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THE CIRCULAR RESTRICTED 4-BODY PROBLEM WITH THREE EQUAL PRIMARIES IN THE COLLINEAR CENTRAL CONFIGURATION OF THE 3-BODY PROBLEM

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ABSTRACT. We study the dynamics of the circular restricted 4-body problem with three primaries with equal masses at the collinear configuration of the 3-body problem with an infinitesimal mass. **We calculate the equilibrium points and study their linear stability. By applying the Lyapunov theorem, we prove the existence of periodic orbits bifurcating from the equilibrium points and further, prove that they continue in the full 4-body problem. Moreover, we prove analyticallythe existence of Hill and of comet-like periodic orbits.**

1. INTRODUCTION

The planar Newtonian n-body problem concerns with the motion of n points of mass $m_i > 0$ moving in a plane under their mutual attraction in accordance with Newton's law of gravitation.

The equations of the motion of the n-body problem are

$$\ddot{r}_i = -\sum_{j=1, j\neq i}^{n} \frac{m_j(r_i - r_j)}{r_{ij}^3}, \quad 1 \leq i \leq n,$$

where we have taken the unit of time in such a way that the Newtonian gravitational constant be one, and $r_i \in \mathbb{R}^2$ $(i = 1, \ldots, n)$ denotes the position vector of the i-body, $r_{ij} = |r_i - r_j|$ is the Euclidean distance between the i-body and the j-body.

It is well-known that the 2-body problem has been completely solved (see for instance [9]), whereas the n-body for $n > 2$ is still open and non-integrable, see for instance [3, 10].

For the Newtonian n-body problem the simplest possible motions, are the so called the *homographic solutions*. In these solutions the configuration formed by the n-bodies is constant up to rotations and scaling. Only some special configurations of particles are allowed in the homographic solutions of the n-body problem, called by Wintner [11] *central configurations.*

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It is well-known that for the 3-body problem when the three masses are equal there is a unique collinear central configuration, where the mass in the middle is equidistant from the other two.

The main objective of this paper is to study the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem, that is to describe the motion of the infinitesimal mass with respect to the primaries under the Newtonian gravitational force.

In [4] the authors determine the ejection-collision orbits of this circular restricted 4-body problem. Here we determine the equilibria, and study aspects of the dynamics mainly about its periodic orbits.

The paper is organized as follows: in section 2 we obtain the equations of motion in a rotating system of coordinates while in section 3 we calculate the equilibrium points. In section 4 we calculate the Jacobian integral and Hill's regions . Further, in Section 5 we study the linear stability of the equilibrium points. We mention that the results in sections 3, 4 and 5 have been considered previously in [7] and also the content of these sections can be seen as a special case of results given in [1], see also [2, 8]. In section 6 we determine some periodic orbits: first we calculate the linear expression of the periodic orbits bifurcating from the equilibrium points L_i, $i = 1, 2, 3, 4$; next we prove the existence of these periodic orbits using the Lyapunov center theorem. We also prove the existence of Hill's and comets orbits. In the last subsection we prove that any elementary periodic solution of the circular collinear restricted 4-body problem whose period is not a multiple of 2π can be continued into the full 4-body problem with one small mass.

2. Equations of motion

Taking the unit of mass equal to the masses of the three primaries and since a central configuration is invariant under rotations and homotheties through its center of mass without loss of generality in a convenient rotating system with angular velocity $\omega = \sqrt{5}/2$ the position vectors q_i's for $i = 1, 2, 3$ of the three primaries are

$$q_1 = (-1, 0),\ q_2 = (0, 0),\ q_3 = (1, 0).$$

See for more details [6, 12, 13]. Denoting the position of the infinitesimal mass by $q_4 = (x, y)$, then the Hamiltonian of the infinitesimal mass in the rotating system of coordinates for the circular restricted collinear 4-body problem is

$$H = \frac{1}{2}(p_x^2 + p_y^2) + \omega(p_x y - p_y x) - \frac{1}{r_{14}} - \frac{1}{r_{24}} - \frac{1}{r_{34}}, \tag{1}$$

where

$$r_{14}^2 = (x + 1)^2 + y^2,\ r_{24}^2 = x^2 + y^2,\ r_{34}^2 = (x - 1)^2 + y^2.$$

For more details on how to obtain this Hamiltonian see [6, 12, 13].

