,y) = 42.8283..x^2 + 13.0887..xy + 1.62877..x + y^2 + 23.1224..y.
\end{aligned}
$$

The discontinuos piecewise differential system (32) has two limit cycles intersecting the cubic $c_5 = 0$ in four points satisfying system (23) and one limit cycle intersecting the cubic $c_5 = 0$ in two points satisfying system (6) with $i = 5$ and H_3 instead of H_1, because all the real solutions of these two systems are

$$
\begin{aligned}
(\alpha_1, \beta_1, \gamma_1, \delta_1, a_1, b_1, c_1, d_1) = &\ (0.0457703.., 0.0468061.., 0.380929.., -0.44764..,\\
&\ -0.0477081.., -0.0465562.., -0.322071.., -0.265182..),\\
(\alpha_2, \beta_2, \gamma_2, \delta_2, a_2, b_2, c_2, d_2) = &\ (0.105172.., 0.110564.., 0.468911.., -0.568314..,\\
&\ -0.134399.., 0.125042.., -0.476399.., -0.344724..),\\
(\alpha_3, \beta_3, \gamma_3, \delta_3) = &\ (-0.388977.., 0.304056.., -0.647556.., -0.384435..),
\end{aligned}
$$

where $\alpha_i = r_i^2 - 1$, $\beta_i = r_i(r_i^2 - 1)$, $\gamma_i = s_i^2 - 1$, $\delta_i = s_i(s_i^2 - 1)$, $a_i = f_i^2 - 1$, $b_i = f_i(f_i^2 - 1)$, $c_i = h_i^2 - 1$, $d_i = h_i(h_i^2 - 1)$ for $i = 1, 2$. Then the discontinuous piecewise differential system (32) has exactly three limit cycles, see Figure 15.

This completes the proof of statement (b) of Theorem 3.

4. The appendix

Here we provide the values of C, D and E that appear in the proof of Theorem 2.

