Neural Network Controller Design for Fractional-Order Systems with Input Nonlinearities and Asymmetric Time-Varying Pseudo-State Constraints

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Abstract. This article considers the neural adaptive control issues of a category of non-integer-order non-square plants with actuator Nonlinearities and Asymmetric Time-Varying pseudo-State Constraints. First, the original non-square non-affine system with input nonlinearities is transformed into an equivalent affine-in-control square model by defining a set of auxiliary variables and by employing the mean-value theorem. Second, Neural networks and Nussbaum functions are exploited to obviate the requirement of a complete knowledge of the system dynamics and the control directions, respectively. Third, a novel adaptive dynamic surface control method based on Caputo fractional derivative definitions and fractional order filters is developed to overcome the *"explosion of complexity"* problem in the traditional backstepping design process and to determine the parameter update laws and control signals, concurrently. Then, Asymmetric Barrier Lyapunov Functions with error variables are adopted to ensure the uniform stability of the closed-loop system and to prevent the violation of the full pseudo-State constraints. The novelties and contributions of this article are : (1) through the introduction of new technical Lemmas and corollaries, existing control design and stability theories linked to integer-order square systems are developed and extended to non-square non-integer-order ones. (2) all signals, including variables and errors in the closed-loop system are semi-global practical finite-time stability whereas the the tracking errors are asymptotically driven to zero without transgression of the constraints. Finally, the effectiveness and potential of the proposed control approach are substantiated by two example simulations.

Keywords: Adaptive control, Neural Networks, Non-integer-order non-square plants, Backstepping design process, dynamic surface control method, Nussbaum functions, Asymmetric Barrier Lyapunov functions.

1. Introduction

In the last decade, there has been a surge in interest in adaptive control schemes for nonlinear plants due to their exceptional adaptability and their potential applications to handle the structural and parametric system uncertainties existing in various research branches of physics, mathematics, engineering and sciences (see [1-28] and the references therein). In the open adaptive control literature, two techniques are used to design the adaptive controller: indirect and direct techniques [27-56]. Subsequently, in the control research areas, several adaptive control approaches based on the Lyapunov stability theory have been developed and provided for both linear and nonlinear dynamical systems, such as Barrier Lyapunov Functions-based adaptive control [1-3, 18, 21, 24, 28, 34, 40, 43, 45, 52, 55], Adaptive Synchronization Control [4, 13, 26, 31, 57, 58], Robust adaptive control [1, 35, 38, 39, 44], Distributed coordination adaptive control [40], Variable structure adaptive control [7, 22], Backstepping adaptive control [8, 22, 23, 33, 36-38], Data-driven adaptive control [49], Feedback adaptive control [30, 42, 54], Decentralized adaptive control [5, 50], Observer-based adaptive control [9, 11, 19, 27, 29, 47, 59], Sliding mode adaptive control [13, 14, 60], Adaptive model predictive control [15], Finite-Time adaptive control [50, 58], Dynamic surface adaptive control [1, 23, 33, 51] and Bipartite consensus control approaches [52]. Traditional adaptive control methods can only be applied to plants whose dynamics are partially (or almost) known [1-58]. In order to overcome this issue, intelligent adaptive controls (such as fuzzy-logic system (FLS) and neural network (NN)-based controls) have received significant research attention and achieved considerable evolution [3-5, 7-12, 15, 16, 19-21, 23-34, 39, 41-44, 46-49, 51, 54, 55, 61-63]. The main idea of these approximation-based adaptive control methods (i.e., intelligent control methods) is that with little *a priori knowledge* of systems, universal approximators like neural networks and fuzzy systems are used to approximate uncertain nonlinear dynamics or to identify ideal control signals, whereas adaptive techniques are employed to build the controllers on the basis of Lyapunov's synthesis methods and tracking errors [16, 19-21, 23-34, 39, 41-44, 46-49, 51]. In view of the fact that the universal approximation capabilities of neural networks and fuzzy systems only hold over compact sets, nearly all current approximation-based adaptive control schemes can only ensure the semi-global stability of closed-loop control systems [23-34, 39, 41-44, 46-49, 51, 54, 55]. Universal approximators associated with the development of well-organized algorithms to adjust the control parameters have paved the way for the successful utilization of intelligent adaptive control methods in different branches of engineering and science, such as optimization on graphs, circuit design, brain modeling activities, financial systems, image processing, biomedical systems, synchronization of chaotic systems and many others [3-5, 7-12, 15, 16, 19-21, 23-34, 39]. Several intelligent adaptive control schemes do not require any offline tuning (or preliminary training phase) to deal with unstructured (or nonlinearly parameterized) and unmatched uncertainties existing in real plants [23-34, 39, 41-44, 46-49]. The foremost limitations of some abovementioned research on adaptive control approaches are as follows: (1) Controlled systems are supposedly square (with an equal number of outputs and inputs) and affine; (2) The effects of the input (actuator) nonlinearities and the state constraints are omitted; (3) The control directions (or the control gain signs) are assumed to be known *a priori* (strictly negative or strictly positive without loss of generality).

In the real world, there is no doubt that non-square plants with actuator nonlinearities, Constraints, non- affine structures and unknown control directions are more widespread than square affine ones and that the analysis of their stability and designs of their controllers are extremely difficult tasks [21]. Constraints, uncertainties, nonlinearities, unknown control directions and non-affine structures are arbitrary factors existing in practical plants that cause considerable troubles, hinder systems from attaining their targets and make the corresponding control issue rather difficult, which has drawn research' attention, and led to major developments [21]. The foremost types of constraints encountered in practical applications and systems, and performed as safety specifications, physical stoppages, and nonlinear saturation are constraints on incremental control variations, control amplitudes, and outputs or states [8, 43]. However, the control direction is the sign of the partial derivative with respect to the “control variable” in non-affine plants or the sign of the “control variable” gain in affine plants [9, 14, 16, 21, 23-28]. Despite this progress, it should be noted that breach of the constraints and ignoring or being unaware of the effects of control directions and input nonlinearities can be sources of instability or poor performance for the controlled systems and can give rise to serious problems, notably poorer transient responses, longer time responses, excessive steady state errors, and large overshoots [21]. From both the engineering and theoretical point of views, the vast majority of control approaches are not directly applied to non-square plants in the presence of actuator nonlinearities, constraints, non-affine forms and unknown control directions due to their special structures [21]. So far, five approaches have mainly been employed in the literature to address the problems with non-affine forms, specifically [6, 7, 10, 21, 24-28]: (1) the implicit function theorem, (2) intelligent affine models (fuzzy logic systems or neural networks), (3) Taylor-series expansions, (4) differentiation of the original state-space systems and (5) the mean value theorem. Furthermore, to prevent system outputs or system states from violating the constraints, several constraint-handling techniques are applicable in the literature [1-3, 18, 21, 24, 28, 34, 40, 43, 45, 52, 55], such as (1) error transformation techniques, (2) reference governors, (3) constrained models, (4) predictive controls, (5) override controls, (6) barrier Lyapunov Functions, and so on. Moreover, there are numerous methods employing continuous functions to alleviate the harmful effects of input nonlinearities [8, 16, 20-22, 24, 26, 27, 28, 31, 57], such as (1) direct decomposition methods, (2) inverse compensation methods, and so on. Additionally, there are popular approaches utilized in the adaptive control literature to counteract the lack of a priori knowledge of control directions [14, 16, 21, 23-28], namely: (1) approach employing Nussbaum- type functions; (2) approach employing hysteresis-type functions; (3) approach based on the identification of the unknown control direction; (4) correction-vector approach; and (5) approach employing switching and monitoring functions. Nevertheless, the traditional methods to deal with the problems arising from non-square structures are commonly squaring (-down or -

up) methods based on the addition or removal of suitable manipulated variables (inputs) or controlled variables (outputs) [21]. From the aforementioned studies [1, 2, 8, 12, 18, 21, 24, 28, 46], it is apparent that: (1) approaches that address state and output constraints cannot solve the issues of input nonlinearities, and (2) the backstepping design technique has been employed as a methodical tool for the conception of regulation and tracking schemes, adapted to a large group of state feedback linearizable nonlinear systems. During the adaptive backstepping control procedure, the "explosion-of-terms" issue is commonly encountered because of the repeated computation of virtual signals' derivatives [8, 22, 23, 33, 36, 37, 38]. In recent years, three techniques have been developed to cope with the above issue with the classical adaptive backstepping design [1, 23, 33], namely: (1) command filtering technology, (2) Dynamic surface control (DSC), (3) estimation of the virtual control inputs and their derivatives in every step using neural networks or/and fuzzy systems. The Dynamic surface control technique does not take into account the issue of errors resulting from the filters, which might cause system damage and lead to poorer system performance or even closed-loop instability [1, 23]. As is well known, the use of the command filtering technique based on an introduced error compensation mechanism provides an effective technique in the literature for dealing with the drawback of the Dynamic surface control technique [1]. Note that most of the above-mentioned control schemes [1-49, 51-57] ensure the asymptotic stability of systems, implying that system trajectories will converge towards the equilibrium when time tends to infinity. Compared with asymptotic stability, finite-time stability habitually offers better robustness, higher tracking precision, greater accuracy, and faster convergence rates [50, 58]. For this reason, the study of finite-time control has attracted considerable academic attention in recent decades [50]. Specifically, approximation-based finite-time adaptive control methods have alleviated the drawbacks of traditional control schemes and have ensured the semi global practical finite time stability of nonlinear integer order systems [58, 64]. That is, most of the above research [1-12, 14-16, 18-20, 22] took no account of control techniques of fractional order systems, which might restrict their application to various fields of engineering, mathematics, physics, and science.

Recently, the use of fractional-order (Non-Integer Order) calculus has attracted much attention in many fields of physics, engineering and applied mathematics as it provides outstanding tools for explaining numerous physical phenomena and for expressing the hereditary, memory and genetic characteristics of real world systems as opposed to classical integer-order calculus [13, 17, 21, 23-28, 31, 33, 41, 50, 52, 53, 58, 64-74]. It is a fact that with the development of modern information technology, fractional-order (Non-Integer Order) integral and differential equations have enhanced the accuracy of modeling and controlling physical systems and that they can be regarded as generalized conventional integer-order models [23-28, 31, 33, 41, 50, 52, 53, 58, 64-74]. It is well known that the fundamental characteristics of fractional-order derivative are nonlocal and possess singular kernels, which leads to infinite memory and greater degrees of freedom for fractional plants [23-28, 31, 33, 41, 50, 52, 53, 72-74]. From the mathematical view, fractional calculus is regularly viewed as an extension and superset of integer-order ones [23-28, 31, 33, 41, 50, 72-74]. In addition, various physical plants and science systems from the real world, such as diffusion processes, COVID-19 transmission dynamics, viscoelastic systems, electromagnetic waves, dielectric polarization, flexible structures, biological systems, electrochemical processes, finance systems and so on, illustrate fractional order dynamic behaviors and are well defined by differential fractional equations, i.e. equations comprising both non-integer (fractional) derivatives and integer -order derivatives [13, 17, 21, 23-28, 31, 33, 41, 50, 52, 53, 58, 64-74]. It has been noted that fractional calculus can cope with problems with derivatives and integrations of arbitrary orders and can ameliorate the stability margins, robustness and performance properties of classical integer-order systems, observers and controllers [21, 23-28, 31, 33, 41, 50, 52, 53, 58, 64-79]. What's more, fractional-order controllers are outstanding methods for disturbance elimination and are extremely robust for modeling uncertainties [23-28], such as fractional-order adaptive control, fractional order sliding mode control, fractional-order optimal control, fractional-order Proportional–integral–derivative (PID) control, and so on. It is hard to generalize the traditional methods of stability analysis and control of integer order plants to those of fractional (non-integer) order ones, because Fractional calculus is a theory of integrals and derivatives of non-integer orders characterized by a group of remarkable, unusual and atypical properties such as transgression of the usual Leibniz rule, deformations of the usual chain rule and violation of the semi-group property [66-68]. There are various outstanding approaches to deal with these problems and to study the properties of fractional order systems, including observability, controllability, and stability, such as: (1) the method based on Matignon's stability theorem [72]; (2) the method based on Gronwall-Bellman-Bihari type inequalities [73]; (3) the method based on Volterra-type Lyapunov functions [67]; (4) the method based on equivalent frequency distributed models [26, 27]; (5) the method based on Mittag-Leffler functions [33, 50, 52, 58, 64-67, 69, 70, 73, 74]; (6) the method based on convex Lyapunov functions [21, 23]; (7) the Laplace transform method [17, 33, 41, 50, 52, 53, 66, 67, 71], (8) the linear matrix inequality (LMI) method [33, 50, 58]; and so on. Despite the increasing amount of research on fractional calculus, there have been very few control engineering practices until recently [23-28, 31, 33, 41, 50, 52, 53, 58, 64-79]. To date and to the best of the authors' knowledge, there have been no effective research studies on neural approximation-Based adaptive tracking controls produced for non-square non-integer order systems in the presence of input nonlinearities, asymmetric Time-Varying pseudo-State Constraints, non- affine structures or unknown control directions.

