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Riesz bases of exponentials for convex polytopes with symmetric faces

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Abstract. We prove that for any convex polytope $\Omega \subset \mathbb{R}^d$ which is centrally symmetric and whose faces of all dimensions are also centrally symmetric, there exists a Riesz basis of exponential functions in the space $L^2(\Omega)$. The result is new in all dimensions d greater than one.

Keywords. Riesz bases, sampling and interpolation, convex polytopes

1. Introduction

1.1.

Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set of positive measure. When is it possible to find a countable set $\Lambda \subset \mathbb{R}^d$ such that the system of exponential functions

$$E(\Lambda) = \{e_\lambda\}_{\lambda \in \Lambda}, \quad e_\lambda(x) = e^{2\pi i \langle \lambda, x \rangle},$$

constitute a basis in the space $L^2(\Omega)$?

The answer depends on what we mean by a "basis". The best one can hope for is to have an *orthogonal basis* of exponentials. The problem of which domains admit an orthogonal basis $E(\Lambda)$ has been extensively studied and goes back to Fuglede [2], and it is well known that many reasonable domains do not have such a basis. For example, it was recently proved in [8] that a convex domain $\Omega \subset \mathbb{R}^d$ admits an orthogonal basis of exponential functions if and only if Ω is a convex polytope which can tile the space by translations (meaning that one can fill the whole space by translated copies of Ω with non-overlapping interiors). In particular, a disk or a triangle in the plane does not have an orthogonal basis $E(\Lambda)$ (this was shown already in [2]).

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If a domain $\Omega \subset \mathbb{R}^d$ does not have an orthogonal basis of exponentials, then a *Riesz basis* is the next best thing one can hope for. A Riesz basis can be defined as the image of an orthonormal basis under a bounded and invertible linear map (there are also other, equivalent definitions, see Section 2) and it shares many of the qualities of an orthonormal basis. In particular, if $E(\Lambda)$ is a Riesz basis in $L^2(\Omega)$ then any function f from the space has a unique and stable Fourier series expansion $f = \sum_{\lambda \in \Lambda} c_\lambda e_\lambda$.

In sharp contrast to the situation with orthogonal bases, no single example is known of a set $\Omega \subset \mathbb{R}^d$ which does not have a Riesz basis of exponentials. At the same time, for most domains it remains unknown whether one can construct a Riesz basis of this type. One of the relatively few results obtained in this direction says that any finite union of intervals in $\mathbb{R}$ has a Riesz basis of exponentials [6] (see also [7] for a multi-dimensional version of this result). However it is still an open problem of whether, say, a disk or a triangle in the plane admits a Riesz basis $E(\Lambda)$.

1.2.

The present paper is concerned with the existence of Riesz bases of exponentials for *convex polytopes* in $\mathbb{R}^d$. Our main result can be stated as follows:

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^d$ be a convex polytope which is centrally symmetric and all of whose faces of all dimensions are also centrally symmetric. Then there is a set $\Lambda \subset \mathbb{R}^d$ such that the system of exponential functions $E(\Lambda)$ is a Riesz basis in $L^2(\Omega)$.*

The result is new in all dimensions d greater than one.

In [9], Lyubarskii and Rashkovskii established the existence of a Riesz basis of exponentials for convex, centrally symmetric polygons in $\mathbb{R}^2$ such that all the vertices of the polygon lie on the integer lattice $\mathbb{Z}^2$ (one may alternatively assume that the vertices lie on some other lattice, due to the invariance of the problem under affine transformations).[1] The approach in [9] involves methods from the theory of entire functions of two complex variables. The paper [9] contains also a weaker result for convex, centrally symmetric polygons whose vertices fail to lie on a lattice, but in this case the result does not amount to the construction of a Riesz basis of exponentials.

A similar result in higher dimensions was obtained in [3, Corollary 3], where the existence of a Riesz basis of exponentials was established for centrally symmetric polytopes in $\mathbb{R}^d$ with centrally symmetric facets, such that all the vertices of the polytope lie on the lattice $\mathbb{Z}^d$. The proof is based on the fact that such a polytope *multi-tiles* the space by lattice translates (in connection with this result, see also [4, 5]).

In this paper, our goal is to prove the existence of a Riesz basis of exponentials for convex, centrally symmetric polytopes with centrally symmetric faces, *without imposing any extra constraints*. This is the content of our main result, Theorem 1.1.

[1] The assumption that all the vertices of the polygon lie on a lattice was stated in [9] in a different (equivalent) form, by imposing a system of arithmetic constraints given in [9, Proposition 3.1 (v)].

1.3.

Our approach to the proof of Theorem 1.1 is inspired by the paper [13] due to Walnut. In that paper, the author applies a technique outlined in [1] in order to construct a system of exponentials $E(\Lambda)$ that is shown to be *complete* in the space $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^2$ is a convex, centrally symmetric polygon. The set Λ constructed in [13] is the union of a finite number of shifted lattices in $\mathbb{R}^2$. It is shown that if the convex polygon satisfies certain extra arithmetic constraints given in [13, Theorem 4.2], then $E(\Lambda)$ is not only a complete system, but is in fact a Riesz basis, in $L^2(\Omega)$.