$$
\begin{aligned}
C = &\ r_1^4r_2^2r_3 - r_1^2r_2^4r_3 - r_1^4r_2r_3^2 + r_1r_2^4r_3^2 + r_1^2r_2r_3^4 - r_1r_2^2r_3^4 + r_1^3r_2^2r_3s_1 - r_1r_2^4r_3s_1\\
&-r_1^3r_2r_3^2s_1 + r_2^4r_3^2s_1 + r_1r_2r_3^4s_1 - r_2^2r_3^4s_1 + r_1^2r_2^2r_3s_1^2 - r_2^4r_3s_1^2 - r_1^2r_2r_3^2s_1^2\\
&+r_2r_3^4s_1^2 + r_1r_2^2r_3s_1^3 - r_1r_2r_3^2s_1^3 + r_2^2r_3s_1^4 - r_2r_3^2s_1^4 + r_1^4r_2r_3s_2 - r_1^2r_2^3r_3s_2\\
&-r_1^4r_3^2s_2 + r_1r_2^3r_3^2s_2 + r_1^2r_3^4s_2 - r_1r_2r_3^4s_2 + r_1^3r_2r_3s_1s_2 - r_1r_2^3r_3s_1s_2 - r_1^3r_3^2s_1s_2\\
&+r_2^3r_3^2s_1s_2 + r_1r_3^4s_1s_2 - r_2r_3^4s_1s_2 + r_1^2r_2r_3s_1^2s_2 - r_2^3r_3s_1^2s_2 - r_1^2r_3^2s_1^2s_2 + r_3^4s_1^2s_2\\
&+r_1r_2r_3s_1^3s_2 - r_1r_3^2s_1^3s_2 + r_2r_3s_1^4s_2 - r_3^2s_1^4s_2 + r_1^4r_3s_2^2 - r_1^2r_2^2r_3s_2^2 + r_1r_2^2r_3^2s_2^2\\
&-r_1r_3^4s_2^2 + r_1^3r_3s_1s_2^2 - r_1r_2^2r_3s_1s_2^2 + r_2^2r_3^2s_1s_2^2 - r_3^4s_1s_2^2 + r_1^2r_3s_1^2s_2^2 - r_2^2r_3s_1^2s_2^2\\
&+r_1r_3s_1^3s_2^2 + r_3s_1^4s_2^2 - r_1^2r_2r_3s_2^3 + r_1r_2r_3^2s_2^3 - r_1r_2r_3s_1s_2^3 + r_2r_3^2s_1s_2^3 - r_2r_3s_1^2s_2^3\\
&-r_1^2r_3s_2^4 + r_1r_3^2s_2^4 - r_1r_3s_1s_2^4 + r_3^2s_1s_2^4 - r_3s_1^2s_2^4 + r_1^4r_2^2s_3 - r_1^2r_2^4s_3 - r_1^4r_2r_3s_3\\
&+r_1r_2^4r_3s_3 + r_1^2r_2r_3^3s_3 - r_1r_2^2r_3^3s_3 + r_1^3r_2^2s_1s_3 - r_1r_2^4s_1s_3 - r_1^3r_2r_3s_1s_3 + r_2^4r_3s_1s_3\\
&+r_1r_2r_3^3s_1s_3 - r_2^2r_3^3s_1s_3 + r_1^2r_2^2s_1^2s_3 - r_2^4s_1^2s_3 - r_1^2r_2r_3s_1^2s_3 + r_2r_3^3s_1^2s_3 + r_1r_2^2s_1^3s_3\\
&-r_1r_2r_3s_1^3s_3 + r_2^2s_1^4s_3 - r_2r_3s_1^4s_3 + r_1^4r_2s_2s_3 - r_1^2r_2^3s_2s_3 - r_1^4r_3s_2s_3 + r_1r_2^3r_3s_2s_3\\
&+r_1^2r_3^3s_2s_3 - r_1r_2r_3^3s_2s_3 + r_1^3r_2s_1s_2s_3 - r_1r_2^3s_1s_2s_3 - r_1^3r_3s_1s_2s_3 + r_2^3r_3s_1s_2s_3\\
&+r_1r_3^3s_1s_2s_3 - r_2r_3^3s_1s_2s_3 + r_1^2r_2s_1^2s_2s_3 - r_2^3s_1^2s_2s_3 - r_1^2r_3s_1^2s_2s_3 + r_3^3s_1^2s_2s_3\\
&+r_1r_2s_1^3s_2s_3 - r_1r_3s_1^3s_2s_3 + r_2s_1^4s_2s_3 - r_3s_1^4s_2s_3 + r_1^4s_2^2s_3 - r_1^2r_2^2s_2^2s_3 + r_1r_2^2r_3s_2^2s_3
\end{aligned}
$$