The equations of motion are

$$
\begin{aligned}
\dot{x} &= p_x + \omega y, \\
\dot{y} &= p_y - \omega x, \\
\dot{p}_x &= \frac{\partial}{\partial x}\left(\frac{1}{r_{14}} + \frac{1}{r_{24}} + \frac{1}{r_{34}}\right) + \omega p_y, \\
\dot{p}_y &= \frac{\partial}{\partial y}\left(\frac{1}{r_{14}} + \frac{1}{r_{24}} + \frac{1}{r_{34}}\right) - \omega p_x.
\end{aligned} \tag{2}
$$

The Newtonian equations of motion are obtained by eliminating the momenta from equations (2)

$$
\ddot{x} - 2\omega\dot{y} = \frac{\partial \Omega}{\partial x}, \quad \ddot{y} + 2\omega\dot{x} = \frac{\partial \Omega}{\partial y}, \tag{3}
$$

where

$$
\Omega = \frac{1}{2}\omega^2(x^2 + y^2) + \frac{1}{r_{14}} + \frac{1}{r_{24}} + \frac{1}{r_{34}},
$$

3. Equilibrium points

We now compute the equilibrium points of equations (3). Calculating the derivatives of Ω and equal them to zero the equilibrium points coordinates are determined by

$$
\begin{aligned}
\frac{\partial \Omega}{\partial x} &= \omega^2 x - \frac{x+1}{r_{14}^3} - \frac{x}{r_{24}^3} - \frac{x-1}{r_{34}^3} = 0, \\
\frac{\partial \Omega}{\partial y} &= y\left(\omega^2 - \frac{1}{r_{14}^3} - \frac{1}{r_{24}^3} - \frac{1}{r_{34}^3}\right) = 0.
\end{aligned} \tag{4}
$$

From the second equation we have two cases.

The first case is when $y = 0$. Substituting this into the first equation we get:

$$
\frac{5}{4}x - \frac{x+1}{|x+1|^3} - \frac{x}{|x|^3} - \frac{x-1}{|x-1|^3} = 0 \tag{5}
$$

and we obtain the following four equations:

$$
\begin{aligned}
&5x^7 - 10x^5 + 12x^4 + 5x^3 + 4 = 0, \text{ if } x < -1, \\
&5x^7 - 10x^5 + 4x^4 + 21x^3 - 8x^2 + 4 = 0, \text{ if } -1 < x < 0, \\
&5x^7 - 10x^5 - 4x^4 + 21x^3 + 8x^2 - 4 = 0, \text{ if } 0 < x < 1, \\
&5x^7 - 10x^5 - 12x^4 + 5x^3 - 4 = 0, \text{ if } x > 1.
\end{aligned} \tag{6}
$$

The first equation in (6) has the single real root $x_1 = -1.7576799791694022..$, the second equation in (6) has the single real root $x_2 = -0.4946664910173645..$, the third equation in (6) has the single real root $x_3 = 0.4946664910173645..$, and the fourth equation in (6) has the single real root $x_4 = 1.7576799791694022...$ **These roots have been computed using the Newton method, and their existence is guaranteed by the bisection method.**

The second case $y \neq 0$ reduces to the equation

$$
\omega^2 - \frac{1}{r_{14}^3} - \frac{1}{r_{24}^3} - \frac{1}{r_{34}^3} = 0. \tag{7}
$$

Multiplying this equation by $-x$ and adding it to the first equation of (4) we obtain

$$r_{14} = r_{34} = \sqrt{y^2+1},\ r_{24} = \pm y,\ x = 0.$$

Then from (7) we have the following equations for y

$$\frac{2}{(y^2+1)^{3/2}} \pm \frac{1}{y^3} = \frac{5}{4}$$

with the real roots $y_{5,6} = \pm 1.1394282249562009...$

In summary we have obtained six equilibrium points $L_i = (x_i, 0)$ for $i = 1, 2, 3, 4$, and $L_j = (0, y_j)$ for $j = 5, 6$.

If we calculate the Hessian of the function Ω at these equilibria we get

$$H(L_1) = H(L_4) = H(\pm 1.7576799791694022.., 0) = \begin{pmatrix} 6.31171.. & 0 \\ 0 & -1.28086.. \end{pmatrix},$$

so L_1 and L_4 are saddles of Ω;

$$H(L_2) = H(L_3) = H(\pm 0.4946664910173645.., 0) = \begin{pmatrix} 33.8708.. & 0 \\ 0 & -15.0604.. \end{pmatrix},$$

so L_2 and L_3 are also saddles of Ω;

$$H(L_5) = H(L_6) = H(0, \pm 1.1394282249562009..) = \begin{pmatrix} 0.749266.. & 0 \\ 0 & 3.00073.. \end{pmatrix},$$

so L_5 and L_6 are minima of Ω.