Motivated by the above insight, this research mainly investigates the neural adaptive control problems for a group of fractional-order non-square systems with unknown control directions, Input Nonlinearities and Asymmetric Time-Varying pseudo-State Constraints. In the process of designing the controller, the original non-square non-affine system with input nonlinearities is first converted into an equivalent affine-in-control square form by introducing auxiliary variables and using the mean-value theorem. Second, Neural Networks are utilized as universal approximators to estimate the uncertain parameters and unknown functions. Third, the difficulties from the unknown control directions are handled by Nussbaum functions. Then, a dynamic surface control approach based on fractional order filters and Caputo fractional derivative definitions is applied to solve the *"explosion of complexity"* issue via the traditional backstepping design procedure and to calculate the control signals and parameter update laws, simultaneously. Moreover, Barrier Lyapunov Functions in asymmetric forms are used to address the issues of pseudo-State constraints and to guarantee the uniform stability of the closed-loop system. Based on the Lyapunov stability theory, it is proved that for any bounded initial conditions, the semi-global practical finite-time stability of the overall closed-loop system is achieved and the the tracking errors asymptotically approach zero without violating the constraints. This paper makes the following contributions:

(1) Through the medium of new corollaries and lemmas about fractional calculus, approaches and techniques concerning the controller design process and the stability

analysis of conventional integer-order plants are extended and applied to a more general category of non-square fractional-order systems with non-affine forms, Input Nonlinearities, unknown control directions, and Asymmetric Time-Varying pseudo-State Constraints.

(2) In comparison with existing works [19, 35, 43, 55], this research does not need frequent restrictive Assumptions and conditions connected to system functions, estimation errors, input nonlinearities, uncertain nonlinear dynamics, desired trajectories and unknown control directions.

(3) Compared with papers [17, 33, 41], this article provides a new finite time control design method for non-square fractional-order systems with unknown control directions, Asymmetric Time-Varying pseudo-State Constraints and Input Nonlinearities. Therefore, the proposed control strategy successfully improves the robustness of a closed-loop system. It also allows reduction of the convergence rate and avoids possible issues raised by the chattering phenomenon and control singularity.

The remainder of this article is structured as follows. Descriptions of the system and preliminaries are presented in Section 2. The system transformation, controller design, and stability analysis are provided in Section 3, followed by some example simulations that show the feasibility of the proposed controller in Section 4. The conclusions are presented in Section 5.

2. Description of Preliminaries and the Problem

2.1. Preliminaries

The following Definitions 1-3, Properties 1-4, Lemmas 1-12 and Corollaries 1-2 will play a significant role in the subsequent developments.

Definition 1 (Riemann–Liouville integral**)** [21, 23, 41, 58, 64, 74]**.** Let $f(.)$ be an integrable function. Then, its fractional integral of order $0<\delta<1$ is defined by

$$D^{-\delta} f(t)=\frac{1}{\Gamma(\delta)} \int_{0}^{t} \frac{f(\tau)}{(t-\tau)^{1-\delta}} d\tau \tag{D1.1}$$

with $\Gamma(\delta)=\int_{0}^{+\infty} \tau^{\delta-1} \exp(-\tau)\, d\tau$ being the Euler Gamma function. $\exp(.)$ represents the natural exponential function.

Definition 2 (Caputo derivative) [17, 21, 23, 31, 33, 41, 50, 52, 58, 74].The Caputo fractional-order derivative of a differentiable function $f(.)$ is defined as

$$D^{\delta} f(t)=\frac{1}{\Gamma(1-\delta)} \int_{0}^{t} \frac{f'(\tau)}{(t-\tau)^{\delta}} d\tau \tag{D2.1}$$

with $f'(.)$ being the first integer-order derivative of $f(.)$. $0<\delta\leq 1$ representing the order of the derivative.

Definition 3 [9, 16, 23]**.** A continuous function $N(.)$ is called a Nussbaum-type function when it has the following properties:

$$\lim_{s\to\pm\infty} \sup \frac{1}{s} \int_{0}^{s} N(\zeta) d\zeta=+\infty \text{ and } \lim_{s\to\pm\infty} \inf \frac{1}{s} \int_{0}^{s} N(\zeta) d\zeta=-\infty, \tag{D3.1}$$

where $s\to\pm\infty$ includes $s\to-\infty$ and $s\to+\infty$.

Property 1 [23]. Let $f(.)$ be a continuously differentiable and integrable function. Afterwards, the subsequent mathematical equations are satisfied for any time instant $t \geq 0$ and constants $0 < \delta < 1$, $\ell \in \sim$, $\ell_1 > 0$

$$\begin{cases} D^1 f(t) = \dfrac{df(t)}{dt} = f'(t) = D^{\delta}\left[D^{1-\delta} f(t)\right] = D^{1-\delta}\left[D^{\delta} f(t)\right] \\ D^{\delta}\left[D^{-\delta} f(t)\right] = f(t) \\ D^{\delta}\left[f(t)\right] = D^1\left[D^{\delta-1} f(t)\right] \\ \\ D^{-1} f(t) = D^{\delta-1}\left[D^{-\delta} f(t)\right] = D^{-\delta}\left[D^{\delta-1} f(t)\right] = \int_0^t f(\tau)\, d\tau \\ D^{-\delta}\left[D^{\delta} f(t)\right] = f(t) - f(0) \\ D^{\delta}\ell = 0 \\ D^{\delta} t^{\ell_1} = \dfrac{\Gamma(1+\ell_1)}{\Gamma(\ell_1 - \delta + 1)} t^{\ell_1 - \delta} \end{cases} \quad \text{(P1.1)}$$

with $\Gamma(.)$ being Euler's Gamma function.

Property 2 [13, 17, 21, 23-28, 31, 33, 41, 50, 52, 53, 58, 64-74]. In common with the integer-order differentiation and integration, the Caputo fractional integral and derivative operators are linear operators

$$\begin{cases} D^{\delta}\left[\ell_1 f_1(t) + \ell_2 f_2(t)\right] = \ell_1 D^{\delta}\left[f_1(t)\right] + \ell_2 D^{\delta}\left[f_2(t)\right] \\ D^{-\delta}\left[\ell_1 f_1(t) + \ell_2 f_2(t)\right] = \ell_1 D^{-\delta}\left[f_1(t)\right] + \ell_2 D^{-\delta}\left[f_2(t)\right] \end{cases}, \quad \text{(P2.1)}$$

where $f_1(t)$ and $f_2(t)$ are continuously differentiable and integrable functions. $0 < \delta \leq 1$, $\ell_1 \in \sim$ and $\ell_2 \in \sim$ are constants.

Property 3 [23-28, 31, 33, 41, 74]. For any continuous and integrable function $f(t) \geq 0$, the following inequality holds

$$D^{-\delta}\left[f(t)\right] \geq 0 \quad \text{(P3.1)}$$

with $0 < \delta \leq 1$ being a constant.

Property 4 [17, 21, 23-28, 31, 33]**.** Let $D^{\delta}\left[g(t)\right] \leq 0$ and $g(0)=0$, then $g(t) \leq 0$, for any time instant $t \geq 0$ and constant $0<\delta \leq 1$.

Lemma 1 [15, 19, 21, 23, 32, 34, 46, 47, 74]**.** Let $f(x): \sim^{\mathrm{n}} \rightarrow \sim$ be a continuous or bounded function defined on a sufficiently large compact set $\Omega \subset \sim^{n}$. Then, for any strictly positive scalar ε, there is a neural network $w^{T}\xi(x)$ such that

$$\sup_{x\in\Omega}\left|f(x)-w^{*T}\xi(x)\right| \leq \varepsilon \tag{L1.1}$$

where $w^{*} \in \sim^{q}$ is the ideal constant weight vector, q is the number of neurons in the hidden layer (the number of neural network nodes,), $\xi(.) \in \sim^{q}$ is an activation function vector.

Lemma 2 [21, 23-28]**.** For any $\lambda_1 \in \sim$ and $\lambda_2 \in \sim_{+}^{*}$, the following relation holds

$$0 \leq \left|\lambda_1\right| - \frac{\lambda_1^{\,2}}{\sqrt{\lambda_1^{\,2}+\lambda_2^{\,2}}} \leq \lambda_2 \tag{L2.1}$$

where $\sim_{+}^{*}$ is the set of strictly positive real numbers.

Lemma 3. Let $f(x_1,\ldots,x_n): \sim^{n} \rightarrow \sim$ be a function of class C^2. Assume that the function $f(.)$ is convex or concave with respect to the variables $x_1,\ldots,x_n$ (i.e., its Hessian matrix

$$H_0=\begin{pmatrix} \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_1^{\,2}} & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_2^{\,2}} & \cdots & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_n^{\,2}} \end{pmatrix}$$

is positive semi-definite or negative semi-definite for all $x=\left[x_1,\ldots,x_n\right]^{T} \in \sim^{n}$). Then, for any time instant $t \geq 0$ and constant $0<\delta<1$, the subsequent inequality is satisfied

$$\operatorname{sgn}\left(\operatorname{tr}\left(H_0\right)\right)\sum_{i=1}^{n}\frac{\partial f\left(x_1(t),\dots,x_n(t)\right)}{\partial x_i}\Bigg|_{t=0} D^{\delta}x_i(t) \le \operatorname{sgn}\left(\operatorname{tr}\left(H_0\right)\right)D^{\delta} f\left(x_1(t),\dots,x_n(t)\right)$$

$$\le \operatorname{sgn}\left(\operatorname{tr}\left(H_0\right)\right)\sum_{i=1}^{n}\frac{\partial f\left(x_1(t),\dots,x_n(t)\right)}{\partial x_i}D^{\delta}x_i(t) \qquad \text{(L3.1)}$$

where $\operatorname{sgn}(.)$ stands for the sign function. $\operatorname{tr}(.)$ denotes the trace of a matrix.

Remark 1. Functions are said to be of class C^2 if their first and second derivatives both exist and are continuous.

Lemma 4 [9, 16, 23]**.** Let $V(t)$ be a positive definite, radially unbounded continuously differentiable function satisfying the following inequality

$$V(t) \le \ell_0 + \int_0^t \left(1+g_\ell(\tau)N(\zeta)\right)\dot{\zeta}(\tau)\,d\tau \qquad \text{(L4.1)}$$

where $N(\zeta)$ is a smooth Nussbaum-type function. $g_\ell(.)$ is a nonzero bounded function. $\zeta(.)$ is a continuously differentiable function. ℓ_0 is a constant. Then, $V(t)$, $\zeta(t)$ and $\int_0^t \left(1+g_\ell(\tau)N(\zeta)\right)\dot{\zeta}(\tau)\,d\tau$ are all bounded for any time instant $t \ge 0$. Note that $\lim_{t\to+\infty} V(t)$, $\int_0^{+\infty} \left(1+g_\ell(\tau)N(\zeta)\right)\dot{\zeta}(\tau)\,d\tau$ and $\lim_{t\to+\infty}\zeta(t)$ exist and are finite.

Lemma 5 [3, 4, 16, 21, 23]**.** Let $f(.) \in \sim$ be a uniformly continuous function for $t \ge 0$. Then, $\lim_{t\to+\infty} f(t) = 0$ if $\lim_{t\to+\infty}\int_0^t f(\tau)\,d\tau$ exists and is finite.

Lemma 6 [3, 28, 34, 43, 74]**.** For any strictly positive functions $\left(k_{x_1}, k_{x_2}\right)$, if $-k_{x_1} < x < k_{x_2}$, the following inequality is satisfied,

$$\hbar\log\left(\frac{{k_{x_1}}^2}{{k_{x_1}}^2 - x^2}\right) + (1-\hbar)\log\left(\frac{{k_{x_2}}^2}{{k_{x_2}}^2 - x^2}\right) \le \left[\frac{\hbar}{{k_{x_1}}^2 - x^2} + \frac{1-\hbar}{{k_{x_2}}^2 - x^2}\right]x^2 \qquad \text{(L6.1)}$$

with $\hbar = \begin{cases} 1, \text{if } x \le 0 \\ 0, \text{if } x > 0 \end{cases}$. $\log(.)$ represents the natural logarithm of $(.)$.

