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Convexity and symmetry of central configurations in the five-body problem: Lagrange plus two

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Abstract

We study convexity and symmetry of central configurations in the five body problem when three of the masses ara located at the vertices of an equilateral triangle, that we call *Lagrange plus two central configurations*. First, we prove that the two bodies out of the vertices of the triangle cannot be placed on certain lines. Next, we give a geometrical characterization of such configurations in the sense as that of Dziobek, and we describe the admissible regions where the two remaining bodies can be placed. Furthermore, we prove that any Lagrange plus two central configuration is concave. Finally, we show numerically the existence of non-symmetric central configurations of the five body problem.

1 Introduction and main results

Central configurations of the planar n-body problem correspond to configurations $\mathbf{r} = (\mathbf{r}_1, \ldots, \mathbf{r}_n)$, $\mathbf{r}_i \in \mathbb{R}^2$, $i = 1, \ldots, n$, for positive masses $m_1, \ldots, m_n$ moving under the Newtonian gravitational attraction such that the acceleration at each mass point is a constant multiple of the relative position with respect to the center of mass $\mathbf{C}$:

$$\lambda(\mathbf{r}_i - \mathbf{C}) = \ddot{\mathbf{r}}_i = \sum_{\substack{j=1\\ j\neq i}}^{n} \frac{m_j}{r_{ij}^3}(\mathbf{r}_j - \mathbf{r}_i), \tag{1}$$

for all $i = 1, \ldots, n$ (in suitable units so the gravitational constant $G = 1$), where $r_{ij} = |\mathbf{r}_i - \mathbf{r}_j|$ is the Euclidean distance between the bodies at $\mathbf{r}_i$ and $\mathbf{r}_j$.

One of the reasons for the importance of the central configurations is that they lead to the so called homographic solutions of the planar n-body problem: the initial shape of the configuration is preserved as time varies. The first homographic solutions were found in the three-body problem, the collinear solution of Euler (1767, consisting in three bodies aligned) and the equilateral triangle solution of Lagrange (1772, consisting in three bodies of arbitrary mass located at the vertices of an equilateral triangle).

Counting up rotations and translations in the plane there are exactly five classes of central configurations for each choice of positive masses when $n = 3$. The finiteness of the number of classes of central configurations was a question posed by Chazy in [4], Wintner in [27], and reformulated by Smale in [25] as a challenge question for the 21st century. In a computer assisted proof, Hampton and Moeckel, in [16], gave an affirmative answer when $n = 4$ and any choice of the masses. This result was obtained analytically by Albouy and Kaloshin in [2], and extended the result to $n = 5$ for almost all choice of the masses. The question about the finiteness of the number of classes of central configurations is open for $n > 5$. We refer the reader to [19, 22, 24], and the references therein, for a wider introduction on central configurations, their usefulness and open problems.

The study of the geometry of central configurations is a complementary matter to the finiteness conjecture. An often studied geometric property is convexity. A configuration is convex if the polygon defined by the configuration have all interior angles less than or equal to 180 degrees, while is strictly convex if all interior angles are strictly less than 180 degrees. Otherwise the configuration is concave.

MacMillan and Bartky in 1932 ([20]) showed that for any four positive masses and any ordering of the bodies there exist a convex central configuration, and using the Perpendicular Bisector Theorem (see Theorem 3) it must be strictly convex. MacMillan and Bartky also derived a geometrical condition which is necessary and sufficient to achieve non trivial configurations in the four-body problem, remaining to check the positivity of the masses. This geometrical condition is named sometimes as Dziobek's condition, see [10] and references therein. Recently, Corbera, Cors and Roberts [7] have classified the full set of convex central configurations in the Newtonian planar four-body problem.

In the five-body problem Williams ([26]) and Chen and Hsiao ([6]) study the existence of convex central configurations and give some geometric properties. Moreover, Llibre and Gidea ([13]) and Chen and Hsiao ([5]) show that in the five-body problem it is possible to achieve a convex but non-strictly convex central configurations, in contrast with the four-body problem. But questions concerning the characterization or classification of the set of convex central configurations in the five-body problem still remain open. See for example [1, 6] and references therein.

Hampton in [15] introduces the concept of *stacked* central configurations in which a proper subset is also a central configuration. We follow the notation (n, k)-stacked central configuration introduced in [11], where n represents the total number of bodies and k, with $k = 1, \dots, n-3$, the bodies that can be removed maintaining the configuration central. Hampton shows a family of concave central configurations where three bodies are at the vertices of an equilateral triangle and the other two placed symmetrically respect to one perpendicular bisector line of the triangle. Since then, several authors have paid attention to this kind of central configurations in the five-body problem, see for instance, [8, 9, 11, 12, 13, 17, 18, 21, 23]. Several questions arise. One inquiry posted recently by Cornelio, Álvarez-Ramírez and Cors in [9] is the following: For a given central configuration of n-body problem, what is the number of (n, k)-stacked central configurations, for all $k = 1, \dots, n-3$? When $n = 5$ and excluding collinear configurations, the authors give the answer: there can be none, one or two. Another question posed is: are there convex central configurations of the n-body problem for which is possible to add a body and yet to have a convex central configuration? For $n = 4$ the authors of [12] gave a negative answer. A different proof is given in [6].

In all the discussed studies about stacked central configurations of the five-body problem we observe two main characteristics: all of them are symmetric and none of them are strictly convex. From that observation, one interesting question arise: are those two conditions necessary to have a stacked central configuration? In the case of the $(5, 1)$–stacked central configurations, the answer is affirmative. Fernandes and Mello, in [12], show that there exists only one $(5, 1)$–stacked central configuration consisting in a square of equal masses at the vertices plus a body with arbitrary mass at the center of square. Therefore, such a central configuration is symmetric and concave. In the case of the $(5, 2)$-stacked central configurations, Gidea and Llibre in [13] prove the existence of a symmetric non-strictly convex stacked central configuration where three of the bodies are in Euler configuration.

In the present work, we study the existence of central configurations in the five-body problem assuming that three bodies are located at the vertices of an equilateral triangle, that we call *Lagrange plus two* central configurations. In particular, all of these configurations are $(5, 2)$-stacked central configurations. Let $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_5)$ be a configuration out of the collision set, i.e., we assume $r_{ij} > 0$, for all $i \neq j$. An equivalent set of equations for central configurations (1) is given by the *Andoyer's equations* (see the references [3] and [14])

$$f_{ij} = \sum_{\substack{k=1 \\ k \neq i,j}}^{n} m_k \left(R_{ik} - R_{jk}\right) \Delta_{ijk} = 0, \tag{2}$$

for $1 \leq i < j \leq n$, where $R_{ij} = r_{ij}^{-3}$ and $\Delta_{ijk} = (\mathbf{r}_i - \mathbf{r}_j) \wedge (\mathbf{r}_i - \mathbf{r}_k)$ is twice times the oriented area defined by the triangle with vertices at $\mathbf{r}_i$, $\mathbf{r}_j$ and $\mathbf{r}_k$. Notice, for a given configuration $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$, system (2) is a system of linear equations $A(\mathbf{r}) \cdot M = 0$ for the mass vector $M = (m_1 \ldots m_5)^t$.

