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A survey on the Kovalevskaya exponents and their applications

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Abstract The Kovalevskaya exponents are widely used in the theory of the integrability of dynamical systems. There are many works about this subject. In this survey we list the main results on the Kovalevskaya exponents in integrability theory and nonlinear dynamics. The survey is divided into seven sections. Each section contains a different topic where the Kovalevskaya exponents play a role. For each result we give an example to illustrate its application for a better understanding.

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1 Introduction

In the theory of ordinary differential equations one of the classical relevant problems is to study the integrability and nonintegrability of nonlinear differential equations. Many applications involve differential equations which appear in the description of many physical phenomena such as mathematical physics, celestial mechanics, control engineering and other fields. The global dynamical behavior of these phenomena has attracted the interest of numerous mathematicians. To obtain global dynamical information we need to analyze the local behavior at the singular points of the differential equations. One of the most basic tools is to study the linearized system at singular points and their eigenvalues. Another commendable method to investigate the local and global dynamical behavior is to obtain explicit or implicitly some solutions of these differential systems. A third method consists in finding first integrals of the differential systems. The central problem in the theory of integrability is to decide whether a given differential system has or not first integrals. Many different methods have been developed for studying the theory of integrability for example Noether symmetries [43], Lie symmetries [40], the Darboux theory of integrability [12, 13], the Lax pairs [31], the Painlevé analysis [7], the Kovalevskaya exponents [48, 49], etc. The main purpose of this paper is to summarize the results on the Kovalevskaya exponents and their applications.

Here we provide a brief history of the Kovalevskaya exponents and refer the reader to the survey [18] for more information about its history. In 1889, Kovalevskaya did pioneering work on the rigid body motion. In [26, 27] she investigated the following Euler equations for the motion of a rigid body with a fixed point

$$\boldsymbol{I}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\boldsymbol{I}\boldsymbol{\omega}) = \boldsymbol{X} \times \boldsymbol{\gamma}, \quad \dot{\boldsymbol{\gamma}} + \boldsymbol{\omega} \times \boldsymbol{\gamma} = 0, \tag{1.1}$$

where $\boldsymbol{I} = (I_1, I_2, I_3)$ are the eigenvalues of the inertia tensor, $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ is the angular velocity, the vector $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ describes the orientation of the top, and the vector $\boldsymbol{X} =$

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(X_1, X_2, X_3) is the center of mass of the body. Kovalevskaya shows that if the general solution of equations (1.1) can be represented by Laurent series with five free parameters of the form

$$\boldsymbol{\omega} = t^{\boldsymbol{p}} \sum_{i=0}^{\infty} \boldsymbol{a}_i t^i, \quad \boldsymbol{\gamma} = t^{\boldsymbol{q}} \sum_{i=0}^{\infty} \boldsymbol{b}_i t^i \tag{1.2}$$

with $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{Z}^3, \boldsymbol{a}_i, \boldsymbol{b}_i \in \mathbb{C}^3$, then they must satisfy the following conditions:
(i) The series (1.2) provide a formal solution and they converge in a punctured disk.
(ii) Five coefficients should be left undetermined.

The Kovalevskaya's method is based in finding the scale of the invariant solutions (or equivalently the balances, see Section 2) of system (1.2), and introduce a matrix K (i.e. called now the Kovalevskaya matrix, see Section 2) for each of these solutions. She analysed when the eigenvalues of the matrix K (called now the Kovalevskaya exponents) are integers, and found the three known integrable cases and a new case, called now the Kovalevskaya case $I_1 = I_2 = 2I_3$ and $X_3 = 0$ with the fourth first integral

$$\{(\omega_1 + i\omega_2)^2 + X_1 (\gamma_1 + i\gamma_2)\}\{(\omega_1 - i\omega_2)^2 + X_1 (\gamma_1 - i\gamma_2)\} = C_4.$$

Additionally Kovalevskaya shows that there are no other cases, except the three already known cases and the Kovalevskaya case, for which their solutions can be expressed in terms of single-valued functions. Indeed, Kovalevskaya's proof tell us that the general Euler equations are not integrable within the class of single-valued functions.

However at the beginning Kovalevskaya's method was thought to be peculiar for the Euler equations and that it cannot be extended to other differential systems. So that it seems to be forgotten and will not be applied to other physical systems until 1980s when integrability theory was recovered. In 1983, Yoshida proved a necessary condition for existence of first integrals by using the method of Kovalevskaya and the result was applied to some well-known differential systems in mathematical physics [48, 49]. The terminology "Kovalevskaya exponents" was first named by Yoshida. After Yoshida the Kovalevskaya exponents play an important role in the study of the integrability of differential systems. Except integrability problems, Kovalevskaya exponents also have a direct link to the complex dynamic behavior of the system such as fractal structures, blowup, the Melnikov integrals, see [9, 20] and the references therein. Over the past two decades the Kovalevskaya exponents have been the focus of intensive research in the theory of integrability, and a lot of valuable results has been obtained.

In this paper we present a survey of the main results on the Kovalevskaya exponents in integrability theory and nonlinear dynamics. We divide these results into seven topics, see Sections 2-8, respectively.

2 The Classic work

A polynomial differential system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{P}(\boldsymbol{x}), \quad \boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{C}^n, \tag{2.1}$$

with $\boldsymbol{P}(\boldsymbol{x}) = (P_1(\boldsymbol{x}), \ldots, P_n(\boldsymbol{x}))$ and $P_i \in \mathbb{C}[x_1, \ldots, x_n]$ for $i = 1, \ldots, n$. As usual $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ will denote the sets of positive integers, real and complex numbers, respectively. The integer $m = \max\{\deg P_1, \ldots, \deg P_n\}$ is the *degree* of the polynomial differential system (2.1). The system (2.1) is *quasi-homogeneous* if there exist $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathbb{N}^n$ and $d \in \mathbb{N}$ such that for arbitrary $\alpha \in \mathbb{R}^+ = \{a \in \mathbb{R}, a > 0\}$,

$$P_i(\alpha^{s_1} x_1, \ldots, \alpha^{s_n} x_n) = \alpha^{s_i - 1 + d} P_i(x_1, \ldots, x_n),$$

for $i=1,\ldots,n$. We call $\boldsymbol{s}=(s_1,\ldots,s_n)$ the *weighted exponent* of system (2.1), and d the *weighted degree* with respect to the weighted exponent $\boldsymbol{s}$, or simply we say that system (2.1) is $(s_1,\ldots,s_n)$-d type. In the particular case that $\boldsymbol{s}=(1,\ldots,1)$ system (2.1) is the classical *homogeneous polynomial differential system of degree d*.

We note that if system (2.1) is quasi-homogeneous of weighted degree $d>1$ with respect to the weighted exponent $\boldsymbol{s}$, then the system is invariant under the change $x_i\to\alpha^{w_i}x_i, t\to\alpha^{-1}t$, where $w_i=s_i/(d-1)$.

A point $\boldsymbol{c}=(c_1,\ldots,c_n)\in\mathbb{C}^n\backslash\{0\}$ is a *balance* of system (2.1) if it satisfies the algebraic equation $\boldsymbol{P}(\boldsymbol{c})+\boldsymbol{wc}=0$, where $\boldsymbol{w}:=\boldsymbol{s}/(d-1)=(s_1/(d-1),\ldots,s_n/(d-1))$ and $\boldsymbol{wc}:=(w_1c_1,\ldots,w_nc_n)$.

To each balance $\boldsymbol{c}$, we introduce a matrix

$$K(\boldsymbol{c})=D\boldsymbol{P}(\boldsymbol{c})+\operatorname{diag}(w_1,\ldots,w_n)\,, \tag{2.2}$$

where $D\boldsymbol{P}(\boldsymbol{c})$ is the Jacobian matrix of $\boldsymbol{P}$ evaluated at $\boldsymbol{c}$ and diag$(w_1,\ldots,w_n)$ is the diagonal matrix with diagonal elements $(w_1,\ldots,w_n)$. The matrix $K(\boldsymbol{c})$ and its eigenvalues are called the *Kovalevskaya matrix* and the *Kovalevskaya exponents*, respectively. The Kovalevskaya exponents of the balance $\boldsymbol{c}$ are denoted by $\boldsymbol{\rho}=(\rho_1,\ldots,\rho_n)$. It can be shown that for any balance $\boldsymbol{c}$ there always exists a Kowalevskaya exponent equal to -1 with the associated eigenvector $\boldsymbol{wc}$, see [14] or [19] for more details.

Let U be an open set of $\mathbb{C}^n$. A non-locally constant function $I:U\to\mathbb{C}$ is called a *first integral* of system (2.1) if it is constant along any solution curve of system (2.1) contained in U. If $I(\boldsymbol{x})$ is differentiable, then $I(\boldsymbol{x})$ is a first integral of system (2.1) if and only if

$$\sum_{i=1}^{n}P_i\frac{\partial I}{\partial x_i}\Big|_{\boldsymbol{x}\in U}=\langle\boldsymbol{P}(\boldsymbol{x}),\nabla I(\boldsymbol{x})\rangle\Big|_{\boldsymbol{x}\in U}\equiv 0, \tag{2.3}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product of two vectors, $\nabla I(\boldsymbol{x})$ is the gradient of $I(\boldsymbol{x})$.

Definition 1. *A function $I(\boldsymbol{x})$ is quasi-homogeneous with respect to $\boldsymbol{s}$ of weighted degree d if $I(\alpha^{\boldsymbol{s}}\boldsymbol{x})=I(\alpha^{s_1}x_1,\ldots,\alpha^{s_n}x_n)=\alpha^dI(x_1,\ldots,x_n)=\alpha^dI(\boldsymbol{x})$ for all $\alpha\in\mathbb{R}^+$ (it will be called a $\boldsymbol{s}$-function of degree $\deg_{\boldsymbol{s}}(I)=d$).*

The functions $I_k:U\to\mathbb{C}$ for $k=1,\ldots,l$ are independent if their gradients are linearly independent, i.e.,

$$\operatorname{rank}(\nabla I_1(\boldsymbol{x}_0),\ldots,\nabla I_l(\boldsymbol{x}_0))=l$$

for all $\boldsymbol{x}_0\in U$ except perhaps in zero Lebesgue measure set.

Sophia Kowalevskaya was the first to introduce the determinant of $K(\boldsymbol{c})$ to compute the Laurent series solutions of the rigid body motion. The next two theorems are due to Yoshida, see [48, 49]. First, he proves that, under certain conditions, the weighted degree of a first integral is a Kowalevskaya exponent. Second, he shows that if one of the Kowalevskaya exponents is not rational, then the system cannot be algebraically integrable.

Theorem 1. *Let $I(\boldsymbol{x})$ be a quasi-homogeneous first integral of weighted degree d for the quasi-homogeneous system (2.1). Assume that the gradient $\nabla I(\boldsymbol{c})$ is not identically zero for at least one choice of a balance $\boldsymbol{c}$. Then d is a Kowalevskaya exponent.*

Example 1. Consider the following Hamiltonian system of three degrees of freedom (see [23]):

$$\begin{aligned}&\dot{x}_1=p_1,\quad \dot{x}_2=p_2,\quad \dot{x}_3=p_3,\quad \dot{p}_1=-4x_1^3-24x_2^2x_1,\\&\dot{p}_2=-64x_2^3-24x_1^2x_2,\quad \dot{p}_3=-4\mu x_3^3,\end{aligned} \tag{2.4}$$

with the Hamiltonian $H = \left(p_1^2+p_2^2+p_3^2\right)/2+\mu x_3^4+x_1^4+12x_2^2x_1^2+16x_2^4$. It is clear that system (2.4) is $(1,1,1,2,2,2)$-2 type system and H is quasi-homogeneous with respect to the weighted exponent $\boldsymbol{s}=(1,1,1,2,2,2)$. Moreover system (2.4) has the following two independent first integrals

$$I_1 = p_3^2+2\mu x_3^4, \quad I_2 = x_2p_1^2 - x_1p_1p_2 - 8x_1^2x_2^3 - 4x_1^3x_2^2. \tag{2.5}$$

The weighted degree of H, I_1, I_2 with respect to $\boldsymbol{s}$ are, respectively, $d_H=4$, $d_1=4$, $d_2=5$. Doing some computations we get that system (2.4) has 24 different balances. We pick 3 different balances of these 24 balances in order to illustrate Theorem 1:

$$\begin{aligned}
\boldsymbol{c}_1 &= \left(\frac{i}{\sqrt{2}},0,\frac{i}{\sqrt{2\mu}},-\frac{i}{\sqrt{2}},0,-\frac{i}{\sqrt{2\mu}}\right), \quad \boldsymbol{\rho}_1=(5,4,4,-2,-1,-1)\,,\\
&\nabla H(\boldsymbol{c}_1)\neq 0, \quad \nabla I_1(\boldsymbol{c}_1)\neq 0, \quad \nabla I_2(\boldsymbol{c}_1)\neq 0;\\
\boldsymbol{c}_2 &= \left(\frac{i}{\sqrt{2}},0,0,-\frac{i}{\sqrt{2}},0,0\right), \quad \boldsymbol{\rho}_2=(5,4,-2,2,-1,1)\,,\\
&\nabla H(\boldsymbol{c}_2)\neq 0, \quad \nabla I_1(\boldsymbol{c}_2)= 0, \quad \nabla I_2(\boldsymbol{c}_2)\neq 0;\\
\boldsymbol{c}_3 &= \left(0,-\frac{i}{4\sqrt{2}},0,0,\frac{i}{4\sqrt{2}},0\right), \quad \boldsymbol{\rho}_3=\left(4,\frac{5}{2},2,-1,1,\frac{1}{2}\right),\\
&\nabla H(\boldsymbol{c}_3)\neq 0, \quad \nabla I_1(\boldsymbol{c}_3)= 0, \quad \nabla I_2(\boldsymbol{c}_3)= 0.
\end{aligned}$$

In the first case the gradients of the first integrals do not vanish evaluated at $\boldsymbol{c}_1$. Hence d_H, d_1 and d_2 are Kowalevskaya exponents. In the second case, ∇I_1 vanishes identically and d_1 is not a Kowalevskaya exponent. In the third case, only $\nabla H(\boldsymbol{c}_3)$ does not vanish and d_H is a Kowalevskaya exponent.