Lemma 7. Suppose that a continuous, positive-definite function $V(t)$ satisfies the inequality:

$$D^\delta V(t) \le -\ell_1 V(t) + \ell_2 \tag{L7.1}$$

where $\ell_1 \in \sim_+^*$, $\ell_2 \in \sim_+^*$ and $0 < \delta < 1$ are all constants.

Then, for any time instant $t \ge 0$, the following inequality holds

$$V(t) \le V(0) + \frac{\ell_2}{\ell_1}\left[1 - \exp\left(-\frac{\ell_1 t^\delta}{\Gamma(1+\delta)}\right)\right] \tag{L7.2}$$

Lemma 8. Assume that there is a Lyapunov function candidate $V(x)$ for the system $D^\delta x = f(x)$ satisfying the following inequality

$$D^\delta V(x) \le -\ell_0 V(x) - \ell_3 V^2(x) - \ell_1 \frac{V(x)}{V(x)+\sigma} + \ell_2 \tag{L8.1}$$

where $0 < \delta < 1$ and $\ell_i > 0$ are constants, for $i = 0,\dots,3$. The strictly positive function σ is continuously differentiable, such that σ, $\dot{\sigma}$ and $\int_0^t \sigma(\tau)d\tau$ are bounded, $\forall\, t \ge 0$. $f(.)$ is a continuous function. The initial condition $V(0)$ is bounded.

Consequently, the system $D^\delta x = f(x)$ is semi-global practical finite-time stable (SGPFS).

Lemma 9. Consider the equation

$$D^\delta \alpha^c(t) = b\left[\alpha(t) - \alpha^c(t)\right] \tag{L9.1}$$

where $\alpha(0) = \alpha^c(0)$. The continuous functions $\alpha(t)$, $D^\delta \alpha(t)$ and $D^{2\delta}\alpha(t)$ are bounded. $b > 0$ and $0 < \delta < 1$ are constants. Then, there are strictly positive constants ℓ and M such that the subsequent properties hold

- $\ell > \dfrac{1}{2\sqrt{b}}$ and $\left|D^{2\delta}\alpha(t)\right| \le M$.
- $\alpha^c(t)$, $D^{\delta}\alpha^c(t)$ and $D^{2\delta}\alpha^c(t)$ are bounded,
- $\left|D^{\delta}\alpha^c(t) - D^{\delta}\alpha(t)\right| \le \sqrt{\left(D^{\delta}\alpha(0)\right)^2 + \dfrac{4\ell^4 M^2}{4b\ell^2 - 1}\left[1 - \exp\left(-\dfrac{\left(4b\ell^2 - 1\right)t^{\delta}}{2\ell^2\Gamma(1+\delta)}\right)\right]}$ for any time instant $t \ge 0$.

Lemma 10. Let $\mu_1(t) \in \sim$ and $\mu_2(t) \in \sim$ be monotonic functions, so, the following inequalities hold for any time instant $t \ge 0$ and constant $0 < \delta \le 1$

$$\begin{cases} \left[\mu_1(0) D^{\delta}\left[\mu_2(t)\right] + \mu_2(0) D^{\delta}\left[\mu_1(t)\right]\right]\operatorname{sgn}\left(\dfrac{\partial \mu_1(t)}{\partial t}\dfrac{\partial \mu_2(t)}{\partial t}\right) \\ \qquad \le \operatorname{sgn}\left(\dfrac{\partial \mu_1(t)}{\partial t}\dfrac{\partial \mu_2(t)}{\partial t}\right) D^{\delta}\left[\mu_1(t)\mu_2(t)\right] \\ \qquad \le \left[\mu_1(t) D^{\delta}\left[\mu_2(t)\right] + \mu_2(t) D^{\delta}\left[\mu_1(t)\right]\right]\operatorname{sgn}\left(\dfrac{\partial \mu_1(t)}{\partial t}\dfrac{\partial \mu_2(t)}{\partial t}\right) \\ \operatorname{sgn}\left(\dfrac{\partial \mu_1(t)}{\partial t}\right)\mu_1(0)\dfrac{t^{\delta}}{\Gamma(\delta+1)} \le \operatorname{sgn}\left(\dfrac{\partial \mu_1(t)}{\partial t}\right) D^{-\delta}\left[\mu_1(t)\right] \\ \qquad \le \operatorname{sgn}\left(\dfrac{\partial \mu_1(t)}{\partial t}\right)\mu_1(t)\dfrac{t^{\delta}}{\Gamma(\delta+1)} \end{cases} \tag{L10.1}$$

Lemma 11 [74]. Let $V(t)$ satisfy $V(t) \le \int_0^t \mu_1(\tau)V(\tau)\, d\tau + \mu_2(t)$ with $\mu_1(.)$ being a real function and $\mu_2(.)$ being a differentiable real function. Afterwards, we have

$$V(t) \le \mu_2(0)\exp\left(\int_0^t \mu_1(\tau)d\tau\right) + \int_0^t \mu_2{}'(\tau)\exp\left(\int_\tau^t \mu_1(\upsilon)d\upsilon\right)d\tau \tag{L11.1}$$

If $\mu_2(t) = \mu_2$ is a constant, therefore

$$V(t) \le \mu_2 \exp\left(\int_0^t \mu_1(\tau)d\tau\right) \tag{L11.2}$$

with $\mu_2{}'(.)$ being the first integer-order derivative of $\mu_2(.)$.

Lemma 12. Let functions V_1 and V_2 satisfy $0 \le V_1 \le V_2$. Then, we have

$$\begin{cases} -\dfrac{V_2}{V_2+\sigma} \le -\dfrac{V_1}{V_1+\sigma} \\ -V_2^{\,2} \le -V_1^2 \end{cases} \tag{L12.1}$$

where σ is a strictly positive function.

Corollary 1. For any constants $\left(0<\delta<1,\, n\in \bullet^*\right)$, time instant $t\ge 0$, and continuously differentiable functions $\left(\mu_1(t)\in \sim,\ \mu_2(t)\in \sim_+^*,\ \mu_3(t)\in \sim^*\right)$, the following inequalities are satisfied

$$\begin{cases} D^\delta \log\left(\dfrac{1}{\mu_2(t)}\right) = -D^\delta \log\left(\mu_2(t)\right) \le -\dfrac{1}{\mu_2(t)} D^\delta \mu_2(t) \\ 2n\mu_1^{2n-1}(0) D^\delta \mu_1(t) \le D^\delta \mu_1^{2n}(t) \le 2n\mu_1^{2n-1}(t) D^\delta \mu_1(t) \\ D^\delta\left(\dfrac{\mu_1^2(t)}{\mu_2(t)}\right) \le 2\dfrac{\mu_1(t)}{\mu_2(t)} D^\delta \mu_1(t) - \left(\dfrac{\mu_1(t)}{\mu_2(t)}\right)^2 D^\delta \mu_2(t) \\ D^\delta\left(\dfrac{\mu_1(t)}{\mu_3(t)}\right)^{2n} \le 2n\left(\dfrac{\mu_1^{2n-1}(t)}{\mu_3^{2n}(t)}\right) D^\delta \mu_1(t) - 2n\mu_3(0)\left(\dfrac{\mu_1^{2n}(t)}{\mu_3^{2n+2}(t)}\right) D^\delta \mu_3(t) \end{cases}$$

(C1.1)

where $\sim^*$ is the set of non-zero real numbers. $\bullet^*$ is the set of non-zero integer numbers.

Corollary 2. For any time instant $t\ge 0$, constants $\left(0<\delta<1,\, n\in \bullet^*\right)$ and continuously differentiable functions $\left(\mu_1(t)\in \sim,\ \mu_3(t)\in \sim^*\right)$ satisfying $0\le \left(\dfrac{\mu_1(t)}{\mu_3(t)}\right)^2 < 1$, the following inequality holds

$$\frac{1}{2n} D^\delta \log\left(\frac{\mu_3^{2n}(t)}{\mu_3^{2n}(t)-\mu_1^{2n}(t)}\right) \le \frac{\mu_1^{2n-1}(t)}{\mu_3^{2n}(t)-\mu_1^{2n}(t)}\left(D^\delta \mu_1(t) - \mu_3(0)\left(\frac{\mu_1(t)}{\mu_3^2(t)}\right) D^\delta \mu_3(t)\right) \tag{C2.1}$$

Remark 2. The proofs of Lemmas 3, 7-10, 12 and Corollaries 1-2 are given in Appendices 1-8, respectively.

2.2. Description of the problem

Consider a category of fractional-order systems with Input nonlinearities and Time-varying full-pseudo-state Constraints in the form of

$$\begin{cases} D^{\delta_{i,j_i}} x_{i,j_i} = f_{i,j_i}\left(x,\ x_{i,j_i+1},\ d\right), \text{ for } j_i = 1,\ldots,n_i - 1, \\ D^{\delta_{i,n_i}} x_{i,n_i} = f_{i,n_i}\left(x,\ v_i,\ d\right), \\ v_i = \Delta_{i,1} \odot \eta_i\left(u_i\right) + \Delta_{i,2} \\ y_i = x_{i,1}, \text{ for } i = 1,\ldots,p \end{cases} \tag{1}$$

where:

- $D^{\delta_{i,j_i}} = \left(\dfrac{d^{\delta_{i,j_i}}}{dt^{\delta_{i,j_i}}}\right)$ is the Caputo fractional differential operator or Caputo fractional derivative of known constant order $0 < \delta_{i,j_i} \le 1$, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- $u = \left[u_1^T,\ldots,u_p^T\right]^T \in \sim^m$, $x = \left[x_1^T,\ldots,x_p^T\right]^T \in \sim^n$ and $y = \left[y_1,\ldots,y_p\right]^T \in \sim^p$ are the system control input, measurable pseudo-state and output vectors, respectively, with $\bar{x}_{i,j_i} = \left[x_{i,1},\ldots,x_{i,j_i}\right]^T \in \sim^{j_i}$, $\bar{x}_{i,n_i} = x_i = \left[x_{i,1},\ldots,x_{i,n_i}\right]^T \in \sim^{n_i}$, $x_{i,j_i+1} \in \sim$, $u_i = \left[u_{i,1},\ldots,u_{i,m_i}\right]^T \in \sim^{m_i}$, $y_i \in \sim$, $j_i = 1,\ldots,n_i - 1$, $i = 1,\ldots,p$, $\sum_{i=1}^{p} m_i = m$ and $\sum_{i=1}^{p} n_i = n$.
- All pseudo-states are required to remain in the compact set, $-\kappa_{1,i,j_i} < x_{i,j_i} < \kappa_{2,i,j_i}$ (**Time-varying pseudo-state Constraints**) with the time-varying barrier functions κ_{1,i,j_i} and κ_{2,i,j_i} being bounded, known and strictly positive. Additionally, the fractional order derivatives of the functions (κ_{1,i,j_i} and κ_{2,i,j_i}) are bonded, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- The external disturbance vector $d \in \sim^n$ is bounded and unknown.

- The unknown functions $f_{i,j_i}(.)$, $j_i = 1, \ldots, n_i$, $i = 1, \ldots, p$ are continuous and differentiable.
- $\eta_i(u_i) = \left[\eta_{i,1}(u_i), \eta_{i,2}(u_i), \ldots, \eta_{i,r_i}(u_i)\right]^T \in \sim^{r_i}$ is a vector of unknown continuous functions, for $i = 1, \ldots, p$.
- $\Delta_{i,1} = \left[\Delta_{i,1,1}, \Delta_{i,1,2}, \ldots, \Delta_{i,1,r_i}\right]^T \in \sim^{r_i}$ is a vector of unknown, nonzero and bounded functions, for $i = 1, \ldots, p$.
- $\odot$ means the Hadamard product (also known as the element-wise, entrywise or Schur product).
- $\Delta_{i,2} = \left[\Delta_{i,2,1}, \Delta_{i,2,2}, \ldots, \Delta_{i,2,r_i}\right]^T \in \sim^{r_i}$ is a vector of unknown bounded functions, for $i = 1, \ldots, p$.
- The vector of unknown continuous functions $v_i = \left[v_{i,1}, v_{i,2}, \ldots, v_{i,r_i}\right]^T \in \sim^{r_i}$ denotes the input nonlinearities, for $i = 1, \ldots, p$.