Our first goal is to give a geometric characterization of Lagrange plus two central configurations in the five-body problem in the sense of Dziobek's condition.

Theorem 1. *Consider a five-body configuration $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$ so that $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, form an equilateral triangle, there are no three bodies lined-up, and none of the two remaining bodies $\mathbf{r}_4$, $\mathbf{r}_5$ is located on a bisector line of the triangle.*

(i) If the system of Andoyer's equations (2) has solution for m_i, $i = 1, \ldots, 5$ with $m_4, m_5 \neq 0$, then the system

$$\left\{ \begin{array}{l} (R_{14} - R_{24})(R_{15} - R_{35})\Delta_{124}\Delta_{135} - (R_{15} - R_{25})(R_{14} - R_{34})\Delta_{125}\Delta_{134} = 0, \\ (R_{14} - R_{24})(R_{25} - R_{35})\Delta_{124}\Delta_{235} - (R_{15} - R_{25})(R_{24} - R_{34})\Delta_{125}\Delta_{234} = 0, \end{array} \right. \tag{3}$$

is satisfied.

(ii) If the configuration $\mathbf{r}$ satisfy the equations (3), then the system of Andoyer's equations (2) has non-trivial solution.

Notice that geometric condition (3) only depends on the relative positions of the bodies. After that, positivity in the masses must be checked from Andoyer's equations to ensure the existence of Lagrange plus two central configurations. The proof of Theorem 1 is given in Section 3. Also, in that section we include a characterization of the admissible regions in the plane where the two bodies out of the equilateral triangle can be placed (Proposition 4).

The second goal is to give an answer to the questions posted above about the convexity and symmetry of Lagrange plus two central configurations. More concretely:

- Are there convex central configurations of the five body problem formed by piling up two bodies to the Lagrange central configuration?
- Are there non-symmetric central configurations of the five body problem formed by piling up two bodies to the Lagrange central configuration?

With respect to the convexity, we prove following result in Section 4.

Theorem 2. *Let $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$ be a planar convex central configuration of the five-body problem. Then there are no three vertices of the pentagon forming an equilateral triangle.*

Next result is then a straightforward consequence of Theorem 2 and the result about (5,1) stacked central configurations of Fernandes and Mello [12].

Corollary 1. *Consider a central configuration of the five–body problem. If the configuration is strictly convex, then the configuration is not stacked.*

We notice that the claims of Theorem 1 and 2 are valid not only for the Newtonian potential, but also for any homogeneous potential $1/r^{\alpha}$.

Finally, with respect to the question whether the symmetry is a necessary condition for the existence of stacked central configurations, the answer is negative. In Section 5 we show numerically the existence of non-symmetric central configurations containing an equilateral triangle. We perform a massive numerical exploration in the admissible regions, given in Proposition 4, to find all non-symmetric Lagrange plus two central configurations. It turns out that all non-symmetric central configurations of five bodies containing an equilateral triangle are placed around the two families of symmetric central configurations already shown by Hampton ([15]) and Llibre, Mello and Pérez-Chavela ([18]).

2 Preliminary results

Given five bodies with masses $m_1, \dots, m_5$ and located at $\mathbf{r}_1, \dots, \mathbf{r}_5$ ($\mathbf{r}_i \in \mathbb{R}^2$, $i = 1, \dots, 5$), we denote by c_{ij} the line that passes through $\mathbf{r}_i$, $\mathbf{r}_j$, and by l_{ij} the bisector line of the segment defined by the points $\mathbf{r}_i$, $\mathbf{r}_j$. See Figure 1.

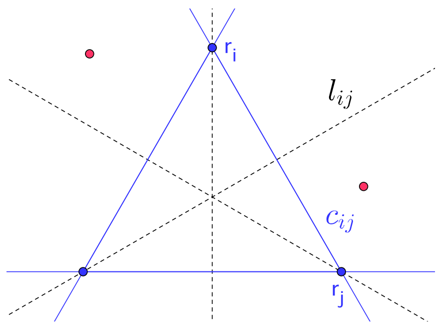

Figure 1: A five body configuration where three bodies are at the vertices of an equilateral triangle, with lines c_{ij} (solid) and l_{ij} (dashed). According to Perpendicular Bisector Theorem applied with respect $\mathbf{r}_i$ and $\mathbf{r}_j$, this configuration cannot be central.

We recall the Perpendicular Bisector Theorem, see [22], which is a geometric tool to check if a configuration can be central (see Figure 1 for an example of a configuration not allowed by this property).

Theorem 3 (Perpendicular Bisector)**.** *Let $\mathbf{r}_i$ and $\mathbf{r}_j$ be two positions of a planar central configuration, and consider the two open double cones determined by the lines c_{ij} and l_{ij}. If one of the double cone has nonempty intersection with the central configuration, then so does the other open double cone.*

We introduce a technical lemma that gives a relation between the areas of the triangles Δ_{ijk} given by any configuration of five points, see for instance [26].

Lemma 1. *Consider five points in the plane $\mathbf{r}_i$, $i = 1, \dots, 5$, and $\Delta_{ijk} = (\mathbf{r}_i - \mathbf{r}_j) \wedge (\mathbf{r}_i - \mathbf{r}_k)$ twice times the oriented area defined by the triangle with vertices at $\mathbf{r}_i$, $\mathbf{r}_j$ and $\mathbf{r}_k$. Then the following equations hold:*

- $\Delta_{345}\Delta_{123} - \Delta_{235}\Delta_{134} + \Delta_{234}\Delta_{135} = 0,$
- $\Delta_{345}\Delta_{124} - \Delta_{245}\Delta_{134} + \Delta_{234}\Delta_{145} = 0,$
- $\Delta_{345}\Delta_{125} - \Delta_{245}\Delta_{135} + \Delta_{235}\Delta_{145} = 0,$

- $\Delta_{145}\Delta_{123} - \Delta_{135}\Delta_{124} + \Delta_{134}\Delta_{125} = 0$,
- $\Delta_{245}\Delta_{123} - \Delta_{235}\Delta_{124} + \Delta_{234}\Delta_{125} = 0$,
- $\Delta_{123} - \Delta_{124} + \Delta_{134} - \Delta_{234} = 0$,
- $\Delta_{123} - \Delta_{125} + \Delta_{135} - \Delta_{235} = 0$,
- $\Delta_{124} - \Delta_{125} + \Delta_{145} - \Delta_{245} = 0$,
- $\Delta_{134} - \Delta_{135} + \Delta_{145} - \Delta_{345} = 0$,
- $\Delta_{234} - \Delta_{235} + \Delta_{245} - \Delta_{345} = 0$.

Proof. It is a straightforward computation from the definition of Δ_{ijk}. □

Consider now that three of the five bodies are located at the vertices of an equilateral triangle, that is, in Lagrange plus two configuration, see Figure 1. In the following propositions, we show that there are forbidden locations for the other two if the whole configuration is central.

