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ON A CLASS OF POLYNOMIAL DIFFERENTIAL SYSTEMS OF DEGREE 4: PHASE PORTRAITS AND LIMIT CYCLES

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ABSTRACT. In this paper we characterize the phase portraits in the Poincaré disc of the class of polynomial differential systems of the form

$$\dot{x} = -y, \qquad \dot{y} = x + ax^4 + bx^2y^2 + cy^4,$$

with $a^2 + b^2 + c^2 \neq 0$, which are symmetric with respect to the x-axis. Such systems have a center at the origin of coordinates. Moreover using the averaging theory of five order we study the number of limit cycles which can bifurcate from the period annulus of this center when it is perturbed inside the class of all polynomial differential systems of degree 4.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULTS

By definition a *polynomial differential system* in $\mathbb{R}^2$ is a differential system of the form

$$\frac{dx}{dt} = \dot{x} = P(x, y), \qquad \frac{dy}{dt} = \dot{y} = Q(x, y), \tag{1}$$

where the dependent variables are x and y, and the independent one (the time) is denoted by t, and $P(x, y)$ and $Q(x, y)$ are polynomials in the variables x and y with real coefficients. We denote by $m = \max\{\deg P, \deg Q\}$ the *degree* of the polynomial differential system.

A singular point p of a polynomial differential system (1) is a *center* if there is a neighborhood U of p such that $U \setminus \{p\}$ is filled by periodic orbits.

A *limit cycle* of a polynomial differential system (1) is a periodic orbit isolated in the set of all periodic orbits of system (1).

The *phase portrait* of a differential system is the decomposition of its domain of definition as union of all its oriented orbits. The phase portraits of the polynomial differential systems are drawn in the Poincaré disc which, roughly speaking, is the closed disc centered at the origin of coordinates with radius one, the interior of this disc is diffeomorphic to $\mathbb{R}^2$ and its boundary $\mathbb{S}^1$ corresponds to the infinity of $\mathbb{R}^2$, each point of $\mathbb{S}^1$ provides a direction for going or coming from infinity. For more details see Chapter 5 of [4], we shall use the conditions and notations introduced in that chapter.

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In the qualitative theory of polynomial differential systems in $\mathbb{R}^2$ two of the main problems are: first the characterization of their centers together with their phase portraits in the Poincaré disc, and second the analysis of the limit cycles that bifurcate from the periodic orbits surrounding those centers when they are perturb inside some class of polynomial differential systems.

The polynomial differential systems of the form

$$\dot{x} = -y, \qquad \dot{y} = x + Q_n(x, y) \tag{2}$$

where Q_n is a homogeneous polynomial of degree n started to be studied by Volokitin and Ivanov in [19] trying to characterize their centers. Later on this characterization was continued by Giné [5], and by Giné, Llibre and Valls in [6, 7].

Polynomial differential systems (2) with $n = 1$ are linear, and of course they can have centers, but linear centers perturbed inside the class of all linear differential systems cannot produce limit cycles, because any linear differential systems has no isolated periodic solutions.

Polynomial differential systems (2) with $n = 2$ is a particular class of quadratic polynomial differential systems, and the centers of the quadratic polynomial differential systems as their phase portraits in the Poincaré disc have been classified by Bautin [1], Kapteyn [9, 10], Schlomiuk [18], Vulpe [20], Żołądek [21], ...

Again polynomial differential systems (2) with $n = 3$ are a particular class of cubic polynomial differential systems without quadratic terms. Volokitin and Ivanov in [19] proved that the centers of this class have their vector fields symmetric with respect to one of the coordinate axes, i.e. are reversible centers. These reversible centers as its phase portraits in the Poincaré disc have been classified by Żołądek [22, 23], and Buzzi, Llibre and Medrado [3]. Previously where characterized the centers without their phase portraits of all the cubic polynomial differential systems without quadratic terms by Malkin [13] and Vulpe and Sibirskii [17]. When systems (2) with $n = 3$ are perturb inside the class of all cubic polynomial differential systems the limit cycles that bifurcate from the periodic orbits of those centers were studied in [8, 11].

Giné in [5] proved that the polynomial differential systems (2) with $n = 4$ have a center at the origin if and only if its vector field is symmetric about one of the coordinate axes. From these centers the ones that are symmetric with respect to the y-axis where classified recently by Benterki and Llibre in [2], together with their phase portraits. So for the polynomial differential systems (2) with $n = 4$ only remains to classify the centers and their phase portraits symmetric with respect to the x-axis. These is the objective of this paper.

More precisely, the first objective of this paper is to study the phase portraits in the Poincaré disc of the centers of systems (2) with $n = 4$ which are symmetric with respect to the x–axis, i.e. the phase portraits of the systems

$$\dot{x} = -y, \qquad \dot{y} = x + ax^4 + bx^2y^2 + cy^4. \tag{3}$$

If $c \neq 0$ then without loss of generality we can assume that $c = 1$ doing the change of variables $x = X/c^{\frac{1}{3}}$, $y = Y/c^{\frac{1}{3}}$. If $c = 0$ we can consider $b = 1$ doing $x = X/b^{\frac{1}{3}}$, $y = Y/b^{\frac{1}{3}}$, and if $c = b = 0$ then we can assume $a = 1$ doing $x = X/a^{\frac{1}{3}}$, $y = Y/a^{\frac{1}{3}}$. Therefore we only need to study the phase portraits of system 3 in the cases:

$$c=1,\begin{cases} b>0,\begin{cases} a>0;\\ a=0;\\ a<0;\end{cases}\\ \\ b=0,\begin{cases} a>0;\\ a=0;\\ a<0;\end{cases}\\ \\ b<0,\begin{cases} a>0;\\ a=0;\\ a<0.\end{cases}\end{cases}\qquad \text{And } c=0,\begin{cases} b=1,\begin{cases} a>0;\\ a=0;\\ a<0;\end{cases}\\ \\ b=0,\ a=1.\end{cases}$$

The second objective is to study the limit cycles that bifurcate from the periodic orbits of the centers of systems (3) when they are perturbed inside the class of all polynomial differential systems of degree 4.

Our first result is the following

Theorem 1. *The phase portraits in the Poincaré disc of systems* (3) *with* $a^2+b^2+c^2\neq 0$ *are topologically equivalent to one of the fourteen phase portraits given in Figure 1.*

We perturb the centers of system (3) as follows

$$\begin{aligned}\dot{x}&=-y+\sum_{i=1}^{5}\varepsilon^i P_i(x,y),\\ \dot{y}&=x+ax^4+bx^2y^2+cy^4+\sum_{i=1}^{5}\varepsilon^i Q_i(x,y),\end{aligned}\tag{4}$$

where $P_i(x,y)$ are polynomials of degree 4. For the definition of the averaging theory of order $k=1,\ldots,5$, see section 2.

In what follows we state our second main result.

Theorem 2. *For* $\varepsilon\neq 0$ *sufficiently small the number of limit cycles of the differential system* (4) *obtained using averaging theory up to order five is two.*

We prove Theorem 2 in section 5.

2. The singular points and their local phase portraits

The singular points finite or infinite are *hyperbolic* if its two eigenvalues have non-zero real part, are *semi-hyperbolic* if one of its eigenvalues is zero and the other is non-zero, are *linearly zero* if its linear part is identically zero. For instance the local phase portraits of hyperbolic and semi-hyperbolic singular points can be obtained from Theorems 2.15 and 2.19 of [4], respectively. The local phase portraits of linearly zero singular points can be studied doing the changes of variables called blow-ups, see Chapters 2 and 3 of [4]. The unique singular points with eigenvalues non-zero but which are not hyperbolic have eigenvalues purely imaginary, and they only can be foci or centers, see [4].