Remark 3. Note that fractional (non-integer) -order calculus (integration and differentiation) is a natural generalization of their classical integer-order counterparts [13, 17, 21, 23-28, 31, 33, 41, 50, 52, 53, 58, 64-73]. It has been demonstrated that several systems in various domains of engineering and science such as Biology, life sciences, psychology, sound wave propagation, seismic wave analysis, astrophysics, geology, physics, electrochemistry, and so on could be better expressed by mathematical models incorporating differential-algebraic and / or fractional differential equations than integer-order models [21, 23-28, 31, 33, 41, 50, 52, 53, 58, 64-79]. Up to now, there have been numerous definitions of non-integer-order calculus, namely the Caputo definition, the Grünwald–Letnikov definition, the Coimbra definition, the Marchaud definition, the Riemann–Liouville definition, the Weyl definition, the Riesz-Feller definition, the Erdelyi–Kober definition, and the Riesz definition [21, 23-28, 52, 53, 58, 64-79]. The Caputo definition is selected in this study because it is more convenient, concise, and effective in diverse branches of engineering, mathematics and sciences than the aforementioned definitions. The foremost benefits of using the Caputo definition are that:

- The initial conditions for non integer-order differential equations with Caputo derivatives take on the same form as those of integer-order differential equations [17, 21, 23-28, 52, 58, 64, 74].
- The Caputo derivative of constant functions is equal to zero [21, 23-28, 65, 74].
- The mass balance error and the hyper-singular improper integral are avoided with the Caputo definition [26].

Remark 4. The numerical simulations of non-integer order systems are as complex as those of ordinary integer-order systems [13, 17, 21, 23-28, 31, 33, 41, 50, 52, 53, 58, 64-79]. For the numerical solutions of fractional-order systems, there are two approximation methods described in the literature [21, 23-28, 41, 53, 58, 72]:

- Approximation Method based on fractional-order system behavior in the frequency domain.
- Approximation Method based on the predictor–correctors scheme.

Remark 5. Note that an improved version of the Adams–Bashforth–Moulton numerical algorithm presented in [13, 21, 23, 41], is used in this study for numerical computer simulations of the Caputo non-integer order differential equations. It is found that this algorithm is based on the predictor–correctors scheme.

Remark 6. Fractional-order plant (1) is said to be in a nonlinear, non-square and nonstrict feedback structure since the number of input variables surpasses the number of output variables and all the system functions $f_{i,j_i}(.)$, $j_i = 1,\dots,n_i$, $i = 1,\dots,p$ contain the whole pseudo-state variables with nonlinear characteristics. Besides, there may be potential algebraic loop issues. In the literature [1-28, 42], there are various physical and practical systems that can be transformed or expressed as particular cases of the general system (1), such as elastic spacecraft systems, cancer immunotherapy systems, communication networks, robotic manipulators, underwater vehicles, electromechanical systems, flexible crane systems, and so on. As mentioned in [13, 17, 21, 23-28, 31, 33, 41, 52, 53], adaptive control designs for fractional-order plants are much more challenging and complex than for integer-order plants. In addition, it is well known that various traditional control methods cannot be directly applied to non-strict feedback nonlinear fractional-order systems with input nonlinearities, asymmetric time-varying pseudo-state constraints and an unequal number of outputs and inputs because of several technical obstacles [21].

Remark 7. Unknown dynamics, constraints and input nonlinearities frequently appear as major factors that degrade closed-loop system performance in real practical and industrial nonlinear systems such as robotic systems, chemical systems, flexible crane systems, permanent magnet synchronous motors, and so on [21]. It should be noted that the input (actuator) nonlinearities in this study are more general than those presented in [8, 16, 19, 20, 22, 25-27, 31, 35, 43, 50, 55, 57, 69, 70] because v_i can satisfy, deadzone, backlash, hysteresis, nonsmooth asymmetric saturation, and sector nonlinearities. In addition, v_{i,l_i} should be a continuous function but $\Delta_{i,1,l_i}$ and $\Delta_{i,2,l_i}$ can be discontinuous functions, for $l_i = 1,\ldots,r_i$, $i = 1,\ldots,p$. Evidently, the study of fractional-order non-square systems with input nonlinearities is more difficult and general than affine square ones [21].

The **control objective of this work** is to construct an adaptive neural network controller u, such that:

- the system output y_i is driven to track a given bounded reference signal y_{di}, having bounded fractional derivatives to a bounded compact set, for $i = 1,\ldots,p$.
- and all the signals in the closed-loop system are bounded, as well as the pseudo-states not being able to violate their constrained bounds.

Let us introduce the successive variables to be employed through the controller design procedure

$$
\begin{cases}
\alpha_{i,0} = y_{di} \\
z_{i,j_i} = x_{i,j_i} - \alpha_{i,j_i-1} \\
E_{i,j_i} = \left[\dfrac{\hbar_{i,j_i}}{\kappa_{3,i,j_i}{}^2 - z_{i,j_i}{}^2} + \dfrac{1-\hbar_{i,j_i}}{\kappa_{4,i,j_i}{}^2 - z_{i,j_i}{}^2} \right] z_{i,j_i} \\
D^{\delta_{i,j_i}} \alpha^c{}_{i,j_i-1} = b_{i,j_i} \left[\alpha_{i,j_i-1} - \alpha^c{}_{i,j_i-1} \right] \\
D^{\delta_{i,j_i}} \alpha_{i,j_i-1} = D^{\delta_{i,j_i}} \alpha^c{}_{i,j_i-1} + F^c{}_{i,j_i}\left(\wp_{i,j_i}\right) \\
u_{i,k_i} = \alpha_{i,n_i}, \text{ for } k_i = 1,\ldots,m_i,\ j_i = 1,\ldots,n_i,\ i = 1,\ldots,p
\end{cases}
\tag{2}
$$

where

- y_{di} and z_{i,j_i} are the reference signal and the tracking error variable, respectively, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- κ_{3,i,j_i} and κ_{4,i,j_i} are strictly positive functions, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- The known constant b_{i,j_i} is a strictly positive design parameter and $\hbar_{i,j_i} = \begin{cases} 1, \text{ if } z_{i,j_i} \le 0 \\ 0, \text{ if } z_{i,j_i} > 0 \end{cases}$, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- α_{i,j_i-1} is the virtual controller and can also be regarded as the input of the fractional-order filter, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- $D^{\delta_{i,j_i}} \alpha^c{}_{i,j_i-1}(t)$ or $b_{i,j_i}\left[\alpha_{i,j_i-1}(t) - \alpha^c{}_{i,j_i-1}(t)\right]$ represents the output of the filter and $\alpha^c{}_{i,j_i-1}(t)$ stands for the state of the command filter with $\alpha_{i,j_i-1}(0) = \alpha^c{}_{i,j_i-1}(0)$, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- $F^c{}_{i,j_i}(.)$ is an unknown continuous function with $\wp_{i,j_i} = \left[x^T, E_{i,1},\ldots,E_{i,j_i}, z_{i,1},\ldots,z_{i,j_i}, \kappa_{3,i,j_i}, \kappa_{4,i,j_i}, \alpha_{i,0},\ldots,\alpha_{i,j_i-1}, \alpha^c{}_{i,0},\ldots,\alpha^c{}_{i,j_i-1}\right]^T \in {}^{\sim c_{i,j_i}}$ and $c_{i,j_i} = n + 4j_i + 2$, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.

Remark 8. It should be noted that:

- κ_{3,i,j_i} and κ_{4,i,j_i} are specified later on.
- The unknown continuous function $F^{c}_{i,j_i}(.)$, the virtual controller α_{i,j_i-1}, the actual controller u_{i,k_i} and the terms (z_{i,j_i}, E_{i,j_i} and $\wp_{i,j_i}$) are defined on the set $\Omega_{z_{i,j_i}} = \left\{ z_{i,j_i} \in \sim \middle| -\kappa_{3,i,j_i} < z_{i,j_i} < \kappa_{4,i,j_i} \right\}$, for $k_i = 1,\ldots,m_i$, $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- The fractional-order filters $\begin{cases} D^{\delta_{i,j_i}} \alpha^{c}_{i,j_i-1}(t) = b_{i,j_i}\left[\alpha_{i,j_i-1}(t) - \alpha^{c}_{i,j_i-1}(t)\right] \\ \alpha_{i,j_i-1}(0) = \alpha^{c}_{i,j_i-1}(0) \end{cases}$, $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$ will be used to overcome the so-called explosion of complexity issue caused by the repeated differentiation of virtual controllers.

To achieve the control goal, the following assumptions are imposed on system (1).

Assumption 1. The functions $\dfrac{\partial f_{i,j_i}\left(x, x_{i,j_i+1}, d\right)}{\partial x_{i,j_i+1}}$, $\sum_{l_i=1}^{r_i} \dfrac{\partial f_{i,n_i}\left(x, v_i, d\right)}{\partial v_{i,l_i}}$ and $\sum_{k_i=1}^{m_i} \dfrac{\partial \eta_{l_i,i}\left(u_i\right)}{\partial u_{i,k_i}}$ are unknown, nonzero and bounded, for $j_i = 1,\ldots,n_i - 1$, $i = 1,\ldots,p$.

Assumption 2. The functions $\alpha_{i,0}$, $\kappa_{3,i,1}$ and $\kappa_{4,i,1}$ satisfy the conditions $0 \le \left|\alpha_{i,0}\right| < \bar{\alpha}_{i,0}$, $\kappa_{3,i,1} = \kappa_{1,i,1} - \bar{\alpha}_{i,0} > 0$ and $\kappa_{4,i,1} = \kappa_{2,i,1} - \bar{\alpha}_{i,0} > 0$, for $i = 1,\ldots,p$.

Remark 9. In contrast to numerous existing studies [1, 4, 7, 9-11, 16, 19, 32, 41- 43, 51, 52, 55, 58, 64], the number of assumptions is low in this study. Recall that **Assumptions 1-2** are common, reasonable and standard because similar assumptions have already appeared in works related to many branches of sciences and engineering, such as mass-spring-damper systems, cancer immunotherapy systems, chemical reactors, and networked systems, and so on [1, 2, 21]. At the same time, **Assumption 1** can be regarded as necessary and sufficient conditions for the controllability of non-integer-order systems with actuator nonlinearities and pseudo-state constraints [1, 2, 7, 10, 11, 21, 32]. **Assumption 2** is the statement concerning the virtual control signals and desired trajectories [1, 2, 21]. For fractional-order systems with pseudo-State Constraints, appropriate bounded desired trajectories and virtual control signals need to be chosen [74].

Remark 10. In the open control literature, disturbances, reference signals and their fractional-order derivatives are assumed to be bounded [13, 21, 23-28].

3. Controller design and System stability analysis

Under **Assumptions 1-2**, the following tools will be exploited during the system controller design and stability analysis:

- the mean-value theorem to convert the original non-square, non-affine nonlinear system into a novel equivalent affine-in-control one with an equal number of inputs and outputs (affine- in-control square system),
- neural networks to estimate the uncertain nonlinear functions,
- Nussbaum functions to tackle the unknown control direction problem,
- A dynamic surface control approach based on fractional order filters and Caputo fractional derivative definitions to avoid the 'explosion of complexity' issue in the backstepping design process and to correctly choose the control signals and parameter update laws,
- and Asymmetric Barrier Lyapunov functions based on new lemmas and corollaries to prevent the pseudo-states from transgressing the limits and to ensure the stability of the transformed system.

