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Periodic solutions for differential systems in $\mathbb{R}^3$ and $\mathbb{R}^4$

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Abstract

We provide sufficient conditions for the existence of periodic solutions for the differential systems

$$x' = y, \quad y' = z, \quad z' = -y - \varepsilon F(t,x,y,z), \quad \text{and}$$

$$x' = y, \quad y' = -x - \varepsilon G(t,x,y,z,u), \quad z' = u, \quad u' = -z - \varepsilon H(t,x,y,z,u),$$

where F, G and H are 2π–periodic functions in the variable t and ε is a small parameter. These differential systems appear frequently in many problems coming from the sciences and the engineering.

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1 Introduction and statement of the main results

In the study of the dynamics of the differential systems after the analysis of their equilibrium points, we must study the existence or not of their periodic orbits. This paper is dedicated to study the periodic orbits of two kind of differential systems which appear frequently in many problems coming from the physics, chemist, economics, engineering, ...

First we shall provide sufficient conditions for the existence of periodic orbits in the differential systems in $\mathbb{R}^3$ of the form

$$x' = y, \quad y' = z, \quad z' = -y - \varepsilon F(t,x,y,z), \tag{1}$$

where F is a 2π–periodic function in the variable t, ε is a small parameter, and the prime denotes derivative with respect to the variable t. These differential systems usually come when we write as a first–order differential system in $\mathbb{R}^3$ the third–order differential equation

$$x''' + x' + \varepsilon F(t,x,x',x'') = 0, \tag{2}$$

taking $y = x'$ and $z = x''$. Third–order differential equations have been studied by many authors, see for instance [1, 3, 8, 9, 12, 14, 15, 17–19]. But as far as we know we are presenting in the next theorem the more general sufficient conditions up to know for determining the existence of periodic orbits of the differential equation (2), or equivalently of the differential system (1).

Theorem 1. *We define*

$$\mathscr{F}_1(x_0,y_0,z_0) = \frac{1}{2\pi}\int_0^{2\pi} F(t,A(t),B(t),C(t))(\cos t - 1)dt,$$
$$\mathscr{F}_2(x_0,y_0,z_0) = \frac{1}{2\pi}\int_0^{2\pi} F(t,A(t),B(t),C(t))\sin t\, dt,$$
$$\mathscr{F}_3(x_0,y_0,z_0) = -\frac{1}{2\pi}\int_0^{2\pi} F(t,A(t),B(t),C(t))\cos t\, dt,$$

where $A(t) = x_0 + y_0\sin t + z_0(1-\cos t)$, $B(t) = y_0\cos t + z_0\sin t$ and $C(t) = -y_0\sin t + z_0\cos t$. If the function $F(t,x,x',x'')$ is 2π-periodic in the variable t, then for every (x_0^*,y_0^*,z_0^*) solution of the system $\mathscr{F}_k(x_0,y_0,z_0) = 0$ for $k = 1,2,3$, satisfying

$$\det\left(\left.\frac{\partial(\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3)}{\partial(x_0,y_0,z_0)}\right|_{(x_0,y_0,z_0)=(x_0^*,y_0^*,z_0^*)}\right) \neq 0,$$

the differential equation (2) has a 2π–periodic solution $x(t,\varepsilon)$ which when $\varepsilon \to 0$ tends to the 2π–periodic solution $x_0(t)$ given by $x_0(t) = x_0^* + y_0^*\sin t + z_0^*(1-\cos t)$ of $x''' + x' = 0$.

Theorem 1 is proved in section 3. Its proof is based in the averaging theory for computing periodic orbits, see section 6.

An application of Theorem 1 is the following.

Corollary 2. *Consider the generalized memory oscillators given by the differential equation (2) with $F(t,x,x',x'') = x'' - (b_0 + b_1x + b_2x^2)\sin t$. If $b_2 \neq 0$, $4 + 3b_1^2 - 12b_0b_1 > 0$ and $-4 + b_1^2 - 4b_0b_2 > 0$, then this differential equation has four periodic solutions $x_k(t,\varepsilon)$ for $k = 1,2,3,4$, tending when $\varepsilon \longrightarrow 0$ to the periodic solutions $x_k(t)$ where*

$$x_{1,2}(t) = \frac{-b_1 \mp 2\sqrt{-4 + b_1^2 - 4b_0b_2}}{2b_2} - \frac{2}{b_2}\sin t \pm \frac{\sqrt{-4 + b_1^2 - 4b_0b_2}}{b_2}(1-\cos t),$$
$$x_{3,4}(t) = -\frac{b_1}{2b_2} - \frac{2 \pm \sqrt{4 + 3b_1^2 - 12b_0b_2}}{3b_2}\sin t,$$

of $x''' + x' = 0$.