Proposition 3. *The polynomial differential systems* (3) *have*

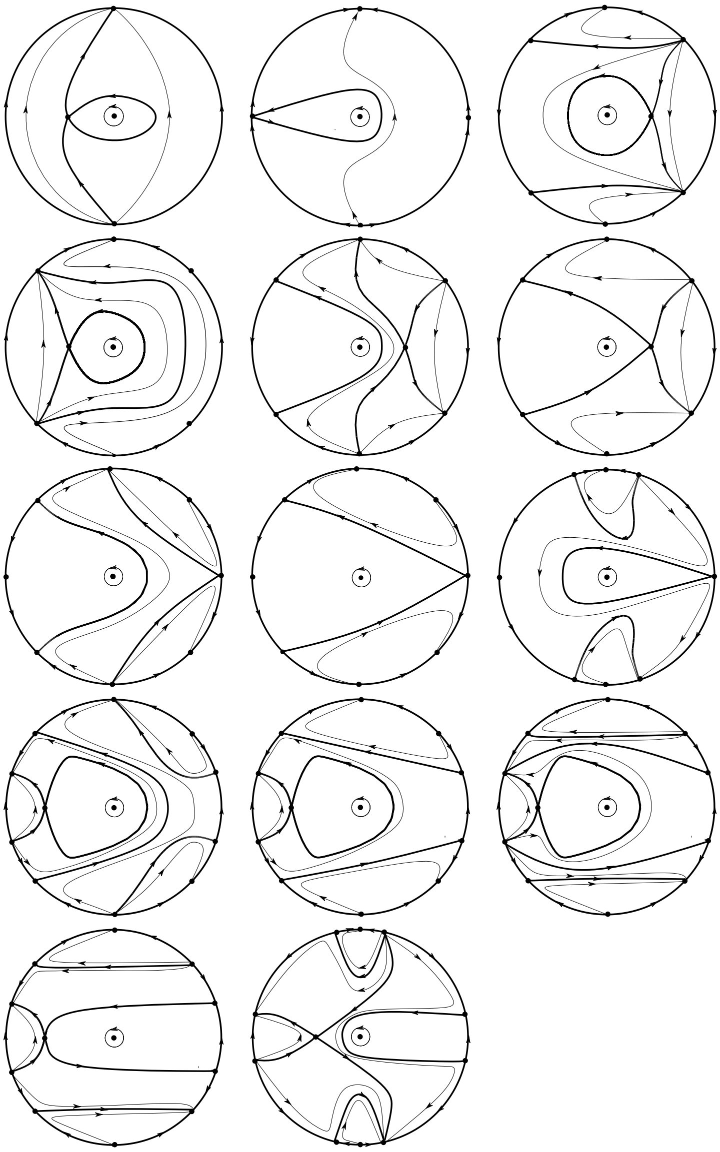

FIGURE 1. All topologically different phase portraits in the Poincaré disc of systems (3).

(a) *two finite singular points if* $a \neq 0$*, a center at* $(0,0)$*, and a saddle at* $(-a^{-1/3}, 0)$*;*
(b) *one finite singular point if* $a = 0$*, a center at* $(0,0)$*.*

Proof. Since the eigenvalues of the linear part at the origin are $\pm i$ and at the point $(-a^{-1/3}, 0)$ are $\pm\sqrt{3}$, and the system is reversible, i.e. it is invariant by

$(x, y, t) \longmapsto (x, -y, -t)$, it follows that the origin is a center and the other is a hyperbolic saddle. $\square$

For studying the infinite singular points in the Poincaré disc, we use the definitions and notations given in chapter 5 of [4], then we have the following result.

Proposition 4. *Let $\Delta = b^2 - 4a$. The polynomial differential systems* (3) *have in the local chart U_1:*

(a) *no infinite points in the following cases:*
  I: $\Delta < 0$;
  II: $\Delta = 0$ *and* $b > 0$;
  III: $\Delta > 0$ *and* $b - \sqrt{\Delta} > 0$;
  V: $c = 0$, $b = 1$ *and* $a > 0$;
  VI: $c = 0$, $b = 0$ *and* $a = 1$;

(b) *one singular point, the origin* $(0, 0)$ *which is a lineary zero and its local phase portrait is a topological saddle in the following cases:*
  I: $c = 1$, $b > 0$ *and* $a = 0$;
  II: $c = 1$, $b = 0$ *and* $a = 0$;
  III: $c = 0$, $b = 1$ *and* $a = 0$;

(c) *two singular points in the following cases:*
  I: *two linearly zero topological saddles at* $\left(\pm\sqrt{-b/2}, 0\right)$ *if* $\Delta = 0$, $c = 1$ *and* $b < 0$;
  II: *two semi-hyperbolic saddle-nodes at* $\left(\pm\sqrt{-(b - \sqrt{\Delta})/2}, 0\right)$ *if* $c = 1$, $\Delta > 0$, $b + \sqrt{\Delta} > 0$, *and* $b - \sqrt{\Delta} < 0$, *moreover these two saddle-nodes have the two hyperbolic sectors in* $v > 0$ *(resp.* $v < 0$*), and the parabolic one in* $v < 0$ *(resp.* $v > 0$*);*
  III: *two semi-hyperbolic saddle-nodes at* $\left(\pm\sqrt{-a}, 0\right)$ *if* $c = 0$, $b = 1$ *and* $a < 0$, *moreover these two saddle-nodes have the two hyperbolic sectors in* $v > 0$ *(resp.* $v < 0$*), and the parabolic one in* $v < 0$ *(resp.* $v > 0$*);*

(d) *three singular points, a linearly zero topological saddle at* $(0, 0)$ *and two semi-hyperbolic saddle-nodes at* $\left(\pm\sqrt{-b}, 0\right)$ *if* $c = 1$, $b < 0$ *and* $a = 0$, *moreover these two saddle-nodes have the two hyperbolic sectors in* $v > 0$ *(resp.* $v < 0$*), and the parabolic one in* $v < 0$ *(resp.* $v > 0$*);*

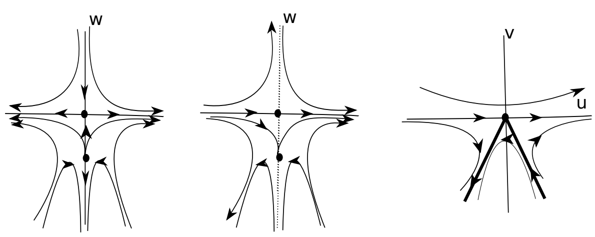

FIGURE 2. Blow-up at the origin of U_1 in all the cases of statement (b) of Proposition 4.

(e) *four semi-hyperbolic saddle-nodes at* $\left(\pm\sqrt{-(b+\sqrt{\Delta})/2}, 0\right)$ *and* $\left(\pm\sqrt{-(b-\sqrt{\Delta})/2}, 0\right)$ *if* $c = 1$, $\Delta > 0$, *and* $b + \sqrt{\Delta} < 0$, *moreover these four saddle-nodes have the two hyperbolic sectors in* $v > 0$ *(resp.* $v < 0$*), and the parabolic one in* $v < 0$ *(resp.* $v > 0$*).*

The polynomial differential systems (3) *have at the origin of the local chart* U_2

(f) *a hyperbolic stable node if* $c = 1$, *and a linearly zero singularity if* $c = 0$, *which is a topological stable node if* $b = 1$, *and if* $b = 0$ *we have a topological stable node if* $a = 1$.

A summary of the results of Proposition 4 is given in Table 1.