In [13] the author does not provide any transparent description as to which convex, centrally symmetric polygons satisfy the extra constraints imposed in [13, Theorem 4.2]. One can verify though that these constraints are satisfied if and only if, possibly after applying an affine transformation, all the vertices of the polygon lie in $\mathbb{Z}^2$. Hence the class of planar convex polygons for which a Riesz basis $E(\Lambda)$ is constructed in [13] coincides with the class covered by the result in [9].

In this paper we will extend the technique from [13] to all dimensions, and also combine it with the Paley–Wiener theorem about the stability of Riesz bases under small perturbations (see Section 2). This will allow us to construct a Riesz basis $E(\Lambda)$ for any convex, centrally symmetric polytope in $\mathbb{R}^d$ with centrally symmetric faces, without imposing any extra arithmetic constraints. The set of frequencies Λ in our construction will no longer be a union of finitely many shifted lattices, but it will rather have a less regular structure.

2. Preliminaries

2.1. Riesz bases

Let H be a separable Hilbert space. A system of vectors $\{f_n\} \subset H$ is called a *Riesz basis* if it is the image of an orthonormal basis under a bounded and invertible linear map. If $\{f_n\}$ is a Riesz basis, then any $f \in H$ admits a unique expansion in a series $f = \sum c_n f_n$, and the coefficients $\{c_n\}$ satisfy $A\|f\|^2 \leqslant \sum |c_n|^2 \leqslant B\|f\|^2$ for some positive constants A, B which do not depend on f. In fact, the latter condition for the system $\{f_n\}$ can serve as an equivalent definition of a Riesz basis.

There are also several other ways to characterize Riesz bases in a separable Hilbert space H. The following characterization, see [14, Section 4.4, Theorem 8], will be used in the present paper:

Proposition 2.1. *A system $\{f_n\} \subset H$ is a Riesz basis if and only if it satisfies the following three conditions:*

(1) *$\{f_n\}$ is a complete system in H,*

(2) *for every $f \in H$ we have $\sum_n |\langle f, f_n\rangle|^2 < \infty$,*

(3) *given any sequence of scalars $\{c_n\}$ such that $\sum_n |c_n|^2 < \infty$, there exists at least one $f \in H$ satisfying $\langle f, f_n\rangle = c_n$ for all n.*

For a discussion about the various properties and characterizations of Riesz bases we refer the reader to [14].

2.2. *Paley–Wiener spaces*

Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set of positive measure. The Paley–Wiener space $\mathrm{PW}(\Omega)$ consists of all functions $F \in L^2(\mathbb{R}^d)$ which are Fourier transforms of functions from $L^2(\Omega)$, namely,

$$F(t) = \int_{\Omega} f(x) e^{-2\pi i \langle t, x \rangle} \, dx, \quad f \in L^2(\Omega).$$

A set $\Lambda \subset \mathbb{R}^d$ is called a *set of uniqueness* for the space $\mathrm{PW}(\Omega)$ if whenever a function F from the space satisfies $F(\lambda) = 0$, $\lambda \in \Lambda$, then F is identically zero. This means that the functions from the space $\mathrm{PW}(\Omega)$ are uniquely determined by their values on Λ.

We say that Λ is a *set of interpolation* for $\mathrm{PW}(\Omega)$ if for any $\{c(\lambda)\} \in \ell^2(\Lambda)$ there exists at least one $F \in \mathrm{PW}(\Omega)$ satisfying $F(\lambda) = c(\lambda)$, $\lambda \in \Lambda$. Such a function F is said to solve the interpolation problem with the set of nodes Λ and with the values $\{c(\lambda)\}$.

The Fourier transform is a unitary map from the space $L^2(\Omega)$ onto $\mathrm{PW}(\Omega)$. This allows to reformulate the uniqueness and interpolation properties of a set $\Lambda \subset \mathbb{R}^d$ with respect to the space $\mathrm{PW}(\Omega)$, in terms of properties of the system of exponential functions $E(\Lambda)$ in the space $L^2(\Omega)$. Thus Λ is a set of uniqueness for $\mathrm{PW}(\Omega)$ if and only if $E(\Lambda)$ is a complete system in $L^2(\Omega)$; while Λ is a set of interpolation for $\mathrm{PW}(\Omega)$ if and only if the system of equations $\langle f, e_\lambda \rangle = c(\lambda)$, $\lambda \in \Lambda$, admits at least one solution $f \in L^2(\Omega)$ whenever the scalars $\{c(\lambda)\}$ belong to $\ell^2(\Lambda)$.

2.3. *Uniformly discrete sets*

A set $\Lambda \subset \mathbb{R}^d$ is said to be *uniformly discrete* if there is $\delta > 0$ such that $|\lambda' - \lambda| \geqslant \delta$ for any two distinct points λ, λ' in Λ. The following proposition may be found for instance in [10, Proposition 2.7].

Proposition 2.2. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set of positive measure, and let $\Lambda \subset \mathbb{R}^d$ be a uniformly discrete set. Then there is a constant $C = C(\Omega, \Lambda)$ such that*

$$\sum_{\lambda \in \Lambda} |F(\lambda)|^2 \leqslant C \|F\|^2_{L^2(\mathbb{R}^d)}$$

for every function $F \in \mathrm{PW}(\Omega)$.