3.1. System transformation

Using the differential Mean Value Theorem [2, 7, 15, 20, 21, 24-28, 47], the unknown functions on the right-hand side of the equation (1) may be expressed as

$$
\begin{cases}
f_{i,j_i}\left(x, x_{i,j_i+1}, d\right) = f_{i,j_i}\left(x, x_{i,j_i+1}, d\right)\Big|_{x_{i,j_i+1}=0} + \dfrac{\partial f_{i,j_i}\left(x,\ x_{i,j_i+1}, d\right)}{\partial x_{i,j_i+1}}\Bigg|_{x_{i,j_i+1}=x^0{}_{i,j_i+1}} x_{i,j_i+1} \\
f_{i,n_i}\left(x, \nu_i, d\right) = f_{i,n_i}\left(x, \nu_i, d\right)\Big|_{\nu_i=0_{r_i}} + \displaystyle\sum_{l_i=1}^{r_i} \dfrac{\partial f_{i,n_i}\left(x, \nu_i, d\right)}{\partial \nu_{i,l_i}}\Bigg|_{\nu_{i,1}=0,\ldots,\ \nu_{i,l_i-1}=0,\ \nu_{i,l_i}=\nu^0{}_{i,l_i}} \nu_{i,l_i} \\
\eta_{i,l_i}\left(u_i\right) = \eta_{i,l_i}\left(u_i\right)\Big|_{u_i=0_{m_i}} + \displaystyle\sum_{k_i=1}^{m_i} \dfrac{\partial \eta_{i,l_i}\left(u_i\right)}{\partial u_{i,k_i}}\Bigg|_{u_{i,1}=0,\ldots,\ u_{i,k_i-1}=0,\ u_{i,k_i}=u^0{}_{i,k_i}} u_{i,k_i},\quad l_i = 1,\ldots,r_i
\end{cases}
\tag{3}
$$

where $x^0{}_{i,j_i+1}$ is some point between zero and x_{i,j_i+1}; $u^0{}_{i,k_i}$ is some point between zero and u_{i,k_i}; $\nu^0{}_{i,l_i}$ is some point between zero and ν_{i,l_i}, $0_{r_i} = [0,\ldots,0]^T \in \sim^{r_i}$, and $0_{m_i} = [0,\ldots,0]^T \in \sim^{m_i}$, for $l_i = 1,\ldots,r_i$, $k_i = 1,\ldots,m_i$, $j_i = 1,\ldots,n_i - 1$, $i = 1,\ldots,p$.

Then, from (1)-(3), the tracking error dynamics can be rewritten as

$$
\begin{cases}
D^{\delta_{i,j_i}} z_{i,j_i} = F_{i,j_i}\left(\wp_{i,j_i}\right) + \hbar_{i,j_i}\kappa_{3,i,j_i}(0)\left(\dfrac{z_{i,j_i}}{\kappa_{3,i,j_i}{}^2}\right) D^{\delta_{i,j_i}}\kappa_{3,i,j_i} + \left(1-\hbar_{i,j_i}\right)\kappa_{4,i,j_i}(0)\left(\dfrac{z_{i,j_i}}{\kappa_{4,i,j_i}{}^2}\right) D^{\delta_{i,j_i}}\kappa_{4,i,j_i} \\
\qquad - g_{i,j_i-1}\dfrac{\left(\kappa_{3,i,j_i}{}^2 - z_{i,j_i}{}^2\right)\left(\kappa_{4,i,j_i}{}^2 - z_{i,j_i}{}^2\right)}{\hbar_{i,j_i}\kappa_{4,i,j_i}{}^2 + \left(1-\hbar_{i,j}\right)\kappa_{3,i,j_i}{}^2 - z_{i,j_i}{}^2} E_{i,j_i-1} \\
\qquad + g_{i,j_i}\alpha_{i,j_i} + g_{i,j_i}\dfrac{\left(\kappa_{3,i,j_i+1}{}^2 - z_{i,j_i+1}{}^2\right)\left(\kappa_{4,i,j_i+1}{}^2 - z_{i,j_i+1}{}^2\right)}{\hbar_{i,j_i+1}\kappa_{4,i,j_i+1}{}^2 + \left(1-\hbar_{i,j_i+1}\right)\kappa_{3,i,j_i+1}{}^2 - z_{i,j_i+1}{}^2} E_{i,j_i+1} \\
u_{i,k_i} = \alpha_{i,n_i},\ \text{for}\ k_i = 1,\ldots,m_i,\ j_i = 1,\ldots,n_i,\ i = 1,\ldots,p
\end{cases}
\tag{4}
$$

where

$$F_{i,j_i}\left(\wp_{i,j_i}\right)=f_{i,j_i}\left(x,\,x_{i,j_i+1},\,d\right)\Big|_{x_{i,j_i+1}=0}-F^{c}{}_{i,j_i}\left(\wp_{i,j_i}\right)-b_{i,j_i}\left[\alpha_{i,j_i-1}-\alpha^{c}{}_{i,j_i-1}\right]$$
$$-\hbar_{i,j_i}\kappa_{3,i,j_i}(0)\left(\frac{z_{i,j_i}}{\kappa_{3,i,j_i}{}^{2}}\right)D^{\delta_{i,j_i}}\kappa_{3,i,j_i}-\left(1-\hbar_{i,j_i}\right)\kappa_{4,i,j_i}(0)\left(\frac{z_{i,j_i}}{\kappa_{4,i,j_i}{}^{2}}\right)D^{\delta_{i,j_i}}\kappa_{4,i,j_i}\,,$$
$$+g_{i,j_i-1}\frac{\left(\kappa_{3,i,j_i}{}^{2}-z_{i,j_i}{}^{2}\right)\left(\kappa_{4,i,j_i}{}^{2}-z_{i,j_i}{}^{2}\right)}{\hbar_{i,j_i}\kappa_{4,i,j_i}{}^{2}+\left(1-\hbar_{i,j_i}\right)\kappa_{3,i,j_i}{}^{2}-z_{i,j_i}{}^{2}}E_{i,j_i-1}$$

$$F_{i,n_i}\left(\wp_{i,n_i}\right)=f_{i,n_i}\left(x,\,\nu_i,\,d\right)\Big|_{\nu_i=0_{r_i}}-F^{c}{}_{i,n_i}\left(\wp_{i,n_i}\right)-b_{i,n_i-1}\left[\alpha_{i,n_i-1}-\alpha^{c}{}_{i,n_i-1}\right]$$
$$-\hbar_{i,n_i}\kappa_{3,i,n_i}(0)\left(\frac{z_{i,n_i}}{\kappa_{3,i,n_i}{}^{2}}\right)D^{\delta_{i,n_i}}\kappa_{3,i,n_i}-\left(1-\hbar_{i,n_i}\right)\kappa_{4,i,n_i}(0)\left(\frac{z_{i,n_i}}{\kappa_{4,i,n_i}{}^{2}}\right)D^{\delta_{i,n_i}}\kappa_{4,i,n_i}$$
$$+\sum_{l_i=1}^{r_i}\frac{\partial f_{i,n_i}\left(x,\,\nu_i,\,d\right)}{\partial\nu_{i,l_i}}\Bigg|_{\nu_{i,1}=0,\ldots,\,\nu_{i,l_i-1}=0,\;\nu_{i,l_i}=\nu^{0}{}_{i,l_i}}\left[\Delta_{i,1,l_i}\,\eta_{i,l_i}\left(u_i\right)\Big|_{u_i=0_{m_i}}+\Delta_{i,2,l_i}\right]\,,$$
$$+g_{i,n_i-1}\frac{\left(\kappa_{3,i,n_i}{}^{2}-z_{i,n_i}{}^{2}\right)\left(\kappa_{4,i,n_i}{}^{2}-z_{i,n_i}{}^{2}\right)}{\hbar_{i,n_i}\kappa_{4,i,n_i}{}^{2}+\left(1-\hbar_{i,n_i}\right)\kappa_{3,i,n_i}{}^{2}-z_{i,n_i}{}^{2}}E_{i,n_i-1}$$

$$g_{i,j_i}=\frac{\partial f_{i,j_i}\left(x,\,x_{i,j_i+1},\,d\right)}{\partial x_{i,j_i+1}}\Bigg|_{x_{i,j_i+1}=x^{0}{}_{i,j_i+1}}$$, and

$$g_{i,n_i}=\sum_{k_i=1}^{m_i}\frac{\partial f_{i,n_i}\left(x,\,\nu_i,\,d\right)}{\partial\nu_{i,l_i}}\sum_{l_i=1}^{r_i}\Delta_{i,1,l_i}\frac{\partial\eta_{i,l_i}\left(u_i\right)}{\partial u_{i,k_i}}\Bigg|_{\substack{\nu_{i,1}=0,\ldots,\,\nu_{i,l_i-1}=0,\;\nu_{i,l_i}=\nu^{0}{}_{i,l_i}\\ u_{i,1}=0,\ldots,\,u_{i,k_i-1}=0,\;u_{i,k_i}=u^{0}{}_{i,k_i}}}$$ are unknown functions, with

$g_{i,0}=E_{i,0}=E_{i,n_i+1}=\kappa_{3,i,n_i+1}=\kappa_{4,i,n_i+1}=\hbar_{i,n_i+1}=z_{i,n_i+1}=0$, for $j_i=1,\ldots,n_i-1,\;i=1,\ldots,p$.

Remark 11. It is obvious that with intermediate control functions and newly pseudo-states defined in (2)-(3), the non-square, non-affine nonlinear system (1) is now transformed into an affine-in-control square system (4) with the input $u_{i,k_i}=\alpha_{i,n_i}$, for $k_i=1,\ldots,m_i,\quad i=1,\ldots,p$. Afterwards, the control purpose of plant (1) may be retained by controlling plant (4).

Remark 12. The biggest differences between the Taylor series expansion and the Mean Value Theorem are that:

- The first is only valid locally around specific points, whereas the second is valid globally [26, 28].
- The Mean Value Theorem can express the continuously differentiable functions over closed bounded intervals free of high-order approximating errors, while the Taylor series expansion can approximate these functions at points with high-order approximating errors [21, 28].
- The Taylor series expansion may frequently provide local stability [7], whereas the Mean Value Theorem can yield the semi-global stability of nonlinear systems as shown in **Theorem 1**.

Remark 13. Note that:

- According to Assumption 1, g_{i,j_i} and g_{i,n_i} are unknown, nonzero and bounded functions, for $j_i = 1,\ldots,n_i - 1$, $i = 1,\ldots,p$.
- The unknown function $F_{i,j_i}\left(\wp_{i,j_i}\right)$ is bounded over a compact set $\Omega_{i,j_i} = \left\{ \wp_{i,j_i} \in \sim^{c_{i,j_i}} \middle| \begin{array}{c} \left\|\wp_{i,j_i}\right\| \le M_{i,j_i} \\ \text{and} - \kappa_{3,i,j_i} < z_{i,j_i} < \kappa_{4,i,j_i} \end{array} \right\}$, with the strictly positive constant M_{i,j_i} being, unknown and arbitrary large, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.

3.2. Neural networks for Approximating the uncertain functions

By use of the universal approximation property of the neural networks (see Remark 13 and Lemma 1), the unknown function $F_{i,j_i}\left(\wp_{i,j_i}\right)$ can be approximated over the compact set Ω_{i,j_i} as follows

$$F_{i,j_i}\left(\wp_{i,j_i}\right) = w_{i,j_i}^{*T} \xi_{i,j_i}\left(\wp_{i,j_i}\right) + \varepsilon_{i,j_i}, \text{ for } j_i = 1,\ldots,n_i, \; i = 1,\ldots,p \tag{5}$$

where

- The vectors $\wp_{i,j_i} \in \sim^{c_{i,j_i}}$ and $\xi_{i,j_i}(.) \in \sim^{q_{i,j_i}}$ denote the input and hidden-layer activation function of the neural network, respectively, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- q_{i,j_i} is the number of neural network nodes, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- $c_{i,j_i} = n + 4\,j_i + 2$ represents the input layer size of the neural network, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- ε_{i,j_i} denotes an approximation error (corresponding reconstruction error) that satisfies $\left|\varepsilon_{i,j_i}\right| \le \varepsilon_{i,j_i}{}^{*}$ over a compact set Ω_{i,j_i} with $\varepsilon^{*}{}_{i,j_i}$ being an unknown strictly positive constant, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- $w_{i,j_i}{}^{*}$ is the ideal constant weight vector (or the optimal constant weight vector of the neural network) defined as $w_{i,j_i}{}^{*} = \arg\min_{w_{i,j_i} \in \sim^{q_{i,j_i}}} \left(\sup_{\wp_{i,j_i} \in \Omega_{i,j_i}} \left| F_{i,j_i}\left(\wp_{i,j_i}\right) - w_{i,j_i}{}^{T} \xi_{i,j_i}\left(\wp_{i,j_i}\right) \right| \right)$, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.