No infinite singular points in U_1 if	$c=1 \begin{cases} \Delta<0; \\ \Delta=0, \text{ and } b>0; \\ \Delta>0,\ b>0 \text{ and } a>0; \end{cases}$
	$c=0 \begin{cases} b=1 \text{ and } a>0; \\ b=0 \text{ and } a=1; \end{cases}$
One infinite singular points in U_1 if	$c=1 \begin{cases} b>0 \text{ and } a=0, \text{ which is saddle}; \\ b=0 \text{ and } a=0, \text{ which is saddle}; \end{cases}$
	$c=0$, $b=1$ and $a=0$, is saddle;
Two infinite singular points in U_1 if	$c=1 \begin{cases} \Delta=0 \text{ and } b<0, \text{ which are saddles}; \\ \Delta>0,\ b+\sqrt{\Delta}>0 \text{ and } b-\sqrt{\Delta}<0, \\ \quad \text{which are semi-hyperbolic saddles-nodes}; \end{cases}$
	$c=0$, $b=1$ and $a<0$, which are semi-hyperbolic saddles-nodes;
Three infinite singular points in U_1 if	$c=1$, $b<0$ and $a=0$, which are saddles;
Four infinite singular points in U_1 if	$c=1$, $\Delta>0$, and $b+\sqrt{\Delta}<0$, which are semi-hyperbolic saddles-nodes.

TABLE 1. The conditions for the existence of the infinite singular points in the local chart U_1 in function of the parameters a, b and c. Here $\Delta = b^2 - 4a$.

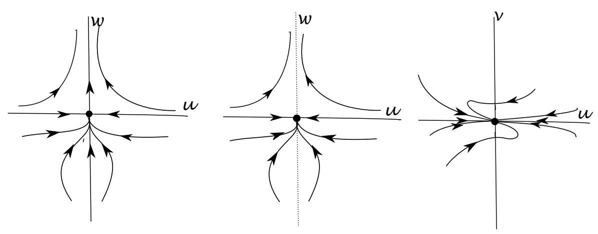

FIGURE 3. Blow-up at the origin of U_2 in the case $c = 0$ and $b = 1$.

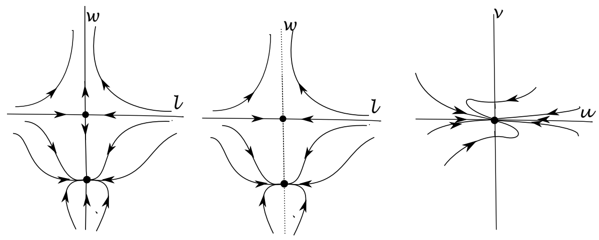

FIGURE 4. Blow-up at the origin of U_2 in the case $c = 0$, $b = 0$ and $a = 1$.

Using the Poincaré compactification on the local chart U_1 and V_1. The system (3) in the local chart U_1 becomes

$$\dot{u} = cu^4 + u^2v^3 + bu^2 + v^3 + a, \qquad \dot{v} = uv^4. \tag{5}$$

For each real root u^* of the polynomial $f(u) = \dot{u}\,|_{v=0} = cu^4 + bu^2 + a$, the point $(u^*, 0)$ is an infinite singular point in U_1. Since

$$f(u) = c(u + r_1)(u - r_1)(u + r_2)(u - r_2),$$

with

$$r_1 = \sqrt{\frac{-(b + \sqrt{\Delta})}{2}}, \qquad r_2 = \sqrt{\frac{-(b - \sqrt{\Delta})}{2}}, \quad \Delta = b^2 - 4a \geq 0, \text{ when } c = 1.$$

If $c = 0$ and $b = 1$ we have $u^* = \pm\sqrt{-a}$, it follows that in the infinity of the local chart U_1 there can be 0, 1, 2, 3 or 4 singularities. The eigenvalues λ_1 and λ_2 for these infinite singular points are:

- for $(u, v) = (\pm r_1, 0)$ we have $\lambda_1 = \mp 2\sqrt{\Delta}r_1$ and $\lambda_2 = 0$;
- for $(u, v) = (\pm r_2, 0)$ we have $\lambda_1 = \pm 2\sqrt{\Delta}r_2$ and $\lambda_2 = 0$;
- for $(u, v) = (\pm\sqrt{-a}, 0)$ we have $\lambda_1 = \pm 2\sqrt{-a}$ and $\lambda_2 = 0$.

Proof of statement (*a*) *of Proposition 4.* The proof follows directly from the fact that the polynomial $f(u)$ has no real roots under the five assumptions of statement (a). □

Proof of statement (*b*) *of Proposition 4.* If $c = 1$, $b > 0$ and $a = 0$ the origin of local chart U_1 is a linearly zero singularity. In the direction u we do the blow up $(u, v) = (l^3, l^2w)$, then system (5) becomes

$$\dot{l} = \frac{1}{3}l^4(l^6w^3 + cl^6 + lw^3 + b), \qquad \dot{w} = \frac{1}{3}l^3(l^6w^4 - 2cl^6w - 2w^4 - 2bw).$$

Rescaling the independent variable we eliminate the common factor $l^3/3$ and we get the new system

$$\dot{l} = l^7w^3 + cl^7 + lw^3 + bl, \qquad \dot{w} = l^6w^4 - 2cl^6w - 2w^4 - 2bw.$$

This new differential system on $l = 0$ has two singular points, the origin which is a saddle, and the point $(0, \sqrt[3]{-b})$ is semi-hyperbolic with eigenvalue 0 and b. In order to obtain the local phase portrait at this semi-hyperbolic infinite singular point we use Theorem 2.19 of [4], and we see that this point is a topological saddle. Going back through the change of variables, we get the local phase portrait at the origin of the chart U_1 shown in Figure 2, i.e. we have at the origin of U_1 a topological saddle.

The proof of the case $a = 0$, $b = 1$, and $c = 0$, is the same as the proof of the previous case.

If $a = 0$, $b = 0$ and $c = 1$ system (3) in the local chart U_1 becomes

$$\dot{u} = u^4 + u^2v^3 + v^3, \qquad \dot{v} = uv^4.$$

In the direction u we consider the blow up $(u, v) = (l^3, l^4w)$, and in the new variables the system writes

$$\dot{l} = \frac{1}{3}l^{10}\left(l^6w^3 + w^3 + 1\right), \qquad \dot{w} = -\frac{1}{3}l^9(l^6w^4 + 4w^4 + 4w).$$

Doing a rescaling of the independent variable we eliminate the common factor $l^9/3$ and we get the system

$$\dot{l} = l(l^6w^3 + w^3 + 1), \qquad \dot{w} = -(l^6w^4 + 4w^4 + 4w).$$

This differential system on $l = 0$ has two singular points, the origin with eigenvalues 1 and -4 which is a saddle, and a semi-hyperbolic point at $(0, -1)$ with eigenvalues 0 and 12. Applying Theorem 2.19 of [4] we conclude that this semi-hyperbolic point is a topological saddle. Going back through the changes of variables, we get the local phase portrait at this point, i.e. the origin of U_1 is a topological saddle. This completes the proof of statement (b). □

Proof of statement (c) *of Proposition 4.* If $\Delta = 0$, $c = 1$ and $b < 0$ the singular points $\left(\pm\sqrt{-b/2}, 0\right)$ of system (3) in the chart U_1 are linearly zero.

The singular point $\left(\sqrt{-b/2}, 0\right)$, by the change of variables $(u, v) \longmapsto (l, v)$ with $u = l + \sqrt{-b/2}$, goes to the origin of U_1, after that we do the blow-up $(l, v) \longmapsto (l, w)$ where w= v/l. Now from the obtained differential system we eliminate the common factor l doing a rescaling of the independent variable we get that the differential system (5) becomes

$$\begin{aligned} \dot{l} &= w^3l^4 + \sqrt{-\tfrac{2b}{c}}w^3l^3 + \tfrac{1}{2c}\left(2c - b\right)w^3l^2 + cl^3 + 2\sqrt{-2bc}l^2 - 2bl, \\ \dot{w} &= \sqrt{-\tfrac{b}{2c}}w^4l^2 + \tfrac{1}{2c}\left(b - 2c\right)w^4l - cwl^2 - 2\sqrt{-2bc}wl + 2bw. \end{aligned}$$

This system has a unique singular point on $l = 0$ the origin which is a saddle with eigenvalues $\pm 2b$. Going back through all the changes of variables, we get the local phase portrait at the singular points $\left(\sqrt{-b/2}, 0\right)$. The same study can be made at the singular point $\left(-\sqrt{-b/2}, 0\right)$. This proves the case **I** of the statement (c). The case **II** can be proved in a similar way.