$$
\begin{aligned}
&-r_1r_3^3s_2^2s_3+r_1^3s_1s_2^2s_3-r_1r_2^2s_1s_2^2s_3+r_2^2r_3s_1s_2^2s_3-r_3^3s_1s_2^2s_3+r_1^2s_1^2s_2^2s_3-r_2^2s_1^2s_2^2s_3\\
&+r_1s_1^3s_2^2s_3+s_1^4s_2^2s_3-r_1^2r_2s_2^3s_3+r_1r_2r_3s_2^3s_3-r_1r_2s_1s_2^3s_3+r_2r_3s_1s_2^3s_3-r_2s_1^2s_2^3s_3\\
&-r_1^2s_2^4s_3+r_1r_3s_2^4s_3-r_1s_1s_2^4s_3+r_3s_1s_2^4s_3-s_1^2s_2^4s_3-r_1^4r_2s_3^2+r_1r_2^4s_3^2+r_1^2r_2r_3^2s_3^2\\
&-r_1r_2^2r_3^2s_3^2-r_1^3r_2s_1s_3^2+r_2^4s_1s_3^2+r_1r_2r_3^2s_1s_3^2-r_2^2r_3^2s_1s_3^2-r_1^2r_2s_1^2s_3^2+r_2r_3^2s_1^2s_3^2\\
&-r_1r_2s_1^3s_3^2-r_2s_1^4s_3^2-r_1^4s_2s_3^2+r_1r_2^3s_2s_3^2+r_1^2r_3^2s_2s_3^2-r_1r_2r_3^2s_2s_3^2-r_1^3s_1s_2s_3^2\\
&+r_2^3s_1s_2s_3^2+r_1r_3^2s_1s_2s_3^2-r_2r_3^2s_1s_2s_3^2-r_1^2s_1^2s_2s_3^2+r_3^2s_1^2s_2s_3^2-r_1s_1^3s_2s_3^2-s_1^4s_2s_3^2\\
&+r_1r_2^2s_2^2s_3^2-r_1r_3^2s_2^2s_3^2+r_2^2s_1s_2^2s_3^2-r_3^2s_1s_2^2s_3^2+r_1r_2s_2^3s_3^2+r_2s_1s_2^3s_3^2+r_1s_2^4s_3^2\\
&+s_1s_2^4s_3^2+r_1^2r_2r_3s_3^3-r_1r_2^2r_3s_3^3+r_1r_2r_3s_1s_3^3-r_2^2r_3s_1s_3^3+r_2r_3s_1^2s_3^3+r_1^2r_3s_2s_3^3\\
&-r_1r_2r_3s_2s_3^3+r_1r_3s_1s_2s_3^3-r_2r_3s_1s_2s_3^3+r_3s_1^2s_2s_3^3-r_1r_3s_2^2s_3^3-r_3s_1s_2^2s_3^3+r_1^2r_2s_3^4\\
&-r_1r_2^2s_3^4+r_1r_2s_1s_3^4-r_2^2s_1s_3^4+r_2s_1^2s_3^4+r_1^2s_2s_3^4-r_1r_2s_2s_3^4+r_1s_1s_2s_3^4-r_2s_1s_2s_3^4\\
&+s_1^2s_2s_3^4-r_1s_2^2s_3^4-s_1s_2^2s_3^4,
\end{aligned}
$$