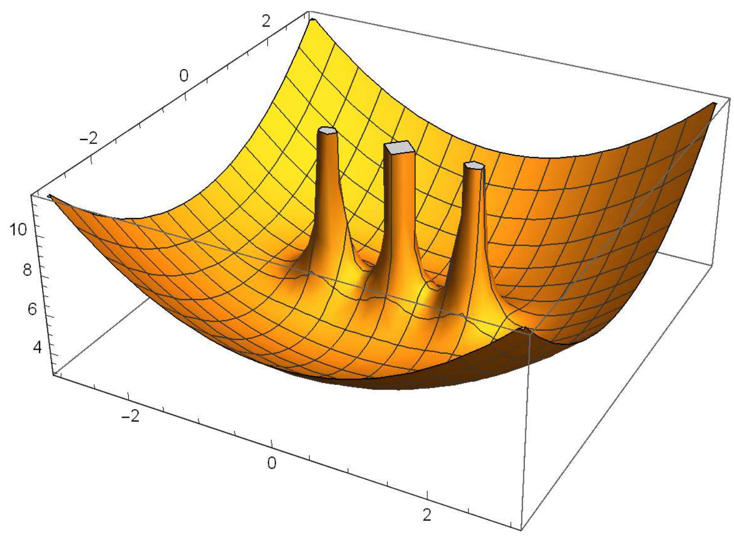


FIGURE 1. Plot of the surface $z = \Omega(x, y)$. In the base of the parallelepiped we have the (x, y)-plane and in the height we have the z-axis.

4. Jacobian integral and Hill's regions

As in the circular restricted three body problem, a direct calculation leads for our circular restricted four body problem to the *Jacobian* first integral

$$C = 2\Omega - v^2,$$

where $v = \sqrt{\dot{x}^2 + \dot{y}^2}$ is the velocity of the infinitesimal mass, and C is a constant.

Since the square of the velocity cannot be negative we have that

$$\Omega(x, y) = \frac{5}{8}(x^2 + y^2) + \frac{1}{\sqrt{(x+1)^2 + y^2}} + \frac{1}{\sqrt{x^2 + y^2}} + \frac{1}{\sqrt{(x-1)^2 + y^2}} \geq \frac{C}{2}.$$

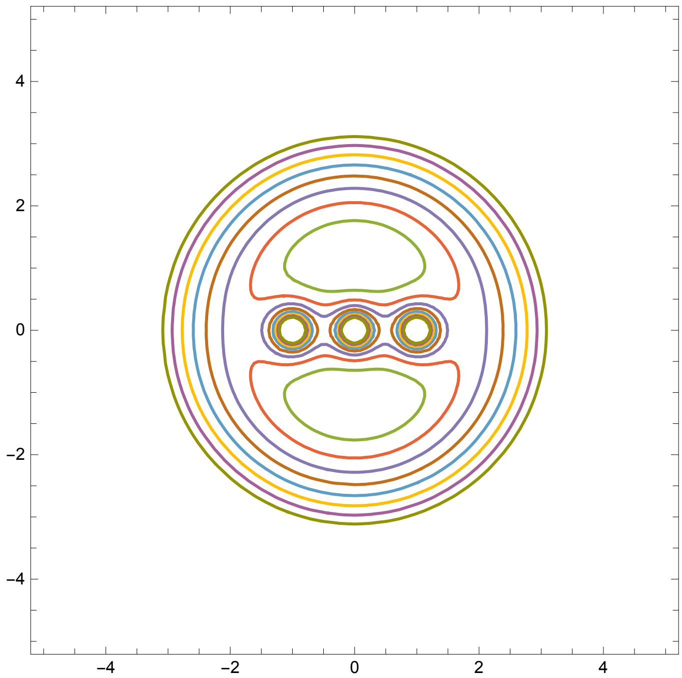


Figure 2. The curves of zero velocity for different values of C in the (x, y)-plane.

Now we study the curves of zero velocity. If $v = 0$ the Jacobian first integral reduces to $2\Omega = C$. When the Jacobi constant varies we obtain different curves of zero velocity. These curves of zero velocity are the boundaries of the Hill's regions, i.e. the set of points (x, y) of the plane where $\Omega(x, y) \geq C/2$, i.e. the regions of the plane where the motion is possible for the value C of the Jacobi constant. This means that the motion cannot take place in the regions where $\Omega(x, y) < C/2$.

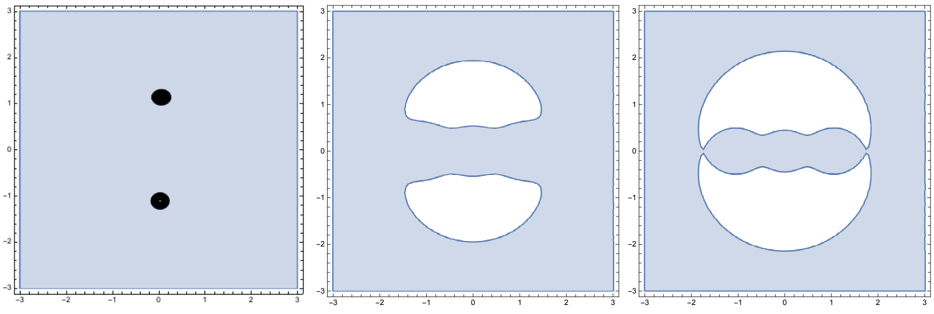
If we compute the values of the function Ω at the equilibrium points we get