If $c = 0$, $b = 1$ and $a < 0$ the singular points $(\pm\sqrt{-a}, 0)$ of system (3) in the chart U_1 with eigenvalues $\pm 2\sqrt{-a}$ and 0 are semi-hyperbolic. Form Theorem 2.19 of [4], they are saddle-nodes. This proves the case **III** of statement (c). Hence the proofs of statement (c) is done. □

Proof of statement (*d*) *of Proposition 4.* When $c = 1$, $b < 0$ and $a = 0$ system (3) has three singular points in the local chart U_1. The origin of this chart is a linearly zero topological saddle, the proof is similar to the proof of the case **I** of statement (b). The singular points $\left(\pm\sqrt{-b}, 0\right)$ are semi-hyperbolic sadlle-nodes using Theorem 2.19 of [4]. Hence the proof of statement (d) is done. □

Proof of statement (*e*) *of Proposition 4.* This proof is similar to the proof of the statement (c). □

Proof of statement (*f*) *of Proposition 4.* The unique possible singular points at infinity which are not covered by the local charts U_1 and V_1 are the origins of coordinates of the local charts U_2 and V_2. So we only need to study if the origin of the chart U_2 is a singular point or not, and if it is we must study its local phase portrait.

System (3) in the local chart U_2 becomes

$$\dot{u} = -au^5 - bu^3 - u^2v^3 - u - v^3, \qquad \dot{v} = -au^4v - bu^2v - uv^4 - v.$$

If $c = 1$ the eigenvalues at the origin are -1 of multiplicity 2, so the origin is a hyperbolic stable node.

If $c = 0$ and $b = 1$ in the direction u we do the blow up $(u, v) \longrightarrow (u, uw)$, and we obtain the system

$$\dot{u} = -u^5w^3 - au^5 - u^3w^3 - u^3, \qquad \dot{w} = u^2w^4.$$

Rescaling the independent variable of the differential system we eliminate the common factor u^2 between $\dot{u}$ and $\dot{w}$. The new differential system becomes

$$\dot{u} = -u^3w^3 - au^3 - uw^3 - u, \qquad \dot{w} = w^4.$$

Applying Theorem 2.19 of [4] to the origin of this system, we get the origin is a semi-hyperbolic saddle-node. Going back through the changes of variables, we obtain the local phase portrait at the origin of U_2, which is the one shown in Figure 3. If $c = 0$, $b = 0$ and $a = 1$ system (3) in the local chart U_2 becomes

$$\dot{u} = -u^5 - u^2v^3 - v^3, \qquad \dot{v} = -u^4v - uv^4.$$

In the direction u we do the blow-up $(u, v) \longmapsto (l^3, l^5w)$, and we obtain the system

$$\dot{l} = -\frac{1}{3}l^{13}\left(l^6w^3 + w^3 + 1\right), \qquad \dot{w} = \frac{1}{3}l^{12}(2l^6w^4 + 5w^4 + 2w).$$

Doing a rescaling of the independent variable we eliminate the common factor l^{12}, and the differential system becomes

$$\dot{l} = -\frac{1}{3}l\left(l^6w^3 + w^3 + 1\right), \qquad \dot{w} = \frac{1}{3}(2l^6w^4 + 5w^4 + 2w).$$

On $l = 0$ this differential system has two singular points, the origin with eigenvalues $-1/3$ and $2/3$, so it is a hyperbolic saddle, and the point $(0, \sqrt[3]{-2/5})$ with eigenvalues $-1/5$ and -2, which is a stable hyperbolic node. Going back through the change of variables we get that the local phase portrait at origin of U_2 is the one shown in Figure 4. So statement (f) is proved. □

3. Phase portraits in the Poincaré disc

We study the phase portraits of system (3) in the Pioncaré disc, first for the case $c = 0$, $b = 0$, $a = 1$, after for the case $c = 0$, $b = 1$; and finally for the case $c = 1$.

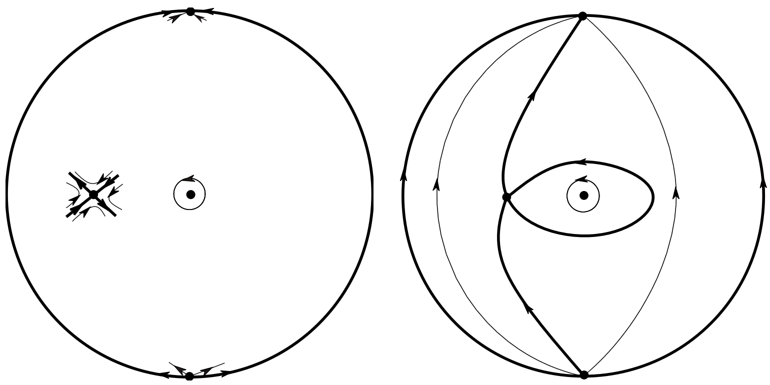

FIGURE 5. Phase portrait for $c = 0$, $b = 0$, and $a = 1$. The global phase portrait has nine separatrices ($S = 9$) and three canonical regions ($R = 3$), see the definitions of separatrix and of a canonical region in section 1.9 of [4].

3.1. **Phase portrait for $c = 0$, $b = 0$ and $a = 1$.** In this case the infinity has no singular points in the chart U_1, and the origins of the charts U_2 and V_2 are singular points. The system has two finite singular points a center and a saddle. From Propositions 3 and 4 the local phase portraits at the singularities of system (2) are given in the left picture of Figure 5. Then from these local phase portraits, there is a unique possible global phase portrait in the Poincaré disc given in the right picture of Figure 5.

3.2. **Phase portrait for $c = 0$, $b = 1$.** Then we draw its diagram of bifurcation on the straight lien in Figure 6, and its singular points in Table 2.

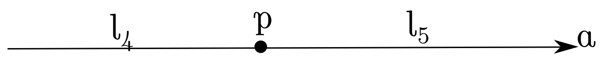

FIGURE 6. Bifurcation diagram in the case c=0 and b=1.

Regions	infinite singular points		finite singular points
	U_1	U_2	
$l_4 :\ a < 0$	two	origin	center + saddle
$p :\ a = 0$	one	origin	center
$l_5 :\ a > 0$	$\emptyset$	origin	center + saddle

TABLE 2. The singular points of system (2) in the cases $c = 0$ and $b = 1$.

3.2.1. *Phase portrait in* l_5 *(*$c = 0$*,* $b = 1$*,* $a > 0$*), see Figure 6.* In this case the local phase portraits of the singularities of system (2) coincide with case of $c = 0$, $b = 0$, $a = 1$. So we get the global phase portrait of Figure 5.

3.2.2. *Phase portrait in* p *(*$c = 0$*,* $b = 1$*,* $a = 0$*),see Figure 6.* From Propositions 3 and 4 the local phase portraits at the singularities of system (2) are given in the left picture of Figure 7. Then from these local phase portraits there is a unique possible global phase portrait in the Poincaré disc given in the right picture of Figure 7.

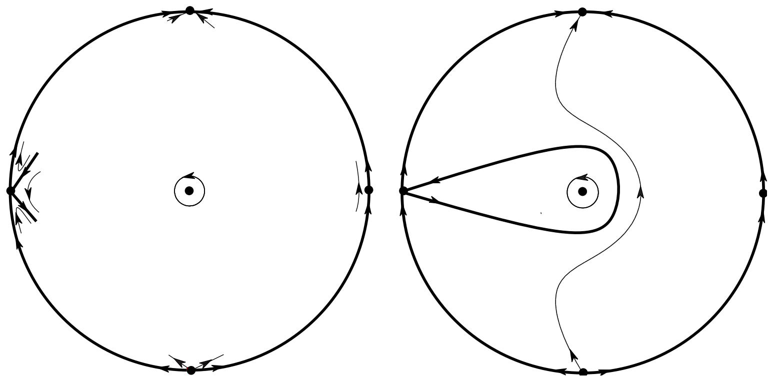

FIGURE 7. Phase portrait in p ($c = 0$, $b = 1$, $a = 0$). The global phase portrait has ten separatrices ($S = 10$) and two canonical regions ($R = 2$).

