This is a preprint of: "Planar Kolmogorov systems coming from spatial Lotka-Volterra systems", Erika Diz Pita, Jaume Llibre, M. Victoria Otero Espinar, *Internat. J. Bifur. Chaos Appl. Sci. Engrg.*, vol. 31(13), Paper No. 2150201, 37, 2021.
DOI: [10.1142/S0218127421502011]

Planar Kolmogorov systems coming from spatial Lotka-Volterra systems

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In this paper we classify the phase portraits in the Poincaré disc of all the Kolmogorov systems

$$\dot{y} = y\,(b_0 + b_1 yz + b_2 y + b_3 z)\,,$$
$$\dot{z} = z\,(c_0 - \mu(b_1 yz + b_2 y + b_3 z))\,,$$

which depend on six parameters. These systems are provided by a general 3-dimensional Lotka–Volterra system with a rational first integral of degree two of the form $H = x^i y^j z^k$, restricted to each surface $H(x,y,z) = h$ varying $h \in \mathbb{R}$, with the additional assumption that they have a Darboux invariant of the form $y^\ell z^m e^{st}$.

Keywords: Kolmogorov system, Lotka–Volterra system, phase portrait, Poincaré disc.

1. Introduction

The Lotka-Volterra systems have been used for modelling many natural phenomena, such as the time evolution of conflicting species in biology [Llibre & Martínez , 2020; Llibre & Dongmei , 2014], chemical reactions [Hering , 1990], plasma physics [Laval & Pellat , 1975], hydrodynamics [Busse , 1981], just as other problems from economical science [Gandolfo , 2009, 2008; Wijeratne *et al.*, 2009] or social science, as the evolution of the internet users [Fu *et al.*, 2017].

These systems, which are polynomial differential equations of degree two, were initially proposed, independently, by Alfred J. Lotka in 1925 and Vito Volterra in 1926, both in the context of competing species. Later on Lotka-Volterra systems were generalized and considered in arbitrary dimension, i.e.

$$\dot{x_i} = x_i \left(a_{i0} + \sum_{j=1}^{n} a_{ij} x_j \right), \quad i = 1, ..., n.$$

Consequently the applications of these systems started to multiply. Moreover Kolmogorov in [Kolmogorov , 1936] extended the Lotka-Volterra systems as follows

$$\dot{x_i} = x_i P_i(x_1, \ldots, x_n), \quad i = 1, ..., n.$$

where P_i are polynomials of degre at most m. These kind of systems are now known as Kolmogorov systems. They have in particular all the applications of the Lotka-Volterra systems as for instance in the study of the black holes in cosmology, see [Alavez-Ramírez *et al.*, 2012; Breitenlohner *et al.*, 1998].

The global qualitative dynamics of the Lotka-Volterra systems in dimension two has been completely studied in [Schlomiuk & Vulpe , 2012], where all the possible phase portraits on the Poincaré disc have been classified.

There are few results about the global dynamics of the Lotka-Volterra systems in dimension three. Our objective is to study the phase portraits of the 3-dimensional Lotka-Volterra systems

$$\begin{aligned} \dot{x} &= x\ (\ a_0 + a_1 x + a_2 y + a_3 z\ ), \\ \dot{y} &= y\ (\ b_0 + b_1 x + b_2 y + b_3 z\ ), \\ \dot{z} &= z\ (\ c_0 + c_1 x + c_2 y + c_3 z\ ), \end{aligned} \tag{1}$$

which have a rational first integral of degree two of the form $x^i y^j z^k$. We have used the Darboux theory of integrability to obtain a characterization of these systems. As a result, we have reduced the initial problem to a problem in dimension two, the study of the global dynamics of two families of Kolmogorov systems. We carried out the study of the first family in [Diz-Pita *et al.*, 2021]. In this paper we focus on the second family, which is

$$\begin{aligned} \dot{y} &= y\ (\ b_0 + b_1 yz + b_2 y + b_3 z\ ), \\ \dot{z} &= z\ (\ c_0 + c_1 yz + c_2 y + c_3 z\ ). \end{aligned} \tag{2}$$

Kolmogorov systems (2) depend on eight parameters, and this is a big number in order to classify all their distinct topological phase portraits. Then we require that Kolmogorov systems (2) have a Darboux invariant of the form $y^\ell z^m e^{st}$, then these systems are reduced to study the Kolmogorov systems

$$\begin{aligned} \dot{y} &= y\,(b_0 + b_1 yz + b_2 y + b_3 z))\,, \\ \dot{z} &= z\,(c_0 - \mu(b_1 yz + b_2 y + b_3 z))\,, \end{aligned} \tag{3}$$

which now depend on six parameters. For these Kolmogorov systems we give the topological classification of all their phase portraits in the Poincaré disc. Roughly speaking the Poincaré disc is the closed unit disc centered at the origin of $\mathbb{R}^2$. Its interior is identified with $\mathbb{R}^2$ and the circle of its boundary is identified with the infinity of $\mathbb{R}^2$. In the plane $\mathbb{R}^2$ we can go or come from the infinity in as many directions as points have the circle. The polynomial differential systems can be extended to the closed Poincaré disc, i.e. they can be extended to infinity and in this way we can study their dynamics in a neighborhood of infinity. This extension is called the Poincaré compactification, for more details see Subsection 2.1.

Thus our main result is the following.

Theorem 1. *Kolmogorov systems* (3) *have* 106 *topologically distinct phase portraits in the Poincaré disc, given in Figure 16.*

Note that systems (3) must have $b_1 \neq 0$, otherwise they will be Lotka-Volterra instead of Kolmogorov systems.

In Section 3 we use the Darboux theory of integrability to reduce the Lotka-Volterra system (1) to the Kolmogorov systems (3). In Section 4 we give some properties of the systems obtained. In Section 5 we study the local phase portrait of the finite singular points, and in Sections 6 and 7 we do the same with the infinite singular points, applying the blow-up technique. Finally in Section 8 we prove Theorem 1.

2. Preliminaries

2.1. *Poincaré Compactification*

In order to study the behavior of the trajectories of our polynomial differential systems near the infinity we will use the Poincaré compactification. We provide a short summary about this method, more details can be found in [Dumortier *et al.*, 2006, Chapter 5].

Let $X = (P(x,y), Q(x,y))$ be a polynomial vector field of degree d defined in $\mathbb{R}^2$. Consider the *Poincaré sphere* $\mathbb{S}^2 = \left\{y \in \mathbb{R}^3 : y_1^2 + y_2^2 + y_3^2 = 1\right\}$ and its tangent plane at the point $(0,0,1)$ which is identified with $\mathbb{R}^2$.

We consider the central projections $f^+ : \mathbb{R}^2 \to \mathbb{S}^2$ and $f^- : \mathbb{R}^2 \to \mathbb{S}^2$. By definition, $f^+(x)$ is the intersection of the straight line passing through the point x and the origin with the northern hemisphere of $\mathbb{S}^2$, and respectively for $f^-(x)$ with the southern hemisphere. The differential Df^+ and respectively Df^- send the vector field X into a vector field $\overline{X}$ on $\mathbb{S}^2 n \mathbb{S}^1$. Note that the points at infinity of $\mathbb{R}^2$ are in bijective correspondence with the points of the equator $\mathbb{S}^1$ of $\mathbb{S}^2$.

The vector field $\overline{X}$ can be extended analytically to a vector field on $\mathbb{S}^2$ multiplying $\overline{X}$ by y_3^d. We denothe this vector field by $\rho(X)$, and it is called the *Poincaré compactification* of the vector field X on $\mathbb{R}^2$.

For studying the dynamics of X in the neighborhood of the infinity, we must study the dynamics of $\rho(X)$ near $\mathbb{S}^1$. The sphere $\mathbb{S}^2$ is a 2-dimensional manifold so we need to know the expressions of the vector field $\rho(X)$ in the local charts (U_i, ϕ_i) and (V_i, ψ_i), where $U_i = \left\{y \in \mathbb{S}^2 : y_i > 0\right\}$, $V_i = \left\{y \in \mathbb{S}^2 : y_i < 0\right\}$, $\phi_i : U_i \longrightarrow \mathbb{R}^2$ and $\psi_i : V_i \longrightarrow \mathbb{R}^2$ for $i = 1, 2, 3$ with $\phi_i(y) = -\psi_i(y) = (y_m/y_i, y_n/y_i)$ for $m < n$ and $m, n \neq i$.

In the local chart (U_1, ϕ_1) the expression of $\rho(X)$ is

$$\dot{u} = v^d \left[-u\, P\left(\frac{1}{v}, \frac{u}{v}\right) + Q\left(\frac{1}{v}, \frac{u}{v}\right)\right],\ \dot{v} = -v^{d+1}\, P\left(\frac{1}{v}, \frac{u}{v}\right). \tag{4}$$

In the local chart (U_2, ϕ_2) the expression of $\rho(X)$ is

$$\dot{u} = v^d \left[P\left(\frac{1}{v}, \frac{u}{v}\right) - uQ\left(\frac{1}{v}, \frac{u}{v}\right)\right],\ \dot{v} = -v^{d+1}\, P\left(\frac{1}{v}, \frac{u}{v}\right), \tag{5}$$

and in the local chart (U_3, ϕ_3) the expression of $\rho(X)$ is

$$\dot{u} = P(u,v),\ \dot{v} = Q(u,v). \tag{6}$$

In the charts (V_i, ψ_i), with $i = 1, 2, 3$, the expression for $\rho(X)$ is the same as in the charts (U_i, ϕ_i) multiplied by $(-1)^{d-1}$.

The equator $\mathbb{S}^1$ is invariant by the vector field $\rho(X)$ and all the singular points of $\rho(X)$ which lie in this equator are called the *infinite singular points* of X. If $y \in \mathbb{S}^1$ is an infinite singular point, then $-y$ is also an infinite singular point and they have the same (respectively opposite) stability if the degree of vector field is odd (respectively even).

The image of the closed northern hemisphere of $\mathbb{S}^2$ onto the plane $y_3 = 0$ under the orthogonal projection π is called the *Poincaré disc* $\mathbb{D}^2$. Since the orbits of $\rho(X)$ on $\mathbb{S}^2$ are symmetric with respect to the origin of $\mathbb{R}^3$, we only need to consider the flow of $\rho(X)$ in the closed northern hemisphere, and we can project the phase portrait of $\rho(X)$ on the northern hemisphere onto the Poincaré disc. We shall present the phase portraits of the polynomial differential systems (3) in the Poincaré disc.

2.2. *Topological equivalence of two polynomial vector fields*

Two polynomial vector fields X_1 and X_2 on $\mathbb{R}^2$ are *topologically equivalent* if there exists a homeomorphism on the Poincaré disc which preserves the infinity $\mathbb{S}^1$ and sends the trajectories of the flow of $\pi(\rho(X_1))$ to the trajectories of the flow of $\pi(\rho(X_2))$, preserving or reversing the orientation of all the orbits.

A *separatrix* of the Poincaré compactification $\pi(\rho(X))$ is an orbit at the infinity $\mathbb{S}^1$, or a finite singular point, or a limit cycle, or an orbit on the boundary of a hyperbolic sector at a finite or an infinite singular point. The set of all separatrices of $\pi(\rho(X))$ is closed and we denote it by Σ_X.

An open connected component of $\mathbb{D}^2 n\Sigma_X$ is a *canonical region* of $\pi(\rho(X))$. The *separatrix configuration* of $\pi(\rho(X))$ is the union of an orbit of each canonical region with the set Σ_X, and it is denoted by $\Sigma_X^{'}$. We denote by S (respectively R) the number of separatrices (respectively canonical regions) of a vector field $\pi(\rho(X))$.

We say that two separatrix configurations $\Sigma_{X_1}^{'}$ and $\Sigma_{X_2}^{'}$ are *topologically equivalent* if there is a homeomorphism $h : \mathbb{D}^2 \to \mathbb{D}^2$ such that $h(\Sigma_{X_1}^{'}) = \Sigma_{X_2}^{'}$.

The following theorem of Markus [Markus , 1954], Neumann [Neumann , 1975] and Peixoto [Peixoto , 1973] allows to investigate only the separatrix configuration of a polynomial differential ssytem in order to determine its phase portrait in the Poincaré disc.

Theorem 2. *The phase portraits in the Poincaré disc of two compactified polynomial vector fields $\pi(\rho(X_1))$ and $\pi(\rho(X_2))$ with finitely many separatrices are topologically equivalent if and only if their separatrix configurations $\Sigma_{X_1}^{'}$ and $\Sigma_{X_2}^{'}$ are topologically equivalent.*

2.3. *Blow-up technique*

There exist classification theorems for hyperbolic and semi-hyperbolic singular points, and also for nilpotent singular points which can be found in [Dumortier *et al.*, 2006, Chapters 2,3]. The centers are more difficult to study, see for instance [Dumortier *et al.*, 2006, Chapter 4]. Whereas to study a singular point for which the Jacobian matrix is identically zero, the only possibility is studying each singular point case by case. The main technique to perform the desingularization of a linearly zero singular point is the blow-up technique. We give a short summary about this method, more details can be found in [Álvarez *et al.*, 2011].

Roughly speaking the idea behind the blow up technique is to explode, through a change of variables that is not a diffeomorphism, the singularity to a line. Then, for studying the original singular point, one studies the new singular points that appear on this line, and this is simpler. If some of these new singular points are linearly zero, the process is repeated. Dumortier proved that this iterative process of desingularization is finite, see [Dumortier , 1977].

Consider a real planar polynomial differential system of the form

$$\begin{aligned} \dot{x} &= P(x,y) = P_m(x,y) + \ldots, \\ \dot{y} &= Q(x,y) = Q_m(x,y) + \ldots, \end{aligned} \tag{7}$$

where P and Q are coprime polynomials, P_m and Q_m are homogeneous polynomials of degree $m \in \mathbb{N}$ and the dots mean higher order terms in x and y. Note that we are assuming that the origin is a singular point because $m > 0$. We define the *characteristic polynomial* of (7) as

$$\mathcal{F}(x,y) := xQ_m(x,y) - yP_m(x,y), \tag{8}$$

and we say that the origin is a *nondicritical* singular point if $\mathcal{F} \not\equiv 0$ and a *dicritical* singular point if $\mathcal{F} \equiv 0$. In this last case $P_m = xW_{m-1}$ and $Q_m = yW_{m-1}$, where $W_{m-1} \not\equiv 0$ is a homogeneous polynomial of degree $m-1$. If $y - vx$ is a factor of W_{m-1} and $v = \tan\theta^*$, $\theta^* \in [0, 2\pi)$, then θ^* is a *singular direction.*

The *homogeneous directional blow up in the vertical direction* (resp. in the horizontal direction) is the mapping $(x,y) \to (x,z) = (x, y/x)$ (resp. $(x,y) \to (z,y) = (x/y, y)$), where z is a new variable. This map transforms the origin of (7) into the line $x = 0$ (resp. $y = 0$), which is called the *exceptional divisor*. The expression of system (7) after the blow up in the vertical direction is

$$\dot{x} = P(x, xz), \quad \dot{z} = \frac{Q(x,xz) - zP(x,xz)}{x}, \tag{9}$$

that is always well-defined since we are assuming that the origin is a singularity. After the blow up, we cancel an appearing common factor x^{m-1} (x^m if $\mathcal{F} \equiv 0$). Moreover, the mapping swaps the second and the third quadrants in the vertical directional blow up and the third and the fourth queadrants in the horizontal blow up, which writes as

$$\dot{z} = \frac{P(yz,y) - zQ(yz,y)}{y}, \quad \dot{x} = P(yz,y). \tag{10}$$

Propositions 2.1 and 2.2 of [Álvarez *et al.*, 2011] provide the relationship between the original singular point of system (7) and the new singularities of system (9). For additional details see [Andronov *et al.*, 1973].

Finally, to study the behavior of the solutions around the origin of system (7), it is necessary to study the singular points of system (9) on the exceptional divisor. They correspond to either characteristic directions in the nondicritical case, or singular directions in the dicritical case. It may happen that some of these singular points are linearly zero, in which case we have to repeat the process. As we said before, it is proved in [Dumortier , 1977] that this chain of blow ups is finite.

2.4. *Indices of Planar Singular Points*

Given an isolated singularity q of a vector field X, defined on an open subset of $\mathbb{R}^2$ or $\mathbb{S}^2$, we define the index of q by means of the Poincaré Index Formula. We assume that q has the finite sectorial decomposition property. Let e, h and p denote the number of elliptic, hyperbolic and parabolic sectors of q, respectively, and suppose that $e + h + p > 0$. Then the *index of* q is $i_q = (e - h)/2 + 1$, and it is always an integer.

We recall that the Poincaré compactification of a vector field in $\mathbb{R}^2$ introduced in subsection 2.1 is a tangent vector field on the sphere $\mathbb{S}^2$, so the next result will be very useful in our study.

Theorem 3 [Poincaré–Hopf Theorem]. *For every tangent vector field on* $\mathbb{S}^2$ *with a finite number of singular points, the sum of their indices is* 2*.*

2.5. *Invariants and Darboux Theory*

The Darboux Theory of Integrability provides a link between the integrability of polynomial vector fields and the number of invariant algebraic curves that they have. The basic results on dimension two can be found in [Dumortier *et al.*, 2006, Chapter 8], and these results have been extended to $\mathbb{R}^n$ and $\mathbb{C}^n$ in [Llibre & Zhang , 2009,?, 2010].