$$\Omega(\pm 1.7576799791694022.., 0) = 4.18227..,$$

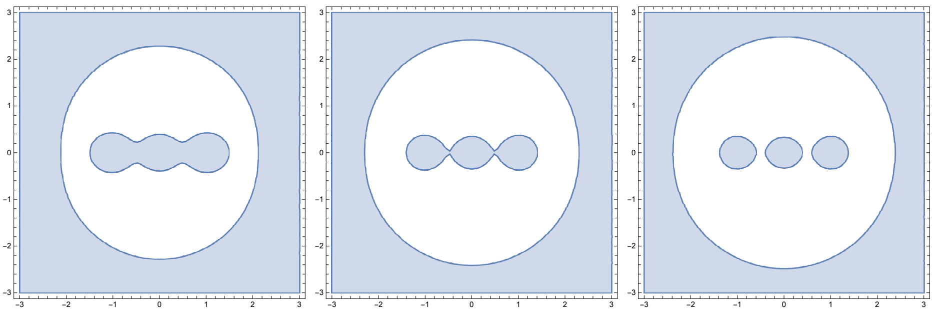
$$\Omega(\pm 0.4946664910173645.., 0) = 4.82244..,$$

$$\Omega(0, \pm 1.1394282249562009..) = 3.00832...$$

This means that no level curves exist when $\Omega < 3.00832..$, and consequently no motion exists for the values of $\Omega < 3.00832...$ The equilibrium points are located at critical values where the curves of zero velocity change their topology. In Figure 3 the Hill's regions where the motion is allowed are the white regions.



(a) $C = 3.00832..$ (b) $C \in (3.00832.., 4.18227..)$. (c) $C = 4.18227..$



(d) $C \in (4.18227.., 4.82244..)$ (e) $C = 4.82244..$ (f) $C > 4.82244..$

FIGURE 3. The curves of zero velocity and their Hill's regions. **The motion is allowed in the white regions.**

5. Stability of the equilibrium points

At the equilibrium points the linearized system corresponding to system (2) has the matrix

$$\begin{pmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ \Omega_{xx} - \omega^2 & \Omega_{xy} & 0 & \omega \\ \Omega_{xy} & \Omega_{yy} - \omega^2 & -\omega & 0 \end{pmatrix}. \tag{8}$$

For the equilibrium points $L_i = (x_i, 0)$ for $i = 1, 2, 3, 4$ on the x-axis, the above matrix becomes

$$\begin{pmatrix} 0 & \sqrt{5}/2 & 1 & 0 \\ -\sqrt{5}/2 & 0 & 0 & 1 \\ 2\alpha & 0 & 0 & \sqrt{5}/2 \\ 0 & -\alpha & -\sqrt{5}/2 & 0 \end{pmatrix},$$

where

$$\alpha = \frac{1}{|x_i+1|^3} + \frac{1}{|x_i|^3} + \frac{1}{|x_i-1|^3}, \quad \text{for } i = 1, 2, 3, 4.$$

The eigenvalues of this matrix at the equilibrium points L_1 and L_4 are

$$-1.68164..i, \quad 1.68164..i, \quad -1.69079.., \quad 1.69079..,$$

and the eigenvalues at the equilibrium points L_2 and L_3 are

$$-4.08808..i, \quad 4.08808..i, \quad -5.52474.., \quad 5.52474...$$

Since for each one of these equilibrium points one eigenvalue is real positive, all these equilibrium points are unstable.

For the equilibrium points $L_j = (0, y_j)$ for $j = 5, 6$ on the y-axis, the matrix (8) becomes

$$\begin{pmatrix} 0 & \sqrt{5}/2 & 1 & 0 \\ -\sqrt{5}/2 & 0 & 0 & 1 \\ \beta & 0 & 0 & \sqrt{5}/2 \\ 0 & \gamma & -\sqrt{5}/2 & 0 \end{pmatrix},$$

where

$$\beta = \frac{6 - 2(1+y_i^2)}{(1+y_i^2)^{\frac{5}{2}}} - \frac{1}{|y_i|^3}, \quad \text{for } i = 5, 6,$$

and

$$\gamma = \frac{6y_i^2 - 2(1+y_i^2)}{(1+y_i^2)^{\frac{5}{2}}} + \frac{2}{|y_i|^3}, \quad \text{for } i = 5, 6.$$

The eigenvalues of this matrix at the equilibrium points L_5 and L_6 are

$$0.661228.. + 1.03064..i, \qquad 0.661228.. - 1.03064..i,$$

and

$$-0.661228.. + 1.03064..i, \qquad -0.661228.. - 1.03064..i,$$

respectively. Hence the equilibrium points L_5 and L_6 are unstable.

In summary we have proved the next result.