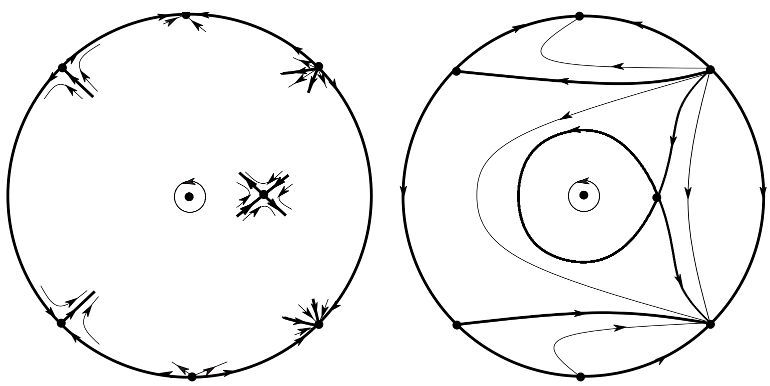

FIGURE 8. Phase portrait in l_4 ($c = 0$, $b = 1$, $a < 0$). The global phase portrait has nineteen separatrices ($S = 19$) and five canonical regions ($R = 5$).

3.2.3. *Phase portrait in* l_4 *(*$c = 0$*,* $b = 1$*,* $a < 0$*), see Figure 6.* From Propositions 3 and 4 the local phase portraits at the singularities of system (2) are given in the left picture of Figure 8. Then from these local phase portraits there is a unique possible global phase portrait in the Poincaré disc given in the right picture of Figure 8.

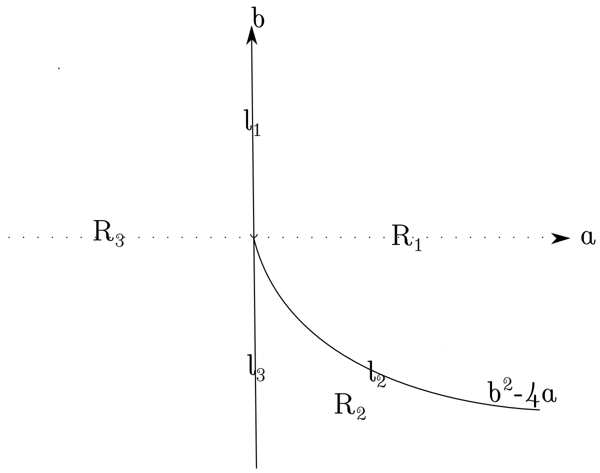

FIGURE 9. Bifurcation diagram in the case c=1.

3.3. **Phase portrait for** $c = 1$. According with Table 1, we draw in Figure 9 the diagram of bifurcation on the plane (a, b) the case $c = 1$.

We illustrate the singular points corresponding to the bifurcation diagram of Figure 9 in Table 3 :

Regions	infinite singular points		finite singular points
	U_1	U_2	
$R_1:\ (b > -2\sqrt{a}) \wedge (a > 0)$	$\emptyset$	origin	center + saddle
$l_2:\ (b = -2\sqrt{a}) \wedge (a > 0)$	two	origin	center + saddle
$R_2:\ (b < -2\sqrt{a}) \wedge (a > 0)$	four	origin	center + saddle
$l_3:\ (b < 0) \wedge (a = 0)$	three	origin	center
$R_3:\ a < 0$	two	origin	center + saddle
$l_1:\ (b \geq 0) \wedge (a = 0)$	one	origin	center

TABLE 3. The singular points of system (2) in the case $c = 1$.

3.3.1. *Phase portrait in R_1 ($b > -2\sqrt{a}$, $a > 0$), see Figure 9.* In this case the local phase portraits of the singularities of system (2) coincide with case of $c = 0$, $b = 0$, $a = 1$. So we get the global phase portrait of Figure 5.

3.3.2. *Phase portrait in l_1 ($b \geqslant 0$, $a = 0$), see Figure 9.* In this case the local phase portraits of the singularities of system (2) coincide with case of $c = 0$, $b = 1$, $a = 0$. So we get the global phase portrait of Figure 7.

3.3.3. *Phase portrait in l_2 ($b = -2\sqrt{a}$, $a > 0$), see Figure 9.* From Propositions 3 and 4 the local phase portraits at the singularities of system (2) are given in the left picture of Figure 10. Then from these local phase portraits and using the fact that the phase portrait is symmetric with respect to the x-axis there is a unique possible global phase portrait in the Poincaré disc given in the right picture of Figure 10.

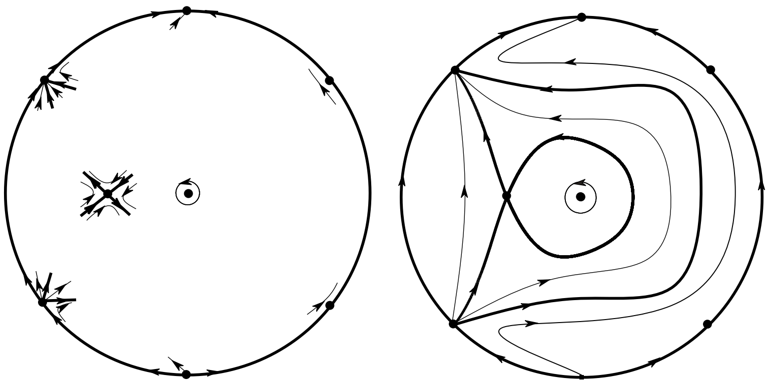

FIGURE 10. Phase portrait in l_2 ($b = -2\sqrt{a}$, $a > 0$). The global phase portrait has eighteen separatrices ($S = 18$) and four canonical regions ($R = 4$).

3.3.4. *Phase portrait in* R_3 *(*$a < 0$*), see Figure 9.* From Propositions 3 and 4 the local phase portraits at the singularities of system (2) are given in Figure 11. Then from these local phase portraits and using the fact that the phase portrait is symmetric with respect to the x-axis there are three possible global phase portraits in the Poincaré disc given in Figure 12.

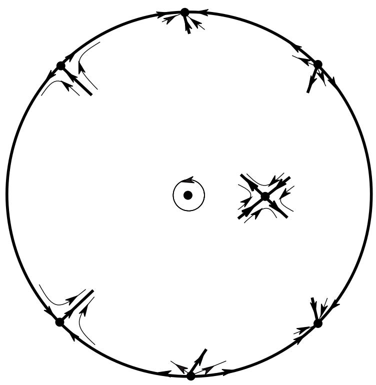

FIGURE 11. Local phase portrait in R_3 ($a < 0$).

3.3.5. *Phase portrait in* l_3 *(*$c = 1$, $b < 0$, $a = 0$*), see Figure 9.* From Propositions 3 and 4 the local phase portraits at the singularities of system (2) are given in Figure 13. Then from these local phase portraits and using the fact that the phase portrait is symmetric with respect to the x-axis there are three possible global phase portraits in the Poincaré disc given in Figure 14.