Corollary 2 is proved in section 4.

Our second result is on the periodic orbits of the differential system in $\mathbb{R}^4$ of the form

$$x' = y, \quad y' = -x - \varepsilon G(t,x,y,z,u), \quad z' = u, \quad u' = -z - \varepsilon H(t,x,y,z,u). \tag{3}$$

where G and H are a 2π–periodic functions in the variable t and ε is a small parameter. These systems are a perturbation of the harmonic oscillator in $\mathbb{R}^4$ and these kind of perturbations appear frequently in the study of the dynamics of a galaxy, in atomic physic, ... see for instance [2, 4, 6, 7, 10, 11].

Theorem 3. *We define*

$$\begin{aligned}
\mathscr{F}_1(x_0,y_0,z_0,u_0) &= \frac{1}{2\pi}\int_0^{2\pi} G(t,A(t),B(t),C(t),D(t))\sin t\,dt,\\
\mathscr{F}_2(x_0,y_0,z_0,u_0) &= -\frac{1}{2\pi}\int_0^{2\pi} G(t,A(t),B(t),C(t),D(t))\cos t\,dt,\\
\mathscr{F}_3(x_0,y_0,z_0,u_0) &= \frac{1}{2\pi}\int_0^{2\pi} H(t,A(t),B(t),C(t),D(t))\sin t\,dt,\\
\mathscr{F}_4(x_0,y_0,z_0,u_0) &= -\frac{1}{2\pi}\int_0^{2\pi} H(t,A(t),B(t),C(t),D(t))\cos t\,dt,
\end{aligned}$$

where $A(t)=x_0\cos t+y_0\sin t$, $B(t)=-x_0\sin t+y_0\cos t$, $C(t)=z_0\cos t+u_0\sin t$ *and* $D(t)=-z_0\sin t+u_0\cos t$. *Then for every* $(x_0^*,y_0^*,z_0^*,u_0^*)$ *solution of the system* $\mathscr{F}_k(x_0,y_0,z_0,u_0)=0$ *for* $k=1,2,3,4$, *satisfying*

$$\det\left(\frac{\partial(\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3,\mathscr{F}_4)}{\partial(x_0,y_0,z_0,u_0)}\bigg|_{(x_0,y_0,z_0,u_0)=(x_0^*,y_0^*,z_0^*,u_0^*)}\right)\neq 0,$$

the differential system (3) *has a* 2π*–periodic solution* $(x(t,\varepsilon),y(t,\varepsilon),z(t,\varepsilon),u(t,\varepsilon))$ *which when* $\varepsilon\to 0$ *tends to the* 2π*–periodic solution* $(x_0(t),y_0(t),z_0(t),u_0(t))$ *given by* $x_0(t)=x_0^*\cos t+y_0^*\sin t$, $y_0(t)=-x_0^*\sin t+y_0^*\cos t$, $z_0(t)=z_0^*\cos t+u_0^*\sin t$, $u_0(t)=-z_0^*\sin t+u_0^*\cos t$, *of the unperturbed system* (3) *with* $\varepsilon=0$.

Theorem 3 is proved in section 5.

An application of Theorem 3 is the following.

Corollary 4. *Consider the differential system* (3) *with* $G(t,x,y,z,u)=(-1-x^2+z^2)\sin t$ *and* $H(t,x,y,z,u)=(1-x^2)\cos t$. *Then this differential system has 8 periodic solutions* $(x_k(t,\varepsilon),y_k(t,\varepsilon),z_k(t,\varepsilon),u_k(t,\varepsilon))$ *for* $k=1,\ldots,8$, *tending when* $\varepsilon\longrightarrow 0$ *to the periodic solutions* $(x_k(t),y_k(t),z_k(t),u_k(t))$ *where*