We consider a real polynomial differential system in dimension three, that is a system of the form

$$\begin{aligned} dx/dt &= \dot{x} = P(x, y, z), \\ dy/dt &= \dot{y} = Q(x, y, z), \\ dz/dt &= \dot{z} = R(x, y, z), \end{aligned} \tag{11}$$

where P, Q and R are polynomials in the variables x, y and z. We denote by $m = \max\{\deg P, \deg Q, \deg R\}$ the degree of the polynomial system, and we always assume that the polynomials P, Q and R are relatively prime in the ring of the real polynomials in the variables x, y and z.

Theorem 4 [Darboux Integrability Theorem]. *Suppose that a polynomial system* (11) *of degree* m *admits* p *irreducible invariant algebraic surfaces* $f_i = 0$ *with cofactors* K_i *for* $i = 1, \ldots, p$*. Then the next statements hold.*

(1) There exist $\lambda_i \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i = 0$ *if and only if the function* $f_1^{\lambda_1} \ldots f_p^{\lambda_p}$ *is a first integral of system* (11)*.*
(2) There exist $\lambda_i \in \mathbb{C}$ *not all zero such that* $\sum_{i=1}^{p} \lambda_i K_i = -s$ *for some* $s \in \mathbb{R}n\,\{0\}$ *if and only if the function* $f_1^{\lambda_1} \ldots f_p^{\lambda_p} exp(st)$ *is a Darboux invariant of system* (11)*.*

3. Reduction of the Lotka–Volterra systems in $\mathbb{R}^3$ to the Kolmogorov systems (3) in $\mathbb{R}^2$

As we said our objective is to study the global dynamics of the Lotka-Volterra systems (1) in dimension three, which have a rational first integral of degree two of the form $x^i y^j z^k$. The Darboux theory of integrability allow us to obtain a characterization of these systems.

We consider the irreducible invariant algebraic surfaces $f_1(x, y, z) = x = 0$, $f_2(x, y, z) = y = 0$ and $f_3(x, y, z) = z = 0$ of system (1), with cofactors K_1, K_2 and K_3, respectively. As K_i is the cofactor of f_i

we have that

$$Xf_i = P\frac{\partial f_i}{\partial x} + Q\frac{\partial f_i}{\partial y} + R\frac{\partial f_i}{\partial z} = K_i f_i.$$

Then for the invariant algebraic surfaces considered we get the cofactors $K_1 = a_0 + a_1x + a_2y + a_3z$, $K_2 = b_0 + b_1x + b_2y + b_3z$ and $K_3 = c_0 + c_1x + c_2y + c_3z$, respectively.

Applying Theorem 4, since we assume that $x^{\lambda_1}y^{\lambda_2}z^{\lambda_3}$ is a first integral of system (1), we get that there exist $\lambda_i \in \mathbb{C}$, with $i \in \{1, 2, 3\}$, not all zero, such that $\sum_{i=1}^{3} \lambda_i K_i = 0$. Apart from the trivial solution $\{\lambda_1 = 0,\ \lambda_2 = 0,\ \lambda_3 = 0\}$, there are the following three solutions of this equation:

$$\begin{aligned}
S_1 &= \{c_0 = 0,\ c_1 = 0,\ c_2 = 0,\ c_3 = 0,\ \lambda_2 = 0,\ \lambda_1 = 0\}\,,\\
S_2 &= \left\{b_0 = -\frac{c_0\lambda_3}{\lambda_2},\ b_1 = -\frac{c_1\lambda_3}{\lambda_2},\ b_2 = -\frac{c_2\lambda_3}{\lambda_2},\ b_3 = -\frac{c_3\lambda_3}{\lambda_2},\ \lambda_1 = 0\right\}, \text{ and}\\
S_3 &= \left\{a_0 = \frac{-b_0\lambda_2 - c_0\lambda_3}{\lambda_1},\ a_1 = \frac{-b_1\lambda_2 - c_1\lambda_3}{\lambda_1},\ a_2 = \frac{-b_2\lambda_2 - c_2\lambda_3}{\lambda_1},\ a_3 = \frac{-b_3\lambda_2 - c_3\lambda_3}{\lambda_1}\right\},
\end{aligned}$$

which give rise to three families of Lotka-Volterra polynomial differential systems of degree two in $\mathbb{R}^3$, with a first integral of the form $x^{\lambda_1}y^{\lambda_2}z^{\lambda_3}$.

If we consider the family given by solution S_1, as the parameters c_i, $i = 0, ..., 3$, are zero, we have that $\dot{z} = 0$ and the Lotka-Volterra system is reduced to:

$$\begin{aligned}
\dot{x} &= x\ (\ a_0 + a_1x + a_2y + a_3z\ ),\\
\dot{y} &= y\ (\ b_0 + b_1x + b_2y + b_3z\ ),\\
\dot{z} &= 0.
\end{aligned}$$

As $\dot{z} = 0$, z is constant and this system has $H = z$ as a first integral. Note that if we consider the first integral $H = x^{\lambda_1}y^{\lambda_2}z^{\lambda_3}$, and apply the conditions given by S_1, it is $\lambda_1 = \lambda_2 = 0$, we obtain $H = z^{\lambda_3}$ with $\lambda_3 = 2$, for getting the degree two, but in this case we will consider the simplest first integral. In each invariant plane with z constant, we have a Lotka-Volterra polinomial differential system in $\mathbb{R}^2$. The phase portrait of these systems has been studied in [Schlomiuk & Vulpe , 2012], so we are not going to deal with this case.

The topological classification of all the phase portraits in the Poincaré disc of the family given by the solution S_2 has been carried out in [Diz-Pita *et al.*, 2021].

In this paper we study the family given by the solution S_3. This solution provides the values of parameters a_i as a function of the parameters λ_i, b_i and c_i, with $i = 0, .., 3$. Replacing them in the expression of $\dot{x}$ we obtain:

$$\dot{x} = x\left(\frac{-b_0\lambda_2 - c_0\lambda_3}{\lambda_1} + \frac{x\,(-b_1\lambda_2 - c_1\lambda_3)}{\lambda_1} + \frac{y\,(-b_2\lambda_2 - c_2\lambda_3)}{\lambda_1} + \frac{z\,(-b_3\lambda_2 - c_3\lambda_3)}{\lambda_1}\right),$$

and if we denote $\lambda = -\lambda_2/\lambda_1$ and $\mu = -\lambda_3/\lambda_1$, then the original Lotka-Volterra system becomes

$$\begin{aligned}
\dot{x} &= x\ (\ b_0\lambda + c_0\mu + x\ (b_1\lambda + c_1\mu) + y\ (b_2\lambda + c_2\mu) + z\ (b_3\lambda + c_3\mu)\ ),\\
\dot{y} &= y\ (\ b_0 + b_1x + b_2y + b_3z\ ),\\
\dot{z} &= z\ (\ c_0 + c_1x + c_2y + c_3z\ ).
\end{aligned}$$

By hypothesis, this system has the first integral $H = x^{\lambda_1}y^{\lambda_2}z^{\lambda_3}$. Thus, we also can consider as a first integral

$$H = (x^{\lambda_1}y^{\lambda_2}z^{\lambda_3})^{-\frac{1}{\lambda_1}} = x^{-1}\, y^{-\frac{\lambda_2}{\lambda_1}}\, z^{-\frac{\lambda_3}{\lambda_1}} = \frac{y^{\lambda}z^{\mu}}{x}.$$

We want H to be rational of degree two so we must take $\lambda = \mu = 1$. In each level $H = 1/h$, with $h \neq 0$, we will have

$$\frac{1}{h} = \frac{yz}{x} \Longrightarrow x = hyz,$$

and then, for each h, the initial Lotka-Volterra system on dimension three is reduced to the differential system on dimension two

$$\dot{y} = y\ (\ b_0 + b_1 hyz + b_2 y + b_3 z\ ),$$
$$\dot{z} = z\ (\ c_0 + c_1 hyz + c_2 y + c_3 z\ ).$$

We must study the phase portrait of this family of differential systems, which is equivalent to study the phase portraits of the following Kolmogorov family on dimension two:

$$\begin{aligned} \dot{y} &= y\ (\ b_0 + b_1 yz + b_2 y + b_3 z\ ),\\ \dot{z} &= z\ (\ c_0 + c_1 yz + c_2 y + c_3 z\ ). \end{aligned} \tag{12}$$

In the particular cases in which H is zero or infinity, the differential system on dimension three is reduced to a Lotka-Volterra system on dimension two, having in each case $z = 0$ and $y = 0$, respectively. We recall that these systems had already been studied in [Schlomiuk & Vulpe , 2012].

Systems (12) depend on eight parameters and the classification of all their distinct topological phase protraits is huge. For this reason, in the same way that it was done in [Diz-Pita *et al.*, 2021], we study the subclass of them having a Darboux invariant of the form $e^{st}y^{\lambda_1}z^{\lambda_2}$. At first, the expression $\lambda_1 K_y + \lambda_2 K_z + s$ must be zero by Theorem 4(ii), with K_y and K_z the cofactors of the invariant planes $y = 0$ and $z = 0$, respectively. We recall that s and $\lambda_1^2 + \lambda_2^2$ can not be zero.

We obtain the cofactors $K_y = b_0 + b_1 yz + b_2 y + b_3 z$ and $K_z = c_0 + c_1 yz + c_2 y + c_3 z$ and solving the equation $\lambda_1 K_x + \lambda_2 K_z + s = 0$, we get the following non-trivial solution

$$\left\{ c_1 = -\frac{b_1\lambda_1}{\lambda_2},\ \ c_2 = -\frac{b_2\lambda_1}{\lambda_2},\ \ c_3 = -\frac{b_3\lambda_1}{\lambda_2},\ \ s = -b_0\lambda_1 - c_0\lambda_2 \right\},$$

which leads to the system

$$\dot{y} = y\left(b_0 + b_1 yz + b_2 y + b_3 z\right),$$
$$\dot{z} = z\left(c_0 - \frac{b_1 yz\lambda_1}{\lambda_2} - \frac{b_2 y\lambda_1}{\lambda_2} - \frac{b_3 z\lambda_1}{\lambda_2}\right).$$

If we denote $\lambda_2 = \lambda$ and $\lambda_1 = \lambda\mu$, then the system becomes

$$\dot{y} = y\left(b_0 + b_1 yz + b_2 y + b_3 z)\right),$$
$$\dot{z} = z\left(c_0 - \mu(b_1 yz + b_2 y + b_3 z)\right),$$

and the Darboux invariant is $y^{\lambda\mu}z^{\lambda}e^{-t\lambda(c_0+b_0\mu)}$. But if this is a Daboux invariant, also it is $y^{\mu}ze^{-t(c_0+b_0\mu)}$. Note that in order that we have a Darboux invariant $c_0 + b_0\mu$ cannot be zero.

4. Properties of system (3)

In this section we state some results that will be used on the classification in order to reduce the number of phase portraits appearing. Note that if $b_1 = 0$, then the system (3) is a Lotka-Volterra system in dimension two. A global topological classification of these systems has been completed in [Schlomiuk & Vulpe , 2012], so we limit our study to the case $b_1 \neq 0$.

We recall that for obtaining system (3) we have supposed that system (12) has the Darboux invariant $y^{\mu}ze^{-t(c_0+b_0\mu)}$, so it is required that $c_0 + b_0\mu \neq 0$. The proofs of the next two propositions are easy and we omit them.

Proposition 1. *Let $(\tilde{y}(t), \tilde{z}(t))$ be a solution of system (3). In the next cases we obtain another system with solution $(-\tilde{y}(-t), -\tilde{z}(-t))$.*

(1) If c_0, b_0 and b_2 are not zero, and we change the sign of all of them.
(2) If $b_0 = 0$ and we change the sign of c_0 and c_1, which are not zero.
(3) If $c_0 = 0$ and we change the sign of b_0 and c_1, which are not zero.

Remark 4.1. By Proposition 1 we can limit our study to Kolmogorov systems (3) with b_0 non-negative. In the cases with this parameter negative, we will obtain phase portraits symmetric to the ones obtained in the positive cases. And when $b_0 = 0$ we will consider also $c_0 > 0$.

Proposition 2. *Consider system* (3) *and suppose that* $(\tilde{y}(t), \tilde{z}(t))$ *is a solution of this system. If we change* b_1 *by* $-b_1$ *and* b_2 *by* $-b_2$ *(respectively* b_3 *by* $-b_3$*), then* $(-\tilde{y}(t), \tilde{z}(t))$ *(respectively* $(\tilde{y}(t), -\tilde{z}(t))$*) is a solution of the obtained system.*

Corollary 4.1. *Consider system* (3) *and suppose* $(\tilde{y}(t), \tilde{z}(t))$ *is a solution. If* $b_2 = 0$ *(respectively* $b_3 = 0$*) and we change* b_1 *by* $-b_1$*, then* $(-\tilde{y}(t), \tilde{z}(t))$ *(respectively* $(\tilde{y}(t), -\tilde{z}(t))$*) is a solution.*

Remark 4.2. In order to classify all the phase portraits of the Kolmogorov systems (3), according with the previous results, it is sufficient to consider $b_2 \geq 0$ and $b_3 \geq 0$. And when $b_2 b_3 = 0$ we will consider also $b_1 > 0$.

Remark 4.3. In short according with the previous results and considerations from now on it will be sufficient to study the Kolmogorov systems (3) with their parameters satisfying

$$(H) = \{b_1 \neq 0, c_0 + b_0\mu \neq 0, b_0 \geq 0, b_2 \geq 0, b_3 \geq 0\}.$$

and also, if $b_0 = 0$ then $c_0 > 0$ and if $b_2 b_3 = 0$ then $b_1 > 0$.

5. Local study of finite singular points

System (3) has the following finite singularities:

$$P_0 = (0,0), \quad P_1 = \left(0, \frac{c_0}{\mu b_3}\right) \text{ if } \mu b_3 \neq 0 \text{ and } P_2 = \left(-\frac{b_0}{b_2}, 0\right) \text{ if } b_2 \neq 0.$$

Moreover if $\mu b_3 = 0$ and $c_0 = 0$ all the points on the z-axis are singular points, and if $b_2^2 + b_0^2 = 0$ all the points on the y-axis are singular points, and in both cases the system can be reduced to a Lotka-Volterra system in dimension 2. Therefore from now on we will consider the hypothesis

$$(H_1) = \left\{b_1 \neq 0, b_0\mu + c_0 \neq 0, b_0 \geq 0, b_2 \geq 0, b_3 \geq 0, (\mu b_3)^2 + c_0^2 \neq 0, b_2^2 + b_0^2 \neq 0\right\}.$$

Assuming (H_1) there are 4 different cases according to the finite singular points existing for system (3), which are given in Table 1. Then we study the possible local phase portraits in each one of the finite singular points under the hypothesis (H_1).

Table 1. The different cases for the finite singular points.

Case	Conditions	Finite singular points
1	$\mu b_3 \neq 0$, $b_2 \neq 0$.	P_0, P_1, P_2.
2	$\mu b_3 \neq 0$, $b_2 = 0$, $b_0 \neq 0$.	P_0, P_1.
3	$\mu b_3 = 0$, $c_0 \neq 0$, $b_2 \neq 0$.	P_0, P_2.
4	$\mu b_3 = 0$, $c_0 \neq 0$, $b_2 = 0$, $b_0 \neq 0$.	P_0.

The origin is always an isolated singular point for system (3), and we have the next classification for its phase portraits: If $b_0 c_0 \neq 0$ the singularity is hyperbolic and two cases are possible, the origin is a saddle point if $c_0 < 0$, and it is an unstable node if $c_0 > 0$. If $b_0 c_0 = 0$ the origin is a semi-hyperbolic saddle-node.

When P_1 is a singular point of system (3), it can present different phase portraits. If $c_0 \neq 0$ then P_1 is hyperbolic and it can present the following phase portraits: If $c_0\mu(b_0\mu + c_0) > 0$ then P_1 is a saddle, if $c_0 > 0$, $\mu < 0$ and $b_0\mu + c_0 > 0$ it is a stable node, and finally if $c_0 < 0$ and $\mu(b_0\mu + c_0) > 0$ it is an unstable node. The singular point P_1 collides with the origin if $c_0 = 0$.

When P_2 is a singular point of system (3), it is a saddle if $b_0\mu + c_0 > 0$ and it is a stable node if $b_0\mu + c_0 < 0$. If $b_0 = 0$ then P_2 collides with the origin.

Lemma 1. *Asumming hypothesis* (H_1) *there are* 22 *different cases according to the local phase portrait of the finite singular points of system* (3)*, which are given in Tables 2 - 5.*

Proof. We have to analyse cases 1 to 4 in Table 1 and determine the local phase portraits of the singular points existing in each one of them, according to their individual classification.

We start with the first one, in which the conditions $\mu b_3 \neq 0$ and $b_2 \neq 0$ hold. The singular points are P_0, P_1 and P_2.

Consider case $c_0 < 0$, $b_0 \neq 0$ in which the origin is a saddle. Then P_1 can be a saddle if $\mu(b_0\mu + c_0) < 0$ or an unstable node if $\mu(b_0\mu + c_0) > 0$. If P_1 is a saddle, then P_2 is a stable node, because if it was a saddle with $b_0\mu + c_0 > 0$, then $\mu < 0$ and so $b_0\mu < 0$ and $b_0\mu + c_0 < 0$, wich is a contradiction. If P_1 is an unstable node, then P_2 can be either a saddle or a stable node. This leads to cases 1.1 to 1.3 in Table 2.