It is well known that if $\Omega \subset \mathbb{R}^d$ is a bounded, measurable set of positive measure, and if $\Lambda \subset \mathbb{R}^d$ is a set of interpolation for the space $\mathrm{PW}(\Omega)$, then Λ must be a uniformly discrete set, see e.g. [10, Section 4.2.1]. Due to Propositions 2.1 and 2.2, this implies the following characterization of Riesz bases of exponentials in the space $L^2(\Omega)$:

Proposition 2.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set of positive measure, and let $\Lambda \subset \mathbb{R}^d$. The system of exponentials $E(\Lambda)$ is a Riesz basis in $L^2(\Omega)$ if and only if Λ is a set of both uniqueness and interpolation for the space* $\mathrm{PW}(\Omega)$.

2.4. Stability

We will need the following result which goes back to Paley and Wiener [11] about the stability of Riesz bases of exponentials under small perturbations:

Proposition 2.4. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set of positive measure, and let $\Lambda = \{\lambda_n\}$ be a sequence of points in $\mathbb{R}^d$ such that the system $E(\Lambda)$ is a Riesz basis in $L^2(\Omega)$. Then there is a constant $\eta = \eta(\Omega, \Lambda) > 0$ such that if a sequence $\Lambda' = \{\lambda'_n\}$ satisfies $|\lambda'_n - \lambda_n| \leqslant \eta$ for all n, then also $E(\Lambda')$ is a Riesz basis in $L^2(\Omega)$.*

For a proof of this result the reader may consult [10, Section 4.3].

2.5. Convex polytopes

A set $\Omega \subset \mathbb{R}^d$ is called a *convex polytope* if Ω is the convex hull of a finite number of points. Equivalently, a convex polytope is a bounded set which can be represented as the intersection of finitely many closed halfspaces.

A convex polytope $\Omega \subset \mathbb{R}^d$ is said to be *centrally symmetric* if the set $-\Omega$ is a translate of Ω. In this case, there exists a unique point $x \in \mathbb{R}^d$ such that $-\Omega + x = \Omega - x$, and we say that Ω is symmetric with respect to the point x.

A *zonotope* in $\mathbb{R}^d$ is a set Ω which can be represented as the Minkowski sum of a finite number of line segments, that is, $\Omega = S_1 + S_2 + \cdots + S_n$, where each one of the sets $S_1, S_2, \dots, S_n$ is a line segment in $\mathbb{R}^d$. A zonotope in $\mathbb{R}^d$ can be equivalently defined as the image of a cube in $\mathbb{R}^n$ under an affine map from $\mathbb{R}^n$ to $\mathbb{R}^d$.

A zonotope is a convex, centrally symmetric polytope, and all its faces of all dimensions are also zonotopes. In particular, all the faces of a zonotope are centrally symmetric. The following proposition shows that the converse is also true:

Proposition 2.5 (see [12, Theorem 3.5.2]). *Let Ω be a convex polytope in $\mathbb{R}^d$. Then the following conditions are equivalent:*

(1) *Ω is a zonotope (i.e. Ω is the Minkowski sum of finitely many line segments).*

(2) *Ω is centrally symmetric and all its faces of all dimensions are also centrally symmetric.*

(3) *all the two-dimensional faces of Ω are centrally symmetric.*

Thus, for example, any convex centrally symmetric polygon $\Omega \subset \mathbb{R}^2$ is a zonotope.

3. Cylindric sets

3.1.

We denote a point in $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ as (x, y), where $x \in \mathbb{R}^{d-1}$ and $y \in \mathbb{R}$.

Definition 3.1. A bounded, measurable set $\Sigma \subset \mathbb{R}^d$ will be called a *cylindric set* if there exists a bounded, measurable set $\Pi \subset \mathbb{R}^{d-1}$ and a bounded, measurable function

$\varphi : \Pi \to \mathbb{R}$ such that

$$\Sigma = \{(x, y) : x \in \Pi,\ \varphi(x) \leqslant y \leqslant \varphi(x) + 1\}. \tag{3.1}$$

In other words, Σ is a cylindric set if $\Sigma \subset \Pi \times \mathbb{R}$ and for every $x \in \Pi$, the set $\{y : (x, y) \in \Sigma\}$ is an interval of length exactly 1, where the position of this interval is allowed to depend on x. The set Π will be called the *base* of the cylindric set Σ.

In the special case when the function φ in (3.1) is constant, the cylindric set Σ has a cartesian product structure, namely $\Sigma = \Pi \times I$ where I is an interval of length 1. In this case it is well known that if there is a set $\Gamma \subset \mathbb{R}^{d-1}$ such that the system $E(\Gamma)$ is a Riesz basis in $L^2(\Pi)$, then the system $E(\Gamma \times \mathbb{Z})$ is a Riesz basis in $L^2(\Sigma)$.

We will use the fact that the latter assertion remains valid for arbitrary cylindric sets of the form (3.1), not only for those with a cartesian product structure:

Lemma 3.2. *Let $\Sigma \subset \mathbb{R}^d$ be a cylindric set with base Π, and suppose that there is a set $\Gamma \subset \mathbb{R}^{d-1}$ such that the system of exponentials $E(\Gamma)$ is a Riesz basis in $L^2(\Pi)$. Then the system $E(\Gamma \times \mathbb{Z})$ is a Riesz basis in $L^2(\Sigma)$.*

The proof of this lemma for arbitrary cylindric sets is similar to the one for cartesian products, which should be well known. We were not able to find the proof in the literature though, so we include it below for completeness.

3.2.