Definition 2. *System (2.1) is algebraically integrable if there exist $(n-1)$ independent algebraic first integrals $I_i\,(i=1,\ldots,n-1)$.*

Theorem 2. *If system (2.1) is algebraically integrable, then all Kovalevskaya exponents are rational.*

Example 2. Consider the system

$$\dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad \dot{x}_3 = x_1x_2. \tag{2.6}$$

Note that it is a $(2,3,4)$-2 type system. System (2.6) has the following two independent polynomial first integrals :

$$I_1 = x_3 - \frac{x_1^2}{2}, \quad I_2 = x_3x_1 - \frac{x_2^2}{2} - \frac{x_1^3}{2},$$

see [25]. Thus, system (2.6) is algebraically integrable and $\boldsymbol{c}=(12,-24,72)$ is the only balance of system (2.6). The Kovalevskaya exponents $\boldsymbol{\rho}=(6,4,-1)$ are rational. So this differential system satisfies Theorem 2.

Example 3 (Jouanolou system)**.** We consider the Jouanolou system

$$\dot{x}_1 = x_3^n, \quad \dot{x}_2 = x_1^n, \quad \dot{x}_3 = x_2^n, \tag{2.7}$$

where $n\geq 2$ is a natural number. This system is homogeneous of degree n. The balance $\boldsymbol{c}=(c_1,c_2,c_3)$ satisfies

$$c_3^n + \frac{c_1}{n-1}=0, \quad c_1^n+\frac{c_2}{n-1}=0, \quad c_2^n+\frac{c_3}{n-1}=0.$$

Note that $c_i\neq 0$ for $i=1,2,3$. The matrix (2.2) is

$$K(\boldsymbol{c}) = \begin{pmatrix} \frac{1}{n-1} & 0 & nc_3^{n-1}\\ nc_1^{n-1} & \frac{1}{n-1} & 0\\ 0 & nc_2^{n-1} & \frac{1}{n-1}\end{pmatrix} = \begin{pmatrix} \frac{1}{n-1} & 0 & -\frac{nc_1}{(n-1)c_3}\\ -\frac{nc_2}{(n-1)c_1} & \frac{1}{n-1} & 0\\ 0 & nc_2^{n-1} & \frac{1}{n-1}\end{pmatrix}.$$

Doing some computations, the Kovalevskaya exponents

$$\boldsymbol{\rho} = \left(-1, \frac{\left(1 - i\sqrt{3}\right) n + 2}{2(n-1)}, \frac{\left(1 + i\sqrt{3}\right) n + 2}{2(n-1)} \right)$$

are independent of the all balances $\boldsymbol{c}$. Using Theorem 2 we can obtain that system (2.7) is not algebraically integrable. This means that for system (2.7) does not exist two independent algebraic first integrals. Moulin-Ollagnier et al. [39] used Bezout theorem to prove that system (2.7) has no Darboux polynomials. So it has no rational first integrals.

The following relationship between the degrees of the first integrals and the Kovalevskaya exponents was given by Goriely [17].

Theorem 3. *If the quasi-homogeneous system (2.1) has k independent algebraic first integrals $I_1, \cdots, I_k$ of weighted degrees $d_1, \cdots, d_k$ and Kovalevskaya exponents $\rho_1 = -1, \rho_2, \cdots, \rho_n$, then there exists a $k \times n$ matrix $M = (m_{ij})$ with integer entries so that*

$$\sum_{j=2}^{n} m_{ij}\rho_j = d_i, \quad i = 1, \cdots, k. \tag{2.8}$$

This relations (2.8) can view as resonance relations between the Kovalevskaya exponents the same as in the normal form theory, see for instance [8].

Example 4. Consider the Liouville integrable Hamiltonian system

$$\dot{x}_1 = 3p_1^2 + x_1^2, \quad \dot{x}_2 = 2p_2x_1, \quad \dot{p}_1 = -2p_1x_1 - p_2^2 - x_2^2, \quad \dot{p}_2 = -2x_1x_2, \tag{2.9}$$

with Hamiltonian $H = p_1\left(p_1^2 + x_1^2\right) + x_1\left(p_2^2 + x_2^2\right)$, and having the second first integral $I_1 = p_2^2 + x_2^2$, see [5]. The system (2.9) has three different balances $\boldsymbol{c}_1 = (-1, 0, 0, 0)$, $\boldsymbol{c}_2 = (1/2, 0, -i/2, 0)$ and $\boldsymbol{c}_3 = (1/2, 0, i/2, 0)$. The corresponding Kovalevskaya exponents are $\boldsymbol{\rho}_1 = (3, 1+2i, 1-2i, -1)$ and $\boldsymbol{\rho}_2 = \boldsymbol{\rho}_3 = (3, 1+i, 1-i, -1)$. The weighted degree of H and I_1 are respectively $d_H = 3$ and $d_1 = 2$. One can check that the Kovalevskaya exponents satisfy the following relations:

$$M\mathcal{R} = \begin{pmatrix} d_H \\ d_1 \end{pmatrix},$$

where

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix} \text{ and } \mathcal{R} = \left(\boldsymbol{\rho}_1^T, \boldsymbol{\rho}_2^T, \boldsymbol{\rho}_3^T\right).$$

The next two corollaries follow immediately from Theorem 3.

Definition 3. *The numbers $\rho_1, \ldots, \rho_n$ are $\mathbb{Z}$-independent (or $\mathbb{N}$-independent) if there do not exist $k_1, \ldots, k_n \in \mathbb{Z}$ (or $\mathbb{N}$) such that*

$$\sum_{i=1}^{n} k_i\rho_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \mid k_i \mid \neq 0.$$

Corollary 1. *If the Kovalevskaya exponents are $\mathbb{Z}$-independent, then there is no rational first integral.*

Example 5. Consider a three-dimensional quasi-homogeneous polynomial system

$$\dot{x} = \sqrt{3}xy, \quad \dot{y} = \sqrt{2}y^2 + xz, \quad \dot{z} = yz + x. \tag{2.10}$$

Note that it is a $(3,2,1)$-3 type system. This system has three different balances $\mathbf{c}_1 = \left(0, -\sqrt{2}/2, 0\right)$,

$$\mathbf{c}_2 = \left(\frac{i}{2}\left(\sqrt{3}-3\right)\sqrt{\frac{\sqrt{6}-2}{6-2\sqrt{3}}}, -\frac{\sqrt{3}}{2}, -i\sqrt{\frac{3\sqrt{6}-6}{6-2\sqrt{3}}}\right)$$

and

$$\mathbf{c}_3 = \left(-\frac{i}{2}\left(\sqrt{3}-3\right)\sqrt{\frac{\sqrt{6}-2}{6-2\sqrt{3}}}, -\frac{\sqrt{3}}{2}, i\sqrt{\frac{3\sqrt{6}-6}{6-2\sqrt{3}}}\right).$$

The corresponding Kovalevskaya exponents are $\boldsymbol{\rho}_1 = \left(-1, \left(3-\sqrt{6}\right)/2, \left(1-\sqrt{2}\right)/2\right)$ and $\boldsymbol{\rho}_2 = \boldsymbol{\rho}_3 =$

$$\left(-1, \frac{5-\sqrt{3}-2\sqrt{6}+i\left(\sqrt{3}+1\right)\sqrt{8\sqrt{6}-3\sqrt{3}-7}}{4}, \frac{5-\sqrt{3}-2\sqrt{6}-i\left(\sqrt{3}+1\right)\sqrt{8\sqrt{6}-3\sqrt{3}-7}}{4}\right).$$

Using Corollary 1 we can conclude that system (2.10) has no rational first integrals.

Corollary 2. *If the Kovalevskaya exponents are $\mathbb{N}$-independent, then there is no polynomial first integral.*

Example 6 (Halphen system)**.** The quadratic homogeneous system

$$\dot{x}_1 = x_2x_3 - x_1\left(x_2+x_3\right), \quad \dot{x}_2 = x_3x_1 - x_2\left(x_1+x_3\right), \quad \dot{x}_3 = x_1x_2 - x_3\left(x_1+x_2\right), \tag{2.11}$$

is the Darboux-Brioschi-Halphen system, see [37]. There is a unique balance $\boldsymbol{c} = (1,1,1)$ and the Kovalevskaya exponents are $\boldsymbol{\rho} = (-1,-1,-1)$. Therefore, the Halphen system (2.11) has no polynomial first integrals. In [37] Maciejewski et al. also proved that the Halphen system (2.11) has no rational first integral.

3 Kovalevskaya exponents of quasi-homogeneous differential systems

In this section we introduce some important results about the Kowalevskaya exponents of quasi-homogeneous polynomial differential systems.

Furta [14] and Goriely [17] independently proved the existence of the following link between the Kowalevskaya exponents of the system and the degree of their quasi-homogeneous polynomial first integrals.

Theorem 4. *Consider the quasi-homogeneous polynomial differential system (2.1) of weighted exponent $\boldsymbol{s}$. For each balance $\mathbf{c}$, let $\rho_1, \ldots, \rho_n$ be the Kowalevskaya exponents associated with $\mathbf{c}$. If system (2.1) has a quasi-homogeneous polynomial first integral of weighted degree l with respect to the weighted exponent $\boldsymbol{s}$, then there exist non-negative integers $k_1, \ldots, k_n$ satisfying $k_1+\cdots+k_n \leq l$ such that $k_1\rho_1 + \cdots + k_n\rho_n = l$.*

Example 7 (Hénon-Heiles system [6, 22, 49])**.** Consider the Hamiltonian system

$$\dot{q}_1 = p_1, \quad \dot{q}_2 = p_2, \quad \dot{p}_1 = -2q_1q_2, \quad \dot{p}_2 = -\varepsilon q_2^2 - q_1^2, \tag{3.1}$$

with the Hamiltonian $H = \left(p_1^2+p_2^2\right)/2 + q_1^2q_2 + \varepsilon q_2^3/3$. For $\varepsilon = 6$, there exists a polynomial first integral $I_1 = q_1^4 + 4q_2^2q_1^2 - 4p_1\left(p_1q_2 - p_2q_1\right)$ independent of H. This system is a $(2,2,3,3)$-2 type system and has three different balances $\boldsymbol{c}_1 = (0,-1,0,2)$, $\boldsymbol{c}_2 = (-6i,-3,12i,6)$ and $\boldsymbol{c}_3 = (6i,-3,-12i,6)$. The corresponding Kovalevskaya exponents are $\boldsymbol{\rho}_1 = (6,4,-1,1)$ and $\boldsymbol{\rho}_2 = \boldsymbol{\rho}_3 =$

$(8,6,-3,-1)$. Note that the weighted degree of H and I_1 are respectively $d_H=6$ and $d_1=8$. We have that

$$\begin{aligned}
&\boldsymbol{k}_1=(0,2,2,0)\,, \quad \boldsymbol{k}_2=(1,0,0,1)\,, \quad \boldsymbol{k}_1\boldsymbol{\rho}_1^T=d_H, \quad \boldsymbol{k}_2\boldsymbol{\rho}_1^T=d_1;\\
&\boldsymbol{k}_1=(1,0,0,2)\,, \quad \boldsymbol{k}_2=(0,2,0,4)\,, \quad \boldsymbol{k}_1\boldsymbol{\rho}_2^T=\boldsymbol{k}_1\boldsymbol{\rho}_3^T=d_H, \quad \boldsymbol{k}_2\boldsymbol{\rho}_2^T=\boldsymbol{k}_2\boldsymbol{\rho}_3^T=d_1.
\end{aligned}$$

Llibre and Zhang in [35] proved the following new relation.

Theorem 5. *Consider the quasi-homogeneous polynomial differential system (2.1) of weighted degree $d>1$ with respect to the weighted exponent $\boldsymbol{s}$. For each balance $\boldsymbol{c}$, let $\rho_1,\cdots,\rho_n$ be the Kowalevskaya exponents associated with $\boldsymbol{c}$. If system (2.1) has a quasi-homogeneous polynomial first integral of weighted degree l with respect to the weighted exponent $\boldsymbol{s}$, then there exist nonnegative integers $k_1,\ldots,k_n$ satisfying $k_1+\cdots+k_n\le l$ such that $k_1\rho_1+\cdots+k_n\rho_n=l/(d-1)$.*

Example 8. Consider the $(4,3,5)$-6 type quasi-homogeneous polynomial system

$$\dot{x}=y^3, \quad \dot{y}=x^2, \quad \dot{z}=z^2, \tag{3.2}$$

with the first integral $H=x^3/3-y^4/4$. It has three different balances $\boldsymbol{c}_1=(0,0,-1)$, $\boldsymbol{c}_2=\left(\sqrt[5]{540}/5,-\sqrt[5]{1200}/5,-1\right)$ and $\boldsymbol{c}_3=\left(\sqrt[5]{540}/5,-\sqrt[5]{1200}/5,0\right)$. The corresponding Kowalevskaya exponents are $\boldsymbol{\rho}_1=(-1,4/5,3/5)$, $\boldsymbol{\rho}_2=(-1,-1,12/5)$ and $\boldsymbol{\rho}_3=(-1,1,12/5)$. Note that the weighted degree of H is 12. The system satisfies Theorem 5.

Moreover, Llibre and Zhang studied $(1,\cdots,1)$-2 type quasi-homogeneous systems (2.1), i.e., quadratic homogeneous systems, see [35]. They proved that the coefficient corresponding to the Kowalevskaya exponent -1 in Theorem 5 can be taken to be 0 and obtained the following result.

Theorem 6. *Consider a quadratic homogeneous polynomial differential system (2.1). For each balance $\boldsymbol{c}$, let $\rho_1=-1,\rho_2,\cdots,\rho_n$ be the Kowalevskaya exponents associated with $\boldsymbol{c}$. If system (2.1) has a homogeneous polynomial first integral of degree l, then there exist nonnegative integers $k_2,\ldots,k_n$ satisfying $k_2+\cdots+k_n\le l$ such that $k_2\rho_2+\cdots+k_n\rho_n=l$.*

Example 9. Consider the quadratic homogeneous polynomial differential system

$$\dot{x}_1=x_3^2, \quad \dot{x}_2=x_4^2, \quad \dot{x}_3=-x_1^2, \quad \dot{x}_4=-x_2^2 \tag{3.3}$$

with the homogeneous polynomial first integral $H=\left(x_1^3+x_2^3+x_3^3+x_4^3\right)/3$. The balance $\boldsymbol{c}=(c_1,c_2,c_3,c_4)$ satisfies

$$c_1+c_3^2=0, \quad c_2+c_4^2=0, \quad c_3-c_1^2=0, \quad c_4-c_2^2=0.$$

For each balance $\boldsymbol{c}$, the Kovalevskaya exponents can be computed and we get that they are $\boldsymbol{\rho}_1=(-1,-1,3,3)$ and $\boldsymbol{\rho}_2=(-1,1,1,3)$. So this example satisfies Theorem 6.