Theorem 1. *The six equilibrium points of the circular restricted* 4*-body problem with three equal primaries in the collinear central configuration of the* 3*-body problem are unstable.*

6. Periodic orbits

The study of the periodic orbits of the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem of the subsections 6.2 to 6.5 are follow the methodology presented for the study of the periodic orbits of the circular restricted 3-body problem in the book of Meyer, Hall and Offin [6].

6.1. **The linear expression of the periodic orbits bifurcating from the equilibrium points L_i, $i = 1, 2, 3, 4$.** At these four equilibrium points we have $\Omega_{xx} > 0$, $\Omega_{xy} = 0$ and $\Omega_{yy} < 0$ and we denote by $\pm\lambda$ their two real eigenvalues, and by $\pm\beta i$ their two purely imaginary eigenvalues. The linearised equations of motion (3), called the linear variational equations, are

$$\ddot{x} - 2\omega\dot{y} = x\Omega_{xx} + y\Omega_{xy}, \quad \ddot{y} + 2\omega\dot{x} = x\Omega_{xy} + y\Omega_{yy}, \tag{9}$$

where the second order partial derivatives of Ω are calculated at the equilibrium points L_i for $i = 1, 2, 3, 4$. The solution of these linear variational equations is

$$\begin{aligned} x(t) &= A_1 e^{\lambda t} + A_2 e^{-\lambda t} + A_3 e^{\beta t i} + A_4 e^{-\beta t i}, \\ y(t) &= B_1 e^{\lambda t} + B_2 e^{-\lambda t} + B_3 e^{\beta t i} + B_4 e^{-\beta t i}. \end{aligned} \tag{10}$$

The coefficients $A_1, \ldots, A_4$, $B_1, \ldots, B_4$ are not independent because from the first equation of (9) we have

$$\begin{aligned} B_1 &= \frac{\lambda^2 - \Omega_{xx}|_{L_1-L_4}}{2\omega\lambda} A_1 = \alpha_1 A_1, \\ B_2 &= -\frac{\lambda^2 - \Omega_{xx}|_{L_1-L_4}}{2\omega\lambda} A_2 = -\alpha_1 A_2, \\ B_3 &= \frac{\beta^2 + \Omega_{xx}|_{L_1-L_4}}{2\omega\beta} A_3 i = \alpha_3 A_3 i, \\ B_4 &= -\frac{\beta^2 + \Omega_{xx}|_{L_1-L_4}}{2\omega\beta} A_4 i = -\alpha_3 A_4 i, \end{aligned} \tag{11}$$

taking into account that $\Omega_{xy} = 0$ at the equilibrium points L_i for $i = 1, 2, 3, 4$. As in [12] we can see that the four initial conditions of (9) will completely determine the coefficients $A_1, \ldots, A_4$, $B_1, \ldots, B_4$. Thus we have

$$\begin{aligned} x_0 = x(t_0) &= A_1 e^{\lambda t_0} + A_2 e^{-\lambda t_0} + A_3 e^{\beta t_0 i} + A_4 e^{-\beta t_0 i}, \\ \dot{x}_0 = \dot{x}(t_0) &= \lambda A_1 e^{\lambda t_0} - \lambda A_2 e^{-\lambda t_0} + \beta A_3 e^{\beta t_0 i} i - \beta A_4 e^{-\beta t_0 i} i, \\ y_0 = y(t_0) &= \alpha_1 A_1 e^{\lambda t_0} - \alpha_1 A_2 e^{-\lambda t_0} + \alpha_3 A_3 e^{\beta t_0 i} i - \alpha_3 A_4 e^{-\beta t_0 i} i, \\ \dot{y}_0 = \dot{y}(t_0) &= \alpha_1 \lambda A_1 e^{\lambda t_0} + \alpha_1 \lambda A_2 e^{-\lambda t_0} - \alpha_3 \beta A_3 e^{\beta t_0 i} - \alpha_3 \beta A_4 e^{-\beta t_0 i}. \end{aligned} \tag{12}$$

Taking into account (11) the determinant of this system is

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda & -\lambda & \beta i & -\beta i \\ \alpha_1 & -\alpha_1 & \alpha_3 i & -\alpha_3 i \\ \alpha_1\lambda & \alpha_1\lambda & -\alpha_3\beta & -\alpha_3\beta \end{vmatrix} = \frac{\Omega_{xx}|_{L_1-L_4}(\lambda^2+\beta^2)^2}{\omega^2\lambda\beta} i \neq 0.$$