For the proof of Lemma 3.2 we will use the following characterization of the exponential systems $E(\Lambda)$ that form a Riesz basis in the space $L^2(\Omega)$.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^d$ be a bounded, measurable set of positive measure, and let $\Lambda \subset \mathbb{R}^d$. The system of exponentials $E(\Lambda)$ is a Riesz basis in $L^2(\Omega)$ if and only if there exists a constant M such that the following two conditions hold:*

(1) $\sum_\lambda |c(\lambda)|^2 \leqslant M \|\sum_\lambda c(\lambda) e_\lambda\|^2_{L^2(\Omega)}$ *whenever $\{c(\lambda)\}$, $\lambda \in \Lambda$, is a sequence of scalars with only finitely many nonzero elements,*

(2) $\|f\|^2_{L^2(\Omega)} \leqslant M \sum_\lambda |\langle f, e_\lambda\rangle|^2$ *for any $f \in L^2(\Omega)$.*

We note that condition (1) holds if and only if Λ is a set of interpolation for the space $\mathrm{PW}(\Omega)$, see [10, Sections 4.1 and 4.2]. It follows from (1) that Λ must be a uniformly discrete set.

If condition (2) holds, then Λ is said to be a *set of stable sampling* for the space $\mathrm{PW}(\Omega)$, see [10, Section 2.5]. This condition implies in particular that Λ is a set of uniqueness for the space.

3.3. Proof of Lemma 3.2

We suppose that $\Sigma \subset \mathbb{R}^d$ is a cylindric set of the form (3.1), and that $\Gamma \subset \mathbb{R}^{d-1}$ is a set such that the system of exponentials $E(\Gamma)$ is a Riesz basis in $L^2(\Pi)$. We must prove that the system $E(\Gamma \times \mathbb{Z})$ is a Riesz basis in $L^2(\Sigma)$. We will show that the two conditions (1) and (2) in Proposition 3.3 are satisfied.

We will use γ to denote an element of Γ, and n to denote an element of $\mathbb{Z}$.

First we check that (1) holds. Let $\{c(\gamma, n)\}$ be a sequence of scalars with only finitely many nonzero elements. We must show that $\sum_{\gamma,n} |c(\gamma, n)|^2 \leqslant M \|f\|^2_{L^2(\Sigma)}$, where

$$f(x, y) := \sum_{\gamma,n} c(\gamma, n) e_\gamma(x) e_n(y).$$

Let $\psi_n(x) := \sum_\gamma c(\gamma, n) e_\gamma(x)$ and $I(x) := [\varphi(x), \varphi(x) + 1]$. Then we have

$$\int_{I(x)} |f(x, y)|^2 \, dy = \int_{I(x)} \Big| \sum_n \psi_n(x) e_n(y) \Big|^2 dy = \sum_n |\psi_n(x)|^2$$

since $E(\mathbb{Z})$ is an orthonormal basis in $L^2(I(x))$ for every x. In turn, this implies

$$\|f\|^2_{L^2(\Sigma)} = \int_\Pi \int_{I(x)} |f(x, y)|^2 \, dy \, dx = \sum_n \|\psi_n\|^2_{L^2(\Pi)} \geqslant \frac{1}{M} \sum_n \sum_\gamma |c(\gamma, n)|^2,$$

where the last inequality holds for a certain constant $M = M(\Pi, \Gamma)$ since the system $E(\Gamma)$ is a Riesz basis in $L^2(\Pi)$. This confirms that condition (1) is indeed satisfied.

Next we check that (2) holds. Let $f \in L^2(\Sigma)$, then we have

$$\|f\|^2_{L^2(\Sigma)} = \int_\Pi \int_{I(x)} |f(x, y)|^2 \, dy \, dx = \int_\Pi \Big(\sum_n |\phi_n(x)|^2 \Big) dx,$$

where

$$\phi_n(x) := \int_{I(x)} f(x, y) \overline{e_n(y)} \, dy.$$

This is due to the fact that $E(\mathbb{Z})$ is an orthonormal basis in $L^2(I(x))$. It follows that

$$\|f\|^2_{L^2(\Sigma)} = \sum_n \|\phi_n\|^2_{L^2(\Pi)} \leqslant M \sum_n \sum_\gamma |\langle \phi_n, e_\gamma \rangle|^2 = M \sum_n \sum_\gamma |\langle f, e_{\gamma,n} \rangle|^2, \quad (3.2)$$

where we denote $e_{\gamma,n}(x, y) := e_\gamma(x) e_n(y)$. Observe that the inequality in (3.2) holds since $E(\Gamma)$ is a Riesz basis in $L^2(\Pi)$. This establishes (2) and so the lemma is proved. ■

4. Decomposition of functions with a zonotope spectrum

4.1.

Suppose that we are given n vectors $u_1, u_2, \dots, u_n$ in $\mathbb{R}^d$. The origin-symmetric zonotope generated by these vectors is the set

$$\Omega_n = \Big\{ \sum_{j=1}^n t_j u_j : t_1, t_2, \dots, t_n \in [-\tfrac{1}{2}, \tfrac{1}{2}] \Big\}. \quad (4.1)$$

This set is the Minkowski sum of the line segments $[-\frac{1}{2} u_j, \frac{1}{2} u_j]$, $1 \leqslant j \leqslant n$, and so it is indeed a zonotope in $\mathbb{R}^d$.