Liu et al. [32] show the next theorem that Theorem 6 holds not only for $(1,\cdots,1)$-2 type systems but also for any $(s_1,\ldots,s_n)$-d type systems.

Theorem 7. *For each balance $\boldsymbol{c}$ of an $(s_1,\ldots,s_n)$-d type polynomial system (2.1),$d\ge 2$, let $\rho_1=-1,\rho_2,\cdots,\rho_n$ be the Kowalevskaya exponents associated with $\boldsymbol{c}$. If system (2.1) has an $(s_1,\ldots,s_n)$-l type polynomial first integral, then there exist non-negative integers $k_2,\ldots,k_n$, satisfying $k_2+\cdots+k_n\le l$ such that $k_2\rho_2+\cdots+k_n\rho_n=l/(d-1)$.*

Example 10. We consider the $(4,3,5,5)$-6 type quasi-homogeneous polynomial system

$$\dot{x}_1=x_2^3, \quad \dot{x}_2=x_1^2, \quad \dot{x}_3=x_3^2, \quad \dot{x}_4=x_4^2 \tag{3.4}$$

with the first integral $H=x_1^3/3-x_2^4/4$. System (3.4) has 6 different balances. We pick 2 different balances of these 6 balances in order to illustrate Theorem 7: $\boldsymbol{c}_1=(0,0,-1,0)$ and $\boldsymbol{c}_2=(0,0,-1,-1)$. The Kovalevskaya exponents are $\boldsymbol{\rho}_1=(-1,1,4/5,3/5)$ and $\boldsymbol{\rho}_2=(-1,-1,4/5,3/5)$. Then $1+4/5+3/5=12/5$ and $-1\times 2+4\times 4/5+2\times 3/5=12/5$, as described in Theorem 7.

In [35] Llibre and Zhang also considered the planar system

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{3.5}$$

where $P(x, y)$ and $Q(x, y)$ are polynomial, and proved the next result.

Theorem 8. *Assume that the planar system (3.5) is a quasi-homogeneous polynomial system of weighted degree $d = 2$, $P(x, y)\, Q(x, y) \neq 0$, and P and Q are co-prime in $\mathbb{C}[x, y]$. The following statements hold.*

(i) If system (3.5) has two independent balances with Kowalevskaya exponents $(-1, \rho_1)$ and $(-1, \rho_2)$, then it has a quasi-homogeneous polynomial first integral of weighted degree $l \in \mathbb{N}$ if and only if $\rho_1^{-1} + \rho_2^{-1} \leq 1$, $l/\rho_i \in \mathbb{N}$, for $i = 1, 2$.

(ii) If system (3.5) has a unique balance with the Kowalevskaya exponents $(-1, \rho)$, $\rho \neq 0$, then it has a quasi-homogeneous polynomial first integral of weighted degree $l \in \mathbb{N}$ if and only if $\rho^{-1} \leq 1$, $l/\rho \in \mathbb{N}$

(iii) If system (3.5) has no balance or has infinitely many balances, then it has no quasi-homogeneous polynomial first integrals.

The next two examples are due to Llibre and Zhang [35].

Example 11. Consider the quasi-homogeneous polynomial system of weighted degree 2

$$\dot{x} = ax^2 + by, \quad \dot{y} = dx^3 + gxy, \tag{3.6}$$

with $b \neq 0$. The balance $\boldsymbol{c} = (c_1, c_2)$ satisfies

$$c_2 = -\frac{1}{b}c_1 - \frac{a}{b}c_1^2, \quad \left(d - \frac{ag}{b}\right)c_1^2 - \frac{2a+g}{b}c_1 - \frac{2}{b} = 0.$$

If $d = 0$ and $2a \neq g$, then system (3.6) has the first integral $H = \left[(2a - g)\, x^2 + 2by\right]^g y^{-2a}$ and two independent balances $\boldsymbol{c}_1 = \left(-a^{-1}, 0\right)$ and $\boldsymbol{c}_2 = \left(-2\,(2a - g)\, b^{-1} g^{-2}, -2g^{-1}\right)$. The corresponding Kowalevskaya exponents are $\boldsymbol{\rho}_1 = \left(-1, (2a - g)\, a^{-1}\right)$ and $\boldsymbol{\rho}_2 = \left(-1, -2\,(2a - g)\, g^{-1}\right)$. The system satisfies statement (i) of Theorem 8.

If $d = 0$ and $2a = g$, then system (3.6) has the balance $\left(-a^{-1}, 0\right)$ and the first integral $H = b^{-1} g x^2 y^{-1} - \ln\left(-2by^2\right)$. The corresponding Kowalevskaya exponents are $\boldsymbol{\rho} = (-1, 0)$. So in this case statement (ii) of Theorem 8 holds.

Example 12. Consider the system

$$\dot{x} = ax^2, \quad \dot{y} = bx^{n+1} + dxy, \tag{3.7}$$

with $n > 2$, $n \in \mathbb{N}$, $a \neq 0$ and $b^2 + d^2 \neq 0$.

If $n = d/a$ and $b \neq 0$, then system (3.7) has no balances, and it has the first integral $H = b \ln x - ax^{-d/a} y$.

If $n = d/a$ and $b = 0$, then system (3.7) has infinitely many balances $(-1/a, c_2)$ and a first integral $H = x^{-n} y$, where c_2 is an arbitrary constant. In these two cases statement (iii) of Theorem 8 is fulfilled.

In the next theorem Liu et al. [32] generalized Theorem 8 from (s_1, s_2)-2 type systems to systems of (s_1, s_2)-d type with $d \geq 2$, and additionally they extend Theorem 8 from polynomial first integrals to rational first integrals, see the next result.

Theorem 9. *Let the planar quasi-homogeneous polynomial differential system* (3.5) *be* (s_1, s_2)*-d type with* $d \geq 2$*, where* P *and* Q *are co-prime in* $\mathbb{C}[x, y]$*. Then the system always has at least one and at most* $d + 1$ *balances, and it has a polynomial (resp. rational) first integral if and only if for every balance* $\boldsymbol{c}$ *of the system with the corresponding Kowalevskaya exponent* $(-1, \rho)$*,* $\rho \in \mathbb{Q}^+ \, (resp. \rho \in \mathbb{Q} \backslash \{0\})$.

Example 13. Consider the $(3, 2)$-4 type quasi-homogeneous polynomial system

$$\dot{x} = x^2 + y^3, \quad \dot{y} = axy, \tag{3.8}$$

with $a \neq 0$.

If $a = 2/3$, then system (3.8) has only the balance $\boldsymbol{c} = (-1, 0)$ with Kowalevskaya exponent $\boldsymbol{\rho} = (-1, 0)$.

If $a \neq 2/3$, then system (3.8) has the two independent balances $\boldsymbol{c}_1 = (-1, 0)$ and $\boldsymbol{c}_2 = \left(-2/3a, \sqrt[3]{6\,(3a-2)\,a}/3a\right)$. The corresponding Kowalevskaya exponents are $\boldsymbol{\rho}_1 = (-1, 2/3 - a)$ and $\boldsymbol{\rho}_2 = (-1, 2\,(1 - 2/3a))$.

Applying Theorem 9 we have that the first integral of system (3.8) is rational if and only if $a \in \mathbb{Q} \backslash \{3/2\}$, and the first integral is polynomial if and only if $a \in \mathbb{Q}^-$.

From [15] the system has first integral $H = y^2 \left(2y^3 + (2 - 3a)x^2\right)^{-a}$ if $a \neq 2/3$, and $H = y \exp\left(-\frac{x^2}{3y^3}\right)$ if $a = 2/3$.

Remark 1. *The approach of [32] is completely different from the approach of [35]. By the transformation of variables* $(X, Y) = (x^{s_2}, y^{s_1})$*, the quasi-homogeneous polynomial systems* (3.5) *can be changed into a homogeneous system. Hence, studying the integrability of the quasi-homogeneous polynomial systems* (3.5) *can be changed into studying the integrability of its corresponding homogeneous systems.*

The existence of the following link between the Kowalevskaya exponents and the rational first integrals of quasi-homogeneous system (2.1) is found in [28]. It is known that a rational function $H = M(\boldsymbol{x})/G(\boldsymbol{x})$ is a first integral of system (2.1) if and only if there is a polynomial $R(\boldsymbol{x})$ such that $\langle \boldsymbol{P}(\boldsymbol{x}), \nabla M(\boldsymbol{x})\rangle = MR$ and $\langle \boldsymbol{P}(\boldsymbol{x}), \nabla G(\boldsymbol{x})\rangle = GR$.

Theorem 10. *Consider the quasi-homogeneous polynomial differential system* (2.1) *of weighted degree* $d > 1$ *with respect to the weighted exponent* $\boldsymbol{s}$*. Let* $H = M(\boldsymbol{x})/G(\boldsymbol{x})$ *be a rational first integral of system* (2.1)*. If* $\langle \boldsymbol{P}(\boldsymbol{x}), \nabla M(\boldsymbol{x})\rangle = MR$*,* $M(\boldsymbol{c}) = 0$ *and* $\nabla M(\boldsymbol{c}) \neq 0$ *for a balance* $\boldsymbol{c}$*, then* $\deg_{\boldsymbol{s}}(M)/(d-1) + R(\boldsymbol{c})$ *is a Kovalevskaya exponent.*

Example 14 (Lotka-Volterra system)**.** Consider the Lotka-Volterra system

$$\dot{x} = x\,(Cy + z)\,, \quad \dot{y} = y\,(Az + x)\,, \quad \dot{z} = z\,(Bx + y)\,, \tag{3.9}$$

where A, B and C are parameters, see [38]. If $A = B = C = 1$, then system (3.9) has rational first integral $H = (x - y)(y - z)/y$. We have $M = (x - y)(y - z)$, $\deg M = 2$ and $R = x + z$.

In this case, the system (3.9) has four different balances $\boldsymbol{c}_1 = (-1, -1, 0)$, $\boldsymbol{c}_2 = (-1, 0, -1)$, $\boldsymbol{c}_3 = (-1/2, -1/2, -1/2)$ and $\boldsymbol{c}_4 = (0, -1, -1)$. Only two balances $\boldsymbol{c}_1 = (-1, -1, 0)$ and $\boldsymbol{c}_4 = (0, -1, -1)$ satisfy conditions $M(\boldsymbol{c}) = 0$ and $\nabla M(\boldsymbol{c}) \neq 0$. By Theorem 10, $2 + R(\boldsymbol{c}_1) = 2 + R(\boldsymbol{c}_4) = 1$ is a Kovalevskaya exponent. In fact, the corresponding Kowalevskaya exponents of system (3.9) are $\boldsymbol{\rho}_1 = \boldsymbol{\rho}_2 = \boldsymbol{\rho}_4 = (-1, -1, 1)$ and $\boldsymbol{\rho}_3 = (-1, 1/2, 1/2)$.

Denoted by $\boldsymbol{F} = (P, Q)^T$ the polynomial vector field associated to system (3.5). The system (3.5) is called *irreducible* if P and Q are coprime. A polynomial $I(\boldsymbol{x})$ is said to be $\boldsymbol{s}$*-d type quasi-homogeneous* if $I(\alpha^{\boldsymbol{s}}\boldsymbol{x}) = I(\alpha^{s_1}x_1, \ldots, \alpha^{s_n}x_n) = \alpha^d I(x_1, \ldots, x_n) = \alpha^d I(\boldsymbol{x})$ for all $\alpha \in \mathbb{R}^+$ (it will be called $\boldsymbol{s}$-polynomial of degree $\deg_{\boldsymbol{s}}(I) = d$). The vector space of $\boldsymbol{s}$-d type quasi-homogeneous polynomials is denoted by $\mathcal{P}_d^{\boldsymbol{s}}$. The vector field $\boldsymbol{F} = (P, Q)$ is a *$\boldsymbol{s}$-d type quasi-homogeneous vector field* if $P \in \mathcal{P}_{s_1+d-1}^{\boldsymbol{s}}$ and $Q \in \mathcal{P}_{s_2+d-1}^{\boldsymbol{s}}$. We denote the vector space of the $\boldsymbol{s}$-d type quasi-homogeneous polynomial vector fields by $\mathcal{Q}_d^{\boldsymbol{s}}$.

The quasi-homogeneous vector field $\boldsymbol{F}_d \in \mathcal{Q}_d^{\boldsymbol{s}}$ can be decomposed into a sum of two quasi-homogeneous fields, a conservative one (having zero-divergence) and a dissipative one, see the next lemma proved in [1].

Lemma 1. *Every vector field $\boldsymbol{F}_d \in \mathcal{Q}_d^{\boldsymbol{s}}$ can be expressed as*

$$\boldsymbol{F}_d = \frac{1}{d + |\boldsymbol{s}| - 1} \left[\boldsymbol{X}_h + \operatorname{div}(\boldsymbol{F}_d)\, \boldsymbol{D}_0\right], \tag{3.10}$$

where $\boldsymbol{D}_0(x, y) = (s_1 x, s_2 y)^T$ (a dissipative $\boldsymbol{s}$-1 type vector field), $\operatorname{div}(\boldsymbol{F}_d) \in \mathcal{P}_d^{\boldsymbol{s}}$ (the divergence of $\boldsymbol{F}_d$), $h = s_1 x Q - s_2 y P \in \mathcal{P}_{d+|\boldsymbol{t}|-1}^{\boldsymbol{t}}$ (the wedge product of $\boldsymbol{D}_0$ and $\boldsymbol{F}_d$) being $|\boldsymbol{s}| = s_1 + s_2$.

This decomposition is known as *the conservative-dissipative splitting of a quasi-homogeneous vector field*. Using this decomposition, Algaba et al. [2] obtained the Kowalevskaya exponents and relate the rational integrability of quasi-homogeneous vector fields to their Kowalevskaya exponents, see the following four results.

Proposition 1. *If $\boldsymbol{c}$ is a balance of a quasi-homogeneous system (3.5), then $h(\boldsymbol{c}) = 0$. Furthermore, if P and Q are coprime and $PQ \neq 0$, the following statements hold.*

(i) If x is a factor of h, then $(0, c_2)$, with $c_2^{(d-1)/s_2} = -\frac{s_2}{(d-1)Q(0,1)}$, is a balance of system (3.5).

(ii) If y is a factor of h, then $(c_1, 0)$, with $c_1^{(d-1)/s_1} = -\frac{s_1}{(d-1)P(1,0)}$, is a balance of system (3.5).