Therefore the solution of system (12) is

$$\begin{aligned} A_1 &= \frac{e^{-\lambda t_0}}{2}\Big[\frac{\alpha_3\beta x_0}{\alpha_3\beta+\alpha_1\lambda} - \frac{\alpha_3\dot{x}_0}{\alpha_1\beta-\lambda\alpha_3} + \frac{\beta y_0}{\alpha_1\beta-\lambda\alpha_3} + \frac{\dot{y}_0}{\alpha_3\beta+\alpha_1\lambda}\Big], \\ A_2 &= \frac{e^{\lambda t_0}}{2}\Big[\frac{\alpha_3\beta x_0}{\alpha_3\beta+\alpha_1\lambda} + \frac{\alpha_3\dot{x}_0}{\alpha_1\beta-\lambda\alpha_3} - \frac{\beta y_0}{\alpha_1\beta-\lambda\alpha_3} + \frac{\dot{y}_0}{\alpha_3\beta+\alpha_1\lambda}\Big], \\ A_3 &= \frac{e^{-i\beta t_0}}{2}\Big[\frac{\alpha_1\lambda x_0}{\alpha_3\beta+\alpha_1\lambda} - i\frac{\alpha_1\dot{x}_0}{\alpha_1\beta-\lambda\alpha_3} + i\frac{\lambda y_0}{\alpha_1\beta-\lambda\alpha_3} - \frac{\dot{y}_0}{\alpha_3\beta+\alpha_1\lambda}\Big], \\ A_4 &= \frac{e^{i\beta t_0}}{2}\Big[\frac{\alpha_1\lambda x_0}{\alpha_3\beta+\alpha_1\lambda} + i\frac{\alpha_1\dot{x}_0}{\alpha_1\beta-\lambda\alpha_3} - i\frac{\lambda_1 y_0}{\alpha_1\beta-\lambda\alpha_3} - \frac{\dot{y}_0}{\alpha_3\beta+\alpha_1\lambda}\Big]. \end{aligned} \tag{13}$$

Since we are looking for a periodic solution, we must have $A_1 = A_2 = 0$. If we choose the initial conditions so that $A_1 = A_2 = 0$, then from the first two relations of (13) we get that, if x_0, y_0 are arbitrarily selected, then $\dot{x}_0 = \beta y_0/\alpha_3$ and $\dot{y}_0 = -\alpha_3\beta x_0$. Evaluating A_3 and A_4 from (13) and substituting them into equations (10) we obtain the periodic orbit

$$\begin{aligned} x(t) &= x_0\cos\big(\beta(t-t_0)\big) + (y_0/\alpha_3)\sin\big(\beta(t-t_0)\big), \\ y(t) &= y_0\cos\big(\beta(t-t_0)\big) - x_0\alpha_3\sin\big(\beta(t-t_0)\big). \end{aligned} \tag{14}$$

This orbit is the ellipse

$$x^2 + \frac{y^2}{\alpha_3^2} = x_0^2 + \frac{y_0^2}{\alpha_3^2}, \tag{15}$$

where $\alpha_3^2 = 5.90773..$ at the equilibrium points L_1 and L_4, and $\alpha_3^2 = 30.6199..$ at the equilibrium points L_2 and L_3.

Taking into account that at the equilibrium points L_5 and L_6 all the eigenvalues are complex with real part different than zero, we conclude that there are no periodic orbits bifurcating from these equilibrium points.

6.2. **Lyapunov families at the equilibrium points L_i, $i = 1, 2, 3, 4$.** A first integral F is called *nondegenerate* on a point x_0 if its gradient $\nabla F(x_0) \neq 0$.

We note that the Hamiltonian (1) is always a nondegenerate first integral on a nonequilibrium solution because $\nabla H(x_0) = 0$ implies that x_0 is an equilibrium.

Theorem 2 (Lyapunov Center Theorem)**.** *Assume that the differential system $\dot{x} = f(x)$ admits a nondegenerate integral and has an equilibrium point with exponents $\pm\omega i, \lambda_3, \ldots, \lambda_m$, where $i\omega \neq 0$ is pure imaginary. If λ_j/ω is never an integer* *for $j = 3, \ldots, m$, then there exists an one-parameter family of periodic orbits emanating from the equilibrium point. Moreover, when approaching the equilibrium point along the family, the periods tend to $2\pi/\omega$ and the nontrivial multipliers tend to $\exp(2\pi\lambda_j/\omega)$ for $j = 3, \ldots, m$.*

For a proof of Theorem 2 see Theorem 9.2.1 of [6].

Theorem 3. *The existence of the periodic orbits near the linear approximated periodic orbits closed to the equilibrium points L_i for $i = 1, 2, 3, 4$ of the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem studied in subsection 6.1 is proved using the Lyapunov center theorem.*

Proof. At the equilibrium points L_1 and L_4 we have a pair of purely imaginary eigenvalues $\pm 1.68164..i$ and a pair of real eigenvalues $\pm 1.42144..$, so the Lyapunov center theorem implies that there is a one-parameter family of periodic solutions emanating from each of these equilibrium points, points whose linear approximations are given in (14) and (15).