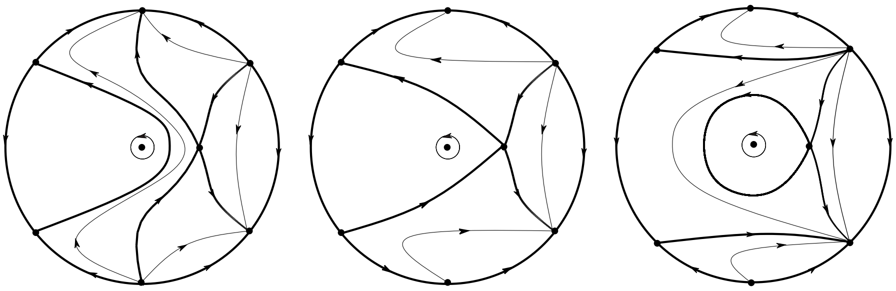

FIGURE 12. The global phase portraits in R_3 ($c = 1$, $a < 0$). Going from the left to the right these phase portraits have 19, 18, 19 separatrices and 5, 4, 5 canonical regions, respectively. The phase portrait on the left can be obtained by taking $a = -1$ and $b = 2$, the one on the right can be obtained with $a = -1$ and $b = -5$. The phase portrait in the middle exists by continuity on the parameter b.

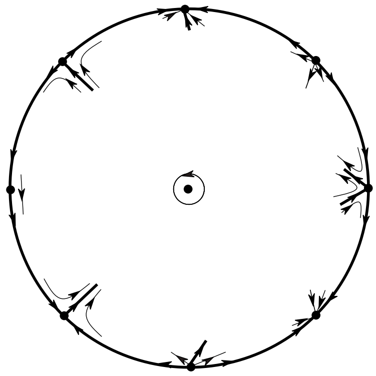

FIGURE 13. The local Phase portrait in l_3 ($c = 1$, $b < 0$, $a = 0$).

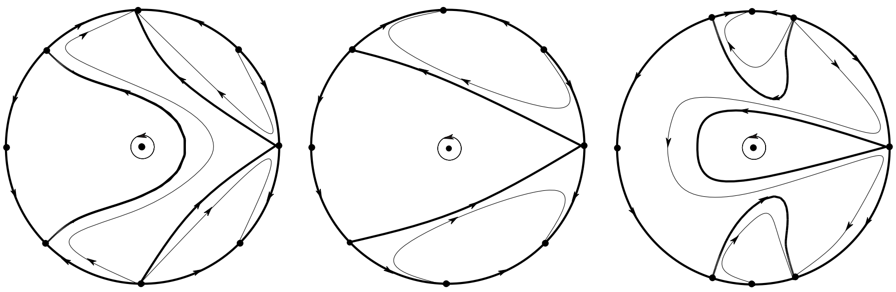

FIGURE 14. The global phase portraits in l_3 ($c = 1$, $b < 0$, $a = 0$). Going from the left to the right these phase portraits have 20, 19, 20 separatrices, and 4, 3, 4 canonical regions, respectively. The phase portrait on the left can be obtained by taking $b = -1$ and the one on the right can be obtained with $b = -5$. The phase portrait in the middle exists by continuity on the parameter b.

3.3.6. *Phase portrait in R_2 ($c = 1$, $b < -2\sqrt{a}$, $a > 0$), see Figure 9.* From Propositions 3 and 4 the local phase portraits at the singularities of system (2) are given in Figure 15. Then from these local phase portraits and using the fact that the phase portrait is symmetric with respect to the x-axis there are five possible global phase portraits in the Poincaré disc given in Figure 16.

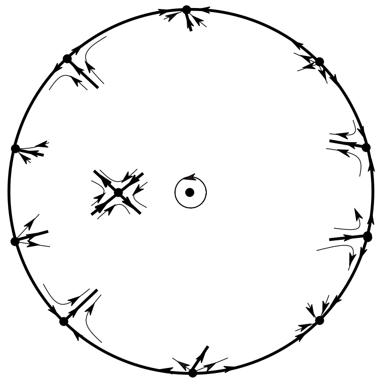

FIGURE 15. Local phase portrait in R_2 ($c = 1$, $b < -2\sqrt{a}$, $a > 0$).

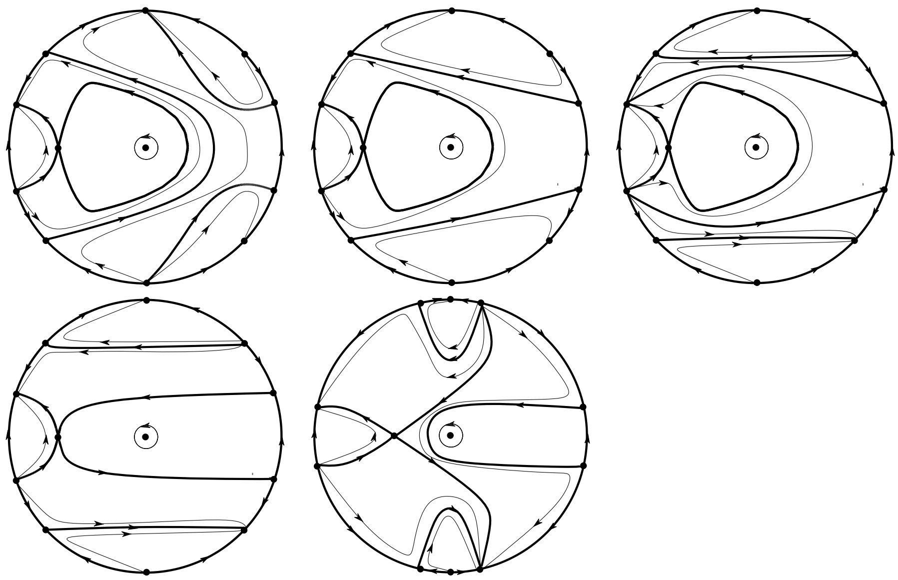

FIGURE 16. The global Phase portraits in R_2 ($c = 1$, $b < -2\sqrt{a}$, $a > 0$). Going from the upper left to the down right these phase portraits have 28, 27, 29, 27, 29 separatrcies, and 6, 5, 7, 6, 7 canonical regions, respectively. The phase portrait on the upper right can be obtained by taking $c = 1$, $b = -4.1$, and $a = \frac{1}{10}$, and the down left phase portrait can be obtained for $c = 1$, $b = -4.2$, $a = \frac{1}{10}$. The second, third and fourth global portraits exist by continuity on the parameter b.

4. The averaging theory up to order 5

In this section we present the basic results from the averaging theory that we shall need for proving the main results of this paper.

We consider the differential system of the form

$$\dot{z}(t) = \sum_{i=0}^{k} \varepsilon^i F_i(t,z) + \varepsilon^{k+1} R(t,z,\varepsilon), \tag{6}$$

where $F_i : \mathbb{R} \times D \longrightarrow \mathbb{R}$, $R : \mathbb{R} \times D \times (-\varepsilon_f, \varepsilon_f) \longrightarrow \mathbb{R}$ are continuous functions, $T-$periodic in the first variable, and D is an open subset of $\mathbb{R}$, and ε a small parameter. From [12] we define the following functions $y_i(t,z)$ for $k = 1,2,3,4,5$ related to system (6) :

$$y_1(s,z) = \int_0^s F_1(t,z)dt, \quad y_2(s,z) = \int_0^s [2F_2(t,z) + 2\partial F_1(t,z)y_1(t,z)]dt,$$

$$y_3(s,z) = \int_0^s [6F_3(t,z) + 6\partial F_2(t,z)y_1(t,z) + 3\partial^2 F_1(t,z)y_1(t,z)^2 + 3\partial F_1(t,z)y_2(t,z)]dt,$$

$$\begin{aligned} y_4(s,z) &= \int_0^s [24F_4(t,z) + 24\partial F_3(t,z)y_1(t,z) + \\ &\quad 12\partial^2 F_2(t,z)y_1(t,z)^2 + 12\partial F_2(t,z)y_2(t,z) + 12\partial^2 F_1(t,z)y_1(t,z)y_2(t,z) + \\ &\quad 4\partial^3 F_1(t,z)y_1(t,z)^3 + 4\partial F_1(t,z)y_3(t,z)]dt, \end{aligned}$$

$$\begin{aligned} y_5(s,z) &= \int_0^s [120F_5(t,z) + 120\partial F_4(t,z)y_1(t,z) + \\ &\quad 60\partial^2 F_3(t,z)y_1(t,z)^2 + 60\partial F_3(t,z)y_2(t,z) + 60\partial^2 F_2(t,z)y_1(t,z)y_2(t,z) + \\ &\quad 20\partial^3 F_2(t,z)y_1(t,z)^3 + 20\partial F_2(t,z)y_3(t,z) + 20\partial^2 F(s,z)y_1(t,z)y_3(t,z) + \\ &\quad 15\partial^2 F_1(t,z)y_2(t,z)^2 + 30\partial^3 F_1(t,z)y_1(t,z)^2 y_2(t,z)) + \\ &\quad 5\partial^4 F_1(t,z)y_1(t,z)^4 + 5\partial F_1(t,z)y_4(t,z)]dt. \end{aligned}$$