We continue with the case $c_0 > 0$ and $b_0 > 0$, in which P_0 is an unstable node. Now P_1 can be a saddle or a stable node. If P_1 is a saddle, then P_2 can be a saddle or a stable node, but if P_1 is a stable node then $b_0\mu + c_0 > 0$ and so P_2 is always a saddle. This leads to cases 1.4 to 1.6.

If P_0 is a saddle-node with $c_0 = 0$ and $b_0 > 0$, then P_1 coincides with the origin and P_2 can be a saddle or a stable node. If P_0 is a saddle-node with $b_0 = 0$ and $c_0 > 0$, then P_2 coincides with the origin and P_1 can be a saddle or a stable node. We recall that by the hypothesis it is not possible to have b_0 and c_0 zero simultaneously, b_0 is non negative and if $b_0 = 0$ then we consider $c_0 > 0$ by Proposition 1, so this allow us to conclude the classification in case 1 of Table 1.

Now we study case 2 of Table 1, in which $\mu b_3 \neq 0$, $b_2 = 0$ and $b_0 \neq 0$. The singular points are P_0 and P_1. If P_0 is a saddle, then P_1 can be only a saddle or an unstable node, by the sign of the coefficient c_0. Likewise the sign of c_0 determines that if P_0 is an unstable node, P_1 can be a saddle or a stable node. This leads to cases 2.1 to 2.4 of Table 3. At last, if $c_0 = 0$ then P_1 coincides with the origin and it is a saddle-node.

We address the case 3 of Table 1 in which $\mu b_3 = 0$, $c_0 \neq 0$ and $b_2 \neq 0$. The origin can be a saddle or an unstable node and in both cases P_2 can be a saddle or a stable node, as $b_0 \neq 0$. If $b_0 = 0$ then P_2 coincides with the origin and it is a saddle-node.

Finally in case 4 of Table 1 we have the conditions $\mu b_3 = 0$, $c_0 \neq 0$, $b_2 = 0$ and $b_0 \neq 0$. The unique singular point is the origin and as $c_0 b_0$ cannot be zero, it is either a saddle or an unstable node. ■

Table 2. Classification in case 1 of Table 1 according with the local phase portraits of finite singular points.

Case 1: $\boldsymbol{\mu b_3 \neq 0}$, $\boldsymbol{b_2 \neq 0}$.

Sub.	Conditions	Classification
1.1	$b_0 > 0$, $c_0 < 0$, $\mu > 0$, $(b_0\mu + c_0) < 0$.	P_0 saddle, P_1 saddle, P_2 stable node.
1.2	$b_0 > 0$, $c_0 < 0$, $\mu > 0$, $b_0\mu + c_0 > 0$.	P_0 saddle, P_1 unstable node, P_2 saddle.
1.3	$b_0 > 0$, $c_0 < 0$, $\mu < 0$, $b_0\mu + c_0 < 0$.	P_0 saddle, P_1 unstable node, P_2 stable node.
1.4	$b_0 > 0$, $c_0 > 0$, $\mu > 0$, $b_0\mu + c_0 > 0$.	P_0 unstable node, P_1 saddle, P_2 saddle.
1.5	$b_0 > 0$, $c_0 > 0$, $\mu < 0$, $b_0\mu + c_0 < 0$.	P_0 unstable node, P_1 saddle, P_2 stable node.
1.6	$b_0 > 0$, $c_0 > 0$, $\mu < 0$, $b_0\mu + c_0 > 0$.	P_0 unstable node, P_1 stable node, P_2 saddle.
1.7	$c_0 = 0$, $\mu > 0$.	$P_0 \equiv P_1$ saddle-node, P_2 saddle.
1.8	$c_0 = 0$, $b_0 > 0$, $\mu < 0$.	$P_0 \equiv P_1$ saddle-node, P_2 stable node.
1.9	$b_0 = 0$, $\mu > 0$.	$P_0 \equiv P_2$ saddle-node, P_1 saddle.
1.10	$b_0 = 0$, $c_0 > 0$, $\mu < 0$.	$P_0 \equiv P_2$ saddle-node, P_1 stable node.

Table 3. Classification in case 2 of Table 1 according with the local phase portraits of finite singular points.

Case 2: $\boldsymbol{\mu b_3 \neq 0}$, $\boldsymbol{b_2 = 0}$, $\boldsymbol{b_0 \neq 0}$.

Sub.	Conditions	Classification
2.1	$b_0 > 0$, $c_0 < 0$, $\mu > 0$, $b_0\mu + c_0 < 0$.	P_0 saddle, P_1 saddle.
2.2	$b_0 > 0$, $c_0 < 0$, $\mu(b_0\mu + c_0) > 0$.	P_0 saddle, P_1 unstable node.
2.3	$b_0 > 0$, $c_0 > 0$, $\mu(b_0\mu + c_0) > 0$.	P_0 unstable node, P_1 saddle.
2.4	$b_0 > 0$, $c_0 > 0$, $\mu < 0$, $b_0\mu + c_0 > 0$.	P_0 unstable node, P_1 stable node.
2.5	$c_0 = 0$, $b_0 > 0$.	$P_0 \equiv P_1$ saddle-node.

Table 4. Classification in case 3 of Table 1 according with the local phase portraits of finite singular points.

Case 3: $\mu b_3 = 0$, $c_0 \neq 0$, $b_2 \neq 0$.

Sub.	Conditions	Classification
3.1	$b_0 > 0$, $c_0 < 0$, $\mu > 0$, $b_0\mu + c_0 > 0$.	P_0 saddle, P_2 saddle.
3.2	$b_0 > 0$, $c_0 < 0$, $b_0\mu + c_0 < 0$.	P_0 saddle, P_3 stable node.
3.3	$b_0 > 0$, $c_0 > 0$, $b_0\mu + c_0 > 0$.	P_0 unstable node, P_2 saddle.
3.4	$b_0 > 0$, $c_0 > 0$,$\mu < 0$, $b_0\mu + c_0 < 0$.	P_0 unstable node, P_2 stable node.
3.5	$b_0 = 0$, $c_0 > 0$.	$P_0 \equiv P_3$ saddle-node.

Table 5. Classification in case 4 of Table 1 according with the local phase portraits of finite singular points.

Case 4: $\mu b_3 = 0$, $c_0 \neq 0$, $b_2 = 0$, $b_0 \neq 0$.

Sub.	Conditions	Classification
4.1	$b_0 c_0 < 0$.	P_0 saddle.
4.2	$b_0 > 0$, $c_0 > 0$.	P_0 unstable node.

6. Local study of infinite singular points in the chart U_1

In order to study the behavior of the trajectories of system (3) near infinity we consider its Poincaré compactification. For the moment we assume the same hypothesis (H_1) than in previous sections. From Section 2 it is enough to study the singular points over $v = 0$ in the chart U_1 and the origin of the chart U_2. We will deal with the study of the origin of chart U_2 in Section 7, so now we focus on the chart U_1. According to equation (4) we get the compactification in the local chart U_1, where system (3) writes

$$\begin{aligned} \dot{u} &= -b_3(\mu+1)u^2v + (c_0 - b_0)uv^2 - b_1(\mu+1)u^2 - b_2(\mu+1)uv, \\ \dot{v} &= -b_3uv^2 - b_0v^3 - b_1uv - b_2v^2. \end{aligned} \tag{13}$$

Taking $v = 0$ we get $\dot{u}\mid_{v=0} = -b_1(\mu+1)u^2$ and $\dot{v}\mid_{v=0} = 0$. Therefore if $\mu = -1$ all points at infinity are singular points, and we will not deal with this situation in this paper. In other case, if $\mu \neq -1$, the only singular point is the origin of U_1, which we denote by O_1. As the linear part of system (13) at the origin is identically zero we use the blow-up technique to study it, leading to the next result, which is proved in Subsections 6.1 and 6.2.

From now on we include the condition $\mu \neq -1$ in our hypothesis, so we will work under the conditions

$$(H_2) = \left\{ b_1 \neq 0, b_0\mu + c_0 \neq 0, b_0 \geq 0, b_2 \geq 0, b_3 \geq 0, (\mu b_3)^2 + c_0^2 \neq 0, b_2^2 + b_0^2 \neq 0, \mu \neq -1 \right\}.$$

Lemma 2. *Asumming hypothesis (H_2) the origin of the chart U_1 is an infinite singular point of system (3), and it has 27 distinct local phase portraits described in Figure 1.*

In the following subsections we prove Lemma 2. For system (13) the characteristic polynomial is $\mathcal{F} = b_2\mu uv^2 + b_1\mu u^2 v$, so the origin is a nondicrital singular point if $\mu \neq 0$ and it is dicritical if $\mu = 0$, so we will study this two cases separately.

We introduce the new variable w_1 by means of the variable change $uw_1 = v$, and get the system

$$\begin{aligned} \dot{u} &= (c_0 - b_0)u^3w_1^2 - b_3(\mu+1)u^3w_1 - b_2(\mu+1)u^2w_1 - b_1(\mu+1)u^2, \\ \dot{w_1} &= b_3\mu u^2w_1^2 - c_0u^2w_1^3 + b_2\mu uw_1^2 + b_1\mu uw_1. \end{aligned} \tag{14}$$

In the nondicritical case we have to cancel the common factor u obtaining

$$\begin{aligned} \dot{u} &= (c_0 - b_0)u^2w_1^2 - b_3(\mu+1)u^2w_1 - b_2(\mu+1)uw_1 - b_1(\mu+1)u, \\ \dot{w_1} &= b_3\mu uw_1^2 - c_0uw_1^3 + b_2\mu w_1^2 + b_1\mu w_1. \end{aligned} \tag{15}$$

In the dicritical case, when $\mu = 0$, we must cancel the common factor u^2 and we obtain the system

$$\begin{aligned} \dot{u} &= (c_0 - a_0)uw_1^2 - b_3uw_1 - b_2w_1 - b_1, \\ \dot{w_1} &= -c_0w_1^3. \end{aligned} \tag{16}$$

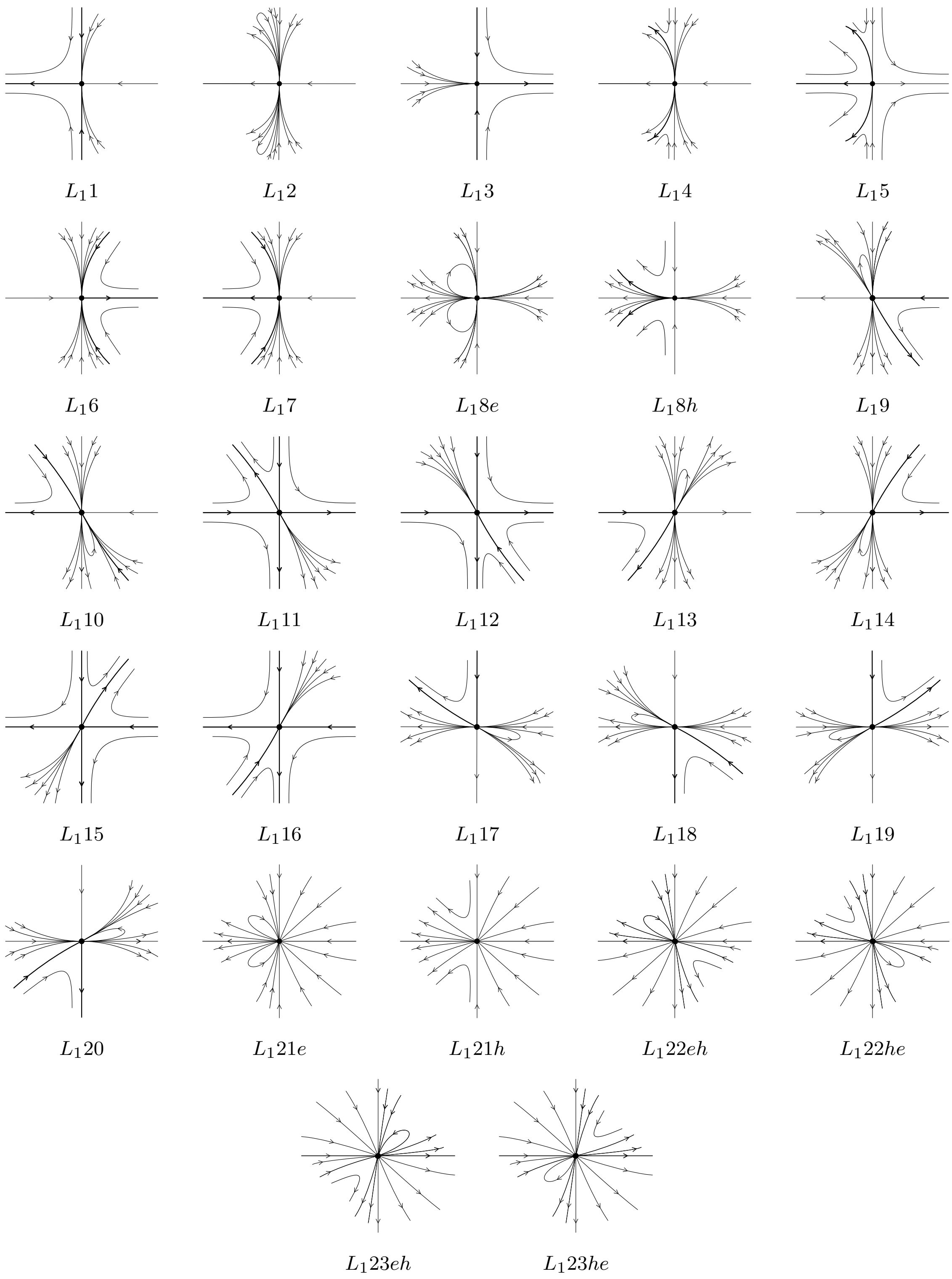


Fig. 1. Local phase portraits of the infinite singular point O_1.

6.1. *Nondicritical case*

At first, it is necessary to study the singular points of system (15) on the exceptional divisor. The origin is always a singular point, and we denote it by Q_0. When $b_2 \neq 0$ there is another singular point, $Q_1 = (0, -b_1/b_2)$.

The origin, Q_0, is always hyperbolic. It is a saddle if $\mu \in (-\infty,-1)\cup(0,+\infty)$, a stable node if $\mu \in (-1,0)$ and $b_1>0$, and an unstable node if $\mu\in(-1,0)$ and $b_1<0$. The singular point Q_1 is always a semi-hyperbolic saddle-node. These conditions come together in the next five subcases.

(A) If $b_2=0$, $b_0\neq 0$ and $\mu\in(-\infty,-1)\cup(0,+\infty)$, then the only singular point on the exceptional divisor is Q_0 which is a saddle. In this case the vertical blow up done does not provide a well determined phase portrait, so it is necessary to apply an horizontal blow up. In order to do that, we introduce the variable $w_1=u/v$ on system (13) obtaining:

$$\begin{aligned}\dot{w}_1 =& -\mu b_3 v^2 w_1^2 - \mu b_1 v w_1^2 + c_0 v^2 w_1,\\ \dot{v} =& -b_3 v^3 w_1 - b_1 v^2 w_1 - b_0 v^3,\end{aligned} \tag{17}$$

and eliminating a common factor v we get the system

$$\begin{aligned}\dot{w}_1 =& -\mu b_3 v w_1^2 - \mu b_1 w_1^2 + c_0 v w_1,\\ \dot{v} =& -b_3 v^2 w_1 - b_1 v w_1 - b_0 v^2,\end{aligned} \tag{18}$$

for which the only singular point on the exceptional divisor is the origin, and it is linearly zero, so we have to repeat the process.
Now the characteristic polynomial is $\mathcal{F}=b_1(\mu-1)w_1^2v-(b_0+c_0)w_1v^2$, so the origin is always nondicritical. Note that, as $b_1\neq 0$, $\mathcal{F}\equiv 0$ if and only if $c_0=-b_0$ and $\mu=1$, but in that case $b_0\mu+c_0=0$ which contradicts hypothesis (H_2). Now we introduce the new variable $w_2=w_1/v$, obtaining the system

$$\begin{aligned}\dot{w}_2 =& -b_3(\mu-1)w_2^2v^2 - b_1(\mu-1)w_2^2 v + (b_0+c_0)w_2 v,\\ \dot{v} =& -b_3 w_2 v^3 - b_1 w_2 v^2 - b_0 v^2,\end{aligned} \tag{19}$$

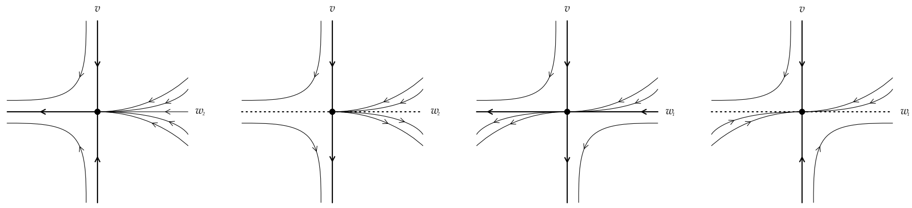
and eliminating a common factor v, we get

$$\begin{aligned}\dot{w}_2 =& -b_3(\mu-1)w_2^2v - b_1(\mu-1)w_2^2 + (b_0+c_0)w_2,\\ \dot{v} =& -b_3 w_2 v^2 - b_1 w_2 v - b_0 v.\end{aligned} \tag{20}$$

The singular points of system (20) on the exceptional divisor are the origin S_0 and the singular point $S_1=((b_0+c_0)/(b_1(\mu-1)),0)$ if $\mu\neq 1$ (note that it coincides with the origin in $b_0+c_0=0$). We determine their local phase portraits obtaining the following classification. We start with the cases in which the only singular point on the exceptional divisor is the origin:

(A1) If $b_0+c_0=0$ and $\mu\neq 1$, then the origin is a semi-hyperbolic saddle-node. The relative position and orientation of the hyperbolic and parabolic sectors depends on the sign of μ and $\mu-1$. Thus we deal with the following subcases.