At the equilibrium points L_2 and L_3 we have a pair of purely imaginary eigenvalues $\pm 4.08808..i$ and a pair of real eigenvalues $\pm 5.52474..$, so the Lyapunov center theorem implies that there is an one-parameter family of periodic solutions emanating from each of these equilibrium points, points whose linear approximations are given in (14) and (15). □

6.3. **Hill's Orbits.** In this subsection we consider the case when the infinitesimal particle moves very close to one of the primaries. This is the Hill's problem. By symmetry, it is sufficient to study the motion of the infinitesimal mass near the masses m_1 and m_2.

Theorem 4. *For each primary there exist two one-parameter families of nearly circular periodic solutions of the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem that encircle the corresponding primary.*

Proof. We introduce a small parameter considered the infinitesimal particle to be very close to one of the primaries. We start with the primary situated at the origin of the coordinates system, i.e. the primary of mass m_2. We introduce a scale parameter ε by changing coordinates by $(x, y) = \varepsilon^2(\xi_1, \xi_2)$, $(p_x, p_y) = \varepsilon^{-1}(\eta_1, \eta_2)$ which is a symplectic change of coordinates with multiplier ε^{-1}. The Hamiltonian (1) becomes

$$H = \varepsilon^{-3}\Big[\frac{\|\eta\|^2}{2} - \frac{1}{\|\xi\|}\Big] - \omega\xi^T K\eta + \mathcal{O}(\varepsilon),$$

where we have dropped the constant terms because they do not affect the equations of motion and

$$K = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \|\xi\|^2 = \xi_1^2 + \xi_2^2, \quad \|\eta\|^2 = \eta_1^2 + \eta_2^2.$$

We scaled the time by $t \to \varepsilon^{-3}t$ and $H \to \varepsilon^3 H$, and the Hamiltonian becomes

$$H = \frac{\|\eta\|^2}{2} - \frac{1}{\|\xi\|} - \varepsilon^3\omega\xi^T K\eta + \mathcal{O}(\varepsilon^4).$$

For the other case when the infinitesimal particle is very close to one of the primaries which are symmetric with respect the origin, if we translate first the origin in one of these primaries and we apply the same change of coordinates as above we get the same Hamiltonian up to the terms of order three in ε.

If we introduce the polar symplectic coordinates (r, θ, R, Θ) as follows

$$\xi_1 = r\cos\theta, \qquad \xi_2 = r\sin\theta,$$
$$\eta_1 = R\cos\theta - \frac{\Theta}{r}\sin\theta, \;\; \eta_2 = R\sin\theta + \frac{\Theta}{r}\cos\theta,$$

we get the Hamiltonian

$$H = \frac{1}{2}\Big(R^2 + \frac{\Theta^2}{r^2}\Big) - \frac{1}{r} - \varepsilon^3\omega\Theta + \mathcal{O}(\varepsilon^4),$$

and its Hamiltonian equations are

$$\dot r = R, \qquad \dot R = \frac{\Theta^2}{r^3} - \frac{1}{r^2},$$
$$\dot\theta = \frac{\Theta}{r^2} - \varepsilon^3\omega, \qquad \dot\Theta = 0,$$

where the terms of order ε^4 have been omitted. We have two solutions $\Theta = \pm c, R = 0, r = c^2$, where c is a constant, which are periodic solutions for the above system, of period $2\pi c^3/(\omega c^3\varepsilon^3 \pm 1)$. If we linearize the r and R equations about these solutions we get

$$\dot r = R, \quad \dot R = -\frac{r}{c^6}.$$

These linear equations have solutions of the form $\exp(\pm it/c^3)$, and so the nontrivial multipliers of the circular orbits of the system are $\exp(\mp 2\pi i/(\omega c^3\varepsilon^3 \pm 1)) = 1 + 2\pi i\omega c^3\varepsilon^3 + \mathcal{O}(\varepsilon^6)$. Using the Poincaré continuation method, as in section 9.4 of [6], we can conclude that for each primary there exist two one-parameter families of nearly circular elliptic periodic solutions of the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem that encircle the corresponding primary. □

6.4. **Orbits close to infinity.** Now we study the orbits that are close to infinity.

Theorem 5. *There exist two one-parameter families of nearly circular periodic solutions of the circular restricted* 4*-body problem with three equal primaries in the collinear central configuration of the* 3*-body problem which tend to infinity.*

Proof. We scale the variables by $(x, y) = \varepsilon^{-2}(\xi_1, \xi_2)$, $(p_x, p_y) = \varepsilon^1(\eta_1, \eta_2)$ which is a symplectic change of coordinates with multiplier ε. The Hamiltonian (1) becomes

$$H = -\omega\xi^T K\eta + \varepsilon^3\Big[\frac{\|\eta\|^2}{2} - \frac{1}{\|\xi\|}\Big] + \mathcal{O}(\varepsilon^5).$$