Proposition 1. *Consider a five-body central configuration $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$ so that three of them form an equilateral triangle. If the two remaining bodies are aligned with one of the vertices of the triangle, then they must be on the bisector line of the other two vertices.*

Proof. Without loss of generality we can suppose that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are at the vertices of the equilateral triangle ordered counterclockwise, and that $\mathbf{r}_4$ and $\mathbf{r}_5$ are aligned with $\mathbf{r}_3$, see Figure 2. Consider the Andoyer's equations (2) corresponding to f_{12}, f_{13} and f_{23}. Using that $R_{12} = R_{23} = R_{13}$ we have that

$$
\begin{aligned}
f_{12} &= m_4(R_{14} - R_{24})\Delta_{124} + m_5(R_{15} - R_{25})\Delta_{125} = 0,\\
f_{13} &= m_4(R_{14} - R_{34})\Delta_{134} + m_5(R_{15} - R_{35})\Delta_{135} = 0,\\
f_{23} &= m_4(R_{24} - R_{34})\Delta_{234} + m_5(R_{25} - R_{35})\Delta_{235} = 0.
\end{aligned}
$$

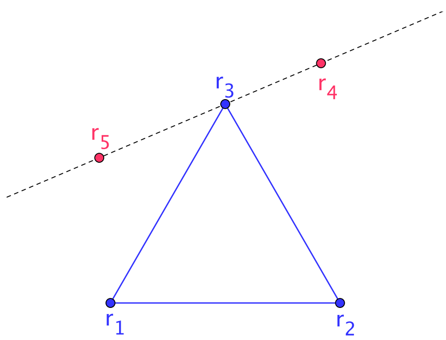

Figure 2: Configuration of five bodies such that three are at the vertices of an equilateral triangle, and the other two are aligned with one of the vertices.

From the first equation of Lemma 1, and the fact that $\Delta_{345} = 0$, we have $\Delta_{234}\Delta_{135} = \Delta_{235}\Delta_{134}$. Using this relation,

$$
\Delta_{124}\left(\Delta_{234}f_{13} - \Delta_{134}f_{23}\right) - \Delta_{234}\Delta_{134}f_{12} = m_5(R_{15} - R_{25})\Delta_{234}(\Delta_{124}\Delta_{135} - \Delta_{134}\Delta_{125}) = 0.
$$

Again from Lemma 1, using the fourth equation, the expression $\Delta_{124}\Delta_{135} - \Delta_{134}\Delta_{125} = \Delta_{145}\Delta_{123}$. Clearly, $\Delta_{123} \neq 0$. On the other hand, if Δ_{145} or Δ_{234} are zero means that four bodies are aligned and contradicts, by Perpendicular Bisector Theorem, the fact the configuration is central. Therefore, $R_{15} = R_{25}$ which is only possible if $\mathbf{r}_4$ and $\mathbf{r}_5$ are on the perpendicular bisector of $\mathbf{r}_1$ and $\mathbf{r}_2$. This concludes the proof. □

In fact, Llibre and Mello [18] consider a five-body configuration where three bodies are on the vertices of an equilateral triangle and the other two bodies are on a perpendicular bisector, and show the existence of three families of central configurations. Proposition 1 says that these three families are the only central configurations when three of the bodies form an equilateral triangle and the two remaining bodies are aligned with one of the vertices.

Proposition 2. *Consider a five-body central configuration* $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$ *so that three of them form an equilateral triangle. Then, none of the two remaining bodies can be aligned with two of the vertices of the triangle.*

Proof. Without loss of generality, we can consider that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are at the vertices of an equilateral triangle ordered counterclockwise, and that $\mathbf{r}_4 \in c_{12}$. Applying the Perpendicular Bisector Theorem with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$, we have that $\mathbf{r}_5 \in c_{12}$ or $\mathbf{r}_5 \in l_{12}$. The first case is not possible, because there are no central configurations of the five body problem with four bodies in a line. Thus $\mathbf{r}_5 \in l_{12}$. We use again the Perpendicular Bisector Theorem twice: first, with respect $\mathbf{r}_1$ and $\mathbf{r}_4$, we have that $\mathbf{r}_5$ must be in a different half plane defined by the line c_{12} than $\mathbf{r}_3$; second, with respect $\mathbf{r}_1$ and $\mathbf{r}_3$, $\mathbf{r}_4$ cannot be located between $\mathbf{r}_1$ and $\mathbf{r}_2$. See Figure 3.

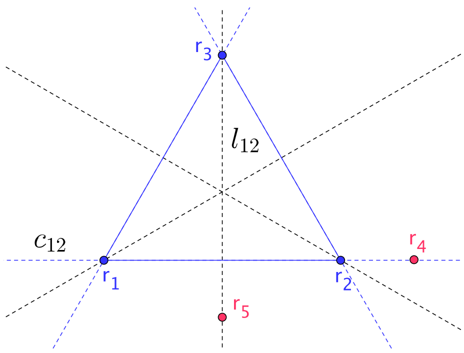

Figure 3: Lagrange plus two configuration where $\mathbf{r}_4 \in c_{12}$

Introducing $\Delta_{124} = 0$, $R_{15} = R_{25}$, $\Delta_{235} = -\Delta_{135}$ into equations f_{13} and f_{23} of Andoyer's equations, we have that

$$f_{13} + f_{23} = m_4\big((R_{14} - R_{34})\Delta_{134} + (R_{24} - R_{34})\Delta_{234}\big) = 0. \tag{4}$$

As $m_4 \neq 0$, we have that the second factor, which depends only on $\mathbf{r}_4$, must vanish. Without loss of generality, we consider that the side of the triangle equals 1 and $\mathbf{r}_4$ is at the right hand side of $\mathbf{r}_2$ at a distance $a > 0$. Therefore,

$$R_{14} = \frac{1}{(1+a)^3}, \quad R_{24} = \frac{1}{a^3}, \quad R_{34} = \frac{1}{(1+a^2+a)^{3/2}},$$

and $\Delta_{134} = -\sqrt{3}\,(1+a)/4$, $\Delta_{234} = -\sqrt{3}\,a/4$. Introducing into the equation (4) we get

$$\frac{1}{(1+a)^2} + \frac{1}{a^2} = \frac{1+2a}{(1+a^2+a)^{3/2}}.$$

Squaring both sides one gets a polynomial with all the coefficients positive, so the equation has no solution for $a > 0$, which proves the claim. □

Proposition 3. *Consider a five-body central configuration* $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$ *so that three of them form an equilateral triangle. If one of the two remaining bodies is on a bisector line of the triangle, then the other one is on the same line.*

Proof. The proof is straightforward using the Perpendicular Bisector Theorem with respect the bisector line that contains two bodies. □

3 Geometric condition for Lagrange plus two central configurations

In this section we give proof of Theorem 1 that states a geometric condition on the configuration of five bodies with three at the vertices of an equilateral triangle, in order to have non-trivial solutions for the masses of Andoyer's equations. We avoid two particular configurations: when three of the bodies are aligned or when any of bodies, excluding the vertices of the triangle, is on a bisector l_{ij} of the triangle. Notice that if a configuration is central, by Propositions 1, 2 and 3, these two particular configurations are excluded except in the case that three of the bodies are on the same bisector line l_{ij}, studied by Llibre and Mello [18]. Therefore, we can exclude them from our study.

Moreover, after the proof of Theorem 1, we study the admissible regions where the two bodies, that are not at the vertices of the equilateral triangle, can be placed.