$$\begin{aligned}
(x_{1,2}(t),y_{1,2}(t),z_{1,2}(t),u_{1,2}(t)) &= (\pm 2\sin t,\pm 2\cos t,-4\cos t,4\sin t),\\
(x_{3,4}(t),y_{3,4}(t),z_{3,4}(t),u_{3,4}(t)) &= \left(\mp\frac{2}{\sqrt{3}}\cos t,\pm\frac{2}{\sqrt{3}}\sin t,-\frac{4}{3}\sin t,-\frac{4}{3}\cos t\right),\\
(x_{5,6}(t),y_{5,6}(t),z_{5,6}(t),u_{5,6}(t)) &= \left(\pm 2\sin t,\pm 2\cos t,-\frac{4}{\sqrt{3}}\sin t,-\frac{4}{\sqrt{3}}\cos t\right),\\
(x_{7,8}(t),y_{7,8}(t),z_{7,8}(t),u_{7,8}(t)) &= \left(\mp\frac{2}{\sqrt{3}}\cos t,\pm\frac{2}{\sqrt{3}}\sin t,-\frac{4}{\sqrt{3}}\cos t,\frac{4}{\sqrt{3}}\sin t\right),
\end{aligned}$$

of $x'=y$, $y'=-x$, $z'=u$, $u'=-z$.

Corollary 4 is proved in section 6.

2 Averaging theory

We want to study the T–periodic solutions of the periodic differential systems of the form

$$\mathbf{x}'=F_0(t,\mathbf{x})+\varepsilon F_1(t,\mathbf{x})+\varepsilon^2 F_2(t,\mathbf{x},\varepsilon), \tag{4}$$

with $\varepsilon>0$ sufficiently small, where $F_0,F_1:\mathbb{R}\times\Omega\to\mathbb{R}^n$ and $F_2:\mathbb{R}\times\Omega\times(-\varepsilon_0,\varepsilon_0)\to\mathbb{R}^n$ are $\mathscr{C}^2$ functions, T–periodic in the variable t, and Ω is an open subset of $\mathbb{R}^n$. We denote by $\mathbf{x}(t,\mathbf{z},\varepsilon)$ the solution of the differential system (4) such that $\mathbf{x}(0,\mathbf{z},\varepsilon)=\mathbf{z}$. We assume that the unperturbed system

$$\mathbf{x}'=F_0(t,\mathbf{x}), \tag{5}$$

has an open set V with $\mathrm{Cl}(V) \subset \Omega$ such that for each $\mathbf{z} \in \mathrm{Cl}(V)$, $\mathbf{x}(t,\mathbf{z},0)$ is T–periodic.

We consider the variational equation

$$\mathbf{y}' = D_{\mathbf{x}}F_0(t,\mathbf{x}(t,\mathbf{z},0))\mathbf{y}, \tag{6}$$

of the unperturbed system on the periodic solution $\mathbf{x}(t,\mathbf{z},0)$, where $\mathbf{y}$ is an $n \times n$ matrix. Let $M_{\mathbf{z}}(t)$ be the fundamental matrix of the linear differential system (6) such that $M_{\mathbf{z}}(0)$ is the $n \times n$ identity matrix. The next result is due to Malkin [13] and Roseau [16], for a shorter and easier proof see [5].

Theorem 5. *Consider the function* $\mathscr{F} : \mathrm{Cl}(V) \to \mathbb{R}^n$

$$\mathscr{F}(\mathbf{z}) = \frac{1}{T}\int_0^T M_{\mathbf{z}}^{-1}(t)F_1(t,\mathbf{x}(t,\mathbf{z},0))dt. \tag{7}$$

If there exists $\alpha \in V$ *with* $\mathscr{F}(\alpha) = 0$ *and* $\det((d\mathscr{F}/d\mathbf{z})(\alpha)) \neq 0$, *then there exists a* T*–periodic solution* $\mathbf{x}(t,\varepsilon)$ *of system* (4) *such that when* $\varepsilon \to 0$ *we have that* $\mathbf{x}(0,\varepsilon) \to \alpha$.

3 Proof of Theorem 1

For $\varepsilon = 0$ all singular points of the differential system (1) are in the x–axis, i.e. $(x,0,0)$ are the singular points of system (1). The eigenvalues of the linearized system at these singular points are $\pm i$, 0. The 2π–periodic solutions $(x(t),y(t),z(t))$ of the unperturbed system (i.e. system (1) with $\varepsilon = 0$) such that $(x(0),y(0),z(0)) = (x_0,y_0,z_0)$ are

$$(x_0 + y_0\sin t + z_0(1-\cos t), y_0\cos t + z_0\sin t, -y_0\sin t + z_0\cos t). \tag{8}$$