Here $\partial^k F_l(s,z)$ means the $k-th$ partial derivative of the function $F_l(s,z)$ with respect to the variable z. Also from [12] we have the functions

$$\begin{aligned} f_1(z) &= \int_0^T F_1(t,z)dt, \\ f_2(z) &= \int_0^T [F_2(t,z) + \partial F_1(t,z)y_1(t,z)]dt, \\ f_3(z) &= \int_0^T [F_3(t,z) + \partial F_2(t,z)y_1(t,z) + \frac{1}{2}\partial^2 F_1(t,z)y_1(t,z)^2 + \frac{1}{2}\partial F_1(t,z)y_2(t,z)]dt, \end{aligned}$$

$$\begin{aligned} f_4(z) &= \int_0^T [F_4(t,z) + \partial F_3(t,z)y_1(t,z) + \\ &\frac{1}{2}\partial^2 F_2(t,z)y_1(t,z)^2 + \frac{1}{2}\partial F_2(t,z)y_2(t,z) + \frac{1}{2}\partial^2 F_1(t,z)y_1(t,z)y_2(t,z) + \\ &\frac{1}{6}\partial^3 F(t,z)y_1(t,z)^3 + \frac{1}{6}\partial F_1(t,z)y_3(t,z)]dt, \end{aligned}$$

$$\begin{aligned} f_5(z) &= \int_0^T [F_5(t,z) + F_4(t,z)y_1(t,z) + \\ &\frac{1}{2}\partial^2 F_3(t,z)y_1(t,z)^2 + \frac{1}{2}\partial F_3(t,z)y_2(t,z) + \frac{1}{2}\partial^2 F_2(t,z)y_1(t,z)y_2(t,z) + \\ &\frac{1}{6}\partial^3 F_2(t,z)y_1(t,z)^3 + \frac{1}{6}\partial F_2(t,z)y_3(t,z) + \frac{1}{6}\partial^2 F_1(s,z)y_1(t,z)y_3(t,z) + \\ &\frac{1}{8}\partial^2 F_1(t,z)y_2(t,z)^2 + \frac{1}{4}\partial^3 F_1(t,z)y_1(t,z)^2 y_2(t,z) + \\ &\frac{1}{24}\partial^4 F_1(t,z)y_1(t,z)^4 + \frac{1}{24}\partial F_1(t,z)y_4(t,z)]dt. \end{aligned}$$

For more details see [12].

5. Proof of theorem 2

Consider system (3) we shall study which periodic solutions of its center become limit cycles when we perturb the center inside the whole class of polynomial differential systems of degree 4 as in system (3). The polynomial P_i and Q_i given in system (3) are

$$\begin{aligned} P_i(x,y) &= \sum_0^4 a_{kj}^{(i)} x^k y^j, \\ Q_i(x,y) &= \sum_0^4 b_{kj}^{(i)} x^k y^j. \end{aligned}$$

This study will be done by applying the averaging theory described in section 4, we introduce a small parameter ε doing the scaling $x = \varepsilon X$, $y = \varepsilon Y$. Thus we obtain a differential system $(\dot{X}, \dot{Y})$. After that we pass the differential system to polar coordinates $(\dot{r}, \dot{\theta})$. Then we take as independent variable the angle θ, and the system $(\dot{r}, \dot{\theta})$ becomes the differential equation $dr/d\theta$. By doing a Taylor expansion truncated at 5-th order in ε we obtain an expression for $dr/d\theta$ similar to the one of the differential system (6). In short we have written our differential system (4) in the normal form (6) for applying the averaging theory. We give only the expression of functions $F_1(r,\theta)$. The explicit expressions of $F_i(r,\theta)$ for $i = 2, ..., 5$ are quite large so we omit them.

The experssion of $F_1(r,\theta)$ is $(A+BC)/D$ where

$$
\begin{aligned}
A &= \sin\theta\cos\theta\big(a_{01}^{(1)}+r(a_{02}^{(1)}+b_{11}^{(1)})\sin\theta+r^2(a_{03}^{(1)}+b_{12}^{(1)})\sin^2\theta+\\
&\quad r^3(a_{04}^{(1)}+b_{13}^{(1)})\sin^3\theta+b_{10}^{(1)}\big)+\\
&\quad \cos^2\theta\big(a_{10}^{(1)}+r(a_{11}^{(1)}+b_{20}^{(1)})\sin\theta+r^2(a_{12}^{(1)}+b_{21}^{(1)})\sin^2\theta+\\
&\quad r^3(a_{13}^{(1)}+b_{22}^{(1)})\sin^3\theta\big)+\\
&\quad r\cos^3\theta\big(a_{20}^{(1)}+r(a_{21}^{(1)}+b_{30}^{(1)})\sin\theta+r^2(a_{22}^{(1)}+b_{31}^{(1)})\sin^2\theta\big)+\\
&\quad r^2\cos^4\theta(a_{30}^{(1)}+r(a_{31}^{(1)}+b_{40}^{(1)})\sin\theta)+a_{40}^{(1)}r^3\cos^5\theta+\\
&\quad \sin^2\theta\big(b_{01}^{(1)}+b_{02}^{(1)}r\sin\theta+b_{03}^{(1)}r^2\sin^2\theta+b_{04}^{(1)}r^3\sin^3\theta\big),\\
B &= -\sin^2\theta\big(a_{01}^{(1)}+a_{02}^{(1)}r\sin(\theta)+a_{03}^{(1)}r^2\sin^2\theta+a_{04}^{(1)}r^3\sin^3\theta\big)+\\
&\quad \sin\theta\cos\theta\big(-a_{10}^{(1)}-r(a_{11}^{(1)}-b_{02}^{(1)})\sin\theta-r^2(a_{12}^{(1)}-b_{03}^{(1)})\sin^2\theta+\\
&\quad r^3(-(a_{13}^{(1)}-b_{04}^{(1)}))\sin^3\theta+b_{01}^{(1)}\big)+\\
&\quad \cos^2\theta\big(-r(a_{20}^{(1)}-b_{11}^{(1)})\sin\theta-r^2(a_{21}^{(1)}-b_{12}^{(1)})\sin^2\theta+\\
&\quad r^3(-(a_{22}^{(1)}-b_{13}^{(1)}))\sin^3\theta+b_{10}^{(1)}\big)+\\
&\quad r\cos^3\theta\big(-r(a_{30}^{(1)}-b_{21}^{(1)})\sin\theta+r^2(-(a_{31}^{(1)}-b_{22}^{(1)}))\sin^2\theta+b_{20}^{(1)}\big)-\\
&\quad r^2\cos^4\theta(r(a_{40}^{(1)}-b_{31}^{(1)})\sin\theta-b_{30}^{(1)})+b_{40}^{(1)}r^3\cos^5\theta,\\
C &= r^4\sin(\theta)\big(a\cos^4(\theta)+b\sin^2(\theta)\cos^2(\theta)+c\sin^4(\theta)\big);\\
D &= \big(ar^3\cos^5(\theta)+br^3\sin^2(\theta)\cos^3(\theta)+cr^3\sin^4(\theta)\cos(\theta)+\sin^2(\theta)+\cos^2(\theta)\big)^2.
\end{aligned}
$$

The differential equation (4) is the normal form (6) for applying the averaging theory up to five order in ε, where in (6) we have now $z=r$, $t=\theta$ and $F_k(\theta,r)$ is the coefficient of ε^k in $dr/d\theta$ for $k=1,2,3,4,5$ we do not write their huge expressions, easy to compute and manipulate with an algebraic manipulator as mathematica or maple.