Remark 14. Note that:

- Neural networks can be grouped as recurrent and feedforward [3, 19, 21, 23-28, 58].
- Multilayer neural networks and linearly parameterized neural networks are the two major types of artificial neural network utilized to learn the unknown dynamics of nonlinear systems and to approximate many unknown functions in the design of control systems [3, 32, 34, 39, 46, 47, 49, 51, 55, 58, 59, 63, 74].
- The artificial neural network $w_{i,j_i}{}^{*T} \xi_{i,j_i}\left(\wp_{i,j_i}\right)$ used in this study is a single-hidden-layer or linearly parameterized neural network and is composed of three different layers: a hidden layer, an input layer, and an output layer [21, 23-28].
- Similar to [23-28], the bias parameters and input layer weights are initially chosen at random and kept fixed during the control system design, while the output layer weight vectors are only adapted and determined online in this study.
- The output layer neurons utilize linear activation functions while the hidden layer neurons regularly have non-linear activation functions, e.g. Radial basis functions (RBF), sigmoid functions, hyperbolic tangent functions and logistic functions [19, 21, 23-28].

- Similar to [21, 23], the hidden-layer activation functions $\xi_{i,j_i}(.)$, $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$ chosen in this research satisfy generalized Lipschitz conditions.
- Increasing the neural network nodes leads to a large number of adjustable variables and a very long online learning time [23, 34, 39, 46, 47, 49, 51, 55, 58, 59, 63, 74].

Remark 15. It is well known that artificial neural networks are broadly employed in diverse fields like system's control, modeling, forecasting, image recognition and processing, and many others [3, 19, 21, 34, 39, 46, 47, 49, 51, 55, 58, 59, 63, 74]. According to [19, 21, 23-28], the efficacy and performance of neural networks generally depend upon the following factors:

- Neuron structure and complexity
- Number of layers
- Number of neurons in each layer
- Type and number of interconnecting weights
- Initialization of weights and training parameters
- Selection of error-correction learning strategy
- Mode of error calculation
- Training algorithms

Remark 16. Various studies have proven that Radial Basis Function (RBF) neural networks are efficacious approaches and have better generalization abilities than multilayer neural networks for the pattern recognition and approximation of any unknown continuous functions in a compact set with arbitrary precisions [3, 19, 21, 23-28, 34, 39, 46, 47, 49, 51, 55, 58, 59]. Compared with other neural networks, RBF neural networks have a simple structure and faster learning speed [3, 19, 21, 23-28, 34, 39, 46, 47, 49, 51, 55, 58, 59, 63, 74]. In fact, an RBF neural network is a linearly parameterized neural network composed of a single hidden layer of nonlinear nodes, centered in such a way that each of them is defined on a special region of the input space [19, 21, 23-28, 34, 39, 46, 47, 49, 51]. Moreover, the hidden-layer neuron activation function is an RBF [19, 21, 23-28, 34, 39, 46]. The desired responses of RBF networks are obtained by updating the weights linking the linear output nodes with the hidden layer and by means of a training process [19, 21, 23-28, 34, 39, 46, 47, 49, 51, 55, 58, 59, 63, 74].

3.3. Controller with Adaptive control laws

By means of the backstepping dynamic surface control method [1, 23, 33], the parameter adaptive laws, virtual control signals and control inputs are constructed as follows

$$
\begin{cases}
\alpha_{i,j_i} = N\left(\zeta_{i,j_i}\right)\left[\Psi_{1,i,j_i} z_{i,j_i} + \Psi_{2,i,j_i} E_{i,j_i} + \chi_{4,i,j_i}\int_0^t E_{i,j_i} dt + \hat{w}_{i,j_i}{}^T \xi_{i,j_i}\left(\wp_{i,j_i}\right)\right], \\
u_{i,k_i} = \alpha_{i,n_i}, \text{for } k_i = 1,\ldots,m_i \\
\dot{\zeta}_{i,j_i} = \left[\Psi_{1,i,j_i} z_{i,j_i} + \Psi_{2,i,j_i} E_{i,j_i} + \chi_{4,i,j_i}\int_0^t E_{i,j_i} dt + \hat{w}_{i,j_i}{}^T \xi_{i,j_i}\left(\wp_{i,j_i}\right)\right] E_{i,j_i}, \\
\dot{\hat{w}}_{i,j_i} = \chi_{2,i,j_i} E_{i,j_i} \xi_{i,j_i}\left(\wp_{i,j_i}\right) - \left[\hat{w}_{i,j_i} - \hat{w}_{i,j_i}(0)\right]\left(\sigma + E_{i,j_i} z_{i,j_i}\right), \\
\dot{\hat{\theta}}_{i,j_i} = \chi_{3,i,j_i} \Phi_{i,j_i} E_{i,j_i}{}^2 - \left[\hat{\theta}_{i,j_i} - \hat{\theta}_{i,j_i}(0)\right]\sigma, \text{ for } j_i = 1,\ldots,n_i,\ i = 1,\ldots,p
\end{cases}
\tag{6}
$$

where

- The strictly positive function σ is known, and continuously differentiable, such that σ, $\dot{\sigma}$ and $\int_0^t \sigma(\tau) d\tau$ are bounded, $\forall\ t \geq 0$.
- The known positive functions $\Phi_{i,j_i} = \dfrac{\rho_{i,j_i}{}^2}{\sqrt{\rho_{i,j_i}{}^2 E_{i,j_i}{}^2 + \sigma^2}} + \dfrac{\rho_{i,j_i}{}^2 z_{i,j_i}{}^2}{\sqrt{\rho_{i,j_i}{}^2 E_{i,j_i}{}^2 z_{i,j_i}{}^2 + \sigma^2}}$, $\Psi_{1,i,j_i} = \chi_{1,i,j_i} + \dfrac{1}{E_{i,j_i} z_{i,j_i} + \sigma}$, and $\Psi_{2,i,j_i} = \hat{\theta}_{i,j_i} \Phi_{i,j_i} + \chi_{5,i,j_i} z_{i,j_i}{}^2$ are continuous with $\rho_{i,j_i} = \left\|\xi_{i,j_i}\left(\wp_{i,j_i}\right)\right\| + \left\|\wp_{i,j_i}\right\| + q_{i,j_i}$ being a known continuous function, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- The design parameter $\chi_{l,i,j_i} > 0$ is a known strictly positive constant, for $l = 1,\ldots,5$, $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.

- $\hat{\theta}_{i,j_i}$ and $\hat{w}_{i,j_i}$ are the estimates of the unknown constant parameters $\theta_{i,j_i}{}^* = 2\left\| w_{i,j_i}{}^* \right\| + \varepsilon_{i,j_i}{}^* + \frac{2\left\| w_{i,j_i}{}^* - \hat{w}_{i,j_i}(0) \right\|^2}{\chi_{2,i,j_i}} + 1$ and $w_{i,j_i}{}^*$, respectively, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- $N\left(\zeta_{i,j_i}\right) = \left(\zeta_{i,j_i}{}^2 + 2\right)\exp\left(\frac{\zeta_{i,j_i}{}^2}{2}\right)\sin\left(\zeta_{i,j_i}\right)$ is a Nussbaum-type function.

Remark 17. It should be emphasized that:

- The functions $\zeta^2\cos(\zeta)$, $\zeta^2\sin(\zeta)$, $\cos\left(\frac{\pi}{2}\zeta\right)\exp\left(\zeta^2\right)$, and $\zeta\cos\left(\sqrt{|\zeta|}\right)$ have been broadly employed in the control literature as Nussbaum-type functions to solve the difficulties arising from unknown control directions [9, 16, 21, 23-28].
- There are also groups of Nussbaum functions described by saturated functions and Mittag_Leffler functions [23].
- Similar to [21, 23-28], the Nussbaum function $N\left(\zeta_{i,j_i}\right) = \left(\zeta_{i,j_i}{}^2 + 2\right)\exp\left(\frac{\zeta_{i,j_i}{}^2}{2}\right)\sin\left(\zeta_{i,j_i}\right)$ is used to fix the problems of actuator nonlinearities, unknown control directions and unknown disturbances.

Remark 18. Note that:

- Similar to [42], the term $\chi_{4,i,j_i}\int_0^t E_{i,j_i}\,dt$ is used to deal with the problems of steady-state errors, for $j_i = 1,\ldots,n_i$, , $i = 1,\ldots,p$.
- According to Lemma 8, the term $\frac{z_{i,j_i}}{E_{i,j_i} z_{i,j_i} + \sigma}$ is used to ensure the semi-globally practical finite-time stability of the system, for $j_i = 1,\ldots,n_i$, , $i = 1,\ldots,p$.

- The term $\Psi_{2,i,j_i} E_{i,j_i}$ is employed to make the system robust against the neural network approximation errors, disturbances, uncertainties and errors caused by the command filters, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$ [21, 23].
- The term $\Psi_{1,i,j_i} z_{i,j_i}$ is used to boost the system performance in all respects while retaining the benefit of ensuring the robustness and asymptotic convergence of tracking errors in the presence of modeling uncertainties, unknown control directions and bounded disturbances, for $j_i = 1,\ldots,n_i$, , $i = 1,\ldots,p$ [21, 23].
- The e-modification term $\left[\hat{w}_{i,j_i} - \hat{w}_{i,j_i}(0)\right] E_{i,j_i} z_{i,j_i}$ and the sigma-modification terms ($\left[\hat{w}_{i,j_i} - \hat{w}_{i,j_i}(0)\right]\sigma$ and $\left[\hat{\theta}_{i,j_i} - \hat{\theta}_{i,j_i}(0)\right]\sigma$) are employed to guarantee the uniform boundedness of the adaptive parameters $\hat{w}_{i,j_i}$ and $\hat{\theta}_{i,j_i}$, for $j_i = 1,\ldots,n_i$, , $i = 1,\ldots,p$ [23-28, 42].
- For the inequality $\hat{\theta}_{i,j_i} \geq 0$ to be valid $\forall\, t \geq 0$, $\hat{\theta}_{i,j_i}(0)$ should be positive, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$ [21, 23-28].
- The possible issues in the control design arising from the control singularity and the chattering phenomenon are avoided because the control inputs, virtual control signals and parameter adaptive laws are continuous [21, 28].
- Similar to [21, 23-28], the controller parameters ($\zeta_{i,j_i}, \hat{w}_{i,j_i}, \hat{\theta}_{i,j_i}$), virtual control signal α_{i,j_i} and control input u_{i,k_i} are tuned online without any pre-training phase (or explicit offline training phase), for $k_i = 1,\ldots,m_i$, $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.

3.4. Stability analysis

Based on the above control design, the main results of this study are summarized in the following **Theorem 1**.

Theorem 1. *Consider the fractional-order system (1) with actuator nonlinearities and asymmetric time-varying pseudo-state constraints under Assumptions 1 and 2. Then, for all* *bounded initial conditions, the virtual controllers, actual controller, and adaptation laws defined by (6) guarantee that:*

- *All the signals in the closed–loop system are semi-globally uniformly ultimately bounded., i.e.,* ζ_{i,j_i}, $\hat{w}_{i,j_i}$, $\hat{\theta}_{i,j_i}$, $\alpha_{i,j_i}, z_{i,j_i}$, $\wp_{i,j_i}$, ρ_{i,j_i} , Ψ_{1,i,j_i}, Ψ_{2,i,j_i} *and* $\Phi_{i,j_i} \in L_\infty$, *for* $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- *The full pseudo-state constraints are never violated, i.e.,* $-\kappa_{1,i,j_i} < x_{i,j_i} < \kappa_{2,i,j_i}$, *for* $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- *The tracking error signals converge to zero asymptotically, i.e.,* $z_{i,j_i} \to 0$, *for* $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$ *as* $t \to +\infty$.

Proof. of Theorem 1

Consider the following asymmetric barrier Lyapunov function candidate

$$V = \sum_{i=1}^{p}\sum_{j_i=1}^{n_i} V_{1,i,j_i} + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i} V_{2,i,j_i} + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i} V_{3,i,j_i} + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i} V_{4,i,j_i} \tag{7}$$

with $V_{1,i,j_i} = \frac{1}{2} D^{\delta_{i,j_i}-1} V_{5,i,j_i}$, $V_{5,i,j_i} = \hbar_{i,j_i} \log\left(\frac{\kappa_{3,i,j_i}{}^2}{\kappa_{3,i,j_i}{}^2 - z_{i,j_i}{}^2}\right) + \left(1-\hbar_{i,j_i}\right)\log\left(\frac{\kappa_{4,i,j_i}{}^2}{\kappa_{4,i,j_i}{}^2 - z_{i,j_i}{}^2}\right)$,

$V_{2,i,j_i} = \frac{\left\|\tilde{w}_{i,j_i}\right\|^2}{2\chi_{2,i,j_i}} = \frac{\tilde{w}_{i,j_i}{}^T \tilde{w}_{i,j_i}}{2\chi_{2,i,j_i}}$, $V_{3,i,j_i} = \frac{\tilde{\theta}_{i,j_i}{}^2}{2\chi_{3,i,j_i}}$, $V_{4,i,j_i} = \frac{\chi_{4,i,j_i}}{2}\left[\int_0^t E_{i,j_i} dt\right]^2$, $\tilde{w}_{i,j_i} = w_{i,j_i}{}^* - \hat{w}_{i,j_i}$ and $\tilde{\theta}_{i,j_i} = \theta_{i,j_i}{}^* - \hat{\theta}_{i,j_i}$, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.