(iii) If $y^{s_1} - \lambda x^{s_2}$ is a factor of h with $\lambda \in \mathbb{C}\backslash\{0\}$, then (c_1, c_2) with $c_1 = u^{s_1}$, $c_2 = u^{s_2}\lambda^{1/s_1}$ and $u^{d-1} = -\frac{s_1}{(d-1)P\left(1,\lambda^{1/s_1}\right)}$, is a balance of system (3.5).

We need the following lemma, see [2].

Lemma 2. *If a quasi-homogeneous system (3.5) is irreducible with $\operatorname{div}(\boldsymbol{F}_d) \not\equiv 0$, then $d + |\boldsymbol{s}| - 1 = k_3 s_1 s_2 + \delta_y s_2 + \delta_x s_1$ with $\delta_x, \delta_y \in \{0, 1\}$ and $k_3 \geq 0$. Then:*

(i) $h(x, y) = x^{\delta_x} y^{\delta_y} h_0(x^{s_2}, y^{s_1})$ with $h_0(x, y)$ a homogeneous polynomial of degree k_3, and

(ii) $\operatorname{div}(\boldsymbol{F}_d)(x, y) = x^{(1-\delta_x)(s_2-1)} y^{(1-\delta_y)(s_1-1)} \mu_0(x^{s_2}, y^{s_1})$ with $\mu_0(x, y)$ a homogeneous polynomial of degree $k_3 - (1 - \delta_x) - (1 - \delta_y)$.

From Lemma 2 we can define the function

$$\eta(x, y) := \frac{\mu_0(x, y)}{x^{\delta_x} y^{\delta_y} h_0(x, y)}.$$

The residue of $\eta(1, y)$ at y_0 is denoted by $\operatorname{Res}[\eta(1, y), y_0]$, see [47] for definition of residue.

Proposition 2. *Assume that quasi-homogeneous system (3.5) is irreducible with $h = \prod_{j=1}^{l+2} f_j^{l_j} \neq 0$. If $w_1 = \infty, w_2 = 0, w_j = \lambda_j, j = 3, \cdots, l+2$ are the poles of $\eta(1, y)$, then $\rho_i = 0$ if $l_i > 1$, otherwise,*

$$\rho_i^{-1} = \frac{d-1}{d + |\boldsymbol{s}| - 1} \left(1 - \frac{s_1 s_2}{\deg_{\boldsymbol{s}}(f_i)} \operatorname{Res}[\eta(1, y), w_i]\right), \tag{3.11}$$

where every $(-1, \rho_i)$ is the Kowalevskaya exponents associated to the factor f_i of h, for $i = 1, \cdots, l+2$.

The next result characterizes the rational (polynomial) integrability of a quasi-homogeneous polynomial system (3.5) through its Kowalevskaya exponents.

Theorem 11. *An irreducible quasi-homogeneous polynomial system* (3.5) *has a rational (resp. polynomial) first integral if and only if* $\operatorname{div}(\boldsymbol{F}_d) \equiv 0$ *or the two following properties hold:*

(i) The $\boldsymbol{s}$*-polynomial* h *has at least two irreducible factors on* $\mathbb{K}[x,y]$*, and all of them are distinct, that is, it can be written as* $h = \prod_{j=1}^{l+2} f_j^{l_j}$*, where* $f_1 = x^{\delta_x}, f_2 = y^{\delta_y}, \delta_x, \delta_y \in \{0,1\}, f_j = y^{s_1} - \lambda_j x^{s_2}$*, for* $j \geq 3$*, and* $j \geq 3$*, and* $l \geq 0$*.*

(ii) For any Kowalevskaya exponent $\rho_j \neq -1$*,* ρ_j *is a rational number. Moreover, in such a case, if we define the rational numbers*

$$r_j = \frac{1}{d-1}\rho_j^{-1},\ j = 1,\ldots,l+2,$$

Then $\prod_{j=1}^{l+2} f_j^{n_j}$ *is a* $\boldsymbol{s}$*-rational (resp. polynomial) first integral of degree* M*, where* $n_j = Mr_j \in \mathbb{Z}$ *(*$\mathbb{N}\cup\{0\}$*, resp.).*

We apply the above results to the following example [2].

Example 15. We consider the $(1,2)$-3 type planar quasi-homogeneous polynomial system given by

$$\dot{x} = (d_1 - 2c)\,xy + d_2x^3, \quad \dot{y} = (c + 2d_1)\,y^2 + 2d_2x^2y - 5cx^4, \tag{3.12}$$

with c, d_1 and d_2 real parameters and $c \neq 0$. The function h associated to system (3.12) is $h\,(x,y) = cx\left(y - x^2\right)\left(y + x^2\right)$. Using Proposition 1, system (3.12) has five balances:

$$\boldsymbol{c}_1 = \left(\frac{1}{\sqrt{2\,(d_1 - 2c - d_2)}}, \frac{1}{2\,(2c - d_1 + d_2)}\right), \quad \boldsymbol{c}_2 = \left(-\frac{1}{\sqrt{2\,(2c - d_1 - d_2)}}, \frac{1}{2\,(2c - d_1 - d_2)}\right),$$
$$\boldsymbol{c}_3 = \left(\frac{1}{\sqrt{2\,(2c - d_1 - d_2)}}, \frac{1}{2\,(2c - d_1 - d_2)}\right), \quad \boldsymbol{c}_4 = \left(-\frac{1}{\sqrt{2\,(d_1 - 2c - d_2)}}, \frac{1}{2\,(2c - d_1 + d_2)}\right),$$
$$\boldsymbol{c}_5 = \left(0, -\frac{1}{c + 2d_1}\right).$$

In this case $\eta\,(x,y) = (d_1y + d_2x)\,/\,(cx\,(y - x)\,(y + x))$. We have

$$\begin{aligned}
\operatorname{Res}[\eta\,(1,y)\,,\infty] &= -\operatorname{Res}[\eta\,(x,1)\,,0] = -\frac{d_1}{c},\\
\operatorname{Res}[\eta\,(1,y)\,,1] &= \frac{d_1 + d_2}{2c},\\
\operatorname{Res}[\eta\,(1,y)\,,-1] &= \frac{d_1 - d_2}{2c}.
\end{aligned}$$

Applying Proposition 2 we get that the Kowalevskaya exponents are

$$\left(-1, \frac{5c}{2\,(c + 2d_1)}\right), \ \left(-1, -\frac{5c}{d_1 + d_2 - 2c}\right) \text{ and } \left(-1, \frac{5c}{2c - d_1 + d_2}\right).$$

By Theorem 21 system (3.12) has a $(1,2)$-polynomial first integral of degree $M > 0$ if and only if

$$\frac{M}{5}\left(1 + \frac{2d_1}{c}\right), \ \frac{M}{5}\left(1 - \frac{d_1 + d_2}{2c}\right) \text{ and } \frac{M}{5}\left(1 + \frac{d_2 - d_1}{2c}\right),$$

are non-negative integer numbers. System (3.12) has a $(1,2)$-rational first integral of degree M if and only if

$$\frac{M}{5}\left(1 + \frac{2d_1}{c}\right), \ \frac{M}{5}\left(1 - \frac{d_1 + d_2}{2c}\right) \text{ and } \frac{M}{5}\left(1 + \frac{d_2 - d_1}{2c}\right),$$

are integer numbers.

4 Kovalevskaya exponents of semi-quasi-homogeneous systems

Let E be identity matrix, $S = \mathrm{diag}\,(s_1, \ldots, s_n)$ and $\alpha^{E-S} = \mathrm{diag}\,(1-s_1, \ldots, 1-s_n)$. We say that system (2.1) is a *semi-quasi-homogeneous system* if it can be expressed as

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{P}_d(\boldsymbol{x}) + \widetilde{\boldsymbol{P}}(\boldsymbol{x}), \tag{4.1}$$

where $\boldsymbol{P}_d(\boldsymbol{x})$ is a quasi-homogeneous vector field of degree d with exponents $\boldsymbol{s} = (s_1, \ldots, s_n)$ and $\alpha^{E-S}\widetilde{\boldsymbol{P}}(\boldsymbol{x}) = \left(\alpha^{1-s_1}\widetilde{P_1}(\boldsymbol{x}), \cdots, \alpha^{1-s_n}\widetilde{P_n}(\boldsymbol{x})\right)$ is the sum of quasi-homogeneous polynomials of degrees all greater than d, or all less than d. In the former case (resp. latter) we say that (2.1) is *positively (resp. negatively) semi-quasi-homogeneous.* Moreover the system

$$\dot{\boldsymbol{x}} = \boldsymbol{P}_d(\boldsymbol{x}). \tag{4.2}$$

is called the *quasi-homogeneous cut* of semi-quasi-homogeneous system (2.1).

If system (2.1) is semi-quasi-homogeneous, then under the transformation

$$\boldsymbol{x} \to \alpha^{\boldsymbol{w}}\boldsymbol{x}, \quad t \to \alpha^{-1}t, \quad \boldsymbol{w} = \frac{\boldsymbol{s}}{d-1} = \left(\frac{s_1}{d-1}, \cdots, \frac{s_n}{d-1}\right),$$

it becomes

$$\dot{\boldsymbol{x}} = \boldsymbol{P}_d(\boldsymbol{x}) + \widetilde{\boldsymbol{P}}(\boldsymbol{x}, \alpha), \tag{4.3}$$

where $\widetilde{\boldsymbol{P}}(\boldsymbol{x}, \alpha)$ is a formal power series either with respect to α (positive semi-quasi-homogeneity) or respect to α^{-1} (negative semi-quasi-homogeneity) without any constant term.

The existence or non-existence of nontrivial first integrals is an important problem in considering integrability and non-integrability for semi-quasi-homogeneous systems. The following result goes back to Poincaré [41], which gave a necessary condition in order that a planar polynomial system has a rational first integral.

Theorem 12 (Poincaré Theorem)**.** *If planar polynomial system* (3.5) *has a rational first integral, then the eigenvalues ρ_1, ρ_2 associated to any singular point of the system must be resonant in the following sense: there exist integers m_1, m_2 with $|m_1| + |m_2| > 0$ such that $m_1\rho_1 + m_2\rho_2 = 0$.*

In 1996, Furta [14] gave an elementary proof of Poincaré's result. Furthermore, Furta studied the integrability of semi-quasi-homogeneous systems and proved the following criterion.

Theorem 13. *Consider that the planar system* (3.5) *is semi-quasi-homogeneous. If the Kovalevskaya matrix of its quasi-homogeneous cut* (4.2) *is diagonalizable and its eigenvalues ρ_1, ρ_2 are nonzero and do not satisfy any resonant condition of form*

$$k_1\rho_1 + k_2\rho_2 = 0, \quad k_1,\ k_2 \in \mathbb{N} \cup \{0\}, \quad k_1 + k_2 \geq 1,$$

then semi-quasi-homogeneous system (3.5) *does not have any polynomial first integral.*

Example 16. Consider the following system ([16])

$$\dot{x} = y\left(ax + by^{2n}\right) + \widetilde{P}(x, y), \quad \dot{y} = cy^2 + \widetilde{Q}(x, y), \tag{4.4}$$

where $\widetilde{P}(x, y) = \sum_{i,j} a_{ij}x^iy^j$ and $\widetilde{Q}(x, y) = \sum_{i,j} b_{ij}x^iy^j$ satisfy $2n(i-1) + j - 1 > 0$ and $2ni + j - 2 > 0$, respectively. This system is positively semi-quasi-homogeneous system with exponents $s_1 = 2n$ and $s_2 = 1$. The corresponding quasi-homogeneous cut is

$$\dot{x} = y\left(ax + by^{2n}\right), \quad \dot{y} = cy^2. \tag{4.5}$$

If $a \neq 2cn$ and $c \neq 0$, then system (4.5) has a balance

$$\boldsymbol{c} = \left(\frac{b}{c^{2n}\,(2cn - a)}, -\frac{1}{c} \right).$$

The associated Kowalevskaya exponents are $\boldsymbol{\rho} = (-1, 2n - a/c)$. Form Theorem 13 it follows that system (4.4) does not have any polynomial first integral if $c\,(2cn - a) \neq 0$ and $a/c \notin \mathbb{Q}^+$.

In [45] the authors presented the following criterion for semi-quasi-homogeneous systems, which is a generalization of Theorem 13.

Theorem 14. *Assume that system* (2.1) *is semi-quasi-homogeneous system. If the Kowalevskaya exponents of its quasi-homogeneous cut* (4.2) $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ *do not satisfy any resonant condition*

$$\sum_{j=1}^{n} k_j \rho_j = 0,\ k_j \in \mathbb{N} \cup \{0\},\ \sum_{j=1}^{n} k_j \geq 1,$$

then semi-quasi-homogeneous system (2.1) *does not have any polynomial first integral.*

To illustrate Theorem 14 we consider the next example given in [45].

Example 17. Consider a four dimensional system of Lotka-Volterra type

$$\begin{aligned} \dot{x}_1 &= x_1\,(\alpha_1 + a x_1 + b x_2)\,, \quad \dot{x}_2 = x_2\,(\alpha_2 + c x_1 + d x_2)\,, \\ \dot{x}_3 &= x_3\,(\alpha_3 + a x_1 + b x_2 + e x_3 + f x_4)\,, \quad \dot{x}_4 = x_4\,(\alpha_1 - f x_4)\,, \end{aligned} \tag{4.6}$$

where $\alpha_j, a, b, c, d, e, f$ are real constants. This system can be regarded as a negative semi-quasi-homogeneous system with exponents $s_1 = s_2 = \cdots = s_n = 1$. Its quasi-homogeneous cut is

$$\begin{aligned} \dot{x}_1 &= x_1\,(a x_1 + b x_2)\,, \quad \dot{x}_2 = x_2\,(c x_1 + d x_2)\,, \\ \dot{x}_3 &= x_3\,(a x_1 + b x_2 + e x_3 + f x_4)\,, \quad \dot{x}_4 = -f x_4^2. \end{aligned} \tag{4.7}$$

Then system (4.7) has the balance

$$\boldsymbol{c} = \left(\frac{b - d}{ad - cb}, \frac{c - a}{ad - cb}, -\frac{1}{e}, \frac{1}{f} \right),$$

if $ad - cb \neq 0$ and $ef \neq 0$. The Kowalevskaya exponents associated to $\boldsymbol{c}$ are

$$\boldsymbol{\rho} = \left(-1, -1, -1, -\frac{(b - d)\,(c - a)}{ad - cb} \right).$$

Using Theorem 14 system (4.6) has no polynomial integral, if for any $k_1, k_2, k_3, k_4 \in \mathbb{N} \cup \{0\}, k_1 + k_2 + k_3 + k_4 \geq 1$,

$$-\,(k_1 + k_2 + k_3) + \left[-\frac{(b - d)\,(c - a)}{ad - cb} \right] \cdot k_4 \neq 0,$$

or equivalently

$$-\frac{(b - d)\,(c - a)}{ad - cb} \notin \mathbb{Q}^-.$$

The function $\Phi\,(\boldsymbol{x})$ is *Laurent polynomial* if it can be represented in the form

$$\Phi\,(\boldsymbol{x}) = \sum_{(k_1, \ldots, k_n) \in \mathcal{A}} \Phi_{k_1 \ldots k_n} x_1^{k_1} \cdots x_n^{k_n},$$

where $\boldsymbol{x} = (x_1, \ldots, x_n)$, $\Phi_{k_1 \ldots k_n} \in \mathbb{C}$ and $\mathcal{A}$ is a finite subset of $\mathbb{Z}^n$.