Using the notion introduced in section 2, we have that $\mathbf{x} = (x,y,z)$, $\mathbf{z} = (x_0,y_0,z_0)$, $F_0(\mathbf{x},t) = (y,z,-y)$, $F_1(\mathbf{x},t) = \Big(0,0,-F\Big)$ and $F_2(\mathbf{x},t,\varepsilon) = (0,0,0)$. The fundamental matrix solution $M_{\mathbf{z}}(t)$ is independent of $\mathbf{z}$ and we shall denote it by $M(t)$. An easy computation shows that

$$M(t) = \begin{pmatrix} 1 & \sin t & 1-\cos t \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix}.$$

According to Theorem 5 we study the zeros $\alpha = (x_0,y_0,z_0)$ of the three components of the function $\mathscr{F}(\alpha)$ given in (7). More precisely we have $\mathscr{F}(\alpha) = (\mathscr{F}_1(\alpha),\mathscr{F}_2(\alpha), \mathscr{F}_3(\alpha))$, such that

$$\mathscr{F}_1(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} F(t,x(t),y(t),z(t))(\cos t - 1)dt,$$
$$\mathscr{F}_2(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} F(t,x(t),y(t),z(t))\sin t\, dt,$$
$$\mathscr{F}_3(\alpha) = -\frac{1}{2\pi}\int_0^{2\pi} F(t,x(t),y(t),z(t))\cos t\, dt,$$

where $x(t)$, $y(t)$, $z(t)$ are given by (8). Now the rest of the proof of Theorem 1 follows directly from the statement of Theorem 5.

4 Proof of Corollary 2

We must apply Theorem 1 with $F(t,x,y,z) = z - \sin t\,(b_0 + b_1 x + b_2 x^2)$. Computing the function $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)$ of Theorem 1 we get

$$\begin{aligned}
\mathscr{F}_1(x_0,y_0,z_0)) &= \frac{1}{4}(2b_1y_0 + 2z_0 + b_2y_0(4x_0 + 5z_0)),\\
\mathscr{F}_2(x_0,y_0,z_0) &= \frac{1}{8}(-4b_0 - 4y_0 - 4b_1(x_0 + z_0) - b_2(4x_0^2 + 3y_0^2 + 8x_0z_0 + 5z_0^2)),\\
\mathscr{F}_2(x_0,y_0,z_0) &= -\frac{1}{4}(2 + b_2y_0)z_0.
\end{aligned}$$

System $\mathscr{F}_1 = \mathscr{F}_2 = \mathscr{F}_2 = 0$ has six solutions (x_0^*, y_0^*, z_0^*) given by

$$\left(\frac{-b_1 \mp 2\sqrt{-4 + b_1^2 - 4b_0b_2}}{2b_2}, -\frac{2}{b_2}, \pm\frac{\sqrt{-4 + b_1^2 - 4b_0b_2}}{b_2}\right),$$
$$\left(-\frac{b_1}{2b_2}, -\frac{2 \pm \sqrt{4 + 3b_1^2 - 12b_0b_2}}{3b_2}, 0\right), \quad \left(-\frac{b_1 \pm \sqrt{b_1^2 - 4b_0b_2}}{2b_2}, 0, 0\right).$$

The last two solutions going to the differential equation $x''' + x' = 0$ provide equilibrium points instead of periodic solutions, so we do not consider them. Since the Jacobian

$$\det\left(\left.\frac{\partial(\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3)}{\partial(x_0,y_0,z_0)}\right|_{(x_0,y_0,z_0)=(x_0^*,y_0^*,z_0^*)}\right)$$

for these remaining four solutions (x_0^*, y_0^*, z_0^*) is $\frac{1}{8}(-4 + b_1^2 - 4b_0b_2)$ and $\frac{1}{72}(4 + 3b_1^2 - 12b_0b_2)(1 + \frac{1}{2}\sqrt{4 + 3b_1^2 - 12b_0b_2})$ respectively, we obtain using Theorem 1 the four periodic solutions given in the statement of the corollary.

5 Proof of Theorem 3

Consider system (3). Its unperturbed system is the system (3) with $\varepsilon = 0$, which has the singular point $(x,y,z,u) = (0,0,0,0)$. The eigenvalues of the linearized system at this singular point are $\pm i$, of multiplicity two. The periodic solutions $(x(t), y(t), z(t), u(t))$ of the unperturbed system such that $(x(0), y(0), z(0), u(0)) = (x_0, y_0, z_0, u_0)$ are

$$(x_0 \cos t + y_0 \sin t, -x_0 \sin t + y_0 \cos t, z_0 \cos t + v_0 \sin t, -z_0 \sin t + v_0 \cos t). \tag{9}$$

Note that all these solutions are periodic with period 2π.