Remark 19. It is clear that:

- $D^{\delta_{i,j_i}-1}$ represents the fractional integer of order $1-\delta_{i,j_i}$, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.

- The Lyapunov functions V_{5,i,j_i}, V_{2,i,j_i}, V_{3,i,j_i} and V_{4,i,j_i} are convex with respect to the variables z_{i,j_i}, $\tilde{w}_{i,j_i}$, $\tilde{\theta}_{i,j_i}$ and $\int_0^t E_{i,j_i} dt$, for $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.
- Over the sets $\Omega_{z_{i,j_i}} = \left\{ z_{i,j_i} \in \sim \left| -\kappa_{3,i,j_i} < z_{i,j_i} < \kappa_{4,i,j_i} \right. \right\}$, $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$, the Lyapunov function V is continuously derivable and positive definite with respect to the error variables, including the tracking errors and parameter estimation errors.
- $$\begin{cases} V \geq 0,\ \forall t \geq 0 \\ V \to +\infty, \text{ if } z_{i,j_i} \to -\kappa_{3,i,j_i},\ z_{i,j_i} \to \kappa_{4,i,j_i}, \left\| \tilde{w}_{i,j_i} \right\| \to +\infty \text{ or } \tilde{\theta}_{i,j_i} \to +\infty \\ V = 0,\ \text{if } z_{i,j_i} = 0, \left\| \tilde{w}_{i,j_i} \right\| = 0 \text{ and } \tilde{\theta}_{i,j_i} = 0, \text{ for } j_i = 1,\ldots,n_i,\ i = 1,\ldots,p \end{cases}$$

Remark 20. It is generally known that Lyapunov-like functions comprising Volterra-type Lyapunov functions, quadratic Lyapunov functions, Convex Lyapunov functions, Lyapunov-like barrier functions, polynomial Lyapunov functions, exponential-type Lyapunov functions and many others have been successfully used in the literature to guarantee the stability of incommensurate and commensurate non integer-order plants [13, 17, 21, 23-28, 31, 33, 41, 50, 52, 53, 64-74].

Remark 21. In the literature, there are diverse forms of Lyapunov barrier functions such as tangent Lyapunov barrier functions, integral Lyapunov barrier functions, and so on [1-3, 8, 18, 21, 24, 28, 34, 43, 46, 55, 74]. The difference between Lyapunov barrier functions and classical Lyapunov functions lies mainly in the feature of the Lyapunov barrier functions that approach infinity when their arguments tend to certain limits [8, 18, 21, 24, 28, 34, 43, 46, 74].

Based on Property 1, Lemma 3, Corollaries 1-2 and Equations (4)-(5), the time derivatives of Lyapunov functions ($\sum_{i=1}^{p}\sum_{j=1}^{n_i} V_{1,i,j_i}$, $\sum_{i=1}^{p}\sum_{j=1}^{n_i} V_{2,i,j_i}$, $\sum_{i=1}^{p}\sum_{j=1}^{n_i} V_{3,i,j_i}$ and $\sum_{i=1}^{p}\sum_{j=1}^{n_i} V_{4,i,j_i}$) can be computed as follows

$$
\begin{cases}
\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\dot{V}_{1,i,j_i} = \frac{1}{2}\sum_{i=1}^{p}\sum_{j_i=1}^{n_i} D^{\delta_{i,j_i}}\left[\hbar_{i,j_i}\log\left(\frac{\kappa_{3,i,j_i}{}^2}{\kappa_{3,i,j_i}{}^2 - z_{i,j_i}{}^2}\right) + \left(1-\hbar_{i,j_i}\right)\log\left(\frac{\kappa_{4,i,j_i}{}^2}{\kappa_{4,i,j_i}{}^2 - z_{i,j_i}{}^2}\right)\right] \\
\qquad \le \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left[D^{\delta_{i,j}} z_{i,j_i} - \hbar_{i,j_i}\kappa_{3,i,j_i}(0)\left(\frac{z_{i,j_i}}{\kappa_{3,i,j_i}{}^2}\right)D^{\delta_{i,j}}\kappa_{3,i,j_i}\right. \\
\qquad\qquad \left. -\left(1-h_{i,j_i}\right)\kappa_{4,i,j_i}(0)\left(\frac{z_{i,j_i}}{\kappa_{4,i,j_i}{}^2}\right)D^{\delta_{i,j_i}}\kappa_{4,i,j_i}\right]E_{i,j_i} \\
\qquad \le \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left[F_{i,j_i}\left(\wp_{i,j_i}\right) + g_{i,j_i}\frac{\left(\kappa_{3,i,j_i+1}{}^2 - z_{i,j_i+1}{}^2\right)\left(\kappa_{4,i,j_i+1}{}^2 - z_{i,j_i+1}{}^2\right)}{\hbar_{i,j+1}\kappa_{4,i,j_i+1}{}^2 + \left(1-\hbar_{i,j_i+1}\right)\kappa_{3,i,j_i+1}{}^2 - z_{i,j_i+1}{}^2}E_{i,j_i+1}\right. \\
\qquad\qquad \left. -g_{i,j_i-1}\frac{\left(\kappa_{3,i,j_i}{}^2 - z_{i,j_i}{}^2\right)\left(\kappa_{4,i,j_i}{}^2 - z_{i,j_i}{}^2\right)}{\hbar_{i,j_i}\kappa_{4,i,j_i}{}^2 + \left(1-\hbar_{i,j_i}\right)\kappa_{3,i,j_i}{}^2 - z_{i,j_i}{}^2}E_{i,j_i-1} + g_{i,j_i}\alpha_{i,j_i}\right]E_{i,j_i} \\
\qquad \le \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left[F_{i,j_i}\left(\wp_{i,j_i}\right) + g_{i,j_i}\alpha_{i,j_i}\right]E_{i,j_i} \\
\qquad \le \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left[w_{i,j_i}{}^{*T}\xi_{i,j_i}\left(\wp_{i,j_i}\right) + \varepsilon_{i,j_i} + g_{i,j_i}\alpha_{i,j_i}\right]E_{i,j_i} \\
\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\dot{V}_{2,i,j_i} = \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\tilde{w}_{i,j_i}{}^T\dot{\tilde{w}}_{i,j_i}}{\chi_{2,i,j_i}} \\
\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\dot{V}_{3,i,j_i} = \sum_{i=1}^{p}\sum_{j=1}^{n_i}\frac{\tilde{\theta}_{i,j_i}\dot{\tilde{\theta}}_{i,j_i}}{\chi_{3,i,j_i}} \\
\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\dot{V}_{4,i,j_i} = \sum_{i=1}^{p}\sum_{j=1}^{n_i}\chi_{4,i,j_i}E_{i,j_i}\int_0^t E_{i,j_i}\,dt
\end{cases}
\tag{8}
$$

Invoking (6) and (8), the derivative of V with respect to time is given by

$$
\begin{aligned}
\dot{V} \le & \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left[w_{i,j_i}{}^{*T}\xi_{i,j_i}\left(\wp_{i,j_i}\right)+\varepsilon_{i,j_i}+g_{i,j_i}\alpha_{i,j_i}\right]E_{i,j_i}+\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\tilde{w}_{i,j_i}{}^{T}\dot{\tilde{w}}_{i,j_i}}{\chi_{2,i,j_i}} \\
& +\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\tilde{\theta}_{i,j_i}\dot{\tilde{\theta}}_{i,j_i}}{\chi_{3,i,j_i}}+\sum_{i=1}^{p}\sum_{j=1}^{n_i}\chi_{4,i,j_i}E_{i,j_i}\int_0^t E_{i,j_i}dt \\
\le & -\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{1,i,j_i}E_{i,j_i}z_{i,j_i}-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{E_{i,j_i}z_{i,j_i}}{E_{i,j_i}z_{i,j_i}+\sigma} \\
& -\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{5,i,j_i}z_{i,j_i}{}^{2}E_{i,j_i}{}^{2}+\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left[1+g_{i,j_i}N\left(\zeta_{i,j_i}\right)\right]\dot{\zeta}_{i,j_i} \\
& +\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\tilde{w}_{i,j_i}{}^{T}\left[\frac{\dot{\tilde{w}}_{i,j_i}}{\chi_{2,i,j_i}}+E_{i,j_i}\xi_{i,j_i}\left(\wp_{i,j_i}\right)\right]+\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\tilde{\theta}_{i,j_i}\left[\frac{\dot{\tilde{\theta}}_{i,j_i}}{\chi_{3,i,j_i}}+\Phi_{i,j_i}E_{i,j_i}{}^{2}\right] \\
& +\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left|\varepsilon_{i,j_i}{}^{*}E_{i,j_i}\right|-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\theta_{i,j_i}{}^{*}\Phi_{i,j_i}E_{i,j_i}{}^{2}
\end{aligned}
\tag{9}
$$

Based on (6)-(7), Lemmas 2, 6, 12 and Young's inequality [21, 23-28], one has

$$\left\{\begin{aligned}
&\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\tilde{w}_{i,j_i}{}^{T}\left[\frac{\dot{\tilde{w}}_{i,j_i}}{\chi_{2,i,j_i}}+E_{i,j_i}\xi_{i,j_i}\left(\wp_{i,j_i}\right)\right]\le-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left(\sigma+E_{i,j_i}z_{i,j_i}\right)\frac{\left\|\tilde{w}_{i,j_i}\right\|^2}{2\chi_{2,i,j_i}}\\
&\qquad+\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left(\sigma+E_{i,j_i}z_{i,j_i}\right)\frac{\left\|w_{i,j_i}{}^{*}-\hat{w}_{i,j_i}\left(0\right)\right\|^2}{2\chi_{2,i,j_i}}\\
&\qquad\le-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left(\sigma+V_{5,i,j_i}\right)V_{2,i,j_i}\\
&\qquad+\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left(\sigma+E_{i,j_i}z_{i,j_i}\right)\frac{\left\|w_{i,j_i}{}^{*}-\hat{w}_{i,j_i}\left(0\right)\right\|^2}{2\chi_{2,i,j_i}}\\
&\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\tilde{\theta}_{i,j_i}\left[\frac{\dot{\tilde{\theta}}_{i,j_i}}{\chi_{3,i,j_i}}+\Phi_{i,j_i}E_{i,j_i}{}^{2}\right]\le-\sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\tilde{\theta}_{i,j_i}{}^{2}}{2\chi_{3,i,j_i}}+\sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left[\theta_{i,j_i}{}^{*}-\hat{\theta}_{i,j_i}\left(0\right)\right]^2}{2\chi_{3,i,j_i}}\\
&\qquad\le-\sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}V_{3,i,j_i}+\sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left[\theta_{i,j_i}{}^{*}-\hat{\theta}_{i,j_i}\left(0\right)\right]^2}{2\chi_{3,i,j_i}}\\
&\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left|\varepsilon_{i,j_i}{}^{*}E_{i,j_i}\right|+\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left\|w_{i,j_i}{}^{*}-\hat{w}_{i,j_i}\left(0\right)\right\|^2}{2\chi_{2,i,j_i}}E_{i,j_i}z_{i,j_i}\\
&\qquad-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\theta_{i,j_i}{}^{*}\Phi_{i,j_i}E_{i,j_i}{}^{2}\le\sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\theta_{i,j_i}{}^{*}\\
&-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{1,i,j_i}E_{i,j_i}z_{i,j_i}\le-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{1,i,j_i}V_{5,i,j_i}\\
&-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{E_{i,j_i}z_{i,j_i}}{E_{i,j_i}z_{i,j_i}+\sigma}\le-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{V_{5,i,j_i}}{V_{5,i,j_i}+\sigma}\\
&-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{5,i,j_i}z_{i,j_i}{}^{2}E_{i,j_i}{}^{2}\le-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{5,i,j_i}V_{5,i,j_i}{}^{2}
\end{aligned}\right. \tag{10}$$