Subcase (A1.1). Let $\mu>1$. This determines the sense of the flow on the w_2-axis, so the phase portrait around this axis for system (20) is the one in Figure 2(a).
To return to system (19) we multiply by v, thus the orbits in the third and fourth quadrants change their orientation. Moreover all the points on the w_2-axis become singular points. The resultant phase portrait is given in Figure 2(b).
When going back to the (w_1,v)-plane the third and fourth quadrants swap from the (w_2,v)-plane, and the exceptional divisor shrinks to a point, and hence the orbits are slightly modified. Attending to the expresions of $\dot{w}_1\mid_{v=0}=-\mu b_1 w_1^2$ and of $\dot{v}\mid_{w_1=0}=-b_0v^2$, we know the sense of the flow along the axes. Following the results mentioned in Subsection 2.3, we get the phase portrait given in Figure 2(c), and multiplying again by v, the one given in Figure 2(d).

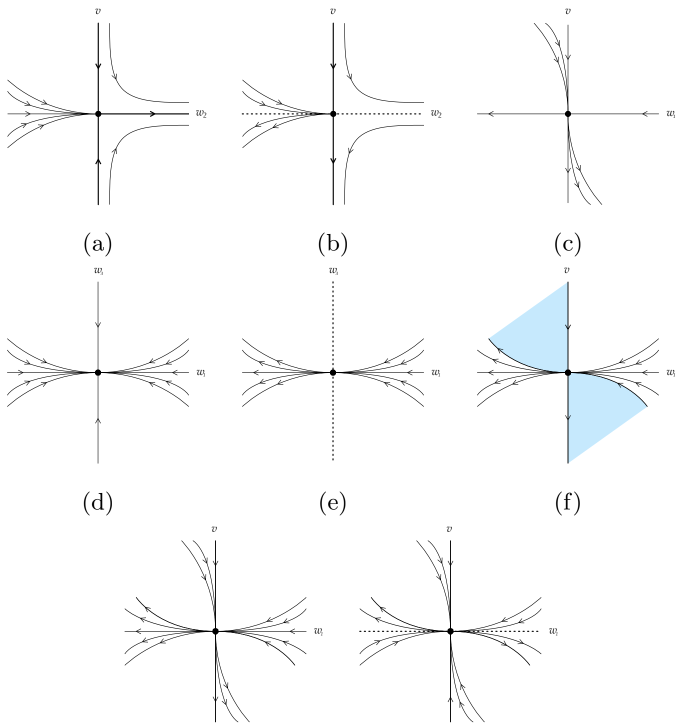


(a) (b) (c) (d)

Fig. 2. Desingularization of the origin of system (13) in nondicritical case (A1.1).

Finally we must go back to the (u, v)-plane, swapping again the third and the fourth quadrants and contracting the exceptional divisor to the origin. The orbits tending to the origin in forward or backward time, became orbits tending to the origin in forward or backward time tangent to the v-axis. We get the local phase portrait at the origin for system (13) given in Figure $1(L_1 1)$.

Subcase (A1.2). Let $\mu \in (0, 1)$. In this case the phase portrait around the w_2-axis for system (20) is the one in Figure 3(a), and multiplying by v, the phase portrait for system (19) is 3(b).



(a) (b) (c)

(d) (e) (f)

(g) (h)

Fig. 3. Desingularization of the origin of system (13) in nondicritical case (A1.2).

Here, if we try go back to the (w_1, v)-plane, swapping the third and fourth quadrants and shrinking the exceptional divisor, the phase portrait is not determined on the first and third quadrants and around the w_1-axis, and the only information we have is on Figure 3(c). We must do a vertical blow up on system (18). We introduce the new variable $w_3 = v/w_1$ and get the system

$$\begin{aligned} \dot{w}_1 &= -b_3 \mu w_1^3 w_3 + c_0 w_1^2 w_3 - \mu b_1 w_1^2, \\ \dot{w}_3 &= b_3(\mu - 1) w_1^2 w_3^2 + b_1(\mu - 1) w_1 w_3. \end{aligned} \tag{21}$$

Now we eliminate a common factor w_1:

$$\begin{aligned} \dot{w}_1 &= -b_3\mu w_1^2 w_3 + c_0 w_1 w_3 - \mu b_1 w_1, \\ \dot{w}_3 &= b_3(\mu-1)w_1 w_3^2 + b_1(\mu-1)w_3. \end{aligned} \tag{22}$$

The only singular point on the exceptional divisor is the origin, which is a stable node. The phase portrait around the w_1-axis for system (22) is the one in Figure 3(d), and multiplying by v, the phase portrait for system (21) is 3(e). Undoing the vertical blow up we obtain 3(f) where the behaviour around the v-axis, in the coloured regions, is not determined. Combinig this phase portrait with the information on 3(c), it is, that there are orbits which go to the origin tangent to the v-axis on the second quadrant, and there are orbit which leave the origin tangent to the v-axis on the fourth quadrant, we can conclude that for system (18) we have phase portrait 3(g). In the second and fourth quadrant it would be possible to have a hyperbolic sector or an elliptic one, but we have proved in the global phase portraits, by applying index theory, that the only feasible option is the one with the elliptic sector.
We must multiply by v and we get the phase portrait given in Figure 2(h) for system (17). Now we undo the first horizontal blow up done and hence we obtain the phase portrait for O_1 given in Figure 1($L_1$2), where the existence of the elliptic sector is proved in the global phase portraits, as we have just mentioned.

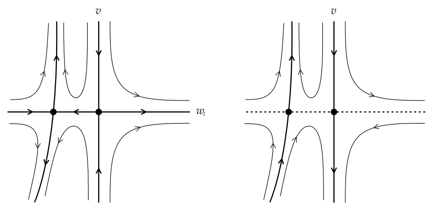
Subcase (A1.3). Taking $\mu < 0$ and similarly to the first subcase, we obtain the local phase portait for O_1 given in Figure 1($L_1$3).

(A2) If $\mu = 1$ and $b_0 + c_0 > 0$, then the origin is a saddle. Here, when undoing the blow ups it is necessary again to do a vertical blow up on system (18), and after that we obtain the same phase portrait as in the first subcase, it is $L_1$1.

(A3) If $\mu = 1$ and $b_0 + c_0 < 0$, then the origin is a stable node. Again, including a vertical blow up on system (18), we obtain phase portait $L_1$2. As in the first case in which we obtained this local phase portait, the blow up does not determine if the elliptic sector are indeed elliptic, but we prove it when analyzing the global phase portraits.

Now we consider the cases with two singular points on the exceptional divisor:

(A4) If $b_0 + c_0 > 0$ and $(b_0\mu + c_0)(\mu - 1) < 0$, then the origin and S_1 are saddles. The phase portrait for system (20) is in Figure 4(a) and multiplying by v, the phase portrait for system (19) is in Figure 4(b).



(a) (b)

Fig. 4. Desingularization of the origin of system (13) in nondicritical case (A4).

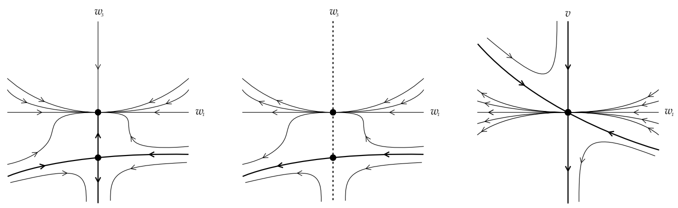
To go back to the (w_1, v)-plane we have to distinguish two cases, one with $\mu < -1$ and other with $\mu \in (0,1)$. In the first one, the subsequent phase portraits are well determined and we arrive easily to final phase portrait in Figure 1($L_1$5). If $\mu \in (0,1)$, the phase portrait for system (18) is not determined. We must do a vertical blow up, introducing the variable $w_3 = v/w_1$. We get the system

$$\begin{aligned} \dot{w}_1 &= -b_3\mu w_1^3 w_3 + c_0 w_1^2 w_3 - \mu b_1 w_1^2, \\ \dot{w}_3 &= b_3(\mu-1)w_1^2 w_3^2 - (b_0+c_0)w_1 w_3^2 + b_1(\mu-1)w_1 w_3. \end{aligned} \tag{23}$$

and eliminating a common factor w_1:

$$\begin{aligned}\dot{w_1} &= -b_3\mu w_1^2 w_3 + c_0 w_1 w_3 - \mu b_1 w_1,\\ \dot{w_3} &= b_3(\mu-1)w_1 w_3^2 - (b_0+c_0)w_3^2 + b_1(\mu-1)w_3.\end{aligned} \tag{24}$$

This system has two singular points on the exceptional divisor: the origin, which is a stable node, and the point $(0, b_1(\mu-1)/(b_0+c_0))$ which is a saddle. Studying the sense of the flow on the axes, we get that the phase portrait for system (24) is the one in Figure 5(a). Multiplying by w_1 and undoing the vertical blow up, we obtain, respectively, the phase portraits on Figure 5(c) and (d). Now the phase portrait for system (18) is well determined, and we can go on undoing the first horizontal blow up, getting the final phase portrait L_14.



(a) (b) (c)

Fig. 5. Vertical blow up on system (18) in nondicritical case (A4).

(A5) If $b_0+c_0 > 0$ and $(b_0\mu+c_0)(\mu-1) > 0$, then the origin is a saddle and S_1 is a stable node. We must distinguish three cases in function of the sign of $\mu-1$, which determines the position of the singular point Q_1 on system (20) and the sense of the flow on the axes on system (18). We do not get any new phase portrait for O_1 in this case. If we take $\mu < 0$ then we obtain the same phase portrait as in subcase A1.3, it is, L_13, and with $\mu > 1$ we obtain the phase portait L_11. At last, with $\mu \in (0,1)$ we obtain the phase portait L_12, but in this case it is necessary to do a vertical blow up on system (18) during the desingularization process.

(A6) If $b_0+c_0 < 0$ and $(b_0\mu+c_0)(\mu-1) > 0$, then the origin is a stable node and S_1 is a saddle. This case is similar to the previous one and we should distinguish three cases: if $\mu < 0$ we obtain the phase portrait L_16, if $\mu > 1$ we obtain phase portrait L_17, and if $\mu \in (0,1)$ we need to do a vertical blow up of system (18) and we obtain phase portait L_12. As we mentioned before, we have proved that the elliptic sectors are always elliptic in the global phase portraits, although it is not provided by the blow ups.
As in the first case in which we obtained this local phase portait, the blow up does not determine if the elliptic sector are indeed elliptic, but we prove it when analyzing the global phase portraits.

(A7) If $b_0+c_0 < 0$ and $(b_0\mu+c_0)(\mu-1) < 0$, then the origin is a stable node and S_1 is an unstable node and we obtain again phase portrait L_12.

(B) If $b_2 = 0$, $b_0 \neq 0$ and $\mu \in (-1,0)$, then Q_0 is a stable node. We obtain phase portraits L_18e and L_18h. Note that the only difference is on the sectors that appear beside the v-axis on the second and third quadrants. On L_18e we have elliptic sectors and on L_18h we have hyperbollic sectors. It will be enough to apply index theory to know which of them appears in a global phase portait, as we will detail on Section 8.

(C) If $b_2 \neq 0$ and $\mu \in (-\infty,-1)\cup(0,+\infty)$, then Q_0 is a saddle and Q_1 a saddle-node. We must distinguish eight cases. At first, the sign of b_1 determines if singular point Q_1 is on the positive or the negative part of w_1-axis. Also we must fix the signs of μ and $b_0\mu+c_0$ as they determine the position and orientation of the sectors at the saddle-node. The different conditions to study and the corresponding results are the following:

Subcase	**Conditions**	**Phase portrait of O_1**
(C.1)	$b_1 > 0,\ \mu > 0,\ b_0\mu + c_0 < 0$	$L_1 9$
(C.2)	$b_1 > 0,\ \mu > 0,\ b_0\mu + c_0 > 0$	$L_1 10$
(C.3)	$b_1 > 0,\ \mu < -1,\ b_0\mu + c_0 > 0$	$L_1 11$
(C.4)	$b_1 > 0,\ \mu < -1,\ b_0\mu + c_0 < 0$	$L_1 12$
(C.5)	$b_1 < 0,\ \mu > 0,\ b_0\mu + c_0 < 0$	$L_1 13$
(C.6)	$b_1 < 0,\ \mu > 0,\ b_0\mu + c_0 > 0$	$L_1 14$
(C.7)	$b_1 < 0,\ \mu < -1,\ b_0\mu + c_0 > 0$	$L_1 15$
(C.8)	$b_1 < 0,\ \mu < -1,\ b_0\mu + c_0 < 0$	$L_1 16$

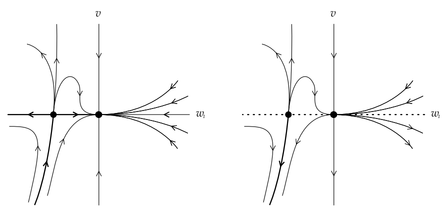
In subcases (C.1), (C.2), (C.5) and (C.6), it is necessary to do an horizontal blow up to completely determine the phase portrait of O_1. The process is the same in the four cases so we describe it only for the first one. If we introduce the variable $w_1 = u/v$ in system (13) we obtain

$$\begin{aligned} \dot{w_1} &= -\mu b_3 w_1^2 v^2 - \mu b_1 w_1^2 v - \mu b_2 w_1 v + c_0 w_1 v^2, \\ \dot{v} &= -b_3 w_1 v^3 - b_1 w_1 v^2 - b_0 v^3 - b_2 v^2, \end{aligned} \tag{25}$$

and eliminating a common factor v we get

$$\begin{aligned} \dot{w_1} &= -\mu b_3 w_1^2 v - \mu b_1 w_1^2 - \mu b_2 w_1 + c_0 w_1 v, \\ \dot{v} &= -b_3 w_1 v^2 - b_1 w_1 v - b_0 v^2 - b_2 v. \end{aligned} \tag{26}$$

The singular points of system (26) on the exceptional divisor are the origin, which in this case is a stable node, and the point $(-b_2/b_1, 0)$ which is a saddle-node. Thus, attending to the sense of the flow on the axes, the phase portrait of this system around the w_1-axis is the one in Figure 6(a). Multiplying by w_1 we get the phase portrait on Figure 6(b). Then we undo the horizontal blow up: we swap the third and fourth quadrants, contract the exceptional divisor into the origin and modify the orbits according the results mentioned in Subection 2.3. We recall that, for example, the separatrix of singular point $(-b_2/b_1, 0)$ which is on the third quadrant, goes into a separatrix on the fourth quadrant that starts from the origin with slope $-b_2/b_1$. Now the phase portrait for system (13) is well determined and as we said it is the one on Figure 1($L_1 9$).



(a) (b)

Fig. 6. Horizontal blow up on system (13) in nondicritical case (C.1).

(D) If $b_2 = 0$, $b_0 \neq 0$ and $\mu \in (-\infty, -1) \cup (0, +\infty)$, then Q_0 is a stable node and Q_1 a saddle-node. We separate two cases as the sign of $b_0\mu + c_0$ determines the position of the saddle-node sectors. If $b_0\mu + c_0 > 0$ we obtain phase portait $L_1 17$ and if $b_0\mu + c_0 < 0$ then we obtain phase portait $L_1 18$.

(E) If $b_2 = 0$, $b_0 \neq 0$ and $\mu \in (-\infty, -1) \cup (0, +\infty)$, then Q_0 is an unstable node and Q_1 a saddle-node. If $b_0\mu > 0$ we obtain phase portait $L_1 19$ and if $b_0\mu < 0$ then we obtain phase portait $L_1 20$.

6.2. *Dicritical case*

Now we study the case with $\mu = 0$, i.e, the dicritical case. System (16) would have a singular point on the on the exceptional divisor if and only if $c_0 = 0$ and $b_2 \neq 0$, but as we are considering $\mu = 0$, it is not possible because then we would have $b_0\mu + c_0 = 0$ which contradicts hypothesis (H_2). Then there are no singular points on the exceptional divisor.

We must consider four cases depending on wheter b_2 is zero or not, and the sign of b_1. If $b_2 = 0$ then the sign of $\dot{u}$ does not change along the w_1-axis, but if $b_2 \neq 0$, $\dot{x}$ changes its sign at the point $(0, -b_1/b_2)$.