We observe that if the linear span of the vectors $u_1, u_2, \dots, u_n$ is the whole $\mathbb{R}^d$, then the zonotope Ω_n has nonempty interior; while if these vectors do not span the whole $\mathbb{R}^d$ then Ω_n is contained in some hyperplane, and it is then a set of measure zero.

Lemma 4.1. *Assume that the first $n-1$ vectors $u_1, u_2, \dots, u_{n-1}$ span the whole $\mathbb{R}^d$, and that we have $u_n = (0, 0, \dots, 0, 1)$. Then there is a cylindric set $\Sigma_n \subset \mathbb{R}^d$ whose base Π_n is a zonotope in $\mathbb{R}^{d-1}$, such that the following holds:*

(1) *Any function $F \in \mathrm{PW}(\Omega_n)$ can be represented in the form*

$$F(x, y) = G(x, y) + H(x, y)\sin(\pi y), \quad (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R}, \tag{4.2}$$

for some $G \in \mathrm{PW}(\Sigma_n)$ and some $H \in \mathrm{PW}(\Omega_{n-1})$. Here we denote by Ω_{n-1} the origin-symmetric zonotope in $\mathbb{R}^d$ generated by the vectors $u_1, u_2, \dots, u_{n-1}$.

(2) *Conversely, for any two functions $G \in \mathrm{PW}(\Sigma_n)$ and $H \in \mathrm{PW}(\Omega_{n-1})$, the function F defined by (4.2) belongs to the space $\mathrm{PW}(\Omega_n)$.*

This is an extension to all dimensions of [13, Lemma 3.1] where the result was proved for a convex, centrally symmetric polygon Ω_n in two dimensions. In the two-dimensional case, the set Σ_n was taken to be a parallelogram that shares with Ω_n its two edges parallel to the vector u_n. In dimensions greater than two, we cannot in general take the set Σ_n in Lemma 4.1 to be a parallelepiped, nor any other type of convex polytope inscribed in Ω_n in a similar way. Instead, the role of the parallelogram will be played in higher dimensions by the cylindric sets introduced in Definition 3.1.

The assumption in Lemma 4.1 that the first $n-1$ vectors $u_1, u_2, \dots, u_{n-1}$ span the whole $\mathbb{R}^d$ is made so as to ensure that the zonotope Ω_{n-1} has nonempty interior.

4.2. *Proof of Lemma 4.1*

By identifying the elements F, G and H of the spaces $\mathrm{PW}(\Omega_n)$, $\mathrm{PW}(\Sigma_n)$ and $\mathrm{PW}(\Omega_{n-1})$ with the Fourier transforms of functions f, g and h from $L^2(\Omega_n)$, $L^2(\Sigma_n)$ and $L^2(\Omega_{n-1})$, respectively, conditions (1) and (2) of Lemma 4.1 can be reformulated as follows: any function $f \in L^2(\Omega_n)$ admits a representation of the form

$$f(x, y) = g(x, y) + \frac{h(x, y+\frac{1}{2}) - h(x, y-\frac{1}{2})}{2i} \quad \text{a.e. } (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R} \tag{4.3}$$

for some $g \in L^2(\Sigma_n)$ and some $h \in L^2(\Omega_{n-1})$; and conversely, for any two functions $g \in L^2(\Sigma_n)$ and $h \in L^2(\Omega_{n-1})$, the function f defined by (4.3) belongs to $L^2(\Omega_n)$.

(Notice that we think of a function from the space $L^2(\Omega_n)$, $L^2(\Sigma_n)$ or $L^2(\Omega_{n-1})$ as a function on the whole space $\mathbb{R}^d$ which is assumed to vanish a.e. outside the set Ω_n, Σ_n or Ω_{n-1}, respectively.)

Let $\Pi_n \subset \mathbb{R}^{d-1}$ be the image of Ω_{n-1} under the map $(x, y) \mapsto x$. If we denote $u_j = (v_j, w_j)$, where $v_j \in \mathbb{R}^{d-1}$ and $w_j \in \mathbb{R}$, then Π_n is the origin-symmetric zonotope generated by the vectors $v_1, v_2, \dots, v_{n-1}$. These vectors span the whole space $\mathbb{R}^{d-1}$ and hence the zonotope Π_n has nonempty interior.

For each $x \in \Pi_n$ we let

$$S(x) := \{y \in \mathbb{R} : (x, y) \in \Omega_{n-1}\}.$$

By the definition of Π_n the set $S(x)$ is nonempty. Since Ω_{n-1} is closed and convex, the set $S(x)$ is a closed interval. We may therefore denote $S(x) := [a(x), b(x)]$. Since

Ω_{n-1} is a convex polytope, each one of $a(x)$, $b(x)$ is a continuous, piecewise linear function on Π_n. (This can be deduced from the representation of Ω_{n-1} as the intersection of finitely many closed halfspaces.)