Shi et al. in [46] studied the nonexistence and existence of Laurent polynomial integrals for semi-quasi-homogeneous systems and proved the following two results.

Theorem 15. *Assume that system (2.1) is a semi-quasi-homogeneous system. If the Kowalevskaya exponents of its quasi-homogeneous cut (4.2)* $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ *are* $\mathbb{Z}$*-independent, i.e., they do not satisfy any resonant condition of the form*

$$\sum_{j=1}^{n} k_j \rho_j = 0, \; k_j \in \mathbb{Z}, \; \sum_{j=1}^{n} |k_j| \geq 1,$$

then system (2.1) does not have any nontrivial Laurent polynomial first integral.

The next example is due to Shi et al., see [46].

Example 18. Consider the following quadratic system in $\mathbb{R}^n$

$$\dot{x}_i = x_i \left(a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n \right), \quad i = 1, 2, \ldots, n, \tag{4.8}$$

where a_i, a_{ij} are real constants. System (4.8) can be seem as a semi-quasi-homogeneous system with exponents $s_1 = s_2 = \cdots = s_n = 1$. If $a_{jj} \neq 0$, then system (4.8) has the balance $\boldsymbol{c} = (0, \ldots, -1/a_{jj}, \ldots, 0)$. For simplicity, we assume that $j = n$. The Kowalevskaya exponents are $\boldsymbol{\rho} = \big(1 - a_{1n}/a_{nn}, 1 - a_{2n}/a_{nn}, \ldots, 1 - a_{(n-1)n}/a_{nn}, -1\big)$. Applying Theorem 15 system (4.8) does not have any Laurent polynomial integral if

$$k_1 \left(1 - \frac{a_{1n}}{a_{nn}} \right) + \cdots + k_n \left(1 - \frac{a_{(n-1)n}}{a_{nn}} \right) - k_n \neq 0, \quad k_j \in \mathbb{Z}, \quad \sum_{j=1}^{n} |k_j| \geq 1,$$

holds for any nonzero integral vector $\boldsymbol{k} \in \mathbb{Z}^n$. This is equivalent to

$$\tilde{k}_1 a_{1n} + \tilde{k}_2 a_{2n} + \cdots + \tilde{k}_n a_{nn} \neq 0, \quad \tilde{k}_j \in \mathbb{Z}, \quad \sum_{j=1}^{n} |\tilde{k}_j| \geq 1.$$

Theorem 16. *Assume that system (2.1) is a semi-quasi-homogeneous system of degree* d *with the weighted exponents* $(s_1, \ldots, s_n)$*, and has* m *nontrivial Laurent polynomial first integrals* $\Phi_1(\boldsymbol{x}), \ldots, \Phi_m(\boldsymbol{x})$*, and that the following conditions hold.*

(i) The Kowalevsky matrix $K(\mathbf{c})$ *of its quasi-homogeneous cut (4.2) is diagonalizable.*

(ii) The Kowalevskaya exponents $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ *associated to the balance* $\mathbf{c}$ *satisfy*

$$\operatorname{rank} G = \operatorname{rank} \left\{ (k_1, \cdots, k_n) \in \mathbb{Z}^n : \sum_{j=1}^{n} k_j \rho_j = 0 \right\} = m.$$

(iii) The Laurent polynomial first integrals $L_1(\boldsymbol{x}, z), \cdots, L_m(\boldsymbol{x}, z)$ *of the linear system*

$$\dot{\boldsymbol{x}} = K \boldsymbol{x}, \quad \dot{z} = -\frac{z}{d-1}, \tag{4.9}$$

are functionally independent.

Then any other nontrivial Laurent polynomial first integral $\Phi(\boldsymbol{x})$ *of system (2.1) is a function of* $\Phi_1(\boldsymbol{x}), \ldots, \Phi_m(\boldsymbol{x})$*.*

Remark 2. *If system (2.1) has a nontrivial Laurent polynomial first integral* $\Phi(\boldsymbol{x})$*, then the linear system (4.9) has also a Laurent polynomial first integral* $L(\boldsymbol{x}, z)$*, see [46] for more details.*

As an application of Theorem 16, the authors in [46] consider the Euler-Poincaré equations on Lie algebras [4] as follows.

Example 19. We consider the following Euler-Poincaré equations

$$\begin{aligned}
\dot{x}_1 &= -s(\boldsymbol{x})(\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3),\\
\dot{x}_2 &= -s(\boldsymbol{x})(\beta_2 x_2 + \gamma_2 x_3),\\
\dot{x}_3 &= -s(\boldsymbol{x})(\beta_3 x_2 + \gamma_3 x_3),\\
\dot{x}_4 &= p(\boldsymbol{x})(\alpha_1 x_1 + \beta_1 x_2 + \gamma_1 x_3) + q(\boldsymbol{x})(\beta_2 x_2 + \gamma_2 x_3) + r(\boldsymbol{x})(\beta_3 x_2 + \gamma_3 x_3),
\end{aligned} \tag{4.10}$$

with the first integral $H = (x_1 p(\boldsymbol{x}) + x_2 q(\boldsymbol{x}) + x_3 r(\boldsymbol{x}) + x_4 s(\boldsymbol{x}))/2$, where $p(\boldsymbol{x}) = ax_1 + ex_2 + fx + 3 + gx + 4$, $q(\boldsymbol{x}) = ex_1 + bx_2 + hx_3 + vx_4$, $r(\boldsymbol{x}) = fx_1 + hx_2 + cx_3 + jx_4$ and $s(\boldsymbol{x}) = gx_1 + vx_2 + jx_3 + dx_4$. System (4.10) is a quasi-homogeneous system as well as a semi-quasi-homogeneous system with exponents $s_1 = s_2 = s_3 = s_4 = 1$. If $a = 0$ and $\alpha_1 g \neq 0$, then system (4.10) has the balance $\boldsymbol{c} = \left(\frac{1}{\alpha_1 g}, 0, 0, 0\right)$ with Kowalevskaya exponents

$$\boldsymbol{\rho} = \left(-1, 2, 1 - \frac{\beta_2 + \gamma_3 - \sqrt{(\beta_2 - \gamma_3)^2 + 4\beta_3\gamma_2}}{2\alpha_1}, 1 - \frac{\beta_2 + \gamma_3 + \sqrt{(\beta_2 - \gamma_3)^2 + 4\beta_3\gamma_2}}{2\alpha_1}\right).$$

Applying Theorem 16, we obtain that if

$$\operatorname{rank} G = \operatorname{rank}\left\{(k_1, k_2, k_3, k_4) \in \mathbb{Z}^4 : k_1\rho_1 + k_2\rho_2 + r_3\rho_3 + r_4\rho_4 = 0\right\} = 1, \tag{4.11}$$

then any other nontrivial Laurent polynomial integral $\Phi(\boldsymbol{x})$ of system (4.10) is a function of H. Moreover equation (4.11) is equivalent to

$$\tilde{k}_1 + \tilde{k}_2 \cdot \frac{\beta_2 + \gamma_3}{2\alpha_1} + \tilde{k}_3 \cdot \frac{\sqrt{(\beta_2 - \gamma_3)^2 + 4\beta_3\gamma_2}}{2\alpha_1} \neq 0$$

for any $\tilde{k}_1, \tilde{k}_2, \tilde{k}_3 \in \mathbb{Z}, |\tilde{k}_1| + |\tilde{k}_2| + |\tilde{k}_3| \geq 1$.

In [44] Shi proved the following criterion of nonexistence of rational first integrals for semi-quasi-homogeneous systems and give a example.

Theorem 17. *Assume that system (2.1) is a semi-quasi-homogeneous system with balance $\boldsymbol{c}$ and $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ are the Kowalevskaya exponents associated to the balance $\boldsymbol{c}$ of its quasi-homogeneous cut (4.2). If system (2.1) has a rational first integral, then there exists a nonzero integral vector $\boldsymbol{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ such that $\sum_{i=1}^n k_i\rho_i = 0$.*

Example 20. Consider the following quadratic system in $\mathbb{R}^n$

$$\dot{x}_i = a_i x_i + x_i(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n), \quad i = 1, 2, \ldots, n, \tag{4.12}$$

where a_i, a_{ij} are real constants. This system can be seem as a negative semi-quasi-homogeneous system with exponents $s_1 = s_2 = \cdots = s_n = 1$. If $a_{jj} \neq 0$, then system (4.12) has the balance $\boldsymbol{c} = (0, \ldots, -1/a_{jj}, \ldots, 0)$. Without loss of generality we assume that $j = n$. In this case the Kowalevskaya exponents are $\boldsymbol{\rho} = \left(1 - a_{1n}/a_{nn}, 1 - a_{2n}/a_{nn}, \ldots, 1 - a_{(n-1)n}/a_{nn}, -1\right)$. By Theorem 17 system (4.12) does not have any rational first integral if

$$k_1\left(1 - \frac{a_{1n}}{a_{nn}}\right) + \cdots + k_n\left(1 - \frac{a_{(n-1)n}}{a_{nn}}\right) - k_n \neq 0$$

holds for any nonzero integral vector $\boldsymbol{k} \in \mathbb{Z}^n$. This is equivalent to

$$\tilde{k}_1 a_{1n} + \tilde{k}_2 a_{2n} + \cdots + \tilde{k}_n a_{nn} \neq 0, \quad \tilde{k}_j \in \mathbb{Z}, \quad \sum_{j=1}^n |\tilde{k}_j| \geq 1.$$

Let $\Phi_1(\boldsymbol{x}), \ldots, \Phi_m(\boldsymbol{x})$ be nontrivial analytic first integrals of a semi-quasi-homogeneous system (2.1). Denote by $\Phi_1^d(\boldsymbol{x}), \cdots, \Phi_m^d(\boldsymbol{x})$ the first quasi-homogeneous terms of $\Phi_1(\boldsymbol{x}), \cdots, \Phi_m(\boldsymbol{x})$, respectively. Then $\Phi_1^d(\boldsymbol{x}), \cdots, \Phi_m^d(\boldsymbol{x})$ are first integrals of the quasi-homogeneous cut system (4.2). Using the Kowalevsky exponents, Kwek et al. [30] gave the following theorem about analytic first integral for general semi-quasi-homogeneous system.

Theorem 18. *Assume that system (2.1) is a semi-quasi-homogeneous system with m nontrivial analytic first integrals $\Phi_1(\boldsymbol{x}), \ldots, \Phi_m(\boldsymbol{x})$, and that the following conditions hold.*

(i) $\Phi_1^d(\boldsymbol{x}), \cdots, \Phi_m^d(\boldsymbol{x})$ *are functionally independent.*

(ii) For a balance $\boldsymbol{c}$, the vectors $\nabla\Phi_1^d(\boldsymbol{c}), \cdots, \nabla\Phi_m^d(\boldsymbol{c})$ are linear independent.

(iii) The last $n-m$ eigenvalues $\rho_{m+1}, \ldots, \rho_n = -1$ of the Kowalevsky matrix K do not satisfy any resonant condition of the form

$$\sum_{j=m+1}^{n} k_j\rho_j = 0,\ |k_n| + \sum_{j=m+1}^{n-1} k_j \geq 1,\ k_n \in \mathbb{Z},\ k_j \in \mathbb{N}\cup\{0\},\ j = m+1, \ldots, n-1.$$

Then any other nontrivial analytic first integral $\Psi(\boldsymbol{x})$ of system (2.1) is a function of $\Phi_1(\boldsymbol{x}), \ldots, \Phi_m(\boldsymbol{x})$, i.e., $\Psi(\boldsymbol{x}) = \mathcal{F}(\Phi_1(\boldsymbol{x}), \ldots, \Phi_m(\boldsymbol{x}))$, where $\mathcal{F}$ is a smooth function.

The following example was shown in [30].

Example 21. Consider the Euler-Poincaré equations (4.10) again. Form Theorem 18 it follows that if

$$\left(1 - \frac{\beta_2 + \gamma_3 - \sqrt{(\beta_2-\gamma_3)^2 + 4\beta_3\gamma_2}}{2\alpha_1}\right)\cdot k_2 + \left(1 - \frac{\beta_2 + \gamma_3 + \sqrt{(\beta_2-\gamma_3)^2 + 4\beta_3\gamma_2}}{2\alpha_1}\right)\cdot k_3 - 1\cdot k_4 \neq 0 \tag{4.13}$$

for any $k_2, k_3 \in \mathbb{N}\cup\{0\}, k_4 \in \mathbb{Z}, k_2 + k_3 + |k_4| \geq 1$, then any other nontrivial analytic first integral $\Phi(\boldsymbol{x})$ of system (4.10) is functionally dependent on H.

In particular, if $\beta_2 = \gamma_3$, and $\beta_3 = 0$, $\gamma_2 \neq 0$ (or $\beta_2 \neq 0, \gamma_2 = 0$), then (4.13) becomes

$$k_2 + k_3 - k_4 - (k_2 + k_3)\frac{\beta_2}{\alpha_1} \neq 0$$

or equivalently, $\beta_2/\alpha_1 \notin \mathbb{Q}$.

The following result is an extension of Theorem 18 and it was proved in [51].