3.1 Proof of Theorem 1

We recall the hypothesis of the theorem. Consider a five-body configuration $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$ so that $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$, form an equilateral triangle, there are no three bodies lined-up, and none of the two remaining bodies $\mathbf{r}_4$, $\mathbf{r}_5$ is located on a bisector line of the triangle. Without loss of generality, we suppose $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ ordered counterclockwise.

First, we want to see that if the system of Andoyer's equations have non-trivial solution for m_i, $i = 1, \ldots, 5$, with $m_4, m_5 \neq 0$, then the system (3) is satisfied.

We consider the system of Andoyer's equations given in (2) as a linear system for the masses

$$A \cdot M = 0, \tag{5}$$

where $A \in \mathcal{M}_{10\times 5}$ and $M = (m_1 \ldots m_5)^t$. Notice that, from the hypothesis, all $\Delta_{ijk} \neq 0$, for all i, j, k. We introduce in the equations the fact that the mutual distances between $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are equal. In particular, the subsystem given by the equations corresponding to f_{12}, f_{13} and f_{23}:

$$\left\{ \begin{array}{rcl} m_4(R_{14} - R_{24})\Delta_{124} + m_5(R_{15} - R_{25})\Delta_{125} &=& 0, \\ m_4(R_{14} - R_{34})\Delta_{134} + m_5(R_{15} - R_{35})\Delta_{135} &=& 0, \\ m_4(R_{24} - R_{34})\Delta_{234} + m_5(R_{25} - R_{35})\Delta_{235} &=& 0, \end{array} \right. \tag{6}$$

must have non-trivial solution. Notice that all the coefficients of m_4 and m_5 in (6) cannot vanish because $\mathbf{r}_4, \mathbf{r}_5 \notin l_{ij}$, $i, j = 1, 2, 3$. Then, for a fixed configuration, system (6) must have rank one, that is,

$$\begin{array}{l} F_1(\mathbf{r}_4, \mathbf{r}_5) := (R_{14} - R_{24})(R_{15} - R_{35})\Delta_{135}\Delta_{124} - (R_{15} - R_{25})(R_{14} - R_{34})\Delta_{134}\Delta_{125} = 0, \\ F_2(\mathbf{r}_4, \mathbf{r}_5) := (R_{14} - R_{24})(R_{25} - R_{35})\Delta_{235}\Delta_{124} - (R_{15} - R_{25})(R_{24} - R_{34})\Delta_{234}\Delta_{125} = 0, \end{array} \tag{7}$$

which is the system of the statement (3).

Second, we want to see that if system (3) is satisfied, then Andoyer's equations have non-trivial solution for m_i, $i = 1, \ldots, 5$. More concretely, we will prove that under the hypothesis the Andoyer's linear system (5) has kernel of dimension at least one. Equations (3) ensure that system (6) has rank one, so we can remove two equations (for example the ones corresponding to f_{13} and f_{23}) from (5) to obtain the linear system

$$\overline{A} \cdot M = 0, \tag{8}$$

with $\overline{A} \in \mathcal{M}_{8\times 5}$. We want to prove that there exist non trivial solutions of (8).

First, using the relations in Lemma 1, five of the Δ_{ijk} can be written in terms of the other five as follows:

$$\begin{array}{rcl}
\Delta_{234} & = & \Delta_{123} - \Delta_{124} + \Delta_{134}, \\
\Delta_{235} & = & \Delta_{123} - \Delta_{125} + \Delta_{135}, \\
\Delta_{145} & = & \dfrac{\Delta_{124}\Delta_{135} - \Delta_{125}\Delta_{134}}{\Delta_{123}}, \\[2ex]
\Delta_{245} & = & \dfrac{\Delta_{123}\Delta_{124} + \Delta_{135}\Delta_{124} - \Delta_{123}\Delta_{125} - \Delta_{125}\Delta_{134}}{\Delta_{123}}, \\[2ex]
\Delta_{345} & = & \dfrac{\Delta_{123}\Delta_{134} - \Delta_{125}\Delta_{134} - \Delta_{123}\Delta_{135} + \Delta_{124}\Delta_{135}}{\Delta_{123}}.
\end{array} \tag{9}$$

For simplicity in what follows we write Δ_{234}, Δ_{235}, Δ_{145}, Δ_{245} and Δ_{345} instead of the corresponding expressions.

Second, we can obtain R_{14} and R_{15} from equations $F_1 = 0$, $F_2 = 0$, as follows. Using the third relation in (9), we write the system (3) as:

$$\begin{array}{rcl}
\Delta_{145}\Delta_{123}R_{14}R_{15} + a_1 R_{14} + b_1 R_{15} + c_1 & = & 0, \\
a_2 R_{14} + b_2 R_{15} + c_2 & = & 0,
\end{array}$$

for a certain coefficients a_i, b_i, c_i, depending on the other R_{ij} and Δ_{ijk}. Then, the above system is solved for R_{14} and R_{15}, and has two solutions:

$$R_{14} = R_{24}, \quad R_{15} = R_{25}, \tag{10}$$

and

$$\begin{array}{rcl}
R_{14} & = & \dfrac{1}{\Delta_{235}\Delta_{145}\Delta_{123}}(\Delta_{135}\Theta_2 R_{24} + \Delta_{125}\Theta_3 R_{34}), \\[2ex]
R_{15} & = & \dfrac{1}{\Delta_{234}\Delta_{145}\Delta_{123}}(\Delta_{134}\Theta_2 R_{25} + \Delta_{124}\Theta_3 R_{35}),
\end{array} \tag{11}$$

where

$$\begin{array}{rcl}
\Theta_2 & = & \Delta_{124}\left(\Delta_{135} + \Delta_{123}\right) - \Delta_{125}(\Delta_{134} + \Delta_{123}), \\
\Theta_3 & = & \Delta_{134}\left(\Delta_{125} - \Delta_{123}\right) - \Delta_{135}\left(\Delta_{124} - \Delta_{123}\right).
\end{array}$$

Recall that $\mathbf{r}_4, \mathbf{r}_5 \notin l_{12}$, so the solution (10) is not admissible. Therefore, R_{14} and R_{15} are given by (11).

Then, we introduce (9) and (11) in equations (8). Using a computer algebra system (concretely the software Mathematica), a generator of the kernel of the matrix is obtained of the form

$$\left(\frac{N_1}{K(R_{24}-R_{34})\Delta_{123}}, \frac{N_2}{K(R_{24}-R_{34})\Delta_{123}\Delta_{234}}, \frac{N_3}{K(R_{24}-R_{34})\Delta_{123}\Delta_{234}}, \frac{-(R_{25}-R_{35})\Delta_{235}}{(R_{24}-R_{34})\Delta_{234}}, 1\right), \tag{12}$$

where N_i, $i = 1, 2, 3$ are expressions depending on R_{ij} and Δ_{ijk} and

$$K = \Delta_{125}\Delta_{134}(R_{12} - R_{25})(R_{12} - R_{34}) - \Delta_{124}\Delta_{135}(R_{12} - R_{24})(R_{12} - R_{35}). \tag{13}$$

The expressions for N_i can be found in the Appendix.

Recall that $\mathbf{r}_4 \notin l_{23}$, so $R_{24} - R_{34} \neq 0$. If $K \neq 0$, the system (8) has kernel of dimension one, with the generator given by (12), so there exists non-trivial solutions of Andoyer's equations.