Using the notations introduced in section 2, we have that $\mathbf{x} = (x,y,z,u)$, $\mathbf{z} = (x_0,y_0,z_0,u_0)$, $F_0(\mathbf{x},t) = (y,-x,u,-z)$, $F_1(\mathbf{x},t) = \Big(0,-G,0,-H\Big)$ and $F_2(\mathbf{x},t,\varepsilon) = (0,0,0,0)$. The fundamental matrix solution $M_{\mathbf{z}}(t)$ corresponding to system (5) for our system (3) is independent of $\mathbf{z}$ and we shall denote it by $M(t)$. An easy computation shows that

$$M(t) = \begin{pmatrix} \cos t & \sin t & 0 & 0\\ -\sin t & \cos t & 0 & 0\\ 0 & 0 & \cos t & \sin t\\ 0 & 0 & -\sin t & \cos t \end{pmatrix}.$$

According to Theorem 5 we study the zeros $\alpha = (x_0, y_0, z_0, u_0)$ of the four components of the function $\mathscr{F}(\alpha)$ given in (7). More precisely we have $\mathscr{F}(\alpha) = (\mathscr{F}_1(\alpha), \mathscr{F}_2(\alpha), \mathscr{F}_3(\alpha), \mathscr{F}_4(\alpha))$, such that

$$\mathscr{F}_1(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} G(t,x(t),y(t),z(t),u(t))\sin t\,dt,$$
$$\mathscr{F}_2(\alpha) = -\frac{1}{2\pi}\int_0^{2\pi} G(t,x(t),y(t),z(t),u(t))\cos t\,dt,$$
$$\mathscr{F}_3(\alpha) = \frac{1}{2\pi}\int_0^{2\pi} H(t,x(t),y(t),z(t),u(t))\sin t\,dt,$$
$$\mathscr{F}_4(\alpha) = -\frac{1}{2\pi}\int_0^{2\pi} H(t,x(t),y(t),z(t),u(t))\cos t\,dt,$$

where $(x(t),y(t),z(t),u(t))$ is periodic solution given in (9). Now the rest of the proof of Theorem 3 follows directly from the statement of Theorem 5.

6 Proof of Corollary 4

We must apply Theorem 1 with $G(t,x,y,z,u) = (-1-x^2+z^2)\sin t$ and $H(t,x,y,z,u) = (1-x^2)\cos t$. Computing the function $\mathscr{F} = (\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3, \mathscr{F}_4)$ of Theorem 5 we obtain

$$\mathscr{F}_1(x_0,y_0,z_0)) = \frac{1}{8}(-4+3u_0^2-x_0^2-3y_0^2+z_0^2),$$
$$\mathscr{F}_2(x_0,y_0,z_0) = \frac{1}{4}(x_0y_0-u_0z_0),$$
$$\mathscr{F}_3(x_0,y_0,z_0) = -\frac{1}{4}x_0y_0,$$
$$\mathscr{F}_4(x_0,y_0,z_0) = \frac{1}{8}(-4+3x_0^2+y_0^2).$$

System $\mathscr{F}_1 = \mathscr{F}_2 = \mathscr{F}_3 = \mathscr{F}_4 = 0$ has sixteen solutions $(x_0^*, y_0^*, z_0^*, u_0^*)$ given by

$$(0,\pm 2,\pm 4,0), \quad \left(\pm\frac{2}{\sqrt{3}},0,\pm\frac{4}{3},0\right), \quad \left(0,\pm 2,0,\pm\frac{4}{\sqrt{3}}\right), \quad \left(\pm\frac{2}{3},0,\pm\frac{4}{\sqrt{3}},0\right).$$

Since the Jacobian

$$\det\left(\left.\frac{\partial(\mathscr{F}_1,\mathscr{F}_2,\mathscr{F}_3),\mathscr{F}_4)}{\partial(x_0,y_0,z_0,u_0)}\right|_{(x_0,y_0,z_0,u_0)=(x_0^*,y_0^*,z_0^*,u_0^*)}\right)$$

for these solutions $(x_0^*, y_0^*, z_0^*, u_0^*)$ is $\frac{1}{4}, \frac{1}{12}, -\frac{1}{4}, -\frac{1}{12}$ respectively, we obtain using Theorem 5 sixteen periodic solutions, but only eight of them are different because all periodic solutions appear repeated when we change $t \to t+\pi$. Hence we obtain the eight periodic solutions given in the statement of the corollary.

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