$$
\begin{aligned}
D= \quad & 4(r_1^4(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2\\
&+s_3)))+r_1^3s_1(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)\\
&+s_3(-s_2+s_3)))+r_1^2(-r_2^4(r_3+s_3)-r_2^3s_2(r_3+s_3)+r_2^2(s_1^2-s_2^2)(r_3+s_3)+r_2(r_3^4+r_3^3s_3\\
&+r_3^2(-s_1^2+s_3^2)+r_3(-s_2^3+s_1^2(s_2-s_3)+s_3^3)+s_3(-s_2^3+s_1^2(s_2-s_3)+s_3^3))+s_2(r_3^4+r_3^3s_3\\
&+r_3^2(-s_1^2+s_3^2)+r_3(-s_2^3+s_1^2(s_2-s_3)+s_3^3)+s_3(-s_2^3+s_1^2(s_2-s_3)+s_3^3)))+r_1(r_2^4(r_3^2\\
&+r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^3s_2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^4+r_3^3s_3\\
&+r_3^2(-s_2^2+s_3^2)+r_3(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3)+s_3(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3))+r_2(r_3^4(s_1\\
&-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1\\
&(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3)))+s_2(r_3^4(s_1-s_2)+r_3^3(s_1\\
&-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3\\
&(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))))+s_1(r_2^4(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))\\
&+r_2^3s_2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^4+r_3^3s_3+r_3^2(-s_2^2+s_3^2)+r_3(-s_1^3+s_1s_2^2\\
&-s_2^2s_3+s_3^3)+s_3(-s_1^3+s_1s_2^2-s_2^2s_3+s_3^3))+r_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3\\
&+s_1s_3^2-s_2s_3^2)+r_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2\\
&-s_3^2)+s_1(-s_2^3+s_3^3)))+s_2(r_3^4(s_1-s_2)+r_3^3(s_1-s_2)s_3+r_3^2(-s_1^3+s_2^3+s_1s_3^2-s_2s_3^2)+r_3(s_1^3\\
&(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3))+s_3(s_1^3(s_2-s_3)+s_2s_3(s_2^2-s_3^2)+s_1(-s_2^3+s_3^3)))))^2\\
&-4(r_1^2(-r_2^3(r_3+s_3)+r_2^2(s_1-s_2)(r_3+s_3)+r_2(r_3^3+r_3(s_1-s_2-s_3)(s_2-s_3)+(s_1-s_2-s_3)\\
&(s_2-s_3)s_3+r_3^2(-s_1+s_3))+s_2(r_3^3+r_3(s_1-s_2-s_3)(s_2-s_3)+(s_1-s_2-s_3)(s_2-s_3)s_3\\
&+r_3^2(-s_1+s_3)))+r_1^3(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)\\
&+s_3(-s_2+s_3)))+r_1(r_2(s_1-s_2)(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2\\
&-s_3)s_3)+(s_1-s_2)s_2(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)(s_2-s_3)s_3)\\
&+r_2^3(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^3+r_3^2(-s_2+s_3)-r_3(s_1-s_3)(s_1-s_2+s_3)\\
&-(s_1-s_3)s_3(s_1-s_2+s_3)))+s_1(r_2(s_1-s_2)(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)\\
&+(s_1-s_3)(s_2-s_3)s_3)+(s_1-s_2)s_2(r_3^3+r_3(s_1-s_3)(s_2-s_3)-r_3^2(s_1+s_2-s_3)+(s_1-s_3)\\
&(s_2-s_3)s_3)+r_2^3(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^3+r_3^2(-s_2+s_3)-r_3(s_1-s_3)(s_1\\
&-s_2+s_3)-(s_1-s_3)s_3(s_1-s_2+s_3))))(r_1^5(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)s_3)\\
&-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))+r_1^4s_1(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2-s_3)\\
&s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))+r_1^3s_1^2(r_2^2(r_3+s_3)+s_2(-r_3^2+r_3(s_2-s_3)+(s_2\\
&-s_3)s_3)-r_2(r_3^2+r_3(-s_2+s_3)+s_3(-s_2+s_3)))+r_1^2(-r_2^5(r_3+s_3)-r_2^4s_2(r_3+s_3)-r_2^3s_2^2(r_3\\
&+s_3)+r_2^2(s_1^3-s_2^3)(r_3+s_3)+r_2(r_3^5+r_3^4s_3+r_3^3s_3^2+r_3^2(-s_1^3+s_3^3)+r_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4)\\
&+s_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4))+s_2(r_3^5+r_3^4s_3+r_3^3s_3^2+r_3^2(-s_1^3+s_3^3)+r_3(-s_2^4+s_1^3(s_2-s_3)\\
&+s_3^4)+s_3(-s_2^4+s_1^3(s_2-s_3)+s_3^4)))+r_1(r_2^5(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^4s_2(r_3^2\\
&+r_3(-s_1+s_3)+s_3(-s_1+s_3))+r_2^3s_2^2(r_3^2+r_3(-s_1+s_3)+s_3(-s_1+s_3))-r_2^2(r_3^5+r_3^4s_3+r_3^3\\
&s_3^2+r_3^2(-s_2^3+s_3^3)+r_3(-s_1^4+s_1s_2^3-s_2^3s_3+s_3^4)+s_3(-s_1^4+s_1s_2^3-s_2^3s_3+s_3^4))+r_2(r_3^5(s_1-s_2)\\
&+r_3^4(s_1-s_2)s_3+r_3^3(s_1-s_2)s_3^2+r_3^2(-s_1^4+s_2^4+s_1s_3^3-s_2s_3^3)+r_3(s_1^4(s_2-s_3)+s_2s_3(s_2^3-s_3^3)\\
&+s_1(-s_2^4+s_3^4))+s_3(s_1^4(s_2-s_3)+s_2s_3(s_2^3-s_3^3)+s_1(-s_2^4+s_3^4)))+s_2(r_3^5(s_1-s_2)+r_3^4(s_1
\end{aligned}
$$