It remains to prove that the same is true when $K = 0$. We proceed as follows:

- First, we isolate one of the R_{ij} from the equation (13), introduce the expression obtained into (8) and compute its kernel.
- If in the expressions obtained, all the denominators are not null, we are done.
- If there are denominators that can be null, we consider those cases apart, repeating the process. At each step, we introduce the new restrictions into (8) and compute again its kernel.

More concretely, the restriction $K = 0$ leads to diferent cases:

- Case 1. From the equation $K = 0$, if $R_{34} \neq R_{12}$ we isolate R_{25}:
$$R_{25} = \frac{\Delta_{125}\Delta_{134}R_{12}(R_{12} - R_{34}) - \Delta_{124}\Delta_{135}(R_{12} - R_{24})(R_{12} - R_{35})}{(R_{12} - R_{34})\Delta_{134}\Delta_{125}}.$$
We introduce the above expression in (8), obtaining the matrix $\overline{A_1}$, which has kernel generated by:
$$\left(\frac{(R_{12} - R_{34})\Delta_{234}}{\Delta_{124}K_1}, \frac{-(R_{12} - R_{34})\Delta_{134}}{(R_{12} - R_{24})\Delta_{124}}, 1, 0, 0\right), \tag{14}$$
where
$$K_1 = R_{12} + \frac{R_{34}\Delta_{125}\Delta_{345} - R_{24}\Delta_{135}\Delta_{245}}{\Delta_{235}\Delta_{145}}.$$
If $K_1 \neq 0$ and $R_{24} \neq R_{12}$ we are done. On the contrary, we have to study two new cases:
  - Case 1a: $K_1 = 0$. We solve this equation for R_{24}:
  $$R_{24} = \frac{R_{12}\Delta_{235}\Delta_{145} + R_{34}\Delta_{125}\Delta_{345}}{\Delta_{135}\Delta_{245}}$$
  We introduce this expression in $\overline{A_1}$, and the resulting matrix has a kernel generated by and the vector $(1, 0, 0, 0, 0)$. Thus, we have a non-trivial solution for Andoyer's equations.
  - Case 1b: $R_{24} = R_{12}$. In this case, substituting the restriction into (13) we also obtain $R_{25} = R_{12}$, and introducing both restrictions into the system (8), we have that all the coefficients corresponding to m_2 are nul. Therefore, $(0, 1, 0, 0, 0)$ is a non-trivial solution.
- Case 2: $K = 0$ and $R_{34} = R_{12}$. Introducing this restriction into (13), we have that
$$\Delta_{124}\Delta_{135}(R_{12} - R_{24})(R_{12} - R_{35}) = 0,$$
where $R_{12} - R_{24} \neq 0$ because $\mathbf{r}_4 \notin l_{23}$. Therefore, we have $R_{35} = R_{12}$, and similarly to the case 1b, introducing the restrictions into into the system (8), we have that all the coefficients corresponding to m_3 are nul. Therefore, $(0, 0, 1, 0, 0)$ is a non-trivial solution.

In all the cases, we have found non-trivial solutions. This concludes the proof of Theorem 1.

Remark. *In the proof of Theorem 1 we obtain a generator of the kernel of the Andoyer's linear system in all cases. But in the case $K = 0$, we point out that any solution of the Andoyer's equations have one or more masses (but not all) that vanish. Clearly, those cases cannot be central configurations.*

In particular, from the proof of Theorem 1 we have the following equivalence:
Under the hypothesis of Theorem 1, Andoyer's equations have solutions with $m_4, m_5 \neq 0$ if and only if the configuration $\mathbf{r}$ satisfies the geometric condition (3) and $K \neq 0$.

3.2 Admissible regions

If we restrict to symmetric solutions, there are two possibilities: when the line of symmetry contains only one of the bodies, which must be a vertex of the triangle, or when the line of symmetry contains three bodies, one vertex and the other two bodies.

In the first case, if we take the line of symmetry to be the bisector l_{12} (similar results are obtained considering the other bisector lines), the two equations in (3) are equal and can be written as
$$(R_{24} - R_{34})\Delta_{234} + (R_{14} - R_{34})\Delta_{143} = 0. \tag{15}$$
If $\mathbf{r}_4, \mathbf{r}_5 \notin l_{12}$, then the equation (15) has two disconnected components. Taking a system of reference such that the axes correspond to c_{12} and l_{12}, we can write $\mathbf{r}_4 = (x, y)$, and $\mathbf{r}_5 = (-x, y)$,

Hampton ([15]) shows that for any x such that $\mathbf{r}_4$ is in the interior of the triangle, there exists one value y such that the configuration of the five bodies is central and $\mathbf{r}_4$, $\mathbf{r}_5$ are located along the curve defined by (15) that connect the vertices $\mathbf{r}_1$ and $\mathbf{r}_2$, and goes through the barycenter of the triangle (inner symmetric central configurations). Llibre, Mello and Pérez-Chavela ([17]) show that only for a certain range of values of x such that $\mathbf{r}_4$ is in the exterior of the triangle there exists one value of y for which a central configuration exists and $\mathbf{r}_4$, $\mathbf{r}_5$ are located along two pieces of a curve outside the triangle that goes through one vertex (outer symmetric central configurations). See Figure 4.

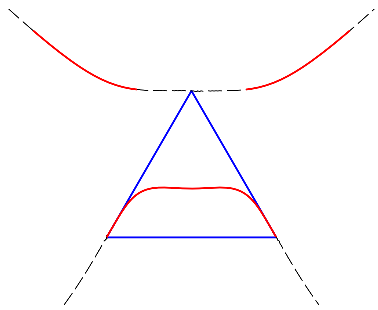

Figure 4: Curves given by equation (15) (dashed black lines) and arches (continuous red lines) that correspond to the location of $\mathbf{r}_4$, $\mathbf{r}_5$ in the inner and outer symmetric central configurations of the five-body problem with three masses at the vertices of an equilateral triangle (see [15, 17]).

If $\mathbf{r}_4, \mathbf{r}_5 \in l_{12}$, then three families of central configurations are obtained, see [18].

In order to study non-symmetric central configurations, using the geometric condition of Theorem 1, we characterize the admissible regions for $\mathbf{r}_4$, $\mathbf{r}_5$, when they are not on any bisector.

Definition 1. *Consider a five-body configuration* $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$ *so that* $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$*, form an equilateral triangle. Let* $r = r_{12} = r_{13} = r_{23}$ *be the side of the triangle,* c_{ij} *be the line that passes through* $\mathbf{r}_i$*,* $\mathbf{r}_j$*, and* l_{ij} *be the bisector line of the segment defined by the points* $\mathbf{r}_i$*,* $\mathbf{r}_j$*. We define the following open regions (see Figure 5):*

- T_1 *is the triangle limited by* l_{13}*,* l_{23} *and* c_{23}*;*
- T_2 *is the unbounded region limited by* c_{23}*,* l_{23}*, and* c_{13}*;*
- T_3 *is the unbounded region limited by* c_{13} *and* l_{12}*;*
- S_1 *is the triangle limited by* l_{13}*,* l_{23} *and* c_{13}*;*
- S_2 *is the unbounded region limited by* c_{23}*,* l_{13} *and* c_{13}*;*
- S_3 *is the unbounded region limited by* c_{23} *and* l_{13}*;*
- S_4 *is the unbounded region limited by* c_{12} *and* l_{13}*.*

Next result shows that any central configuration can be obtained, by symmetries and rotations, from a central configuration where the bodies $\mathbf{r}_4$ and $\mathbf{r}_5$ are located in the regions given in Definition 1.