If $b_2 = 0$ (then we assume $b_1 > 0$ by Remark 4.3), the flow around the w_1 axis is as represented in Figure 7(a). Multiplying by u^2 all the points on the w_1-axis become singular points, but the sense of the flow remains the same in all regions, see Figure 7(b). At last, going back to the (u, v)-plane we get that there are orbits which tend to the origin with any slope on quadrants first and fourth, and there are orbits leaving the origin with any slope on quadrant second and third, but also there are sectors which are not determined in this quadrants near the v-axis. This sectors, coloured on Figure 7(c), can be elliptic or hyperbolic, and this can be determined on the global phase portraits by applying index theory, as we will expain on Section 8. As a result we can have phase portraits on Figure 1($L_1 21e$) or ($L_1 21h$).

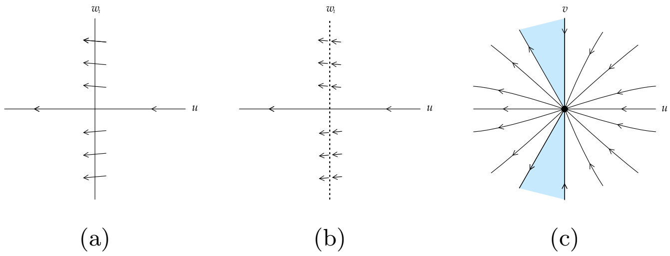


Fig. 7. Desingularization of the origin of system (13) in dicritical case with $b_2 = 0$ and $b_1 > 0$.

If $b_2 > 0$ and $b_1 > 0$, the flow on the w_1 axis changes its direction at the point $(0, -b_1/b_2)$, as represented in Figure 8(a). Multiplying by u^2 the sense of the flow does not change but all the points on the w_1-axis become singular points, see Figure 8(b). Going back to the (u, v)-plane there are again two sectors that are not well determined, the ones coloured on Figure 8(c) on quadrants second and fourth, and they can be either hyperbollic or elliptic. In this case, it can be proved by index theory that there are always a hyperbollic sector and an elliptic sector, but their positions change depending on the global phase portrait we are studying. The two possibilities are the ones on Figure 1($L_1 22eh$) and ($L_1 22he$).

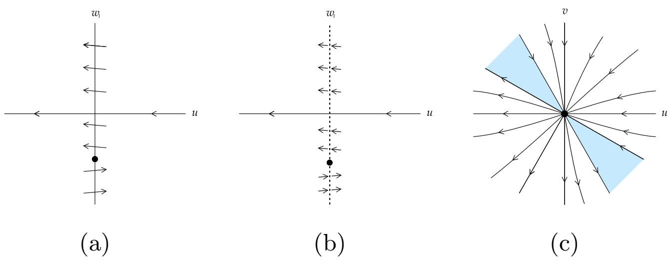


Fig. 8. Desingularization of the origin of system (13) in dicritical case with $b_2 > 0$ and $b_1 > 0$.

If $b_2 > 0$ and $b_1 < 0$, similarly to the previous case we obtain phase portraits on Figure 1($L_1 23eh$) or ($L_1 23he$).

We note that in this classification there are local phase portraits which are topologically equivalent, see for example in Figure 1 the phase potraits $L_1 2$, $L_1 8e$ and $L_1 22e$, but we maintan the distinction here as a result of the application of the blow up technique. In the final global classification thephase portraits which are topologically equivalent are unified.

7. Local study of infinite singular points in the chart U_2

According to equation (5), we get the compactification in the local chart U_2, where system (3) writes

$$\begin{aligned}\dot u &= b_2(\mu+1)u^2v + (b_0-c_0)uv^2 + b_1(\mu+1)u^2 + b_3(\mu+1)uv,\\ \dot v &= b_2\mu uv^2 - c_0v^3 + b_1\mu uv + b_3v^2.\end{aligned} \tag{27}$$

In the local chart U_2 we only need to study its singularity localized at the origin, denoted by O_2, because its other infinite singularities have already been studied in the local chart U_1. As the linear part of system (27) at the origin is identically zero we must use the blow-up technique to study it. We obtain the next result, which is proved below.

Lemma 3. *Asumming hypothesis (H_2) the origin of the chart U_2 is an infinite singular point of system* (3)*, and it has* 26 *distinct local phase portraits described in Figure 9.*

For system (27), the characteristic polynomial is $\mathcal{F} = -b_1u^2v - b_3uv^2 \not\equiv 0$, so the origin is a nondicrital singular point. We introduce the variable w_1 by means of the variable change $uw_1 = v$, and get the system

$$\begin{aligned}\dot u &= (b_0-c_0)u^3w_1^2 + b_2(\mu+1)u^3w_1 + b_3(\mu+1)u^2w_1 + b_1(\mu+1)u^2\\ \dot w_1 &= -b_0u^2w_1^3 - b_2u^2w_1^2 - b_3uw_1^2 - b_1uw_1,\end{aligned} \tag{28}$$

and eliminating a common factor u:

$$\begin{aligned}\dot u &= (b_0-c_0)u^2w_1^2 + b_2(\mu+1)u^2w_1 + b_3(\mu+1)uw_1 + b_1(\mu+1)u\\ \dot w_1 &= -b_0uw_1^3 - b_2uw_1^2 - b_3w_1^2 - b_1w_1.\end{aligned} \tag{29}$$

The singular points on the exceptional divisor $u = 0$ of system (29) are the origin S_0 and the singular point $S_1 = (0, -b_1/b_3)$ if $b_3 \neq 0$. This point S_1 is a semi-hyperbolic saddle-node whenever it exists, while the origin can be a saddle if $\mu > -1$, a stable node if $\mu < -1$ and $b_1 > 0$ and an unstable node if $\mu < -1$ and $b_1 < 0$. Then we must study five different subcases originated by these conditions.

(a) If $b_3 = 0$, $c_0 \neq 0$ and $\mu > -1$, then S_0 is a saddle. Note that once we fix $b_3 = 0$ we can assume $b_1 > 0$ by Remark 4.3. In this case the vertical blow up done does not provide a well determined phase portrait, so we must proceed similarly to the case (A) in Section 6, doing an horizontal blow up. In order to do that, we introduce the variable $w_1 = u/v$ on system (27) obtaining:

$$\begin{aligned}\dot w_1 &= b_2w_1^2v^2 + b_1w_1^2v + b_0w_1v^2,\\ \dot v &= b_2\mu w_1v^3 + b_1\mu w_1v^2 - c_0v^3,\end{aligned} \tag{30}$$

and eliminating a common factor v we get the system

$$\begin{aligned}\dot w_1 &= b_2w_1^2v + b_1w_1^2 + b_0w_1v,\\ \dot v &= b_2\mu w_1v^2 + b_1\mu w_1v - c_0v^2,\end{aligned} \tag{31}$$

for which the only singular point on the exceptional divisor is the origin, and it is linearly zero, so we have to repeat the process. The characteristic polynomial is $\mathcal{F} = b_1(\mu-1)w_1^2v - (b_0+c_0)w_1v^2$, so the origin is always nondicritical by the same reasoning as in (A). Now we introduce the new variable $w_2 = w_1/v$, obtaining the system

$$\begin{aligned}\dot w_2 =& b_2(1-\mu)w_2^2v^2 + b_1(1-\mu)w_2^2v + (b_0+c_0)w_2v,\\ \dot v =& b_2\mu w_2v^3 + b_1\mu w_2v^2 - c_0v^2,\end{aligned} \tag{32}$$

and eliminating a common factor v, we get

$$\begin{aligned}\dot w_2 =& b_2(1-\mu)w_2^2v + b_1(1-\mu)w_2^2 + (b_0+c_0)w_2,\\ \dot v =& b_2\mu w_2v^2 + b_1\mu w_2v - c_0v.\end{aligned} \tag{33}$$

The singular points of system (20) on the exceptional divisor are the origin Q_0 and the singular point $Q_1 = ((b_0+c_0)/(b_1(\mu-1)), 0)$ if $\mu \neq 1$ (note that it coincides with the origin if $b_0 + c_0 = 0$). We determine their local phase portraits obtaining the following classification. We start with the cases in which the only singular point on the exceptional divisor is the origin:

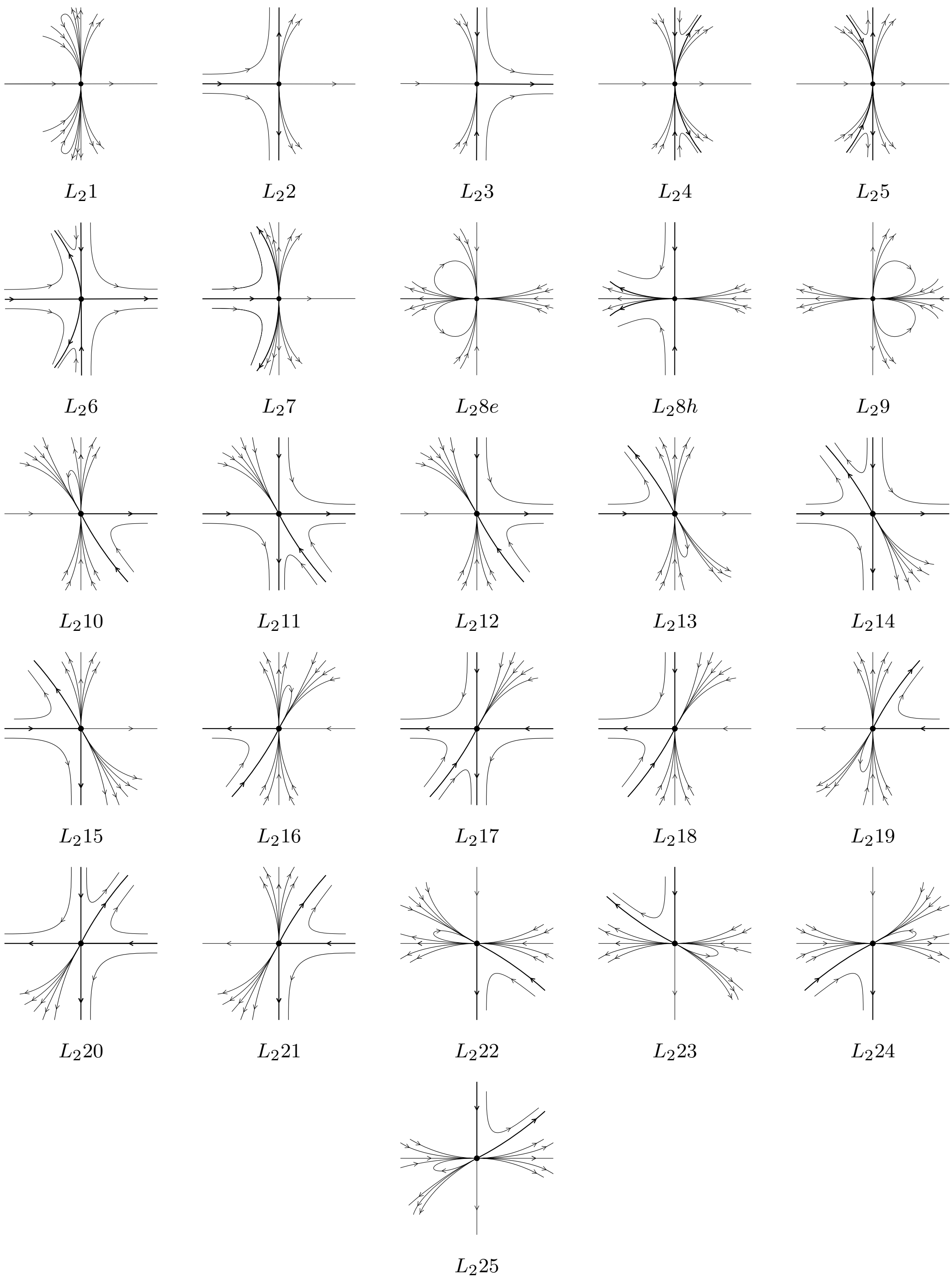


Fig. 9. Local phase portraits of the infinite singular point O_2.

(a1) If $b_0 + c_0 = 0$ and $\mu \neq 1$, then the origin is a semi-hyperbolic saddle-node. If $\mu > 1$, it is necessary to do a vertical blow up of system (31). We omit the details as the process is similar to the explained in case A1.1 for the singular point O_1. The phase portrait obtained for O_2 is given in Figure 1($L_2 1$). If $\mu \in (-1, 1)$ we obtain phase portrait $L_2 2$.

(a2) If $\mu = 1$ and $c_0(b_0 + c_0) > 0$, then the origin is a saddle. If $b_0 + c_0 < 0$ and $c_0 < 0$ we obtain again phase portrait $L_2 2$ and if $b_0 + c_0 > 0$ and $c_0 > 0$ then we obtain phase portrait $L_2 3$. In both cases it is necessary to do a vertical blow up of system (31).

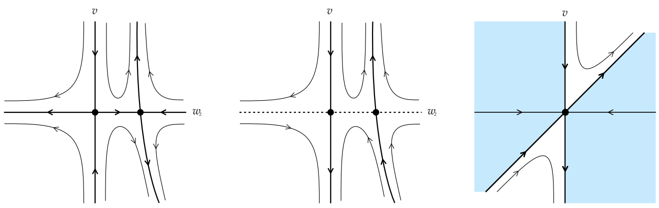
(a3) If $\mu = 1$, $c_0 < 0$, and $b_0 + c_0 > 0$, then the origin is an unstable node. We obtain phase portrait $L_2 1$. Here the same consideration we made in case (a1) about the elliptic sectors applies. The blow ups do not determine if the elliptic sector appearing are indeed elliptic or hyperbollic, but this would be concluded analyzing the gobal phase portraits on Section 8. The same consideration applies also in (a7) when $\mu > 1$.

Now we consider the cases with two singular points on the exceptional divisor:

(a4) If $c_0 > 0$, $b_0 + c_0 > 0$ and $(b_0\mu + c_0)(\mu - 1) < 0$, then the origin is a saddle and Q_1 a stable node. From these conditions we can deduce that $\mu \in (-1, 1)$ and then we obtain only one phase portait which is the same $L_2 3$.

(a5) If $c_0 < 0$, $b_0 + c_0 < 0$ and $(b_0\mu + c_0)(\mu - 1) > 0$, then Q_0 is a saddle and Q_1 is an unstable node. If $\mu > 1$ it is necessary to do a vertical blow up of system (31), and then we obtain phase portrait $L_2 1$. If $\mu \in (-1, 1)$ we obtain phase portrait $L_2 2$.

(a6) If $c_0(b_0 + c_0) > 0$ and $(b_0 + c_0)(b_0\mu + c_0)(\mu - 1) > 0$, then Q_0 and Q_1 are saddles. If $\mu \in (-1, 1)$, $b_0 + c_0 > 0$ and $c_0 > 0$ we easily obtain phase portrait $L_2 6$. If $\mu > 1$, $b_0 + c_0 > 0$ and $c_0 > 0$ we obtain phase portrait $L_2 4$ and if $\mu > 1$, $b_0 + c_0 < 0$ and $c_0 < 0$ we obtain phase portrait $L_2 5$. In this two last cases it is necessary to do a vertical blow up of system (31). We detail it for the first case, it is, with $\mu > 1$, $b_0 + c_0 > 0$ and $c_0 > 0$.
The phase portrait for (33) with the two saddles on the w_2-axis is given in Figure 10(a), and the corresponding for system (32) is 10(b). If we undo the horizontal blow up, we must swap the third and fourth quadrants and shrink the w_2-axis into the origin. As a consequence the separatrices of the saddle Q_1 go into two separatrices with slope $(b_0 + c_0)/(b_1(\mu - 1))$, one of them goes to the origin in the third quadrant and the other leaves the origin in the first quadrant. There are four sectors, coloured in Figure 10(c), in which the behaviour is not determined, so we must do a vertical blow up.



(a) (b) (c)

Fig. 10. Desingularization of the origin of system (27) case (a6) with $\mu > 1$, $c_0 > 0$ and $b_0 + c_0 > 0$.

Let introduce the variable $w_3 = v/w_1$. We get the system

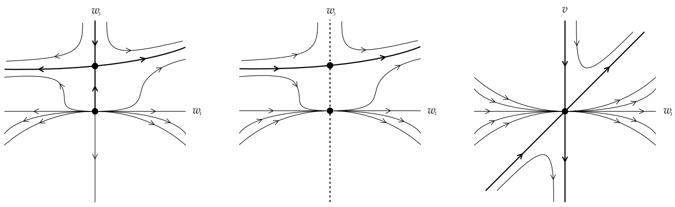
$$\begin{aligned}\dot{w}_1 &= b_2 w_1^3 w_3 + b_0 w_1^2 w_3 + b_1 w_1^2,\\ \dot{w}_3 &= b_2(\mu - 1)w_1^2 w_3^2 - (b_0 + c_0)w_1 w_3^2 + b_1(\mu - 1)w_1 w_3.\end{aligned} \tag{34}$$

Now we eliminate a common factor w_1:

$$\begin{aligned}\dot{w}_1 &= b_2 w_1^2 w_3 + b_0 w_1 w_3 + b_1 w_1,\\ \dot{w}_3 &= b_2(\mu - 1)w_1 w_3^2 - (b_0 + c_0)w_3^2 + b_1(\mu - 1)w_3.\end{aligned} \tag{35}$$

This system has two singular points on the exceptional divisor: the origin which is an unstable node, and the point $(0, b_1(\mu - 1)/(b_0 + c_0))$ which is a saddle. Studying the sense of the flow on the axes, we get that the phase portrait for system (35) is the one in Figure 11(a). Multiplying by w_1

and undoing the vertical blow up, we obtain, respectively, the phase portraits on Figure 11(c) and (d). Now the phase portrait for system (31) is well determined, and we just have to undo the first horizontal blow up, getting the final phase portrait $L_2 4$.