Theorem 19. *Assume that system (2.1) is semi-quasi-homogeneous of degree d associated with the weighted exponents $(s_1, \cdots, s_n)$, and that it has m ($m < n-1$) nontrivial analytic first integrals $\Phi_1(\boldsymbol{x}), \cdots, \Phi_m(\boldsymbol{x})$ in a neighbourhood of the singularity $\boldsymbol{x} = 0$. Denote by $\Phi_1^d(\boldsymbol{x}), \cdots, \Phi_m^d(\boldsymbol{x})$ the first quasi-homogeneous terms of $\Phi_1(\boldsymbol{x}), \cdots, \Phi_m(\boldsymbol{x})$, respectively. Moreover, we suppose that the following conditions hold.*

(i) There exists a balance $\boldsymbol{c}$ such that the corresponding Kowalevskaya exponents $\rho_1, \cdots, \rho_n$ satisfy the conditions: $\rho_1 = 0$, and

$$\operatorname{rank}\left\{(k_2, \cdots, k_n) : \sum_{i=2}^{n} k_i\rho_i = 0,\ \sum_{i=2}^{n} k_i \neq 0,\ k_i \in \mathbb{Z}^+,\ i = 2, \cdots, n\right\} = m.$$

(ii) The eigenspace corresponding to ρ_1 is tangent to the manifold $S = \left\{\Phi_1^d(\boldsymbol{x}) = 0\right\} \cap \ldots \cap \left\{\Phi_m^d(\boldsymbol{x}) = 0\right\}$.

(iii) The functions $\Phi_1^d(\boldsymbol{x}), \cdots, \Phi_m^d(\boldsymbol{x})$ *are independent at the balance* $\boldsymbol{c}$*.*

Then if the balance $\mathbf{c}$ *of the quasi-homogeneous cut* (4.2) *is isolated, any first integral (analytic or formal series) of system* (2.1) *in a neighbourhood of the singularity* $\mathbf{0}$ *is an analytic function or a formal series in* $\Phi_1(\boldsymbol{x}), \cdots, \Phi_m(\boldsymbol{x})$*.*

The next example is inspired by Example 2 of [30].

Example 22. Consider a four-dimensional system of Lotka-Volterra type

$$\begin{aligned}&\dot{x}_1 = x_1(\alpha_1 + ax_1 + bx_3), \quad \dot{x}_2 = -x_2(\alpha_1 + ax_1 + bx_3), \quad \dot{x}_3 = -ax_1x_3,\\ &\dot{x}_4 = x_4(\alpha_2 + ax_1 + dx_2 + ex_3 + fx_4),\end{aligned} \tag{4.14}$$

where $\alpha_1, \alpha_2, a, b, d, e, f$ are real constants. System (4.14) has two first integrals $\Phi_1(\boldsymbol{x}) = ax_1x_3 + bx_3^2/2 + \alpha_1 x_3$ and $\Phi_2(\boldsymbol{x}) = x_1x_2$. This system can be regarded as a negative semi-quasi-homogeneous system with exponents $s_1 = s_2 = \cdots = s_n = 1$. Its quasi-homogeneous cut is

$$\begin{aligned}&\dot{x}_1 = x_1(ax_1 + bx_3), \quad \dot{x}_2 = -x_2(ax_1 + bx_3), \quad \dot{x}_3 = -ax_1x_3,\\ &\dot{x}_4 = x_4(ax_1 + dx_2 + ex_3 + fx_4)\end{aligned} \tag{4.15}$$

with two first integrals $\Phi_1^2(\boldsymbol{x}) = ax_1x_3 + bx_3^2/2$ and $\Phi_2^2(\boldsymbol{x}) = x_1x_2$. System (4.15) has a balance $\boldsymbol{c} = (-1/a, 0, 0, 0)$ with Kowalevsky matrix

$$\begin{pmatrix} -1 & 0 & -\frac{b}{a} & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

So the corresponding Kowalevskaya exponents are $\boldsymbol{\rho} = (0, -1, 2, 2)$. Obviously,

$$\text{rank } \{(k_2, k_3, k_4) : -k_2 + 2k_3 + 2k_4 = 0\} = 2$$

and $\Phi_1^2(\boldsymbol{x}), \Phi_2^2(\boldsymbol{x})$ are independent at the balance $\boldsymbol{c}$. The eigenspace $V = \{(0,0,0,x_4) : x_4 \in \mathbb{R}\}$ corresponding to $\rho_1 = 0$ is tangent to the manifold given by $\left\{\Phi_1^2(\boldsymbol{x}) = 0\right\} \cap \left\{\Phi_2^2(\boldsymbol{x}) = 0\right\}$. By Theorem 19, any first integral (analytic or formal series) of system (4.14) in a neighbourhood of the singularity $\mathbf{0}$ is an analytic function or a formal series in $\Phi_1(\boldsymbol{x})$ and $\Phi_2(\boldsymbol{x})$.

5 Kovalevskaya exponents of Hamiltonian systems

The Hamiltonian systems are very useful in mathematical physics, especially in celestial mechanics, control engineering and other fields. Yoshida was the first to point out an interesting relation between the Kowalevskaya exponent for Hamiltonian systems. The final form of this relation was given by Lochak [36], see the next result.

Proposition 3. *Let system* (2.1) *be a Hamiltonian system with Hamiltonian* H*. If* ρ *is a Kowalevskaya exponent for system* (2.1)*, then* $h - 1 - \rho$ *also is a Kowalevskaya exponent, where* h *is the weighted degree of the Hamiltonian* H*.*

Example 23. Consider the Hamiltonian system

$$\dot{x}_1 = y_1, \quad \dot{x}_2 = y_2, \quad \dot{x}_3 = y_3, \quad \dot{y}_1 = -x_3^3, \quad \dot{y}_2 = -x_3^3, \quad \dot{y}_3 = -3x_3^2(x_1 + x_2) \tag{5.1}$$

with the Hamiltonian $H = \left(y_1^2 + y_2^2 + y_3^2\right)/2 + x_1x_3^3 + x_2x_3^3$. This system is a $(1,1,1,2,2,2)$-2 type system and the weighted degree of H is 4. The Kovalevskaya exponents of all balances $\boldsymbol{c}$ are

$$\boldsymbol{\rho} = \left(-1, 1, \frac{1}{2}\left(3 + i\sqrt{7}\right), 4, 2, \frac{1}{2}\left(3 - i\sqrt{7}\right)\right).$$

The Kovalevskaya exponents of system (5.1) satisfy the relation of Proposition 3.

Form Proposition 3 it follows that the Kowalevskaya exponents always come by pairs for Hamiltonian systems. Yoshida considered Hamiltonian systems of n degree of freedom with a diagonal kinetic contribution and a homogeneous potential of the form

$$H = \frac{1}{2}\left(p_1^2 + \cdots + p_n^2\right) + V\left(q_1, \ldots, q_n\right), \tag{5.2}$$

where $V(\boldsymbol{x})$ is a homogeneous polynomial of degree k with $k \neq 0, \pm 2$. Let $\rho_1, \ldots, \rho_{2n}$ be Kowalevskaya exponents of Hamiltonian (5.2). Then, by Proposition 3, $\rho_i + \rho_{n+i} = (k+2)/(k-2)$ for $i = 1, \ldots, n$. We can define the difference between two exponents of each pair as $\Delta\rho_i = \rho_{i+n} - \rho_i$ for $i = 1, \ldots, n$. In [50] Yoshida proved the following result.

Theorem 20. *Let $\rho_1, \ldots, \rho_{2n}$ be Kowalevskaya exponents of the Hamiltonian system with Hamiltonian (5.2). If $\Delta\rho_1, \ldots, \Delta\rho_n$ are $\mathbb{Q}$-independent, then this Hamiltonian system has no additional first integral besides H.*

Example 24 (Reduce Yang-Mills equations [19]). Consider the Yang-Mills quadratic potential in three dimensions

$$V = q_1^2 q_2^2 + q_2^2 q_3^2 + q_1^2 q_3^2,$$

i.e., Hamiltonian system

$$\begin{aligned} &\dot{p}_1 = 2q_1 q_2^2 + 2q_1 q_3^2, \quad \dot{p}_2 = 2q_2 q_1^2 + 2q_2 q_3^2, \quad \dot{p}_3 = 2q_3 q_1^2 + 2q_2^2 q_3, \\ &\dot{q}_1 = -p_1, \quad \dot{q}_2 = -p_2, \quad \dot{q}_3 = -p_3, \end{aligned} \tag{5.3}$$

with the Hamiltonian $H = \left(p_1^2 + p_2^2 + p_3^2\right)/2 + q_1^2 q_2^2 + q_3^2 q_2^2 + q_1^2 q_3^2$. This system has the balance $\boldsymbol{c} = (0, -i, -i, 0, -i, -i)$. The corresponding Kowalevskaya exponents are

$$\boldsymbol{\rho} = \left(-1, \frac{\sqrt{17}+3}{2}, \frac{3+i\sqrt{7}}{2}, 4, \frac{3-\sqrt{17}}{2}, \frac{3-i\sqrt{7}}{2}\right).$$

The diference of Kovalevskaya exponents is $(\Delta\rho_1, \Delta\rho_2, \Delta\rho_3) = \left(5, -\sqrt{17}, -i\sqrt{7}\right)$. Clearly, these numbers are $\mathbb{Q}$-independent. So system (5.3) has no additional first integrals.

6 Darboux Polynomials and Kovalevskaya exponents

Darboux [12, 13] provided a method to get first integrals of polynomial differential systems via a sufficient number of Darboux polynomials, that is, invariant algebraic curves, or surfaces, or hypersurfaces. The Darboux polynomials play an important role in the Darboux theory of integrability, see [10, 33, 52] and the references cited therein. In this section we describe the relationship between Kovalevskaya exponents and Darboux polynomials.

The polynomial $F(\boldsymbol{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ is a *Darboux polynomial* of system (2.1) if there exists a $k(\boldsymbol{x}) \in \mathbb{C}[x_1, \ldots, x_n]$ such that

$$\sum_{i=1}^{n} P_i \frac{\partial F}{\partial x_i} = \boldsymbol{P} \cdot \nabla F = kF,$$

The polynomial $k(\boldsymbol{x})$ is called the *cofactor* and has degree at most $m-1$ if m is the degree of the polynomial differential system (2.1). As usual, ∇F denotes the gradient of the function F. Then $F = 0$ is an *invariant algebraic hypersurface* of system (2.1) if F is a Darboux polynomial of system (2.1).

We can decompose the polynomial $P_i(\boldsymbol{x})$ for $i = 1, \cdots, n$ as the sum of its quasi-homogeneous parts, that is,

$$P_i(\boldsymbol{x}) = \sum_{j=0}^{m_i} P_i^{(j)}(\boldsymbol{x}), \tag{6.1}$$

where $P_i^{(j)}(\boldsymbol{x})$ is a quasi-homogeneous polynomial of weighted exponent $\boldsymbol{s} = (s_1, \ldots, s_n) \in \mathbb{Q}$ with weighted degree $s_i + q^{(j)} - 1$, i.e.

$$P_i^{(j)}(\alpha^{\boldsymbol{s}} \boldsymbol{x}) = \alpha^{s_i + q^{(j)} - 1} P_i^{(j)}(\boldsymbol{x}), \tag{6.2}$$

for $i = 1, \cdots, n$ and $j = 0, 1, \cdots, m_i$, with $q^{(j)} \in Q$ and $q^{(0)} = 0 < q^{(1)} < \cdots < q^{(\tilde{m})}$, where $\tilde{m}$ is the maximum of the $\{m_1, \cdots, m_n\}$. Thus system (2.1) can be written in the form

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{P}(\boldsymbol{x}) = \sum_{j=0}^{\tilde{m}} \boldsymbol{P}^{(j)}(\boldsymbol{x}), \tag{6.3}$$

where $\boldsymbol{P}^{(j)}(\boldsymbol{x}) = \left(P_1^{(j)}(\boldsymbol{x}), \ldots, P_n^{(j)}\right)$, $P_1^{(j)}(\boldsymbol{x})$ satisfies (6.2) and $\boldsymbol{P}^{(0)}(\boldsymbol{x}) \not\equiv 0$.

Assume that the system $\dot{\boldsymbol{x}} = \boldsymbol{P}^{(0)}(\boldsymbol{x})$ has the particular solution $\boldsymbol{x} = \boldsymbol{c}t^{\boldsymbol{s}} = (c_1 t^{s_1}, \cdots, c_n t^{s_n})$ where the coefficients $\boldsymbol{c} \in \mathbb{C}^n$ are solution of the algebraic systems

$$s_i c_i = P_i^{(0)}(\boldsymbol{c}), \quad i = 1, \ldots, n, \tag{6.4}$$

$t = t - t_*$ for some complex t_*, and $\boldsymbol{s} \in \mathbb{C}^n$ with $|\boldsymbol{s}| = |s_1| + \cdots + |s_n| \neq 0$. Then we say that the polynomial differential system (2.1) has a *dominant balance* $\{\boldsymbol{c}, \boldsymbol{s}\}$. The Kovalevskaya exponents $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ associated with the dominant balance $\{\boldsymbol{c}, \boldsymbol{s}\}$ are the eigenvalues of the Kovalevskaya matrix

$$M(\boldsymbol{c}) = D\boldsymbol{P}^{(0)}(\boldsymbol{c}) - \operatorname{diag}(s_1, \ldots, s_n). \tag{6.5}$$

The following two theorems and examples can be found in [34].

Theorem 21. *Assume that the polynomial differential system* (2.1) *admits the particular solution* $\boldsymbol{x} = \boldsymbol{c}t^{\boldsymbol{s}}$, *i.e.,* $\boldsymbol{x} = (c_1 t^{s_1}, \cdots, c_n t^{s_n})$. *If* $F(\boldsymbol{x})$ *is a quasi-homogeneous Darboux polynomial of weighted degree* d *of system* (2.1), *then* $\nabla F(\boldsymbol{c}) \neq 0$ *and its cofactor* k *cannot be constant.*

Example 25. The four dimensional system

$$\begin{aligned} &\dot{x}_1 = -x_3, \quad \dot{x}_2 = -x_4, \quad \dot{x}_3 = 3i(x_2 - ix_1)(x_2 + ix_1)^2 - i(x_2 + ix_1)^3, \\ &\dot{x}_4 = 3(x_2 - ix_1)(x_2 + ix_1)^2 + i(x_2 + ix_1)^3, \end{aligned} \tag{6.6}$$

has the Darboux polynomials $F_1 = x_3 - ix_4 - (x_2 + ix_1)^2$ with cofactor $k_1 = -2(x_1 - ix_2)$, and $F_2 = x_3 - ix_4 + (x_2 + ix_1)^2$ with cofactor $k_2 = 2(x_1 - ix_2)$. This system is a $(1,1,2,2)$-2 type system. The Darboux polynomials F_1 and F_2 are both quasi-homogeneous polynomials with weighted degree 2. Note that $\nabla F_j = (*, *, 1, -i)$ for $j = 1, 2$. So cofactor k_j cannot be constant and $\nabla F_j(\boldsymbol{c}) \neq 0$ for $j = 1, 2$, as described in Theorem 21.