Proposition 4. *Consider a central configuration of five bodies* $\mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_5)$ *so that* $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$*, form an equilateral triangle and* $\mathbf{r}_4, \mathbf{r}_5 \notin l_{ij}$*,* $i, j = 1, 2, 3$*. Let* T_i*,* $i = 1, 2, 3$ *and* S_j*,* $j = 1, \ldots, 4$ *the regions given in Definition 1.*

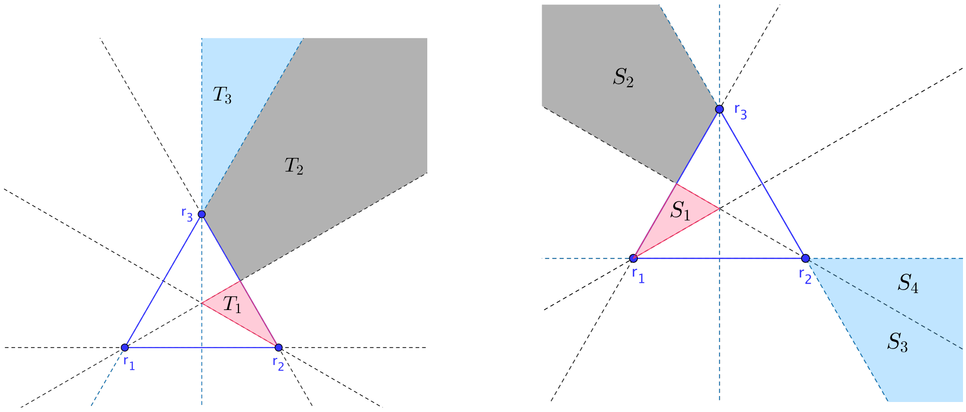

Figure 5: Regions given in Definition 1 when $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are ordered counterclockwise.

1. *There exist a central configuration* $\bar{\mathbf{r}}$ *that can be obtained from* $\mathbf{r}$ *applying rotations of angle* $\pm 2\pi/3$ *and reflections through any of the bisector lines* l_{ij}, $i, j = 1, 2, 3$, *such that* $\bar{\mathbf{r}}_1, \bar{\mathbf{r}}_2, \bar{\mathbf{r}}_3$ *still are at the vertices of an equilateral triangle and* $\bar{\mathbf{r}}_4 \in T_1 \cup T_2 \cup T_3$.

2. *If* $\mathbf{r}_4 \in T_i$, $i = 1, 2, 3$, *then one of the following occurs:*

   (a) *if* $\mathbf{r}_4 \in T_1 \cup T_2$, *then* $\mathbf{r}_5 \in S_1 \cup S_2 \cup S_3$,

   (b) *if* $\mathbf{r}_4 \in T_3$, *then* $\mathbf{r}_5 \in S_4$,

Proof. Without loss of generality, we can suppose that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are ordered counterclockwise.

Clearly applying rotations of angle $\pm 2\pi/3$ and reflections through any of the bisector lines l_{ij}, $i, j = 1, 2, 3$, any central configuration can be transformed to a central configuration such that $\mathbf{r}_4 \in T_1 \cup T_2 \cup T_3$. Notice that the bisectors lines are excluded by hypothesis and lines c_{ij} are not allowed by Proposition 2. This concludes statement *1*.

To prove the statement *2* we use the geometric condition established in Theorem 1. For convenience we write equations (7) as following

$$F_1 = F_{124}F_{135} - F_{134}F_{125} = 0, \quad \text{and} \quad F_2 = F_{124}F_{235} - F_{234}F_{125} = 0, \tag{16}$$

where $F_{klm} = (R_{km} - R_{lm})\Delta_{klm}$. Clearly, when $\mathbf{r}_4 \in T_i$, $i = 1, 2, 3$, $F_{124} < 0$ and $F_{234} > 0$. Moreover, when $\mathbf{r}_4 \in T_1 \cup T_2$, $F_{134} > 0$, whereas if $\mathbf{r}_4 \in T_3$, then $F_{134} < 0$.

First suppose that $\mathbf{r}_4 \in T_1 \cup T_2$. If $F_{125} < 0$, then equations (16) have solution only if $F_{135} > 0$ and $F_{235} > 0$. These three inequalities define three regions (see Figure 6) which share the same double cone defined by the lines c_2 and l_{23} as the region $T_1 \cup T_2$. By the Perpendicular Bisector Theorem that is not allowed. Therefore $F_{125} > 0$, and in that case, the equations (16) have solution only if $F_{135} < 0$ and $F_{235} < 0$. That corresponds to the region $S_1 \cup S_2 \cup S_3$.

When $\mathbf{r}_4 \in T_3$, a similar augment leads to $\mathbf{r}_5 \in S_4$. □

4 On the convexity of Lagrange plus two central configurations

In this section we prove that there no exist convex central configurations of the five body problem containing an equilateral triangle. This result can be proved regardless of the existence of the geometric condition obtained in the previous section.

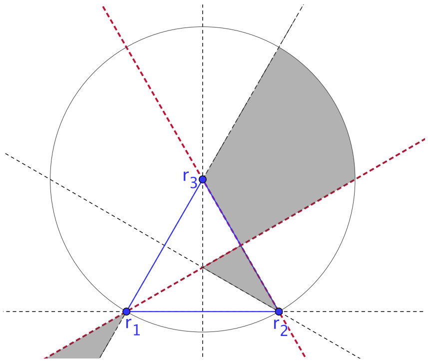

Figure 6: The regions defined by $F_{125} < 0$, $F_{135} > 0$ and $F_{235} > 0$ (see proof of Proposition 4).

We distinguish between strictly convex configurations, i.e., all the interior angles of the polygon are strictly less than 180 degrees, and convex, but non-strictly convex, configurations. For simplicity, the later will be called non-strictly convex configurations.

In the first case, we will use the Theorem of Cheng and Hsiao, (see Theorem 6.1 in [6]), for five-body strictly convex central configurations:

Theorem 4 (Cheng-Hsiao)**.** *For any strictly convex five-body central configuration, all exterior edges are less than $r_0 = (M/\lambda)^{1/3}$, where λ is the multiplier of equation 1 and M is the total mass. The number of interior edges larger than r_0 is either 5, 4, or 3. If there are precisely three interior edges larger than r_0, then the other two interior edges must cross each other.*

We notice that from the proof of the theorem, in the case when three interior edges are larger than r_0, we can obtain that the other two, that must cross each other, do not emanate from the same vertex. That is, the crossing point is not at any vertex $\mathbf{r}_j$.