(a) (b) (c)

Fig. 11. Desingularization of the origin of system (27) case (a6) with $\mu > 1$, $c_0 > 0$ and $b_0 + c_0 > 0$.

(a7) If $c_0 < 0$, $b_0 + c_0 > 0$, and $(b_0\mu + c_0)(\mu - 1) > 0$, then Q_0 is an unstable node and $Q1$ is a saddle. If $\mu > 1$ we obtain phase portrait $L_2 1$ and if $\mu \in (-1, 1)$ we obtain phase portrait $L_2 7$.

(a8) If $c_0 < 0$, $b_0 + c_0 > 0$, and $(b_0\mu + c_0)(\mu - 1) < 0$, then Q_0 is an unstable node and $Q1$ is a stable node, and we obtain phase portrait $L_2 1$.

(b) If $b_3 = 0$, $c_0 \neq 0$ and $\mu < -1$ and $b_1 > 0$ then S_0 is a stable node. If $c_0 > 0$ we obtain phase portraits $L_2 8e$ and $L_2 8h$, which differ on the sectors that appear beside the v-axis on the second and third quadrants. On $L_2 8e$ we have elliptic sectors and on $L_2 8h$ we have hyperbollic sectors. We apply index theory on the global phase portraits to know which of them appears, see Section 8. If $c_0 < 0$ the result is similar and and undetermined sector also appears when undoing the blow up. Nevertheless, in this case we have proved that in all the global phase portraits the undetermined sectors are elliptic, so then we have always phase portrait $L_2 9$.

(c) If $b_3 > 0$ and $\mu > -1$, then S_0 is a saddle and S_1 is a saddle-node. We distinguish four cases in function of the sign of b_1 which determines the position of S_1 and the sign of $b_0\mu + c_0$ which determines the position of the sectors of that singular point. With this signs fixed, we can determine and represent the phase portrait of systems (33) and (32). Then, in each of the four cases, when going back to the (u, v)-plane we must distinguish three cases depending on wheter $\mu = 0$, $\mu \in (-1, 0)$ or $\mu > 0$. In the cases with $\mu > 0$ there appears an undefined sector which could be hyperbollic or elliptic. By doing an horizontal blow up in that cases, it can be determined that those sectors are always elliptic, but we omit the details as the process is the same that has been exposed in other cases. Another posibility is to prove directly on the global phase portrait, by applying index theory, that those sectors can only be elliptic. Now, to avoid repetitions, we simply include the results obtained in each case in the following table.

c1	$b_1 > 0$, $b_0\mu + c_1 > 0$	$\mu > 0$	$L_2 10$
		$\mu \in (-1, 0)$	$L_2 11$
		$\mu = 0$	$L_2 12$
c2	$b_1 > 0$, $b_0\mu + c_1 < 0$	$\mu > 0$	$L_2 13$
		$\mu \in (-1, 0)$	$L_2 14$
		$\mu = 0$	$L_2 15$
c3	$b_1 < 0$, $b_0\mu + c_1 > 0$	$\mu > 0$	$L_2 16$
		$\mu \in (-1, 0)$	$L_2 17$
		$\mu = 0$	$L_2 18$
c4	$b_1 > 0$, $b_0\mu + c_1 < 0$	$\mu > 0$	$L_2 19$
		$\mu \in (-1, 0)$	$L_2 20$
		$\mu = 0$	$L_2 21$

(d) If $b_3 > 0$, $\mu < -1$ and $b_1 > 0$, then S_0 is a stable node and S_1 is a saddle-node. The sign of $b_0\mu + c_0$ determines the position of the sectors of the saddle-node in system (33) so we must distinguish two cases, and undo the blow up in each of them. If $b_0\mu + c_0 > 0$ we obtain phase portrait $L_2 23$ and if $b_0\mu + c_0 > 0$ we obtain $L_2 24$.

(e) If $b_3 > 0$, $\mu < -1$ and $b_1 < 0$, then S_0 is an unstable node and S_1 is a saddle-node. As in the previous case we distinguish the case with $b_0\mu + c_0 > 0$ in which we obtain phase portrait $L_2 25$ and the case with $b_0\mu + c_0 < 0$ in which we obtain phase portrait $L_2 26$.

8. Global phase portraits

In order to prove the global result stated in Theorem 1, we bring together the local information obtained in Sections 5, 6 and 7. We start our classification from the cases in Tables 2 to 5. In Table 6 we give, for each case of the Tables 2 to 5, the local phase portrait of the infinite singular points O_1 and O_2. In some of them the conditions determine only one local phase portrait in each one of the infinite singular points, but in most cases, we shall distinguish several possibilities depending on the parameters. Also in Table 6 we give the global phase portrait on the Poincaré disc obtained. All the global phase portraits are given in Figure 16, where we also indicate the number of separatrices (S) and canonical regions (R) that each of them has. We detail the reasonings in some cases, although they will not be showed in all of them to avoid repetitions. We recall that we are denotig the origins of charts U_1 and U_2 as O_1 and O_2 respectively, and in this section, to simplify the explanations, we will denote by Q_1 the origin of the chart V_1 and by Q_2 the origin of V_2.

8.1. *Cases with a totally-determined local phase portrait at infinity*

Case 1.3. Let consider the conditions $\mu < -1$ and $b_1 > 0$, so the infinite singular point O_1 has the local phase portrait $L_1 12$ given in Figure 1, and O_2 the phase portrait $L_2 23$ given in 9. We must combine the local information to get the global phase portait. The system has an unstable node P_1 wich is on the positive z-axis and a stable node P_2 which is on the negative x-axis. The origin is a saddle and it has the four separatrices over the axes (as the axes are invariant). Also, by the local configurations of O_1 and O_2 we know that the part of the z-axis which connects P_1 with O_1 is a separatrix and the part of the x-axis which connects P_2 with Q_1 is a separatrix. Appart from those, the system has a separatrix leaving the singular point O_2 in the second quadrant and a separatrix which arrives at Q_1 on the third quadrant, see Figure 12. There is only one possible connection for these separatrices: the separatrix which leaves O_2 goes to P_2 and the separatrix which arrives at Q_1 starts at Q_2. Then we obtain phase portrait G9 in Figure 16, which has 19 separatrices and 6 canonical regions.

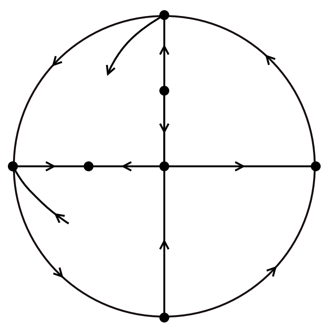


Fig. 12. Separatrices provided by local information in case 1.3 with $\mu < -1$ and $b_1 > 0$.

With the same reasonings we can prove the other subcases in 1.3 and also the following: the subcases in 1.5 and 1.6 with $b_1 < 0$; all subcases in 1.8 and in 1.10; subcases in 2.2, 2.4 and 2.5 with $\mu < -1$;

subcases in 3.2 with $\mu < -1$ or $\mu \in (-1, 0)$; subcases in 3.3, 3.4 and 3.5 with $\mu \in (-1, 0)$; subcases in 4.1 with $\mu < -1$ and subcases in 4.2 with $\mu \in (0, 1)$, $\mu = 1$ or $\mu > 1$.

8.2. *Cases with undetermined sectors at infinity*

Now we will deal with some cases in which the local phase portrait of O_1, O_2 or both was not totally determined in Sections 6 and 7, in the sense that they present certain sectors which can be either hyperbollic or elliptic.

Case 1.1 We consider $b_1 < 0$. The infinite singular point O_1 has the local phase portrait $L_1 13$ given in Figure 1, and O_2 the phase portrait $L_2 19$ given in Figure 9. We must prove here that the elliptic sectors on both phase portraits are indeed elliptic.

The system has a saddle P_1 on the negative z-axis and a stable node P_2 on the negative x-axis. The origin is a saddle and it has the four separatrices over the axes. The repelling separatrices of P_1 are also over the z-axis, but the attracting ones should arrive to P_1, one in the third quadrant and other in the fourth quadrant. Also the system has a separatrix leaving the singular point O_2 in the first quadrant and a separatrix leaving Q_1 on the second quadrant. Then, attending to the local phase portrait of each singular point, the only possible connection is the following: there is a separatrix which goes from Q_1 to the point P_2 and a separatrix from O_2 to O_1, the attracting separatrix of P_1 on the third quadrant starts in Q_1 and the one in the fourth quadrant starts in Q_2. As a result, there is a canonical region delimited by Q_2, the part of the z-axis which connects this point with P_1, P_1, and its separatrix on the fourth quadrant, see Figure 13. If the sector on the local phase portrait of Q_2 were hyperbollic, then there would be any possible alpha or omega-limit on the boundary for the orbits in that region, but as it is not possible to have periodic orbits, this situation is no feasible, and this sector should be elliptic. The same happens with the elliptic sector at O_1. Anyway, this can be proved also anallitical by applying index theory.

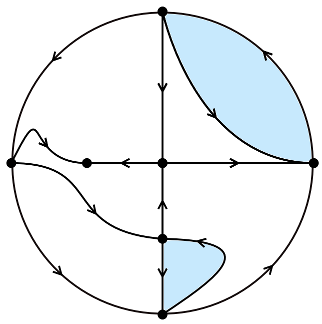


Fig. 13. Separatrix configuration in case 1.1 with $b_1 < 0$ and regions in which local phase portrait was not determined.

By Theorem 3 the sum of the indices of all the singular points on the Poincaré sphere has to be 2. To compute this sum we must consider that the finite singular points on the Poincaré disc appear twice on the sphere (on the northern hemisphere and on the southern hemisphere). Thus if we denote by ind_F the sum of the indices of the finite singular points, and by ind_I the sum of the indices of the infinite singular points, the equality $2ind_F + ind_I = 2$ must be satisfied. In this particular case the finite singular points are two saddles whose index is -1, and a stable node which index is 1, so $ind_F = -1$. We deduce that ind_I must be 4. The infinite singular points are O_1 and Q_1 which have the same index, and O_2 and Q_2 which have also the same index.

The singular point O_1 has a hyperbolic sector so, if the non-determined sector is elliptic, from the Poincaré formula for the index given in Subsection 2.4, O_1 has index 1, and if the non-determined sector were hyperbollic, O_1 would have index 0. The same is valid for the singular point O_2. Then, if at any of these points O_1 or O_2 the sector were hyperbolic, the sum of the index of that point and its symmetric would be zero, and so the sum of the other point and its symmetric should be 4, which is not possible.

In other words,

$$ind_I = 2\left(\frac{e-h}{2}+1\right)+2\left(\frac{\tilde{e}-\tilde{h}}{2}-1\right),$$

where e and h are the number of elliptic and hyperbolic sectors of O_1 and $\tilde{e}$ and $\tilde{h}$ are the number of elliptic and hyperbolic sectors of O_2. As we know that $ind_I = 4$, then we obtain

$$(e+\tilde{e})-(h+\tilde{h})=0,$$

it is, considering both phase portraits of O_1 and O_2 together, there must be the same number of elliptic and hyperbolic sectors. As we have an hyperbolic sector in each of them, the two non-determined sectors must be elliptic.

Case 3.3 We consider $\mu < -1$. The infinite singular point O_1 has the local phase portrait $L_1 11$ given in Figure 1, and O_2 the phase portrait $L_2 8e$, but we must prove that we actually have $L_2 8e$ instead of $L_2 8h$.

The origin is an unstable node and the system has a saddle P_2 on the negative x-axis. The attracting separatrices of P_2 are over the x-axis, but the repelling separatrices should leave P_2, one in the second quadrant and other in the third quadrant. There is also a separatrix leaving the singular point O_1 in the fourth quadrant. There is only one possible way to connect this separatices: the system has a separatrix which goes from P_2 to O_2, another which goes from P_2 to Q_2 and the third one which goes from O_1 to Q_2. Then we have two canonical regions, the ones coloured in Figure 14, in which the only possibility is to have elliptic sectors whose orbits have as alpha and omega-limits the singular point O_2 and Q_2 respectively. We verify this with Theorem 3. As we have a finite saddle with index -1 and a finite node with index 1, then $ind_F = 0$. From the equality $2ind_F + ind_I = 2$ we know that ind_I must be 2.

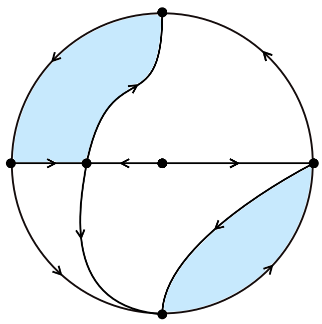


Fig. 14. Separatrix configuration in case 3.3 with $\mu < -1$ and regions in which local phase portrait was not determined.

The local phase portrait of O_1 is well determined and it has four hyperbollic sectors and one parabolic sector, so

$$ind_{O_1} = \frac{e-h}{2}+1 = \frac{0-4}{2}+1 = -1,$$

and also we know that $ind_{O_1} = ind_{Q_1}$ and $ind_{O_2} = ind_{Q_2}$. Then

$$ind_I = 2ind_{O_1} + 2ind_{O_2} \Rightarrow ind_{O_2} = 2,$$

and by the Poincaré formula, if e and h are the number of elliptic and hyperbolic sector at O_2, then

$$\frac{e-h}{2}+1 = 2 \Rightarrow e-h = 2,$$

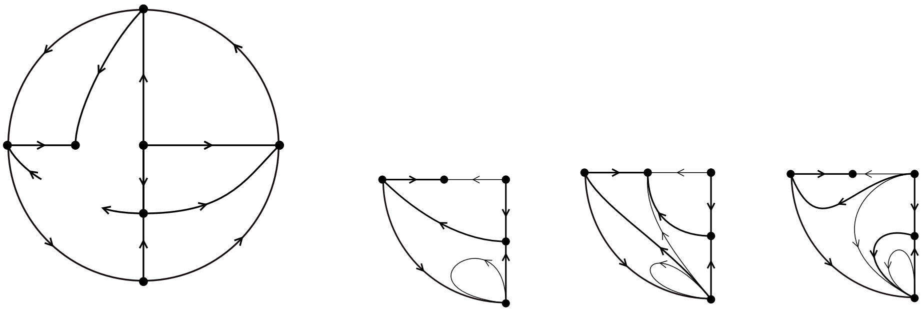
and as we have only two sectors to determine wheter they are elliptic or hyperbolic, the only possibility is that both are elliptic.

With similar reasonings we can conclude and prove the results in all the remaining subcases, except the ones included in the following subsection.

8.3. *Cases with three possible global phase portraits*

Here we focus on the cases in which the separatrices can be connected in three different manners. As can be seen in Table 6 this happens in the subcases of 1.1, 1.2, 1.5, 1.6 and 1.7 in which $b_1 > 0$, in both subcases of 1.4, in the subcase of 1.9 with $b_1 < 0$ and in the subcase of 3.3 with $\mu = 0$, $b_3, b_1 > 0$. We give a detailed explanation in the following case:

Case 1.5. We consider the condition $\mu < -1$ and $b_1 > 0$. The origin is an unstable node, there is a stable node P_2 on the negative z-axis and saddle P_1 on the negative x-axis. The two attracting separatrices of P_1 are over the z-axis, and the repelling ones leave the origin in the third and fourth quadrants respectively. The infinite singular point O_1 has the local phase portrait $L_1 12$ of Figure 1, and O_2 the phase portrait $L_2 23$ of Figure 9. According to this local phase portraits, the positive z-axis, the positive x-axis, and the part of the negative x-axis between Q_1 and P_2 are separatrices. Moreover, there is another separatrix which leaves O_2 in the second quadrant and one that goes to Q_1 in the third quadrant. There is only one possible connection for the separatrices on the second and fourth quadrant, as it is represented in Figure 15(a).



(a) (b) (c) (d)

Fig. 15. Separatrices and possible configurations on the third quadrant on case 1.5 with $\mu < -1$ and $b_1 > 0$.

We focus now on the third quadrant. We know that there is a separatrix leaving the point P_1 and another one going into Q_1. If we analyze the possible ω-limits of the separatrix leaving P_1 there are three options, the singular points Q_1, Q_2 and P_2. If the ω-limit is Q_1, then both separatrices should be the same, as the point Q_1 does not have parabolic sectors in the third quadrant. In that case, the configuration in the third quadrant is the one given in Figure 15(b), and this leads to the global phase portrait G19.

If the ω-limit is P_2, then only possibility for the other separatrix is that its α-limit is Q_2. The configuration in the third quadrant is given in Figure 15(c), and this leads to the global phase portrait G20.

At last, if the ω-limit is Q_2, then the α-limit of the other separatrix is the origin. The configuration in the third quadrant is given in Figure 15(d), and this leads to the global phase portrait G21.

We have proved numerically, by using the program P4 [Dumortier *et al.*, 2006, Chapter 9], that the three global phase portraits are realizable. In Table 7 we give the values of the parameters for which we have found each phase portrait, not only on this subcase but also in all the subcases in which three possibilities appear. In all of them we have check that the three possibilities are realizable.

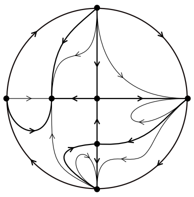
Table 6. Classification of global phase portrais of system (3).