The following theorem [34] characterized the relation between the Kovalevskaya exponents and Darboux polynomials.

Theorem 22. *Assume that the polynomial differential system* (2.1) *has a dominant balance* $\{\boldsymbol{c}, \boldsymbol{s}\}$ *such that the Kovalevskaya matrix* $M(\boldsymbol{c})$ *diagonalizes. Then the following statements hold.*

(i) If the Kovalevskaya exponents $(\rho_1, \cdots, \rho_n)$ *of* $M(\boldsymbol{c})$ *are* $\mathbb{Z}$*-independent and* $F(\boldsymbol{x})$ *is a Darboux polynomial of system* (2.1), *then* $F(\boldsymbol{x})$ *must have a cofactor* k *such that* $k^{(0)}(\boldsymbol{c})$ *is not a rational number.*

(ii) If the Kovalevskaya exponents $(\rho_1, \cdots, \rho_n)$ *of* $M(\boldsymbol{c})$ *are* $\mathbb{N}$*-independent and* $F(\boldsymbol{x})$ *is a Darboux polynomial of system* (2.1), *then* $F(\boldsymbol{x})$ *must have a cofactor* k *such that* $k^{(0)}(\boldsymbol{c})$ *is not a non-negative integer.*

Example 26 (Lorenz system)**.** Consider the Lorenz system

$$\dot{x}_1 = p\,(x_2 - x_1)\,,\ \dot{x}_2 = rx_1 - x_2 - x_1x_3,\ \dot{x}_3 = -bx_3 + x_1x_2, \tag{6.7}$$

where p, r, b are real parameters and $p \neq 0$. If $\boldsymbol{s} = (-1, -2, -2)$, then system (6.7) can be written in the form

$$\dot{\boldsymbol{x}} = \boldsymbol{P}^{(0)} + \boldsymbol{P}^{(1)} + \boldsymbol{P}^{(2)},$$

where $\boldsymbol{P}^{(0)} = (px_2, -x_1x_3, x_1x_2)$, $\boldsymbol{P}^{(1)} = (-px_1, -x_2, -bx_3)$ and $\boldsymbol{P}^{(2)} = (0, rx_1, 0)$. Doing the adequate computations, we get that system (6.7) has two dominant balances $\{\boldsymbol{c}, \boldsymbol{s}\}$ with $\boldsymbol{c} = (-2i, 2i/p, -2/p)$ and $\boldsymbol{s} = (-1, -2, -2)$; and $\{\boldsymbol{c}, \boldsymbol{s}\}$ with $\boldsymbol{c} = (2i, -2i/p, -2/p)$ and $\boldsymbol{s} = (-1, -2, -2)$. The Kovalevskaya exponents of these two dominant balances $\{\boldsymbol{c}, \boldsymbol{s}\}$ are $\boldsymbol{\rho} = (-1, 2, 4)$. These Kowalevskaya exponents are $\mathbb{N}$-dependent. If $p = -n/2$ and $b = 2p$ with $n \in \mathbb{N} \cup \{0\}$, then system (6.7) has the Darboux polynomial $x_1^2 - 2px_3$ with cofactor $k = k^{(0)}(\boldsymbol{c}) = n$. This Lorenz system satisfies statement (ii) of Theorem 22.

We consider the planar vector field $\boldsymbol{F} = (P, Q)^T$ with P, Q analytic functions at the origin, and the origin is an isolated singular point. The curve $f(x, y) = 0$ with $f(x, y) \in \mathbb{C}[[x, y]]$ (ring of formal power series in x, y over $\mathbb{C}$) and $f(0) = 0$, is an *invariant curve at the origin* of the vector field $\boldsymbol{F}$ if there exists $k(x, y) \in \mathbb{C}[[x, y]]$ such that

$$P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y} = kf,$$

where $k(x, y)$ is called the *cofactor*. If the formal power series $f(x, y) \in \mathbb{C}[[x, y]]$ is convergent, then $f(x, y) = 0$ is an *analytic invariant curve*.

If a function $f(x, y) \in \mathbb{C}[[x, y]]$ and $f(x, y) = f_1(x, y)\, f_2(x, y)$ with $f, f_1, f_2 \in \mathbb{C}[[x, y]]$ and $f(0) = f_1(0) = f_2(0) = 0$, then $f(x, y)$ is called *reducible*. Otherwise, $f(x, y)$ is *irreducible*. A formal function $U(x, y)$ is *unit element* if $U(0, 0) \neq 0$.

The analytic vector field $\boldsymbol{F}$ can be written in the form

$$\boldsymbol{F} = \boldsymbol{F}_d + \boldsymbol{F}_{d+1} + \cdots, \tag{6.8}$$

for some integer d, where $\boldsymbol{F}_j \in \mathcal{Q}_j^{\boldsymbol{s}}$ (the vector space of the quasi-homogeneous polynomial vector fields of type $\boldsymbol{s}$ and degree j) and $\boldsymbol{F}_d \not\equiv 0$. The expansion (6.8) is expressed as $\boldsymbol{F} = \boldsymbol{F}_d +$ q-h.h.o.t., where "q-h.h.o.t." means "quasi-homogeneous higher order terms". By Lemma 1, the analytic vector field $\boldsymbol{F}_d$ can be expressed as equation (3.10).

The following result gives a link between the Kowalevskaya exponents of a planar vector field and the existence of analytic invariant curves, see [3].

Theorem 23. *Consider an analytic vector field* $\boldsymbol{F} = \boldsymbol{F}_d + \boldsymbol{F}_{d+1} +$ q-h.h.o.t. *with* $\boldsymbol{F}_d \in \mathcal{Q}_d^{\boldsymbol{s}}$ *and* $\tilde{h} \in \mathbb{C}[x, y]$ *a simple factor of* h*. If the Kowalevskaya exponent* ρ *different from* -1 *associated to* $\tilde{h}$ *satisfies* $\rho^{-1} \notin \mathbb{Q} \cap (-(d-1), 0)$*, then there exists a unique irreducible analytic invariant curve* $f = \tilde{h} +$ q-h.h.o.t. *at the origin.*

Moreover, if $\tilde{f}$ *is an invariant curve starting by* $\tilde{h}^n$*, with* n *a natural number, then* $\tilde{f} = f^n U$*, where* U *is a formal unit element.*

Here for the definition of h *see Lemma 1, and the Kowalevskaya exponents associated to the factor* $\tilde{h}$ *are given in formula* (3.11)*.*

The next example was studied by Algaba et al. in [3].

Example 27. Consider the vector field $\boldsymbol{F} = \left(2x^3 - 7xy^3 + \text{q-h.h.o.t.}, 5x^2y - y^4 + \text{q-h.h.o.t.}\right)$. The lowest-degree quasi-homogeneous term with respect to the type $\boldsymbol{s} = (3, 2)$ are

$$F_7 = \left(2x^3 - 7xy^3, 5x^2y - y^4\right), \; h = 11xy\left(x^2 + y^3\right) \text{ and } \eta\left(x, y\right) = \frac{x - y}{xy\left(x + y\right)}.$$

Using Proposition 2 we get

$$\begin{aligned} &\text{for } \tilde{h} = x, \quad \rho^{-1} = -\frac{6}{11}, \\ &\text{for } \tilde{h} = y, \quad \rho^{-1} = -\frac{12}{11}, \\ &\text{for } \tilde{h} = y^3 + x^2, \quad \rho^{-1} = -\frac{18}{11}. \end{aligned}$$

Thus, by Theorem 23, for the vector field $\boldsymbol{F}$ exists the analytic invariant curve $f = y^3 + x^2 + \text{q-h.h.o.t.}$ at the origin.

Recently, the authors showed that the non-rational Kovalevskayas exponents imply the non-existence of Darboux first integrals, see [24].

7 Global Properties of Kovalevskaya exponents

First of all we recall some notations to introduce related works of this subject. System (2.1) is $\boldsymbol{s}$-homogeneous of weighted degree k if its components $\boldsymbol{P}(\boldsymbol{x}) = (P_1(\boldsymbol{x}), \ldots, P_n(\boldsymbol{x}))$ are quasi-homogeneous and $\deg_{\boldsymbol{s}}(P_i) = s_i + k$, for $i = 1, \ldots, n$ (see Definition 1).

Definition 4. *The point $\mathbf{d} \in \mathbb{C}^n \setminus \{0\}$ is called a Darboux point of the $\boldsymbol{s}$-homogeneous system (2.1) if and only if*

$$P_i(\mathbf{d}) = -\alpha s_i d_i \quad \textit{for } i = 1, \cdots, n, \tag{7.1}$$

for a certain $\alpha \in \mathbb{C}$. If $\boldsymbol{P}(\mathbf{d}) \neq 0$, then the Darboux point $\mathbf{d}$ is called proper, otherwise is nonproper.

The set of all Darboux points of system (2.1) is denoted by $\mathscr{D}(\boldsymbol{P})$, and $\mathscr{D}^*(\boldsymbol{P})$ denotes the set of all proper Darboux points.

Analogously we can define the *Kovalevskaya matrix*

$$K(\boldsymbol{d}) = \frac{1}{\alpha} D\boldsymbol{P}(\boldsymbol{d}) + \operatorname{diag}(s_1, \ldots, s_n)$$

for each Darboux point $\boldsymbol{d}$. The eigenvalues of $K(\boldsymbol{d})$ are also called the *Kowalevskaya exponents* of $\boldsymbol{d}$ and are denoted by $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$. For convenience, we fix $\alpha = 1$ in the above definition.

In [11] Maciejewski et al. studied the global properties of Kovalevskaya exponents, and they proved:

Theorem 24. *Let $\mathbf{d}$ be a Darboux point of a $\boldsymbol{s}$-homogeneous of weighted degree k system (2.1) and $K(\boldsymbol{d})$ the corresponding Kovalevskaya matrix. Then $\rho = -k$ is an eigenvalue of $K(\boldsymbol{d})$ and $\boldsymbol{e} = \boldsymbol{P}(\boldsymbol{d})$ is the corresponding eigenvector.*

Example 28. The system

$$\dot{x} = axy, \quad \dot{y} = b_1x^3 + b_2y^2,$$

is a $(2, 3)$-homogeneous of weighted degree 3 system. This system has two Darboux points $\boldsymbol{d}_1 = (0, -3/b_2)$ and $\boldsymbol{d}_2 = \left(\sqrt[3]{2ab_1^2(3a - 2b_2)}/(ab_1), -2/a\right)$. The Kowalevskaya exponents are

$\boldsymbol{\rho}_1 = (-3, 2-3a/b_2)$ and $\boldsymbol{\rho}_2 = (-3, 6-4b_2/a)$. Then $\rho = -k = -3$ is an eigenvalue of $K(\boldsymbol{d})$. The vectors $\boldsymbol{e}_1 = \boldsymbol{P}(\boldsymbol{d}_1) = (0, 9/b_2)$ and $\boldsymbol{e}_2 = \boldsymbol{P}(\boldsymbol{d}_2) = \left(-2\sqrt[3]{2ab_1^2(3a-2b_2)}/(ab_1), 6/a\right)$ are eigenvectors of $K(\boldsymbol{d}_1)$ and $K(\boldsymbol{d}_2)$, respectively. This system satisfies the statement of Theorem 24.

Let $\boldsymbol{d}$ be a proper Darboux point and $\rho_i(\boldsymbol{d})$ be the corresponding Kovalevskaya exponents for $i = 1, \cdots, n$. Without loss of generality we assume $\rho_n(\boldsymbol{d}) = -k$. The remaining Kovalevskaya exponents will be denoted by $\boldsymbol{\rho}(\boldsymbol{d}) = (\rho_1(\boldsymbol{d}), \cdots, \rho_{n-1}(\boldsymbol{d}))$ and called nontrivial ones. The elementary symmetric polynomial of degree r in its variables is denoted by τ_r.

The following theorem goes back to Przybylska [42]. It shows that the described relations do not depend on a specific form of system (2.1), but only on its degree provided that it is generic.

Theorem 25. *Assume that system (2.1) is a homogeneous polynomial system of degree k, i.e., $\deg(P_i(\boldsymbol{x})) = k+1$ for $i = 1, \dots, n$. Assume that all Darboux points of system (2.1) are proper and simple. Then*

$$\sum_{d \in \mathscr{D}^*(\boldsymbol{P})} \frac{(\rho_1(\boldsymbol{d}) + \cdots + \rho_{n-1}(\boldsymbol{d}))^r}{\rho_1(\boldsymbol{d}) \cdots \rho_{n-1}(\boldsymbol{d})} = (n+k)^r \tag{7.2}$$

and

$$\sum_{d \in \mathscr{D}^*(\boldsymbol{P})} \frac{\tau_r(\boldsymbol{\rho}(\boldsymbol{d}))}{\rho_1(\boldsymbol{d}) \cdots \rho_{n-1}(\boldsymbol{d})} = \sum_{i=0}^{r} \frac{(n-i-1)!}{(n-1-r)!\,(r-i)!}(k+1)^i, \tag{7.3}$$

for $r = 0, \dots, n-1$.