We first suppose that we have a strictly convex configuration with three bodies at the vertices of an equilateral triangle. Without loss of generality we consider that $\mathbf{r}_1, \ldots, \mathbf{r}_5$ are disposed counterclockwise. We examine two possibilities, depending whether the three bodies in the equilateral triangle are correlative or not. Without particularization, we consider the cases:

(i) The position vectors $\mathbf{r}_3$, $\mathbf{r}_4$ and $\mathbf{r}_5$ form an equilateral triangle (correlative);

(ii) The position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_4$ form an equilateral triangle (non-correlative).

In the case (i) we have that $r_{34} = r_{35} = r_{45}$. Let S be the open half-cone defined by the half lines starting at $\mathbf{r}_4$ and passing through $\mathbf{r}_3$ and $\mathbf{r}_5$. As the configuration is strictly convex, it implies that $\mathbf{r}_1$ and $\mathbf{r}_2$ must belong to S outside of the equilateral triangle, see Figure 7, left. Nevertheless, this situation is not admissible by the Perpendicular Bisector Theorem applied at the vertices of $\mathbf{r}_4$ and $\mathbf{r}_5$ (Figure 7, right). So, do not exist any central configuration strictly convex of type (i).

In the case (ii) we have that $r_{12} = r_{14} = r_{24}$, and r_{14}, r_{24} are interior edges. From Theorem 4, since the two diagonals r_{14} and r_{24} are equal to one side (r_{12}), the other three interior edges r_{13}, r_{25} and r_{35} must be greater than r_{12}. See the forbidden region for $\mathbf{r}_3$ and $\mathbf{r}_5$ in Figure 8. Nevertheless, this configuration contradicts Theorem 4, since the diagonals r_{14} and r_{24} do not cross in the interior of the pentagon. So, do not exist any central configuration strictly convex of type (ii).

Next, we suppose that we have a non-strictly convex configuration containing an equilateral triangle. Suppose that the vertices are $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ and ordered counterclockwise. Notice that the Perpendicular Bisector Theorem 3 prevents from four aligned bodies in a central configuration.

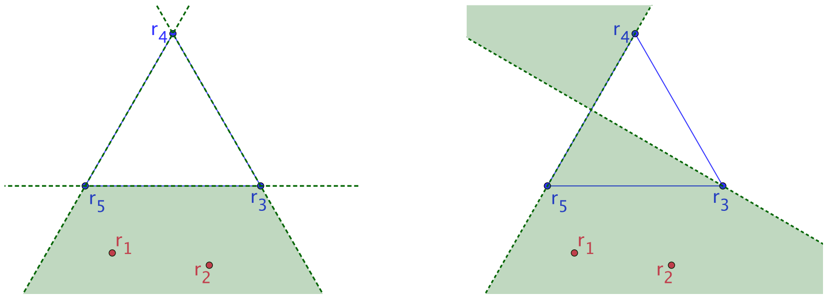

Figure 7: Left: The strictly convex permitted region for r_1 and r_2. Right: The shadow region corresponds to one of the open cones generated by $\mathbf{r}_4$ and $\mathbf{r}_5$, showing that the configuration is not central.

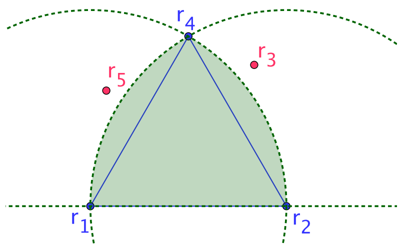

Figure 8: Arrangements of the vector positions in the case (ii) where the vertices of the equilateral triangle are not correlative. The gray region is not permitted.

Therefore, if the configuration is non-strictly convex, three of the bodies must lie on the same line. So, there are two possibilities: the three aligned bodies contain two of the vertices or just one. The first case is not possible from Proposition 2. In the second case, from Proposition 3 the configurations is concave. This ends the proof of Theorem 2.

5 Non-symmetric Lagrange plus two central configurations

In this section we show numerically the existence of non-symmetric central configurations of the five body problem in which three of the masses are located at the vertices of an equilateral triangle. As we mention before, it is known that there exist two families of symmetric central configurations of this problem: a family inside the equilateral triangle (inner central configurations, Hampton [15]), and a family outside the triangle (outer central configurations, Llibre et al. [17]). See Figure 4.

We can consider that $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ are the vertices of the equilateral triangle ordered counterclockwise. By Theorem 1, the admissible configurations for central configurations are those satisfying (3) such that the solution of the Andoyer's equations give all $m_i > 0$. By Proposition 4, it is enough to study their location for $\mathbf{r}_4$ and $\mathbf{r}_5$ in the regions T_i, $i = 1, 2, 3$ and S_j, $j = 1, \ldots, 4$ respectively. See Figure 5. The inner symmetric central configurations are such that $\mathbf{r}_4 \in T_1$ and $\mathbf{r}_5 \in S_1$, whereas the outer ones are such that $\mathbf{r}_4 \in T_2$ and $\mathbf{r}_5 \in S_2$.

We assume a system of reference of axes c_{12} and l_{12} in which $\mathbf{r}_1 = (-1, 0)$, $\mathbf{r}_2 = (1, 0)$, $\mathbf{r}_3 = (0, \sqrt{3})$, $\mathbf{r}_4 = (x, y)$ and $\mathbf{r}_5 = (u, v)$. For each one of the regions T_i, $i = 1, 2, 3$, we take a grid of values (x, y) (by taking a step of 5×10^{-4} and 10^{-3} for variables x and y respectively) within a ball (we will see that for values of (x, y) far away we did not find any solution of the equations). Then, for any fixed (x, y) we solve the system (3) for the variables (u, v) by an iterative scheme. For each solution, we substitute the values (u, v, x, y) into the system of Andoyer's equations (2), where the equations corresponding to f_{13} and f_{23} are removed. The linear system for the masses is solved by using singular value decomposition (SVD) to determine the kernel of the matrix (with a precision up to 10^{-7}). We keep the values for which a generator of the kernel has all the components of the same sign.

By the explorations performed, we have obtained that there exist central configurations only when $(\mathbf{r}_4, \mathbf{r}_5) \in T_1 \times S_1$ and $(\mathbf{r}_4, \mathbf{r}_5) \in T_2 \times S_2$. That is, there only exist non-symmetric central configurations in the regions surrounding the curves of symmetric central configurations. In Figure 9 we show the locations $\mathbf{r}_4$, $\mathbf{r}_5$ for which there exist non-symmetric central configurations. We notice that in the case of outer central configurations (with $\mathbf{r}_4, \mathbf{r}_5$ outside the equilateral triangle), we only find central configurations confined inside a ball of center $\mathbf{r}_3$ and radius the length of the side of the triangle. Therefore we conjecture that Proposition 4 can be improved by deleting T_3, S_3 and S_4 from the statement, and restricting the regions T_2 and S_2 by $r_{3j} \leq r_{12}$.