$$
\begin{aligned}
&-s_2)s_3 + r_3^3(s_1 - s_2)s_3^2 + r_3^2(-s_1^4 + s_2^4 + s_1 s_3^3 - s_2 s_3^3) + r_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4\\
&+s_3^4)) + s_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4)))) + s_1(r_2^5(r_3^2 + r_3(-s_1 + s_3) + s_3(-s_1\\
&+s_3)) + r_2^4 s_2(r_3^2 + r_3(-s_1 + s_3) + s_3(-s_1 + s_3)) + r_2^3 s_2^2(r_3^2 + r_3(-s_1 + s_3) + s_3(-s_1 + s_3))\\
&-r_2^2(r_3^5 + r_3^4 s_3 + r_3^3 s_3^2 + r_3^2(-s_2^3 + s_3^3) + r_3(-s_1^4 + s_1 s_2^3 - s_2^3 s_3 + s_3^4) + s_3(-s_1^4 + s_1 s_2^3 - s_2^3 s_3\\
&+s_3^4)) + r_2(r_3^5(s_1 - s_2) + r_3^4(s_1 - s_2)s_3 + r_3^3(s_1 - s_2)s_3^2 + r_3^2(-s_1^4 + s_2^4 + s_1 s_3^3 - s_2 s_3^3) + r_3(s_1^4(s_2\\
&-s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4)) + s_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4))) + s_2\\
&(r_3^5(s_1 - s_2) + r_3^4(s_1 - s_2)s_3 + r_3^3(s_1 - s_2)s_3^2 + r_3^2(-s_1^4 + s_2^4 + s_1 s_3^3 - s_2 s_3^3) + r_3(s_1^4(s_2 - s_3)\\
&+s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4)) + s_3(s_1^4(s_2 - s_3) + s_2 s_3(s_2^3 - s_3^3) + s_1(-s_2^4 + s_3^4))))),
\end{aligned}
$$

$$
\begin{aligned}
E = \; & r_1^2(-r_2^3(r_3 + s_3) + r_2^2(s_1 - s_2)(r_3 + s_3) + r_2(r_3^3 + r_3(s_1 - s_2 - s_3)(s_2 - s_3) + (s_1 - s_2 - s_3)\\
&(s_2 - s_3)s_3 + r_3^2(-s_1 + s_3)) + s_2(r_3^3 + r_3(s_1 - s_2 - s_3)(s_2 - s_3) + (s_1 - s_2 - s_3)(s_2 - s_3)s_3\\
&+r_3^2(-s_1 + s_3))) + r_1^3(r_2^2(r_3 + s_3) + s_2(-r_3^2 + r_3(s_2 - s_3) + (s_2 - s_3)s_3) - r_2(r_3^2 + r_3(-s_2\\
&+s_3) + s_3(-s_2 + s_3))) + r_1(r_2(s_1 - s_2)(r_3^3 + r_3(s_1 - s_3)(s_2 - s_3) - r_3^2(s_1 + s_2 - s_3)\\
&+(s_1 - s_3)(s_2 - s_3)s_3) + (s_1 - s_2)s_2(r_3^3 + r_3(s_1 - s_3)(s_2 - s_3) - r_3^2(s_1 + s_2 - s_3) + (s_1 - s_3)\\
&(s_2 - s_3)s_3) + r_2^3(r_3^2 + r_3(-s_1 + s_3) + s_3(-s_1 + s_3)) - r_2^2(r_3^3 + r_3^2(-s_2 + s_3) - r_3(s_1 - s_3)(s_1\\
&-s_2 + s_3) - (s_1 - s_3)s_3(s_1 - s_2 + s_3))) + s_1(r_2(s_1 - s_2)(r_3^3 + r_3(s_1 - s_3)(s_2 - s_3) - r_3^2(s_1\\
&+s_2 - s_3) + (s_1 - s_3)(s_2 - s_3)s_3) + (s_1 - s_2)s_2(r_3^3 + r_3(s_1 - s_3)(s_2 - s_3) - r_3^2(s_1 + s_2 - s_3)\\
&+(s_1 - s_3)(s_2 - s_3)s_3) + r_2^3(r_3^2 + r_3(-s_1 + s_3) + s_3(-s_1 + s_3)) - r_2^2(r_3^3 + r_3^2(-s_2 + s_3) - r_3\\
&(s_1 - s_3)(s_1 - s_2 + s_3) - (s_1 - s_3)s_3(s_1 - s_2 + s_3))).
\end{aligned}
$$

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