When ε is small the infinitesimal body is near infinity and the Hamiltonian shows that near infinity the Coriolis force dominates, and the next important force looks like a Kepler problem with all three primaries at the origin. If we change to polar coordinates we get the Hamiltonian

$$H = -\omega\Theta + \varepsilon^3\Big[\frac{1}{2}\Big(R^2 + \frac{\Theta^2}{r^2}\Big) - \frac{1}{r}\Big] + \mathcal{O}(\varepsilon^5),$$

and its Hamiltonian equations are

$$\begin{aligned} \dot{r} &= \varepsilon^3 R, \qquad \dot{R} = \varepsilon^3\Big(\frac{\Theta}{r^3} - \frac{1}{r^2}\Big), \\ \dot{\theta} &= -\omega + \varepsilon^3 \frac{\Theta}{r^2}, \quad \dot{\Theta} = 0, \end{aligned} \tag{16}$$

where the terms of order ε^5 have been omitted. We get $\Theta = \pm 1$, $R = 0$, $r = 1$ are periodic solutions with period $2\pi/(\omega \mp \varepsilon^3)$ of (16). The linearized equations for r and R around these solutions are $\dot{r} = \varepsilon^3 R$, $\dot{R} = -\varepsilon^3 r$. These linear equations have solutions of the form $\exp(\pm i\varepsilon^3 t)$, and so the nontrivial multipliers of the circular orbits of the system are

$$\exp(\mp i\varepsilon^3 2\pi/(\omega \mp \varepsilon^3)).$$

Using the Poincaré continuation method we can conclude that there exist two one-parameter families of nearly circular periodic solutions of the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem which tend to infinity. □

6.5. **From the restricted to the full problem.** We now use the result in [5, 11] to show that the periodic solutions found above persist in the 4 body problem with one small mass. Recall that a periodic solution of a conservative Hamiltonian system always has the characteristic multiplier +1 with algebraic multiplicity at least 2. Thus we have the following definition: a periodic solution of a Hamiltonian conservative systems with no continuous symmetries is called *non-degenerate* or *elementary* periodic solution if the characteristic multiplier +1 has the algebraic multiplicity exactly equal to 2 [5]. If the system is symmetric under the symplectic action of a continuous group (e.g. admits rotational symmetry), then the same definition applies for the reduced Hamiltonian system on a fixed symplectic reduced space [11].

Theorem 6. *Any periodic solution of the circular restricted 4-body problem with three equal primaries in the collinear central configuration of the 3-body problem whose period is not a multiple of 2π can be continued into the full 4-body problem with one small mass and the other three masses equal.*

Taking into account the values of the multipliers found in this section, the periodic orbits at all equilibria are non-degenerate, and the proof of Theorem 6 is a particular case of Proposition 3 of [11].

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References

[1] **M. Arribas, A. Abad, A. Elipe, and M. Palacios,** *Equilibria of the symmetric collinear restricted four-body problem with radiation pressure*, **Astrophys. Space Sci., 361 (2016), 84 pp.**

[2] **A. N. Baltagiannis and K. E. Papadakis,** *Equilibrium Points and Their Stability in the Restricted Four-Body Problem*, **Internat. J. Bifur. Chaos Appl. Sci. Engrg., 21 (2011), 2179–2193.**

[3] **J. Llibre,** *On the extendable regular integrals of the n-body problem*, **J. Differential Equations 54 (1984), 221–230.**

[4] D. Maranhão and J. Llibre, *Ejection-collision orbits and invariant punctured tori in a restricted four-body problem*, Celestial Mech. Dynam. Astronom., **71** (1998), 1–14.

[5] **K.R. Meyer,** *Periodic solutions of the N-body problem*, **Lecture Notes in Mathematics 1719, Springer, 1999.**

[6] K.R. Meyer, R. Hall and D.C. Offin, *Introduction to Hamiltonian Dynamical Systems and the N-Body Problem*, Springer, 2017.

[7] **M. Michalodimitrakis,** *The Circular Restricted Four-Body Problem*, **Astrophys. Space Sci., 75 (1981), 289–305.**

[8] **M. Palacios, M. Arribas, A. Abad, and A, Elipe,** *Symmetric periodic orbits in the Moulton-Copenhagen problem*, **Celestial Mech. Dynam. Astronom., 131 (2019), 16 pp**

[9] **A.E. Roy,** *Orbital motion*, **Adam Hilger Ltd, Bristol, 1982.**

[10] **C.L. Siegel and J.K. Moser,** *Lectures on celestial mechanics*, **Classics in Mathematics. Springer-Verlag, Berlin, 1995.**

[11] C. Stoica, *An approximation theorem in classical mechanics*, J. Geom. Mech., **8** (2016), 359–374.

[12] V. Szebehely, *Theory of orbits. The restricted problem of three bodies*, Academic Press, New York, 1967.

[13] M. Valtonen and H. Karttunen, *The three-body problem*, Cambridge University Press, 2005.

[14] **A. Wintner,** *The Analytical Foundations of Celestial Mechanics*, **Princeton University Press, 1941.**

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