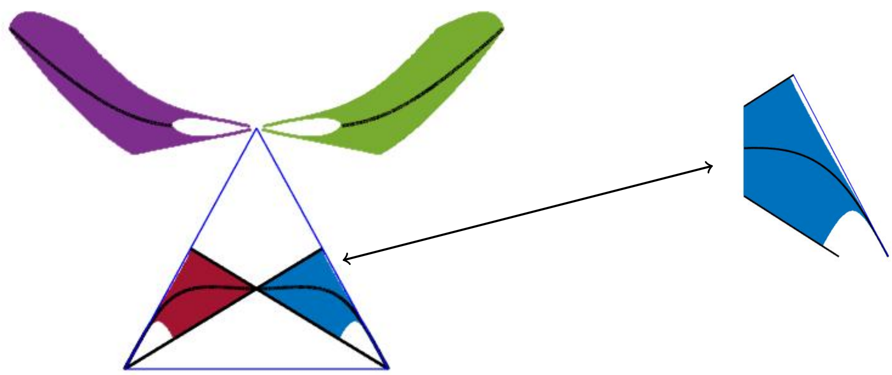

Figure 9: Locations for $\mathbf{r}_4$ and $\mathbf{r}_5$ for which there exists a non-symmetric central configurations of the five body problem containing an equilateral triangle. The black curves represents the symmetric central configurations given by equation (15).

6 Discussion and conclusions

We study central configurations in the five-body problem when a proper subset of the configuration form an equilateral triangle and so is also a central configuration of the three-body problem: Lagrange plus two configuration. In that sense the study can be categorized inside the stacked central configuration class. As far as we know, all the five-body stacked central configurations found till now have two main characteristics: are symmetric and concave, with the exception of one symmetric family that is convex, but non-strictly convex, since three of the bodies are aligned.

Under the previous considerations we give a geometric characterization, similar to Dziobek's condition for four bodies: any Lagrange plus two central configuration must satisfy a system of two equations that depends only on the location of the bodies. Using this geometric condition, we are able to limit the regions where the two bodies out of the Lagrange configuration can be placed.

Furthermore, we prove that is impossible to have convex central configurations in the five-body problem with three bodies at the vertices of an equilateral triangle. In particular, there are no (5,2)-stacked central configurations strictly convex.

Finally, using the geometric characterization of the configuration, we carry out an exhaustive numerical study to explore the existence of non-symmetric Lagrange plus two central configurations. As far as we now these are the first non-symmetric central configurations exhibited in the five-body problem.

Among other things, it is still an open question the existence of non-symmetric five-body central configurations with three aligned bodies.

7 Appendix

The expressions N_i, $i = 1, 2, 3$ in equation (12) are the following:

$$\begin{aligned} N_1 &= \Delta_{235}\Delta_{123}\Delta_{145}\Big((R_{35}-R_{45})(R_{25}R_{34}+R_{12}R_{24}) \\ &+ (R_{45}-R_{34})(R_{24}R_{35}+R_{12}R_{25}) + (R_{34}-R_{35})(R_{24}R_{25}+R_{12}R_{45})\Big) \end{aligned}$$

$$\begin{aligned} N_2 &= \Delta_{125}\Delta_{134}^2(\Delta_{123}-\Delta_{125})R_{34}(R_{25}-R_{35})(R_{34}-R_{45}) \\ &+ \Delta_{124}\Delta_{135}^2(\Delta_{123}-\Delta_{124})R_{35}(R_{24}-R_{34})(R_{35}-R_{45}) \\ &+ \Delta_{124}\Delta_{135}^2\Delta_{134}\Big(R_{24}R_{35}\left(R_{34}-R_{45}\right) - R_{24}R_{25}\left(R_{34}-R_{35}\right) - R_{25}R_{34}\left(R_{35}-R_{45}\right)\Big) \\ &+ \Delta_{123}\Delta_{145}\Delta_{134}(\Delta_{123}-\Delta_{125})R_{12}(R_{25}-R_{35})(R_{34}-R_{45}) \\ &- \Delta_{123}\Delta_{145}\Delta_{135}(\Delta_{123}-\Delta_{124})R_{12}(R_{24}-R_{34})(R_{35}-R_{45}) \\ &- \Delta_{123}\Delta_{145}\Delta_{134}\Delta_{135}R_{12}\Big(R_{24}(R_{35}-R_{45}) - R_{25}(R_{34}-R_{45}) + R_{45}(R_{34}-R_{35})\Big) \\ &+ \Delta_{134}\Delta_{135}R_{34}R_{35}\Big(\Delta_{123}\Delta_{124}\left(R_{35}-R_{45}\right) + \Delta_{125}\left(\Delta_{123}\left(R_{34}-R_{45}\right) + \Delta_{124}\left(R_{34}+R_{35}-2R_{45}\right)\right)\Big) \\ &+ \Delta_{134}\Delta_{135}R_{34}R_{25}\Big(\Delta_{125}\left(\Delta_{124}-\Delta_{123}\right)R_{34} + \left(\Delta_{123}\left(\Delta_{125}-\Delta_{124}\right) + \Delta_{125}\Delta_{134}\right)R_{35} \\ &\quad + \left(\Delta_{123}\Delta_{124} - \Delta_{125}\left(\Delta_{124}+\Delta_{134}\right)\right)R_{45}\Big) \\ &- \Delta_{134}\Delta_{135}R_{24}R_{25}\left(\Delta_{123}\left(\Delta_{124}-\Delta_{125}\right) - \Delta_{125}\Delta_{134}\right)\left(R_{34}-R_{35}\right) \\ &- \Delta_{134}\Delta_{135}R_{24}R_{35}\Big(\Delta_{124}\left(\Delta_{125}-\Delta_{123}\right)R_{35} + \left(\Delta_{123}\left(\Delta_{124}-\Delta_{125}\right) - \Delta_{125}\Delta_{134}\right)R_{34} + \Delta_{125}\Delta_{234}R_{45}\Big) \end{aligned}$$

$$\begin{aligned} N_3 &= \Delta_{123}\Delta_{124}\Delta_{125}\Delta_{345}R_{24}(R_{23}R_{34} - R_{25}R_{35} - R_{35}^2) \\ &+ \Delta_{123}\Delta_{124}\Delta_{145}\Delta_{235}R_{24}R_{35}R_{45} \\ &- \Delta_{123}\Delta_{124}\Delta_{135}\Delta_{245}R_{24}^2\left(R_{25}-R_{35}\right) \\ &+ \Delta_{123}\Delta_{125}R_{25}R_{34}\left(\Delta_{124}\Delta_{345}R_{35} - \Delta_{134}\Delta_{245}R_{25} + \Delta_{145}\Delta_{234}R_{45}\right) \\ &- \Delta_{123}\Delta_{145}R_{12}R_{45}\left(\Delta_{125}\Delta_{234}R_{34} - \Delta_{124}\Delta_{235}R_{35}\right) \\ &- \Delta_{123}\Delta_{145}R_{12}R_{24}\left(-\Delta_{123}\Delta_{245}R_{25} - \Delta_{125}\Delta_{234}R_{45} + \Delta_{124}\Delta_{235}R_{35}\right) \\ &- \Delta_{123}\Delta_{145}R_{12}R_{25}\left(\Delta_{124}\Delta_{235}R_{45} - \Delta_{125}\Delta_{234}R_{34}\right) \\ &+ \Delta_{123}\Delta_{245}R_{24}R_{25}\left(\Delta_{125}\Delta_{134}R_{25} + \Delta_{123}\Delta_{145}R_{45}\right) \end{aligned}$$

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Data Availability Statement The data that support the findings of this study are available from the corresponding author upon reasonable request.

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