The zonotope Ω_n is the Minkowski sum of Ω_{n-1} and the line segment $[-\frac{1}{2}u_n, \frac{1}{2}u_n]$. Since we have assumed that $u_n = (0, 0, \dots, 0, 1)$ this implies that

$$\Omega_n = \{(x, y) : x \in \Pi_n,\ a(x) - \tfrac{1}{2} \leqslant y \leqslant b(x) + \tfrac{1}{2}\}, \tag{4.4}$$

and in particular we have

$$(\Omega_{n-1} - \tfrac{1}{2}u_n) \cup (\Omega_{n-1} + \tfrac{1}{2}u_n) \subset \Omega_n.$$

Let $\varphi : \Pi_n \to \mathbb{R}$ be any bounded, measurable function satisfying

$$a(x) \leqslant \varphi(x) \leqslant b(x), \quad x \in \Pi_n. \tag{4.5}$$

(For example, one may take $\varphi(x) := a(x)$.) We define a cylindric set Σ_n by

$$\Sigma_n := \{(x, y) : x \in \Pi_n, \varphi(x) - \tfrac{1}{2} \leqslant y \leqslant \varphi(x) + \tfrac{1}{2}\}. \tag{4.6}$$

It follows from (4.4), (4.5) and (4.6) that Σ_n is a subset of Ω_n (see Figure 4.1).

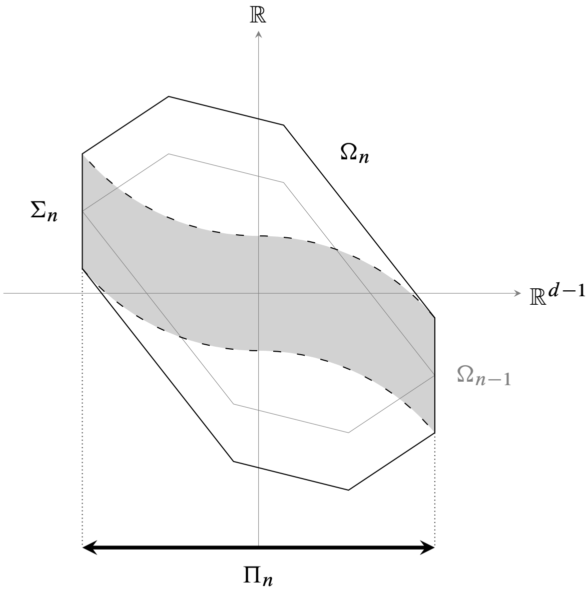

Fig. 4.1. The various sets involved in the statement and proof of Lemma 4.1 are illustrated. The larger polygon represents Ω_n, the smaller polygon is Ω_{n-1}, while the shaded region is the cylindric set Σ_n.

Suppose now that two functions $g \in L^2(\Sigma_n)$ and $h \in L^2(\Omega_{n-1})$ are given, and let f be the function defined by (4.3). Then f is supported on the set

$$\Sigma_n \cup (\Omega_{n-1} - \tfrac{1}{2}u_n) \cup (\Omega_{n-1} + \tfrac{1}{2}u_n)$$

which is contained in Ω_n. This shows that f belongs to $L^2(\Omega_n)$.

Conversely, suppose that we are given a function $f \in L^2(\Omega_n)$. We will show that f admits a representation of the form (4.3) where $g \in L^2(\Sigma_n)$ and $h \in L^2(\Omega_{n-1})$.

First we define the function h by

$$h(x,y) := 2i \sum_{k \geqslant 0} f(x, y-k-\tfrac{1}{2}), \qquad x \in \Pi_n,\ y < \varphi(x), \tag{4.7}$$

$$h(x,y) := -2i \sum_{k \geqslant 0} f(x, y+k+\tfrac{1}{2}), \quad x \in \Pi_n,\ y > \varphi(x), \tag{4.8}$$

and $h(x,y) := 0$ if $x \notin \Pi_n$. (Notice that it is not necessary to define the values of h on the set of points (x,y) such that $x \in \Pi_n$, $y = \varphi(x)$, as this is a set of measure zero.)

We observe that there is a constant M such that the nonzero terms in the sum in either (4.7) or (4.8) correspond only to values of k that are not greater than M. This is due to the assumption that f is supported on Ω_n. Indeed, one can check using (4.4) and (4.5) that it suffices to take $M := \max\,(b(x) - a(x))$, $x \in \Pi_n$. This shows that h is a well-defined function in $L^2(\mathbb{R}^d)$. It also follows from (4.4) that $h(x,y) = 0$ whenever $x \in \Pi_n$ and $y \notin [a(x), b(x)]$. We conclude that h is supported on Ω_{n-1}, so $h \in L^2(\Omega_{n-1})$.

Next we define the function g by

$$g(x,y) := f(x,y) - \frac{h(x, y+\frac{1}{2}) - h(x, y-\frac{1}{2})}{2i}, \quad (x,y) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$

Then g is a function in $L^2(\mathbb{R}^d)$ and (4.3) is satisfied. It remains only to show that

$$g(x,y) = 0 \quad \text{a.e. } (x,y) \in \mathbb{R}^d \setminus \Sigma_n. \tag{4.9}$$

It will be enough to verify this for $x \in \Pi_n$. Since $(x,y) \notin \Sigma_n$ we have two possibilities, either $y < \varphi(x) - \frac{1}{2}$ or $y > \varphi(x) + \frac{1}{2}$. In the former case, we obtain from (4.7) that

$$\begin{aligned}\frac{h(x, y+\frac{1}{2}) - h(x, y-\frac{1}{2})}{2i} &= \sum_{k \geqslant 0} f(x, y-k) - \sum_{k \geqslant 0} f(x, y-k-1) \\ &= f(x,y) \quad \text{a.e.,}\end{aligned}$$

while in the latter case, (4.8) implies that

$$\begin{aligned}\frac{h(x, y+\frac{1}{2}) - h(x, y-\frac{1}{2})}{2i} &= -\sum_{k \geqslant 0} f(x, y+k+1) + \sum_{k \geqslant 0} f(x, y+k) \\ &= f(x,y) \quad \text{a.e.}\end{aligned}$$