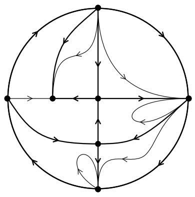
Case	**Conditions**	O_1	O_2	**Global**
1.1	$b_1 > 0$	$L_1 9$	$L_2 13$	G1, G2 or G3
	$b_1 < 0$	$L_1 13$	$L_2 19$	G4
1.2	$b_1 > 0$	$L_1 10$	$L_2 10$	G5, G6 or G7
	$b_1 < 0$	$L_1 14$	$L_2 16$	G8
1.3	$\mu < -1$, $b_1 > 0$	$L_1 12$	$L_2 23$	G9
	$\mu < -1$, $b_1 < 0$	$L_1 16$	$L_2 25$	G10
	$\mu \in (-1, 0)$, $b_1 > 0$	$L_1 18$	$L_2 14$	G11
	$\mu \in (-1, 0)$, $b_1 < 0$	$L_1 20$	$L_2 20$	G12
1.4	$b_1 > 0$	$L_1 10$	$L_2 10$	G13, G14 or G15
	$b_1 < 0$	$L_1 14$	$L_2 16$	G16, G17 or G18
1.5	$\mu < -1$, $b_1 > 0$	$L_1 12$	$L_2 23$	G19, G20 or G21
	$\mu < -1$, $b_1 < 0$	$L_1 16$	$L_2 25$	G22
	$\mu \in (-1, 0)$, $b_1 > 0$	$L_1 18$	$L_2 14$	G23, G24 or G25
	$\mu \in (-1, 0)$, $b_1 < 0$	$L_1 20$	$L_2 20$	G26
1.6	$\mu < -1$, $b_1 > 0$	$L_1 11$	$L_2 22$	G27, G28 or G29
	$\mu < -1$, $b_1 < 0$	$L_1 15$	$L_2 24$	G30
	$\mu \in (-1, 0)$, $b_1 > 0$	$L_1 17$	$L_2 11$	G31, G32 or G33
	$\mu \in (-1, 0)$, $b_1 < 0$	$L_1 19$	$L_2 17$	G34
1.7	$b_1 > 0$	$L_1 10$	$L_2 10$	G35, G36 or G37
	$b_1 < 0$	$L_1 14$	$L_2 16$	G38
1.8	$\mu < -1$, $b_1 > 0$	$L_1 12$	$L_2 23$	G39
	$\mu < -1$, $b_1 < 0$	$L_1 16$	$L_2 25$	G40
	$\mu \in (-1, 0)$, $b_1 > 0$	$L_1 18$	$L_2 14$	G41
	$\mu \in (-1, 0)$, $b_1 < 0$	$L_1 20$	$L_2 20$	G42
1.9	$b_1 > 0$	$L_1 10$	$L_2 10$	G43
	$b_1 < 0$	$L_1 14$	$L_2 16$	G44, G45 or G46
1.10	$\mu < -1$, $b_1 > 0$	$L_1 11$	$L_2 22$	G47
	$\mu < -1$, $b_1 < 0$	$L_1 15$	$L_2 24$	G48
	$\mu \in (-1, 0)$, $b_1 > 0$	$L_1 17$	$L_2 11$	G49
	$\mu \in (-1, 0)$, $b_1 < 0$	$L_1 19$	$L_2 17$	G50
2.1		$L_1 2$	$L_2 13$	G51
2.2	$\mu \in (-1, 0)$	$L_1 8e$	$L_2 14$	G52
	$b_0 + c_0 = 0$, $\mu > 1$	$L_1 1$	$L_2 10$	G53
	$b_0 + c_0 > 0$, $\mu \geq 1$.			
	$b_0 + c_0 \geq 0$, $\mu < -1$.	$L_1 3$	$L_2 23$	G54
	$b_0 + c_0 > 0$, $\mu \in (0, 1)$	$L_1 4$	$L_2 10$	G55
	$b_0 + c_0 < 0$, $\mu < -1$	$L_1 6$	$L_2 23$	G56
	$b_0 + c_0 < 0$, $\mu > 1$	$L_1 7$	$L_2 10$	G57
2.3	$\mu \in (-1, 0)$	$L_1 8e$	$L_2 14$	G58
	$\mu \in (0, 1)$	$L_1 4$	$L_2 10$	G59
	$\mu < -1$	$L_1 3$	$L_2 23$	G60
	$\mu \geq 1$	$L_1 1$	$L_2 10$	G61
2.4	$\mu \in (-1, 0)$	$L_1 8h$	$L_2 11$	G62
	$\mu < -1$	$L_1 5$	$L_2 22$	G63
2.5	$\mu \in (-1, 0)$	$L_1 8e$	$L_2 14$	G64
	$\mu \in (0, 1)$	$L_1 4$	$L_2 10$	G65
	$\mu < -1$	$L_1 3$	$L_2 23$	G66
	$\mu \geq 1$	$L_1 1$	$L_2 10$	G67

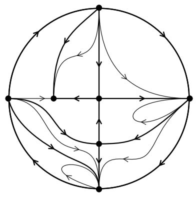
Case	**Conditions**	$\boldsymbol{O_1}$	$\boldsymbol{O_2}$	**Global**
3.1		$L_1 10$	$L_2 1$	G68
3.2	$b_0 + c_0 = 0,\ \mu > 0$	$L_1 9$	$L_2 2$	G69
	$b_0 + c_0 < 0,\ \mu \in (0, 1]$			
	$b_0 + c_0 < 0,\ \mu > 1$	$L_1 9$	$L_2 5$	G70
	$b_0 + c_0 < 0,\ \mu \in (0, 1)$	$L_1 9$	$L_2 7$	G71
	$\mu < -1$	$L_1 12$	$L_2 9$	G72
	$b_0 + c_0 \leq 0,\ \mu \in (-1, 0)$	$L_1 18$	$L_2 2$	G73
	$b_0 + c_0 \leq 0,\ \mu = 0,\ b_3 = 0$	$L_1 22eh$		
	$b_0 + c_0 > 0,\ \mu \in (-1, 0)$	$L_1 18$	$L_2 7$	G74
	$b_0 + c_0 > 0,\ \mu = 0,\ b_3 = 0$	$L_1 22eh$		
	$\mu = 0,\ b_3 > 0,\ b_1 > 0$	$L_1 22eh$	$L_2 15$	G75
	$\mu = 0,\ b_3 > 0,\ b_1 < 0$	$L_1 23eh$	$L_2 21$	G76
3.3	$\mu \in (0, 1]$	$L_1 10$	$L_2 3$	G77
	$\mu > 1$	$L_1 10$	$L_2 4$	G78
	$\mu < -1$	$L_1 11$	$L_2 8e$	G79
	$\mu \in (-1, 0)$	$L_1 17$	$L_2 3$	G80
	$\mu = 0,\ b_3 = 0$	$L_1 22he$		
	$\mu = 0,\ b_3 > 0,\ b_1 > 0$	$L_1 22he$	$L_2 12$	G80, G81 or G82
	$\mu = 0,\ b_3 > 0,\ b_1 < 0$	$L_1 23he$	$L_2 18$	G83
3.4	$\mu < -1$	$L_1 12$	$L_2 8h$	G84
	$\mu \in (-1, 0)$	$L_1 18$	$L_2 6$	G85
3.5	$\mu \in (0, 1]$	$L_1 10$	$L_2 3$	G86
	$\mu > 1$	$L_1 10$	$L_2 4$	G87
	$\mu < -1$	$L_1 11$	$L_2 8e$	G88
	$\mu \in (-1, 0)$	$L_1 17$	$L_2 3$	G89
	$\mu = 0,\ b_3 = 0$	$L_1 22he$		
	$\mu = 0,\ b_3 > 0,\ b_1 > 0$	$L_1 22he$	$L_2 12$	G90
	$\mu = 0,\ b_3 > 0,\ b_1 < 0$	$L_1 23he$	$L_2 18$	G91
4.1	$b_0 + c_0 \leq 0,\ \mu = 0,\ b_3 = 0$	$L_1 21e$	$L_2 2$	G92
	$b_0 + c_0 \leq 0,\ \mu \in (-1, 0)$	$L_1 8e$		
	$b_0 + c_0 = 0,\ \mu \in (0, 1)$	$L_1 2$		
	$b_0 + c_0 < 0,\ \mu \in (0, 1]$	$L_1 2$		
	$b_0 + c_0 > 0,\ \mu = 0,\ b_3 = 0$	$L_1 21e$	$L_2 7$	G93
	$b_0 + c_0 > 0,\ \mu \in (-1, 0)$	$L_1 8e$		
	$b_0 + c_0 > 0,\ \mu \in (0, 1),\ b_0\mu + c_0 < 0$	$L_1 2$		
	$b_0 + c_0 = 0,\ \mu > 1$	$L_1 1$	$L_2 1$	G94
	$b_0 + c_0 > 0,\ \mu \geq 1$			
	$b_0 + c_0 \geq 0,\ \mu < -1$	$L_1 3$	$L_2 9$	G95
	$b_0 + c_0 > 0,\ \mu \in (0, 1),\ b_0\mu + c_0 > 0$	$L_1 4$	$L_2 1$	G96
	$b_0 + c_0 < 0,\ \mu < -1$	$L_1 6$	$L_2 9$	G97
	$b_0 + c_0 < 0,\ \mu > 1,\ b_0\mu + c_0 > 0$	$L_1 7$	$L_2 1$	G98
	$b_0 + c_0 < 0,\ \mu > 1,\ b_0\mu + c_0 < 0$	$L_1 2$	$L_2 5$	G99
	$\mu = 0,\ b_3 > 0$	$L_1 21e$	$L_2 15$	G100
4.2	$\mu = 0,\ b_3 > 0$	$L_1 21h$	$L_2 12$	G101
	$\mu = 0,\ b_3 = 0$	$L_1 21h$	$L_2 3$	G102
	$\mu \in (-1, 0)$	$L_1 8h$		
	$\mu \in (0, 1)$	$L_1 4$		
	$\mu < -1,\ b_0\mu + c_0 > 0$	$L_1 5$	$L_2 8e$	G103
	$\mu < -1,\ b_0\mu + c_0 < 0$	$L_1 3$	$L_2 8h$	G104
	$\mu = 1$	$L_1 1$	$L_2 3$	G105
	$\mu > 1$	$L_1 1$	$L_2 4$	G106

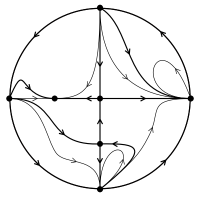
Table 7. Values of the parameters for which each global phase portrait is obtained in cases with three possible configurations.

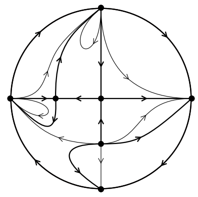
Subcase	Phase portrait	Obtained for the values
1.1 $b_1 > 0$	G1	$b_0 = b_1 = b_2 = b_3 = 1,\ c_0 = -3/2,\ \mu = 1$
	G2	$b_0 = 1/2,\ b_1 = b_2 = b_3 = 1,\ c_0 = -1,\ \mu = 1$
	G3	$b_0 = 1/2,\ b_1 = b_2 = b_3 = 1,\ c_0 = -1,\ \mu = 5/4$
1.2 $b_1 > 0$	G5	$b_1 = b_2 = b_3 = 1,\ c_0 = -1,\ \mu = 5/8$
	G6	$b_0 = b_1 = b_2 = b_3 = 1,\ c_0 = -1/2,\ \mu = 1$
	G7	$b_0 = 1/2,\ b_1 = b_2 = b_3 = 1,\ c_0 = -1,\ \mu = 3$
1.4 $b_1 > 0$	G13	$b_0 = 2,\ b_1 = b_2 = b_3 = c_0 = \mu = 1$
	G14	$b_0 = b_1 = b_2 = b_3 = c_0 = \mu = 1$
	G15	$b_0 = 1/2,\ b_1 = b_2 = b_3 = c_0 = \mu = 1$
1.4 $b_1 < 0$	G16	$b_0 = b_2 = b_3 = c_0 = \mu = 1,\ b_1 = -2$
	G17	$b_0 = 1,\ b_1 = -1,\ b_2 = b_3 = c_0 = \mu = 1$
	G18	$b_0 = b_2 = b_3 = c_0 = \mu = 1,\ b_1 = -1/2$
1.5 $\mu < -1,\ b_1 > 0$	G19	$b_0 = b_2 = b_3 = c_0 = 1,\ b_1 = 2,\ \mu = -2$
	G20	$b_0 = b_1 = b_2 = b_3 = 1,\ c_0 = 3/2,\ \mu = -2$
	G21	$b_0 = b_1 = b_3 = c_0 = 1,\ b_2 = 1/4,\ \mu = -2$
1.5 $\mu \in (-1, 0),\ b_1 > 0$	G23	$b_0 =,\ b_1 = 1/2,\ b_2 = b_3 = c_0 = 1,\ \mu = -1/2$
	G24	$b_0 = 3,\ b_1 = b_3 = c_0 = 1,\ b_2 = 3/2,\ \mu = -1/2$
	G25	$b_0 = b_1 = b_2 = b_3 = 1,\ c_0 = 1/4,\ \mu = -1/2$
1.6 $\mu < -1,\ b_1 > 0$	G27	$b_0 = 1/4,\ b_1 = b_3 = c_0 = 1,\ b_2 = 5/16,\ \mu = -2$
	G28	$b_0 = b_1 = b_2 = b_3 = 1,\ c_0 = 3,\ \mu = -2$
	G29	$b_0 = 1/4,\ b_1 = b_3 = c_0 = 1,\ b_2 = 1/8,\ \mu = -2$
1.6 $\mu \in (-1, 0),\ b_1 > 0$	G31	$b_0 = 3/2,\ b_1 = b_3 = c_0 = 1,\ b_2 = 7/4,\ \mu = -1/2$
	G32	$b_0 = b_1 = b_2 = b_3 = 1,\ c_0 = 2,\ \mu = -1/2$
	G33	$b_0 = 3/2,\ b_1 = b_2 = b_3 = c_0 = 1,\ \mu = -1/2$
1.7 $b_1 > 0$	G35	$b_0 = b_1 = b_2 = \mu = 7/8,\ b_3 = 1,\ c_0 = 0$
	G36	$b_0 = b_1 = b_2 = b_3 = \mu = 1,\ c_0 = 0$
	G37	$b_0 = b_1 = b_2 = \mu = 7/8,\ b_3 = 7/16,\ c_0 = 0$
1.9 $b_1 < 0$	G44	$b_0 = 0,\ b_1 = -1/2,\ b_2 = b_3 = c_0 = \mu = 1$
	G45	$b_0 = 0,\ b_1 = -1,\ b_2 = b_3 = c_0 = \mu = 1$
	G46	$b_0 = 0,\ b_1 = -2,\ b_2 = b_3 = c_0 = \mu = 1$
3.3 $\mu = 0,\ b_3 > 0,\ b_1 > 0$	G80	$b_0 = 2,\ b_1 = b_2 = b_3 = c_0 = 1,\ \mu = 0$
	G81	$b_0 = 1/2,\ b_1 = b_2 = b_3 = c_0 = 1,\ \mu = 0$
	G82	$b_0 = b_1 = b_2 = b_3 = c_0 = 1,\ \mu = 0$

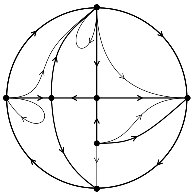

(G1) [R=7, S=20]

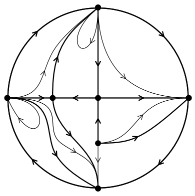

(G2) [R=6, S=19]

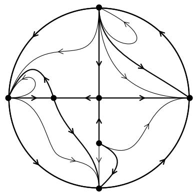

(G3) [R=7, S=20]

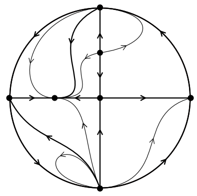

(G4) [R=7, S=20]

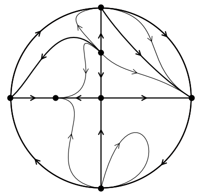

(G5) [R=7, S=20]

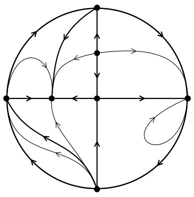

(G6) [R=6, S=19]

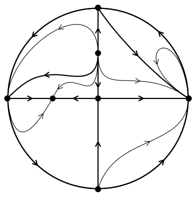

(G7) [R=7, S=20]

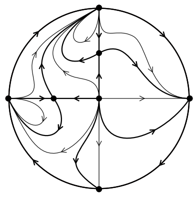

(G8) [R=7, S=20]

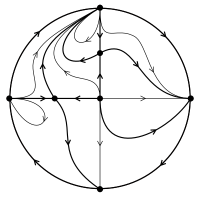

(G9) [R=6, S=19]

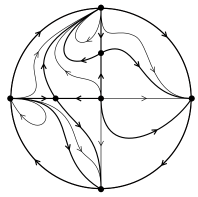

(G10) [R=6, S=19]

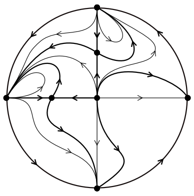

(G11) [R=6, S=19]

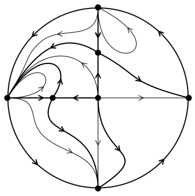

(G12) [R=6, S=19]