Example 29 (Halphen system)**.** Consider the Halphen system

$$\begin{aligned}
\dot{x} &= a_1 x^2 + (1-a_1)(xy + xz - yz),\\
\dot{y} &= a_2 y^2 + (1-a_2)(xy - xz + yz),\\
\dot{z} &= a_3 z^2 + (1-a_3)(xz + yz - xy),
\end{aligned}$$

(see [21]). This system has 7 Darboux points $\boldsymbol{d}_0 = (-1,-1,-1)$, $\boldsymbol{d}_1 = (-1/a_1, 0, 0)$, $\boldsymbol{d}_2 = (0, -1/a_2, 0)$, $\boldsymbol{d}_3 = (0, 0, -1/a_3)$, $\boldsymbol{d}_{3+i} = \boldsymbol{d}_0 - \boldsymbol{d}_i$ for $i = 1,2,3$. The corresponding nontrivial Kowalevskaya exponents are $\boldsymbol{\rho}(\boldsymbol{d}_0) = (-1,-1)$ and $\boldsymbol{\rho}(\boldsymbol{d}_i) = (1, \rho_2(\boldsymbol{d}_i))$ with $\rho_2(\boldsymbol{d}_i) = \rho_2(\boldsymbol{d}_{i+3}) = (a_1 + a_2 + a_3 - 2)/a_i$ for $i = 1,2,3$. Then the following relations hold:

$$\begin{aligned}
&\sum_{i=0}^{6} \frac{1}{\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = \sum_{i=0}^{6} \frac{\tau_0(\rho_1(\boldsymbol{d}_i), \rho_2(\boldsymbol{d}_i))}{\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = 1,\\
&\sum_{i=0}^{6} \frac{\rho_1(\boldsymbol{d}_i) + \rho_2(\boldsymbol{d}_i)}{\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = \sum_{i=0}^{6} \frac{\tau_1(\rho_1(\boldsymbol{d}_i), \rho_2(\boldsymbol{d}_i))}{\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = 4,\\
&\sum_{i=0}^{6} \frac{(\rho_1(\boldsymbol{d}_i) + \rho_2(\boldsymbol{d}_i))^2}{\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = 16,\\
&\sum_{i=0}^{6} \frac{\tau_2(\rho_1(\boldsymbol{d}_i), \rho_2(\boldsymbol{d}_i))}{\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = 7,
\end{aligned}$$

where $\tau_0(x_1, x_2) = 1$, $\tau_1(x_1, x_2) = x_1 + x_2$ and $\tau_2(x_1, x_2) = x_1 x_2$. In this case $k = 1$ and $n = 3$, the above relations are exactly the same as those given in Theorem 25.

Using the Euler-Jacobi-Kronecker formula, Maciejewski et al. [11] extend Theorem 25 to $\boldsymbol{s}$-homogeneous of weighted degree k system, see the following two theorems.

Let $\boldsymbol{h} = (h_1(\boldsymbol{x}), \dots, h_n(\boldsymbol{x})) \in \mathbb{C}[\boldsymbol{x}]^n$ with $h_i(\boldsymbol{x}) = P_i(\boldsymbol{x}) + s_i x_i$ for $i = 1, \dots, n$. The set $\mathscr{V}(\boldsymbol{h})$ of common zeros of polynomials $(h_1, \cdots, h_n)$ is finite and all its points are simple, i.e., $\det D\boldsymbol{h}(\boldsymbol{d}) \neq 0$ for all $\boldsymbol{d} \in \mathscr{V}(\boldsymbol{h})$. We denote also $\mathscr{V}^*(\boldsymbol{h}) = \mathscr{V}(\boldsymbol{h}) \setminus \{\boldsymbol{0}\}$.

Theorem 26. *If all Darboux points of* $\boldsymbol{s}$*-homogeneous of weighted degree* k *system* (2.1) *are proper and simple, then*

$$\sum_{\boldsymbol{d}\in\mathscr{V}^*(\boldsymbol{h})} \frac{(\rho_1(\boldsymbol{d})+\cdots+\rho_{n-1}(\boldsymbol{d}))^r}{k\rho_1(\boldsymbol{d})\cdots\rho_{n-1}(\boldsymbol{d})} = \frac{(k+s_1+\cdots+s_n)^r}{s_1\cdots s_n} \tag{7.4}$$

and

$$\sum_{\boldsymbol{d}\in\mathscr{V}^*(\boldsymbol{h})} \frac{\tau_r(\boldsymbol{\rho}(\boldsymbol{d}))}{k\rho_1(\boldsymbol{d})\cdots\rho_{n-1}(\boldsymbol{d})} = \frac{1}{s_1\cdots s_n}\sum_{i=0}^{r} k^i\tau_{r-i}(\boldsymbol{s}), \tag{7.5}$$

for $r=0,\cdots,n-1$.

Example 30. The three dimensional system

$$\dot{x}=xy^2, \quad \dot{y}=x+y^3, \quad \dot{z}=z^2, \tag{7.6}$$

is $(3,1,2)$-homogeneous of weighted degree 2. It has 9 Darboux points $\boldsymbol{d}_0=(0,0,-2)$, $\boldsymbol{d}_1=(0,-i,-2)$, $\boldsymbol{d}_2=(0,i,-2)$, $\boldsymbol{d}_3=-\boldsymbol{d}_4=(0,-i,0)$, $\boldsymbol{d}_5=\left(-2i\sqrt{3},-i\sqrt{3},-2\right)$, $\boldsymbol{d}_6=\left(2i\sqrt{3},i\sqrt{3},-2\right)$ and $\boldsymbol{d}_7=-\boldsymbol{d}_8=\left(-2i\sqrt{3},-i\sqrt{3},0\right)$. The corresponding nontrivial Kowalevskaya exponents are $\boldsymbol{\rho}(\boldsymbol{d}_0)=(1,3)$, $\boldsymbol{\rho}(\boldsymbol{d}_1)=\boldsymbol{\rho}(\boldsymbol{d}_2)=(-2,2)$, $\boldsymbol{\rho}(\boldsymbol{d}_3)=\boldsymbol{\rho}(\boldsymbol{d}_4)=(2,2)$, $\boldsymbol{\rho}(\boldsymbol{d}_5)=\boldsymbol{\rho}(\boldsymbol{d}_6)=(-6,-2)$ and $\boldsymbol{\rho}(\boldsymbol{d}_7)=\boldsymbol{\rho}(\boldsymbol{d}_8)=(-6,2)$. Then the following relations hold:

$$\begin{aligned}
&\sum_{i=0}^{8}\frac{1}{2\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = \sum_{i=0}^{8}\frac{\tau_0(\rho_1(\boldsymbol{d}_i),\rho_2(\boldsymbol{d}_i))}{2\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = \frac{1}{6},\\
&\sum_{i=0}^{8}\frac{\rho_1(\boldsymbol{d}_i)+\rho_2(\boldsymbol{d}_i)}{2\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = \sum_{i=0}^{8}\frac{\tau_1(\rho_1(\boldsymbol{d}_i),\rho_2(\boldsymbol{d}_i))}{2\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = \frac{4}{3},\\
&\sum_{i=0}^{8}\frac{(\rho_1(\boldsymbol{d}_i)+\rho_2(\boldsymbol{d}_i))^2}{2\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = \frac{32}{3},\\
&\sum_{i=0}^{8}\frac{\tau_2(\rho_1(\boldsymbol{d}_i),\rho_2(\boldsymbol{d}_i))}{2\rho_1(\boldsymbol{d}_i)\rho_2(\boldsymbol{d}_i)} = \frac{9}{2},
\end{aligned}$$

where $\tau_0(x_1,x_2)=1$, $\tau_1(x_1,x_2)=x_1+x_2$ and $\tau_2(x_1,x_2)=x_1x_2$. In this case $\boldsymbol{s}=(3,1,2)$, $k=2$ and $n=3$, the above relations are exactly the same as those given by Theorem 26.

Moreover if $\boldsymbol{d}=(d_1,\ldots,d_n)$ is a single proper Darboux point of $\boldsymbol{s}$-homogeneous of weighted degree k system (2.1), then so is $\boldsymbol{d}_j:=\left(\varepsilon^{js_1}d_1,\ldots,\varepsilon^{js_n}d_n\right)$ with $j=0,\ldots,k-1$, where $\varepsilon^k=1$. Moreover, we have $\boldsymbol{\rho}(\boldsymbol{d}_0)=\cdots=\boldsymbol{\rho}(\boldsymbol{d}_{k-1})$. Thus these k Darboux points have the same Kowalevskaya exponents and form an equivalence class. So we introduce the following equivalence relation

$$\boldsymbol{d}\sim\tilde{\boldsymbol{d}} \text{ if and only if } \tilde{\boldsymbol{d}}=\left(\varepsilon^{js_1}d_1,\ldots,\varepsilon^{js_n}d_n\right)$$

for some $j=0,\ldots,k-1$. The equivalence class of $\boldsymbol{d}$ is denoted by $[\boldsymbol{d}]$. The number of different elements in set $\{\boldsymbol{d}_0,\ldots,\boldsymbol{d}_{k-1}\}$ are denoted by $\kappa(\boldsymbol{d})=\#\{\boldsymbol{d}_0,\cdots,\boldsymbol{d}_{k-1}\}$.

Theorem 27. *If all Darboux points of* $\boldsymbol{s}$*-homogeneous of weighted degree* k *system* (2.1) *are proper and simple, then*

$$\sum_{[d]\in\mathscr{D}^*(\boldsymbol{P})} \frac{\kappa(\boldsymbol{d})}{k}\frac{(\rho_1(\boldsymbol{d})+\cdots+\rho_{n-1}(\boldsymbol{d}))^r}{\rho_1(\boldsymbol{d})\cdots\rho_{n-1}(\boldsymbol{d})} = \frac{(k+s_1+\cdots+s_n)^r}{s_1\cdots s_n},$$

for $r=0,\cdots,n-1$.

Example 31. Consider system (7.6) again. We have $\mathscr{D}^*(\boldsymbol{P}) = \{[\boldsymbol{d}_0], [\boldsymbol{d}_1], [\boldsymbol{d}_3], [\boldsymbol{d}_5], [\boldsymbol{d}_7]\}$ and $\kappa(\boldsymbol{d}_0) = 1$ and $\kappa(\boldsymbol{d}_i) = 2$ for $i = 1, 3, 5, 7$. Then the following relations hold:

$$\frac{\kappa(\boldsymbol{d}_0)}{2\rho_1(\boldsymbol{d}_0)\rho_2(\boldsymbol{d}_0)} + \sum_{i=0}^{3} \frac{\kappa(\boldsymbol{d}_{2i+1})}{2\rho_1(\boldsymbol{d}_{2i+1})\rho_2(\boldsymbol{d}_{2i+1})} = \frac{1}{6},$$

$$\frac{\kappa(\boldsymbol{d}_0)}{2}\frac{\rho_1(\boldsymbol{d}_0)+\rho_2(\boldsymbol{d}_0)}{\rho_1(\boldsymbol{d}_0)\rho_2(\boldsymbol{d}_0)} + \sum_{i=0}^{3} \frac{\kappa(\boldsymbol{d}_{2i+1})}{2}\frac{\rho_1(\boldsymbol{d}_{2i+1})+\rho_2(\boldsymbol{d}_{2i+1})}{\rho_1(\boldsymbol{d}_{2i+1})\rho_2(\boldsymbol{d}_{2i+1})} = \frac{4}{3},$$

$$\frac{\kappa(\boldsymbol{d}_0)}{2}\frac{(\rho_1(\boldsymbol{d}_0)+\rho_2(\boldsymbol{d}_0))^2}{\rho_1(\boldsymbol{d}_0)\rho_2(\boldsymbol{d}_0)} + \sum_{i=0}^{3} \frac{\kappa(\boldsymbol{d}_{2i+1})}{2}\frac{(\rho_1(\boldsymbol{d}_{2i+1})+\rho_2(\boldsymbol{d}_{2i+1}))^2}{\rho_1(\boldsymbol{d}_{2i+1})\rho_2(\boldsymbol{d}_{2i+1})} = \frac{32}{3}.$$

The above relations are exactly the same as those given by Theorem 27.

8 Painlevé property

The representation of the solutions of a differential system in complex time can exhibit different types of behavior. The simplest is to be single-valued, i.e. that all its solutions could be represented as Laurent series of the time parameter. This property is known as *Painlevé property*.

In 2017 Llibre et al. [34] provided some connections between the Painlevé property and Darboux polynomials in the next two results, and show the following two examples.

Theorem 28. *Assume that the polynomial differential system (2.1) satisfies the Painlevé property. Then if the system has a Darboux polynomial, its cofactor k must satisfy $k(\boldsymbol{c}) \in \mathbb{Z}$ for all balances $\boldsymbol{c}$ of the system.*

Example 32. Consider the polynomial differential system

$$\dot{x}_1 = x_1^2, \quad \dot{x}_2 = x_1^2. \tag{8.1}$$

System (8.1) has the Darboux polynomial $F = x_1$ with cofactor $k = x_1$. It is not difficult to get that the general solution is

$$x_1 = -\frac{1}{t+c_1}, \quad x_2 = c_2 - \frac{1}{t+c_1},$$

with c_1 and c_2 constant. This general solution is single-valued on its maximum domain of analytic continuation in $\mathbb{C}$. Hence system (8.1) satisfies the Painlevé property. Moreover, this system has a particular solution of the form $\boldsymbol{c}t^{\boldsymbol{s}}$ with $\boldsymbol{c} = (-1,-1)$ and $\boldsymbol{s} = (-1,-1)$. Therefore, $k(\boldsymbol{c}) = -1 \in \mathbb{Z}$.

Theorem 29. *Assume that the polynomial differential system (2.1) admits a dominant balance $\{\boldsymbol{c}, \boldsymbol{s}\}$ and it has a Darboux polynomial with cofactor k such that $k^{(0)}(\boldsymbol{c}) \notin \mathbb{Z}$. Then system (2.1) cannot satisfy the Painlevé property.*

Example 33. Consider the Lorenz system (6.7) with $p = 1/3$ and $b = 0$. In this case system (6.7) does not satisfy the Painlevé property, see [29] for more details. Furthermore, this system has the Darboux polynomial $F = x_1^4 - 4x_1^2x_3/3 - 4x_2^2/9 - 8x_1x_2/9 + 4rx_1^2/3$ with cofactor $k = -4/3 \notin \mathbb{Z}$.

If $p = 1/2$, $b = 1$ and $r = 0$, then system(6.7) satisfies the Painlevé property, also see [29]. This system has the Darboux polynomials $x_1^2 - 2px_3$ and $x_2^2 + x_3^2$ with cofactors $-2p = -1 \in \mathbb{Z}$ and $-2 \in \mathbb{Z}$, respectively. Thus, the two cases satisfy Theorem 29.

The three dimensions Lotka-Volterra system is defined by

$$\dot{\boldsymbol{x}} = \operatorname{diag}(x_1, x_2, x_3) \begin{pmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{pmatrix} (x_1, x_2, x_3)^T. \tag{8.2}$$

Using Painlevé analysis and more specifically by the use of Kowalevski exponents, Constandinides et al. in [11] obtain a complete classification of system (8.2).

Theorem 30. *The Lotka-Volterra system* (8.2) *in three dimensions satisfy the Kowalevski-Painlevé property if and only if* (a, b, c) *is in the class of (i)* $(1, 0, 1)$*; (ii)* $(1, -1, 1)$*; (iii)* $(1, -1, 2)$*; (iv)* $(1, -2, 3)$*; (v)* $(1, 1, \lambda), \lambda \in \mathbb{Z}\backslash\{0\}$*; (vi)* $(1, 1 + \mu, \mu), \mu \in \mathbb{R}$.

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