Condition (4.9) is therefore established. We have thus constructed the desired representation for the function f, and this completes the proof of Lemma 4.1. ■

5. Construction of Riesz bases for zonotopes

In this section we prove our main result, Theorem 1.1. The theorem asserts that if $\Omega \subset \mathbb{R}^d$ is a convex, centrally symmetric polytope and all its faces of all dimensions are also centrally symmetric, then there is a set $\Lambda \subset \mathbb{R}^d$ such that the system of exponentials $E(\Lambda)$ is a Riesz basis in $L^2(\Omega)$. By Proposition 2.5, a convex polytope $\Omega \subset \mathbb{R}^d$ satisfies the assumptions in Theorem 1.1 if and only if it is a zonotope. Hence Theorem 1.1 can be equivalently stated in the following way:

Theorem 5.1. *Let $\Omega_n \subset \mathbb{R}^d$ be an origin-symmetric zonotope of the form (4.1) which is generated by n vectors $u_1, u_2, \dots, u_n$ that span the whole $\mathbb{R}^d$. Then there is a set $\Lambda_n \subset \mathbb{R}^d$ such that the system $E(\Lambda_n)$ is a Riesz basis in $L^2(\Omega_n)$.*

Proof. We may assume that no two of the vectors $u_1, u_2, \dots, u_n$ are collinear. We will prove the assertion by induction both on the dimension d and on the number n of the vectors generating the zonotope Ω_n.

The induction base case is when $n = d$ and then Ω_n is generated by d linearly independent vectors in $\mathbb{R}^d$. In this case Ω_n is a d-dimensional parallelepiped and so we know that it admits a Riesz basis (in fact, an orthogonal basis) of exponentials.

Suppose now that $n > d$, and therefore the dimension d must be at least two and the vectors $u_1, u_2, \dots, u_n$ are not linearly independent. In this case we may reorder them so that the first $n-1$ vectors $u_1, u_2, \dots, u_{n-1}$ already span the whole space $\mathbb{R}^d$. We may also assume, by applying an invertible linear map, that the last vector $u_n = (0, 0, \dots, 0, 1)$.

We now observe that all the assumptions in Lemma 4.1 are satisfied. It thus follows from the lemma that there exists a cylindric set $\Sigma_n \subset \mathbb{R}^d$ whose base Π_n is a zonotope in $\mathbb{R}^{d-1}$, such that any function $F \in \mathrm{PW}(\Omega_n)$ can be represented in the form (4.2), where $G \in \mathrm{PW}(\Sigma_n)$ and $H \in \mathrm{PW}(\Omega_{n-1})$; and conversely, given any two functions $G \in \mathrm{PW}(\Sigma_n)$ and $H \in \mathrm{PW}(\Omega_{n-1})$, the function F defined by (4.2) belongs to $\mathrm{PW}(\Omega_n)$.

The base Π_n of the cylindric set Σ_n is a zonotope in $\mathbb{R}^{d-1}$ with nonempty interior, hence by the inductive hypothesis there is a set $\Gamma_n \subset \mathbb{R}^{d-1}$ such that the system $E(\Gamma_n)$ is a Riesz basis in the space $L^2(\Pi_n)$. By Lemma 3.2 the system $E(\Gamma_n \times \mathbb{Z})$ is then a Riesz basis in $L^2(\Sigma_n)$.

The zonotope Ω_{n-1} is generated by the $n-1$ vectors $u_1, u_2, \dots, u_{n-1}$ in $\mathbb{R}^d$ whose linear span is the whole $\mathbb{R}^d$. Hence, again by the inductive hypothesis, there is a set $\Lambda_{n-1} \subset \mathbb{R}^d$ such that the system $E(\Lambda_{n-1})$ is a Riesz basis in $L^2(\Omega_{n-1})$.

We now invoke the Paley–Wiener stability result given in Proposition 2.4. The result says that there is a constant $\eta = \eta(\Omega_{n-1}, \Lambda_{n-1}) > 0$ such that if $\Lambda'_{n-1} \subset \mathbb{R}^d$ is any set obtained from Λ_{n-1} by perturbing each element by distance at most η, then the system $E(\Lambda'_{n-1})$ is also a Riesz basis in $L^2(\Omega_{n-1})$.

Recall that we denote a point in $\mathbb{R}^d = \mathbb{R}^{d-1} \times \mathbb{R}$ as (x, y), where $x \in \mathbb{R}^{d-1}$, $y \in \mathbb{R}$. Let a mapping $\phi : \mathbb{R}^d \to \mathbb{R}^d$ be defined by the requirement that if $\phi(x, y) = (x', y')$, then $x' = x$ and y' is a point closest to y in the set

$$\mathbb{R} \setminus \bigcup_{k \in \mathbb{Z}} (k - \eta, k + \eta) \tag{5.1}$$

(notice that if y is an integer, then there are two possible choices for y'). It is obvious that such a mapping ϕ exists whenever $0 < \eta \leqslant \frac{1}{2}$, which we may assume by taking η smaller if necessary. We then consider the set $\Lambda'_{n-1} := \phi(\Lambda_{n-1})$ defined to be the image of Λ_{n-1} under the mapping ϕ. Then each element of Λ'_{n-1} is obtained by perturbing an element of the set Λ_{n-1} by distance at most η. Hence the system $E(\Lambda'_{n-1})$ is a Riesz basis in $L^2(\Omega_{n-1})$ by Proposition 2.4.