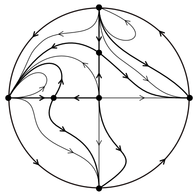

(G13) [R=8, S=21]

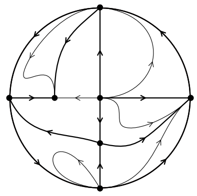

(G14) [R=7, S=20]

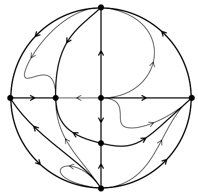

(G15) [R=8, S=21]

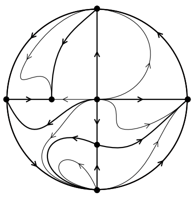

(G16) [R=8, S=21]

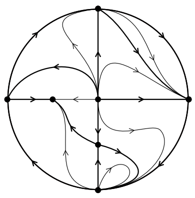

(G17) [R=7, S=20]

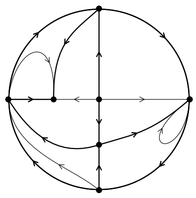

(G18) [R=8, S=21]

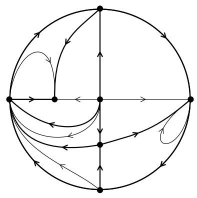

(G19) [R=6, S=19]

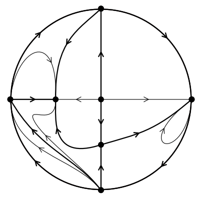

(G20) [R=7, S=20]

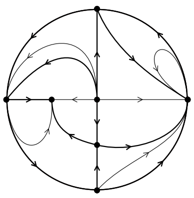

(G21) [R=7, S=21]

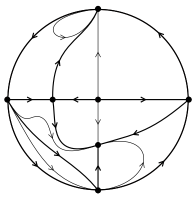

(G22) [R=7, S=20]

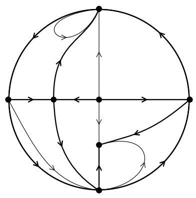

(G23) [R=5, S=18]

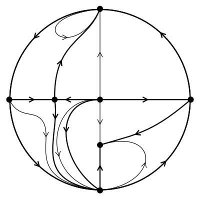

(G24) [R=6, S=19]

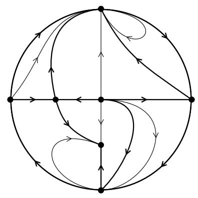

(G25) [R=6, S=19]

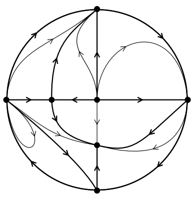

(G26) [R=6, S=19]

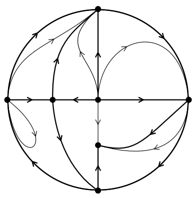

(G27) [R=6, S=19]

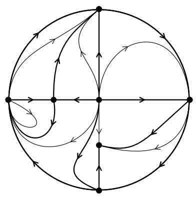

(G28) [R=5, S=18]

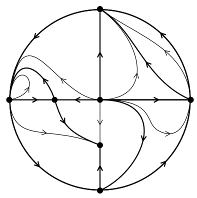

(G29) [R=6, S=19]

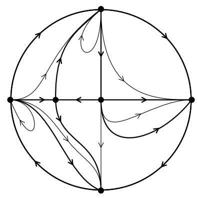

(G30) [R=6, S=19]

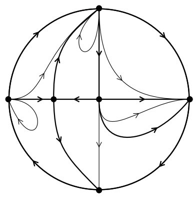

(G31) [R=7, S=20]

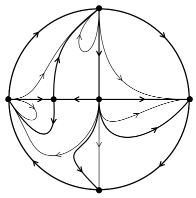

(G32) [R=6, S=19]

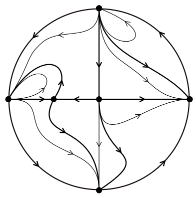

(G33) [R=7, S=20]

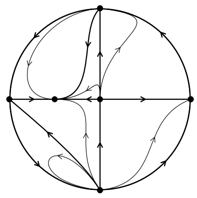

(G34) [R=6, S=19]

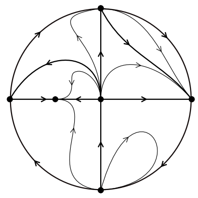

(G35) [R=7, S=18]

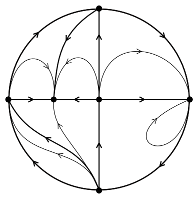

(G36) [R=6, S=17]

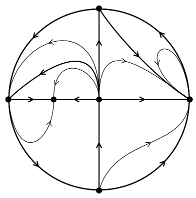

(G37) [R=7, S=18]

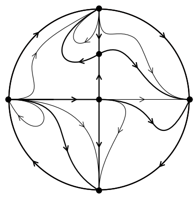

(G38) [R=7, S=18]

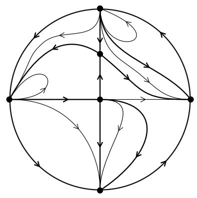

(G39) [R=6, S=17]

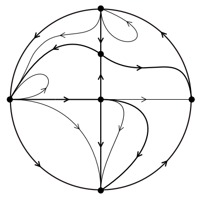

(G40) [R=6, S=17]

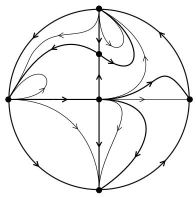

(G41) [R=6, S=17]

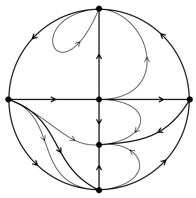

(G42) [R=6, S=17]

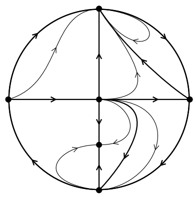

(G43) [R=7, S=18]

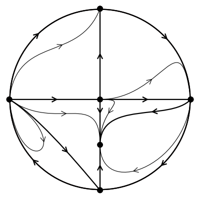

(G44) [R=7, S=18]

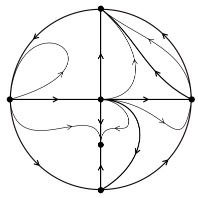

(G45) [R=6, S=17]

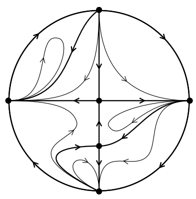

(G46) [R=7, S=18]

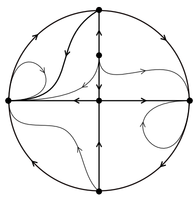

(G47) [R=6, S=17]

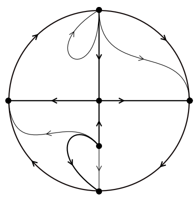

(G48) [R=6, S=17]

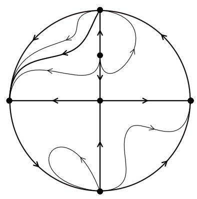

(G49) [R=6, S=17]

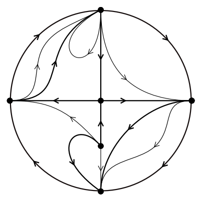

(G50) [R=6, S=17]

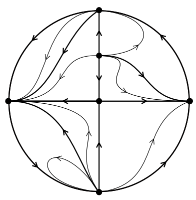

(G51) [R=7, S=18]

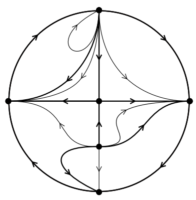

(G52) [R=5, S=16]

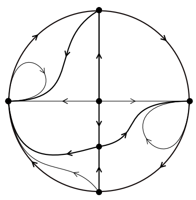

(G53) [R=4, S=15]

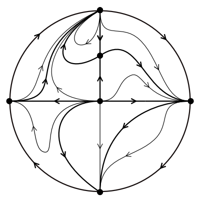

(G54)[R=5, S=16]

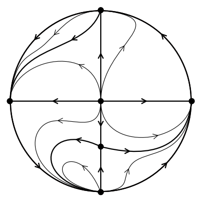

(G55) [R=6, S=17]

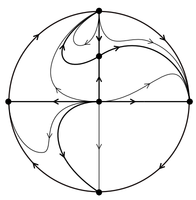

(G56) [R=7, S=18]

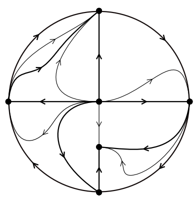

(G57) [R=6, S=17]

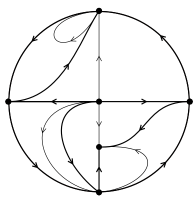

(G58) [R=5, S=16]

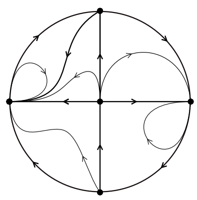

(G59) [R=8, S=19]

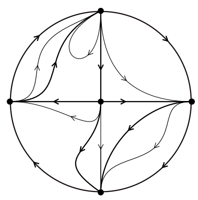

(G60) [R=7, S=18]

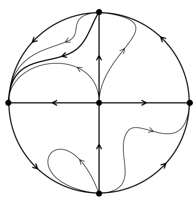

(G61) [R=6, S=17]

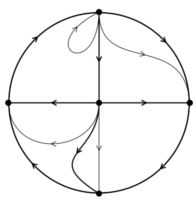

(G62) [R=6, S=17]

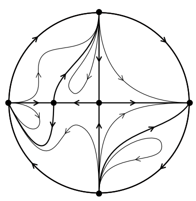

(G63) [R=5, S=16]

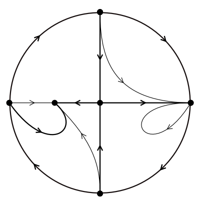

(G64) [R=5, S=14]

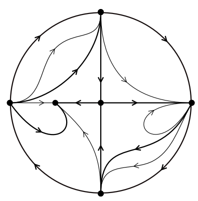

(G65) [R=6, S=15]

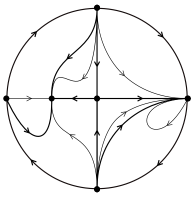

(G66) [R=5, S=14]

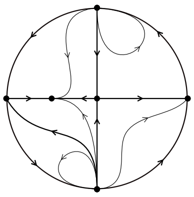

(G67) [R=4, S=13]

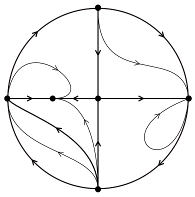

(G68) [R=7, S=18]

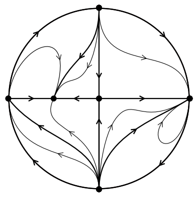

(G69) [R=4, S=15]

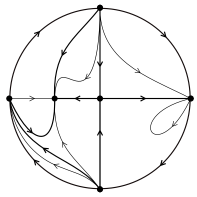

(G70) [R=6, S=17]

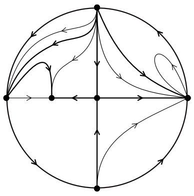

(G71) [R=6, S=17]

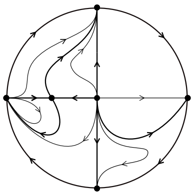

(G72) [R=5, S=16]

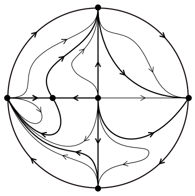

(G73) [R=5, S=16]

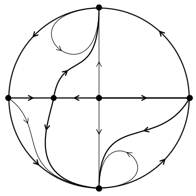

(G74) [R=7, S=18]

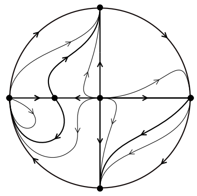

(G75) [R=6, S=17]

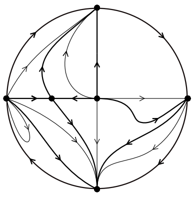

(G76) [R=6, S=17]

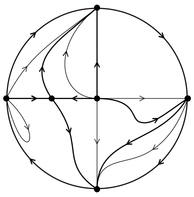

(G77) [R=6, S=17]

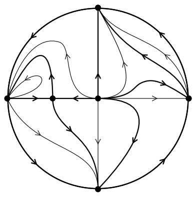

(G78) [R=8, S=19]

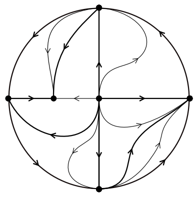

(G79) [R=5, S=16]

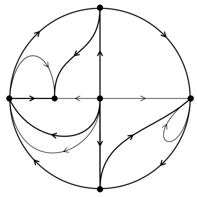

(G80) [R=7, S=18]

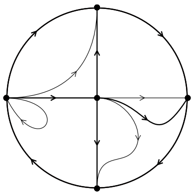

(G81) [R=7, S=18]

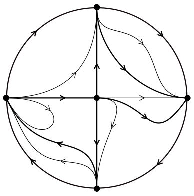

(G82) [R=6, S=17]

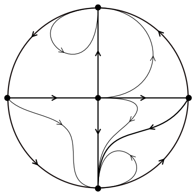

(G83) [R=7, S=18]

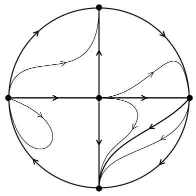

(G84) [R=6, S=17]

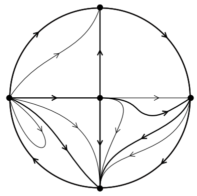

(G85) [R=5, S=16]

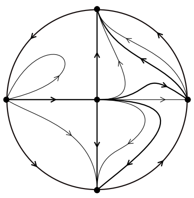

(G86) [R=4, S=13]

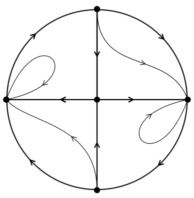

(G87) [R=6, S=15]

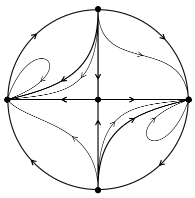

(G88) [R=5, S=14]

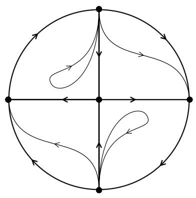

(G89) [R=5, S=14]

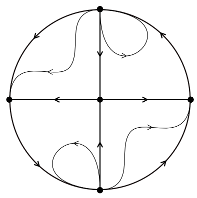

(G90) [R=6, S=15]

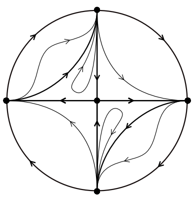

(G91) [R=6, S=15]

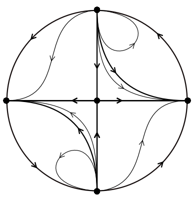

(G92) [R=4, S=13]

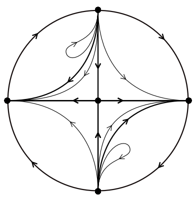

(G93) [R=6, S=15]

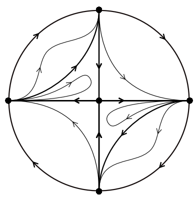

(G94) [R=4, S=13]

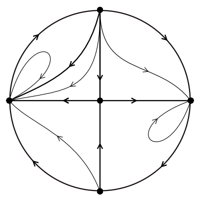

(G95) [R=4, S=13]

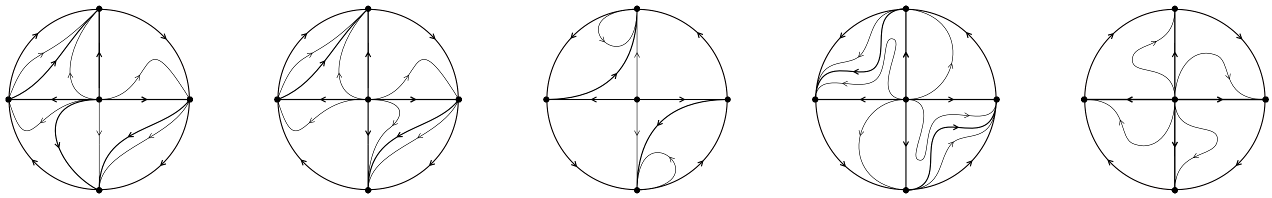

(G96) [R=6, S=15]


(G97) [R=6, S=15]

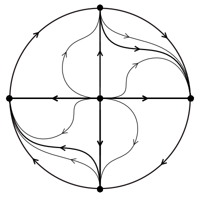

(G98) [R=6, S=15]


(G99) [R=6, S=15]


(G100) [R=5, S=14]



(G101) [R=6, S=15] (G102) [R=6, S=15] (G103) [R=4, S=13] (G104) [R=6, S=15] (G105) [R=4, S=13]



(G106) [R=6, S=15]

Fig. 16. Global phase portraits of system (3) in the Poincaré disc.

Acknowledgments

The first and third authors are partially supported by the Ministerio de Economía, Industria y Competitividad, Agencia Estatal de Investigación (Spain), grant MTM2016-79661-P (European FEDER support included, UE) and the Consellería de Educación, Universidade e Formación Profesional (Xunta de Galicia), grant ED431C 2019/10 with FEDER funds. The first author is also supported by the Ministerio de Educacion, Cultura y Deporte de España, contract FPU17/02125.

The second author is partially supported by the Ministerio de Ciencia, Innovación y Universidades, Agencia Estatal de Investigación grant PID2019-104658GB-100 (FEDER), the Agència de Gestió d'Ajuts Universitaris i de Recerca grant 2017SGR1617, and the H2020 European Research Council grant MSCA-RISE-2017-777911.

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