We compute the function $f_1(r)$ defined in Section 4, and we get

$$f_1(r)=\frac{1}{2}(a_{10}^{(1)}+b_{01}^{(1)})r.$$

Clearly the polynomial $f_1(r)$ cannot have positive roots. For applying the averaging theory of second order we need that $f_1(r)\equiv 0$. So we take $a_{10}^{(1)}=-b_{01}^{(1)}$. Computing the function $f_{20}(r)$ defined in Section 4, we obtain

$$f_2(r)=\frac{1}{2}(a_{10}^{(2)}+b_{01}^{(2)})r.$$

Clearly the polynomial $f_2(r)$ cannot have positive roots. For applying the averaging theory of third order we need that $f_2(r)\equiv 0$. So we take $a_{10}^{(2)}=-b_{01}^{(2)}$. Computing the function $f_3(r)$ defined in Section 4, we obtain

$$f_3(r)=\frac{1}{2}(a_{10}^{(3)}+a_{00}^{(2)}a_{11}^{(1)}-2a_{20}^{(1)}b_{00}^{(2)}+b_{01}^{(3)}+2a_{00}^{(2)}b_{02}^{(1)}-b_{00}^{(2)}b_{11}^{(1)})r+\frac{1}{8}(a_{12}^{(1)}+3a_{30}^{(1)}+3b_{03}^{(1)}+b_{21}^{(1)})r^3.$$

So the polynomial $f_3(r)$ can have at most one positive real root. In order to apply the averaging theory of fourth order we need that $f_3(r) \equiv 0$. So we take

$$\begin{aligned} b_{01}^{(3)} &= -a_{10}^{(3)} - a_{00}^{(2)} a_{11}^{(1)} + 2a_{20}^{(1)} b_{00}^{(2)} - 2a_{00}^{(2)} b_{02}^{(1)} + b_{00}^{(2)} b_{11}^{(1)}, \\ b_{03}^{(1)} &= \frac{1}{3}(-a_{12}^{(1)} - 3a_{30}^{(1)} - b_{21}^{(1)}). \end{aligned}$$

Computing the function $f_4(r)$ defined in Section 4, we obtain

$$\begin{aligned} f_4(r) = {} & \frac{1}{2}(a_{10}^{(4)} + a_{00}^{(3)} a_{11}^{(1)} + a_{00}^{(2)} a_{01}^{(1)} a_{11}^{(1)} + a_{00}^{(2)} a_{11}^{(2)} - 2a_{20}^{(2)} b_{00}^{(2)} - 2a_{20}^{(1)} b_{00}^{(3)} - \\ & 2a_{00}^{(2)} a_{20}^{(1)} b_{01}^{(1)} + a_{11}^{(1)} b_{00}^{(2)} b_{01}^{(1)} + b_{01}^{(4)} + 2a_{00}^{(3)} b_{02}^{(1)} + 2a_{00}^{(2)} a_{01}^{(1)} b_{02}^{(1)} + 2b_{00}^{(2)} b_{01}^{(1)} b_{02}^{(1)} + \\ & 2a_{00}^{(2)} b_{02}^{(2)} + 2a_{20}^{(1)} b_{00}^{(2)} b_{10}^{(1)} - b_{00}^{(3)} b_{11}^{(1)} - a_{00}^{(2)} b_{01}^{(1)} b_{11}^{(1)} + b_{00}^{(2)} b_{10}^{(1)} b_{11}^{(1)} - b_{00}^{(2)} b_{11}^{(2)})r + \\ & \frac{1}{8}(a_{02}^{(1)} a_{11}^{(1)} + a_{12}^{(2)} + a_{11}^{(1)} a_{20}^{(1)} - 3a_{01}^{(1)} a_{30}^{(1)} + 3a_{30}^{(2)} + 2a_{00}^{(2)} b - 2a_{21}^{(1)} b_{01}^{(1)} + \\ & 2a_{02}^{(1)} b_{02}^{(1)} + 3b_{03}^{(2)} - 3a_{30}^{(1)} b_{10}^{(1)} - b_{02}^{(1)} b_{11}^{(1)} - 2b_{01}^{(1)} b_{12}^{(1)} - 2a_{20}^{(1)} b_{20}^{(1)} - b_{11}^{(1)} b_{20}^{(1)} - \\ & a_{01}^{(1)} b_{21}^{(1)} - b_{10}^{(1)} b_{21}^{(1)} + b_{21}^{(2)} + 12a_{00}^{(2)} c)r^3. \end{aligned}$$

So the polynomial $f_4(r)$ can have at most one positive real root. In order to apply the averaging theory of fifth order we need that $f_4(r) \equiv 0$. So we take

$$\begin{aligned} b_{01}^{(4)} = {} & -a_{10}^{(4)} - a_{00}^{(3)} a_{11}^{(1)} - a_{00}^{(2)} a_{01}^{(1)} a_{11}^{(1)} - a_{00}^{(2)} a_{11}^{(2)} + 2a_{20}^{(2)} b_{00}^{(2)} + \\ & 2a_{20}^{(1)} b_{00}^{(3)} + 2a_{00}^{(2)} a_{20}^{(1)} b_{01}^{(1)} - a_{11}^{(1)} b_{00}^{(2)} b_{01}^{(1)} - 2a_{00}^{(3)} b_{02}^{(1)} - \\ & 2a_{00}^{(2)} a_{01}^{(1)} b_{02}^{(1)} - 2b_{00}^{(2)} b_{01}^{(1)} b_{02}^{(1)} - 2a_{00}^{(2)} b_{02}^{(2)} - 2a_{20}^{(1)} b_{00}^{(2)} b_{10}^{(1)} + \\ & b_{00}^{(3)} b_{11}^{(1)} + a_{00}^{(2)} b_{01}^{(1)} b_{11}^{(1)} - b_{00}^{(2)} b_{10}^{(1)} b_{11}^{(1)} + b\,_{00}^{(2)} b_{11}^{(2)}, \\ b_{03}^{(2)} = {} & \frac{1}{3}(-a_{02}^{(1)} a_{11}^{(1)} - a_{12}^{(2)} - a_{11}^{(1)} a_{20}^{(1)} + 3a_{01}^{(1)} a_{30}^{(1)} - 3a_{30}^{(2)} - \\ & 2a_{00}^{(2)} b + 2a_{21}^{(1)} b_{01}^{(1)} - 2a_{02}^{(1)} b_{02}^{(1)} + 3a_{30}^{(1)} b_{10}^{(1)} + b_{02}^{(1)} b_{11}^{(1)} + \\ & 2b_{01}^{(1)} b_{12}^{(1)} + 2a_{20}^{(1)} b_{20}^{(1)} + b_{11}^{(1)} b_{20}^{(1)} + a_{01}^{(1)} b_{21}^{(1)} + b_{10}^{(1)} b_{21}^{(1)} - \\ & b_{21}^{(2)} - 12a_{00}^{(2)} c). \end{aligned}$$

Computing the function $f_5(r)$ defined in Section 4 we obtain that it is of the form

$$f_5(r) = A_1 r + A_2 r^3 + A_3 r^5.$$

We do not give the big expressions of the independent coefficients A_i for $i = 1, 2, 3$. Hence the polynomial $f_5(r)$ can have at most two positive real roots. This completes the proof of the Teorem 2.

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