We now define

$$\Lambda_n := (\Gamma_n \times \mathbb{Z}) \cup \Lambda'_{n-1} \tag{5.2}$$

and claim that $E(\Lambda_n)$ is a Riesz basis in the space $L^2(\Omega_n)$. The proof of this claim will follow the technique from [13]. By Proposition 2.3 it would be enough if we show that Λ_n is a set of both uniqueness and interpolation for the space $\mathrm{PW}(\Omega_n)$.

We start with the uniqueness part of the proof. Let $F \in \mathrm{PW}(\Omega_n)$ be a function such that $F(\lambda) = 0$ for all $\lambda \in \Lambda_n$. The function F has a representation in the form (4.2) where $G \in \mathrm{PW}(\Sigma_n)$ and $H \in \mathrm{PW}(\Omega_{n-1})$. It follows from (4.2) that $F(x, y) = G(x, y)$ for every $(x, y) \in \mathbb{R}^{d-1} \times \mathbb{Z}$, hence using (5.2) this implies that G vanishes on the set $\Gamma_n \times \mathbb{Z}$. But the system $E(\Gamma_n \times \mathbb{Z})$ is a Riesz basis in $L^2(\Sigma_n)$, so in particular $\Gamma_n \times \mathbb{Z}$ is a set of uniqueness for $\mathrm{PW}(\Sigma_n)$. We conclude that G must vanish identically. The expression (4.2) thus becomes

$$F(x, y) = H(x, y) \sin(\pi y), \quad (x, y) \in \mathbb{R}^{d-1} \times \mathbb{R}.$$

Again using (5.2) this now implies that $H(x, y)\sin(\pi y) = 0$ for every $(x, y) \in \Lambda'_{n-1}$. But if (x, y) is a point in Λ'_{n-1}, then y lies in the set (5.1) and hence $\sin(\pi y) \neq 0$, and it follows that $H(x, y) = 0$. Since $E(\Lambda'_{n-1})$ is a Riesz basis in $L^2(\Omega_{n-1})$, in particular Λ'_{n-1} is a set of uniqueness for $\mathrm{PW}(\Omega_{n-1})$, hence H must also vanish identically. We conclude that F is identically zero, and Λ_n is a set of uniqueness for $\mathrm{PW}(\Omega_n)$.

We next turn to the interpolation part of the proof. Let $\{c(\lambda)\}$ be scalar values in $\ell^2(\Lambda_n)$, and we must show that there is $F \in \mathrm{PW}(\Omega_n)$ satisfying $F(\lambda) = c(\lambda)$ for every $\lambda \in \Lambda_n$. We will find the solution F based on the representation (4.2). First we find a function $G \in \mathrm{PW}(\Sigma_n)$ such that $G(\lambda) = c(\lambda)$ for every $\lambda \in \Gamma_n \times \mathbb{Z}$. This is possible since $E(\Gamma_n \times \mathbb{Z})$ is a Riesz basis in $L^2(\Sigma_n)$, and hence $\Gamma_n \times \mathbb{Z}$ is a set of interpolation for the space $\mathrm{PW}(\Sigma_n)$. Now consider the system of values

$$\left\{ \frac{c(x, y) - G(x, y)}{\sin(\pi y)} : (x, y) \in \Lambda'_{n-1} \right\}. \tag{5.3}$$

We claim that these values are in $\ell^2(\Lambda'_{n-1})$. Indeed, observe that by Proposition 2.2 the values of the function G on the uniformly discrete set Λ'_{n-1} are in $\ell^2(\Lambda'_{n-1})$. Furthermore, if (x, y) is a point in Λ'_{n-1}, then y lies in the set (5.1), and we therefore have

$$\inf_{(x,y) \in \Lambda'_{n-1}} |\sin(\pi y)| > 0.$$

It follows from these properties that the values in (5.3) indeed belong to $\ell^2(\Lambda'_{n-1})$. Since $E(\Lambda'_{n-1})$ is a Riesz basis in $L^2(\Omega_{n-1})$, then Λ'_{n-1} is a set of interpolation for $\mathrm{PW}(\Omega_{n-1})$ and hence there is $H \in \mathrm{PW}(\Omega_{n-1})$ such that

$$H(x, y) = \frac{c(x, y) - G(x, y)}{\sin(\pi y)}, \quad (x, y) \in \Lambda'_{n-1}.$$

Using the fact that $\sin(\pi y) = 0$ for every $(x, y) \in \Gamma_n \times \mathbb{Z}$, we can now conclude that the function F defined by (4.2) solves the interpolation problem $F(\lambda) = c(\lambda)$, $\lambda \in \Lambda_n$, and this function belongs to $\mathrm{PW}(\Omega_n)$. We thus obtain that the set Λ_n is also a set of interpolation for $\mathrm{PW}(\Omega_n)$. This completes the proof of Theorem 5.1. ■

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