Based on (10), (9) can be rewritten as

$$\dot{V} \le -\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{1,i,j_i}V_{5,i,j_i} - \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{V_{5,i,j_i}}{V_{5,i,j_i}+\sigma} - \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{5,i,j_i}V_{5,i,j_i}{}^{2}$$
$$-\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left(\sigma+V_{5,i,j_i}\right)V_{2,i,j_i} - \sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}V_{3,i,j_i} + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left[1+g_{i,j_i}N\left(\zeta_{i,j_i}\right)\right]\dot{\zeta}_{i,j_i} \tag{11}$$
$$+\sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left\|w_{i,j_i}{}^{*}-\hat{w}_{i,j_i}(0)\right\|^{2}}{2\chi_{2,i,j_i}} + \sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left[\theta_{i,j_i}{}^{*}-\hat{\theta}_{i,j_i}(0)\right]^{2}}{2\chi_{3,i,j_i}} + \sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\theta_{i,j_i}{}^{*}$$

Integrating (11) over $[0,\, t]$, one gets

$$V(t) + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{1,i,j_i}\int_0^t V_{5,i,j_i}(\tau)d\tau + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{5,i,j_i}\int_0^t V_{5,i,j_i}{}^{2}(\tau)d\tau + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\int_0^t \frac{V_{5,i,j_i}(\tau)}{V_{5,i,j_i}(\tau)+\sigma(\tau)}d\tau$$
$$+\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\int_0^t\left(\sigma(\tau)+V_{5,i,j_i}(\tau)\right)V_{2,i,j_i}(\tau)d\tau + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\int_0^t\sigma(\tau)V_{3,i,j_i}(\tau)d\tau$$
$$\le V(0) + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left\|w_{i,j_i}{}^{*}-\hat{w}_{i,j_i}(0)\right\|^{2}}{2\chi_{2,i,j_i}}\int_0^t\sigma(\tau)d\tau + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\theta_{i,j_i}{}^{*}\int_0^t\sigma(\tau)d\tau \tag{12}$$
$$+\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\int_0^t\left[1+g_{i,j_i}N\left(\zeta_{i,j_i}(\tau)\right)\right]\dot{\zeta}_{i,j_i}(\tau)d\tau + \sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left[\theta_{i,j_i}{}^{*}-\hat{\theta}_{i,j_i}(0)\right]^{2}}{2\chi_{3,i,j_i}}\int_0^t\sigma(\tau)d\tau$$

With the aid of Lemmas 1-12 and Equation (6), we are able to know that $V(t)$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{5,i,j_i}\int_0^t V_{5,i,j_i}{}^{2}(\tau)d\tau$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\int_0^t\left(\sigma(\tau)+V_{5,i,j_i}(\tau)\right)V_{2,i,j_i}(\tau)d\tau$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\int_0^t\sigma(\tau)V_{3,i,j_i}(\tau)d\tau$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\chi_{1,i,j_i}\int_0^t V_{5,i,j_i}(\tau)d\tau$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\int_0^t\left[1+g_{i,j_i}N\left(\zeta_{i,j_i}(\tau)\right)\right]\dot{\zeta}_{i,j_i}(\tau)d\tau$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\int_0^t\frac{V_{5,i,j_i}(\tau)}{V_{5,i,j_i}(\tau)+\sigma(\tau)}d\tau$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\theta_{i,j_i}{}^{*}\int_0^t\sigma(\tau)d\tau$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left\|w_{i,j_i}{}^{*}-\hat{w}_{i,j_i}(0)\right\|^{2}}{2\chi_{2,i,j_i}}\int_0^t\sigma(\tau)d\tau$, $\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left[\theta_{i,j_i}{}^{*}-\hat{\theta}_{i,j_i}(0)\right]^{2}}{2\chi_{3,i,j_i}}\int_0^t\sigma(\tau)d\tau$ are bounded, and accordingly, E_{i,j_i}, z_{i,j_i}, V_{5,i,j_i}, $\|\tilde{w}_{i,j_i}\|$, $\tilde{\theta}_{i,j_i}$, $\dot{\zeta}_{i,j_i}$, $N(\zeta_{i,j_i})$, u_{i,k_i}, α_{i,j_i} and $\wp_{i,j_i}$ are bounded, for $k_i = 1,\dots,m_i$, $j_i = 1,\dots,n_i$, $i = 1,\dots,p$.

Letting the strictly positive constant $\bar{\alpha}_{i,j_i}$ be the upper bound of $|\alpha_{i,j_i}|$, for $j_i = 1,\dots,n_i - 1$, $i = 1,\dots,p$.

From $x_{i,j_i} = z_{i,j_i} + \alpha_{i,j_i-1}$ and $|\alpha_{i,j_i-1}| \le \bar{\alpha}_{i,j_i-1}$, we are able to know that $|x_{i,j_i}| \le |z_{i,j_i}| + |\alpha_{i,j_i-1}|$ and $-\kappa_{1,i,j_i} < x_{i,j_i} < \kappa_{2,i,j_i}$, for $j_i = 1,\dots,n_i$, $i = 1,\dots,p$. Define $\kappa_{3,i,j_i} = \kappa_{1,i,j_i} - \bar{\alpha}_{i,j_i-1} > 0$ and $\kappa_{4,i,j_i} = \kappa_{2,i,j_i} - \bar{\alpha}_{i,j_i-1} > 0$, then, we get $-\kappa_{1,i,j_i} < x_{i,j_i} < \kappa_{2,i,j_i}$, for $j_i = 1,\dots,n_i$, $i = 1,\dots,p$. Consequently, the system pseudo-state constraints are never violated.

Hence, it is clear that $z_{i,j_i} \in L_2$, for $j_i = 1,\dots,n_i$, $i = 1,\dots,p$.

Since Assumptions 1-2 hold, the functions ($F_{i,j_i}(.)$, $F^c{}_{i,j_i}(.)$) and the variable $\wp_{i,j}$ are bounded functions over the compact set Ω_{i,j_i}, then, one has $D^{\delta_{i,j_i}} z_{i,j_i} \in L_\infty$, and $z_{i,j_i} \in L_2 \cap L_\infty$, for $j_i = 1,\dots,n_i$, $i = 1,\dots,p$.

Since $\left[1 + g_{i,j_i} N(\zeta_{i,j_i})\right]\dot{\zeta}_{i,j_i}$, $\int_0^t \sigma(\tau)\,d\tau$, $\dfrac{\left[\theta_{i,j_i}{}^* - \hat{\theta}_{i,j_i}(0)\right]^2}{2\chi_{3,i,j_i}}\sigma$, $\dfrac{\left\|w_{i,j_i}{}^* - \hat{w}_{i,j_i}(0)\right\|^2}{2\chi_{2,i,j_i}}\sigma$, $\theta_{i,j_i}{}^*\sigma$ and $V(0)$ are bounded, therefore, there are strictly positive constants ℓ_5, ℓ_7 and ℓ_{6,i,j_i} such that

$$\begin{cases} \displaystyle\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\left[1 + g_{i,j_i} N(\zeta_{i,j_i})\right]\dot{\zeta}_{i,j_i} + \sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left\|w_{i,j_i}{}^* - \hat{w}_{i,j_i}(0)\right\|^2}{2\chi_{2,i,j_i}} \\ \qquad + \sigma\displaystyle\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\frac{\left[\theta_{i,j_i}{}^* - \hat{\theta}_{i,j_i}(0)\right]^2}{2\chi_{3,i,j}} + \sigma\sum_{i=1}^{p}\sum_{j_i=1}^{n_i}\theta_{i,j_i}{}^* \le \ell_7 \\ \ell_{6,i,j_i} = \sqrt{\left(\dfrac{\chi_{1,i,j_i}}{2\chi_{5,i,j_i}}\right)^2 + \dfrac{\ell_7}{\chi_{5,i,j_i}}} - \dfrac{\chi_{1,i,j_i}}{2\chi_{5,i,j_i}} \\ \int_0^t \sigma(\tau)\,d\tau \le \ell_5 \end{cases}, \text{ for } j_i = 1,\dots,n_i,\ i = 1,\dots,p \tag{13}$$

From Lemma 8, it is straightforward to get

$$V_{5,i,j_i} \le \ell_{6,i,j_i} \quad , \quad \forall\, t \ge \frac{\ell_5}{\ell_{6,i,j_i}} + V(0), \text{ for } j_i = 1,\dots,n_i,\ i = 1,\dots,p \tag{14}$$

The above inequality (14) means that

$$-\kappa_{3,i,j_i}\sqrt{1-\exp\left(-\ell_{6,i,j_i}\right)} \le z_{i,j_i} \le \kappa_{4,i,j_i}\sqrt{1-\exp\left(-\ell_{6,i,j_i}\right)}\ ,\ \forall t \ge \frac{\ell_5}{\ell_{6,i,j_i}} + V(0), \text{ for } j_i = 1,\dots,n_i,\ i = 1,\dots,p \tag{15}$$

Since z_{i,j_i} is a function of the variables ζ_{i,j_i}, $\hat{w}_{i,j_i}$, $\hat{\theta}_{i,j_i}$, α_{i,j_i}, z_{i,j_i}, $\wp_{i,j_i}$, ρ_{i,j_i}, Ψ_{1,i,j_i}, Ψ_{2,i,j_i} and Φ_{i,j_i}, for $j_i = 1,\dots,n_i$, $i = 1,\dots,p$, together with (4), (12), (15), it follows that the solution of the closed loop system is semi-global practical finite-time stable.

Finally, based on the aforementioned results, Lemmas 1- 11 and Corollaries 1-2, we can deduce that :

- The full pseudo-state constraints are not overstepped.
- The signals of the closed loop system are semi globally uniformly bounded.
- Asymptotic tracing is achieved, i.e., $x_{i,1}(t) \to y_{di}(t)$ as $t \to +\infty$, for $i = 1,\dots,p$.
- The tracking errors asymptotically tend to zero, i.e., $\lim_{t \to +\infty} z_{i,j_i}(t) = 0$, for $j_i = 1,\dots,n_i$, $i = 1,\dots,p$.

The proof of Theorem 1 is completed.

□

Remark 22. Note that:

- L_∞ stands for the set of bounded functions.
- L_2 denotes the set of square-integrable functions.

Remark 23. Similar to most studies of adaptive control schemes based on intelligent systems (such as fuzzy systems and neural networks) [3-5, 7-12, 15, 16, 19-21, 23-34, 39, 41-44, 46-49, 51, 54, 55, 61-63], our stability results are semi-global for the reason that the approximation capabilities of the neural networks are only valid over compact sets.

3.5. Implementation and Application of the proposed adaptive controller

It is obvious that the proposed controller scheme can be applicable to diverse practical plants, which possess mathematical structures like (1). To summarize the abovementioned results and analysis, a flowchart and a block diagram describing the practical implementation of the proposed neural adaptive controller strategy are shown in Figures 1 and 2, respectively. Note from Figures 1 and 2 that the proposed controller consists of parameters tuned online without any offline-learning phase and might be implemented in a systematic and methodical manner as follows:

Step 1. Select the design parameters (strictly positive constants b_{i,j_i} and χ_{l,i,j_i}, $l=1,\dots,5$), number of neural network nodes q_{i,j_i}, neural activation function $\xi_{i,j_i}(.)$, initial conditions ($\hat{\theta}_{i,j_i}(0)$, $\zeta_{i,j_i}(0)$, $\hat{w}_{i,j_i}(0)$) and the appropriate function σ employed in the system controller design, for $j_i=1,\dots,n_i$, $i=1,\dots,p$.

Step 2. Determine $\wp_{i,j_i}$ using (2), for $j_i=1,\dots,n_i$, $i=1,\dots,p$.

Step 3. Compute the parameter adaptive laws, virtual control signals and control inputs using (6) with $\kappa_{3,i,j_i}=\kappa_{1,i,j_i}-\bar{\alpha}_{i,j_i-1}>0$, $\kappa_{4,i,j_i}=\kappa_{2,i,j_i}-\bar{\alpha}_{i,j_i-1}>0$, and $\bar{\alpha}_{i,j_i-1}$ being the upper bound of $\left|\alpha_{i,j_i-1}\right|$, for $j_i=1,\dots,n_i$, $i=1,\dots,p$.

Step 4. Apply the proposed controller to system (1).

Step 5. Measure the output y and pseudo-state x of system (1)

Step 6. Go back to Step 2 for the next sampling time interval.

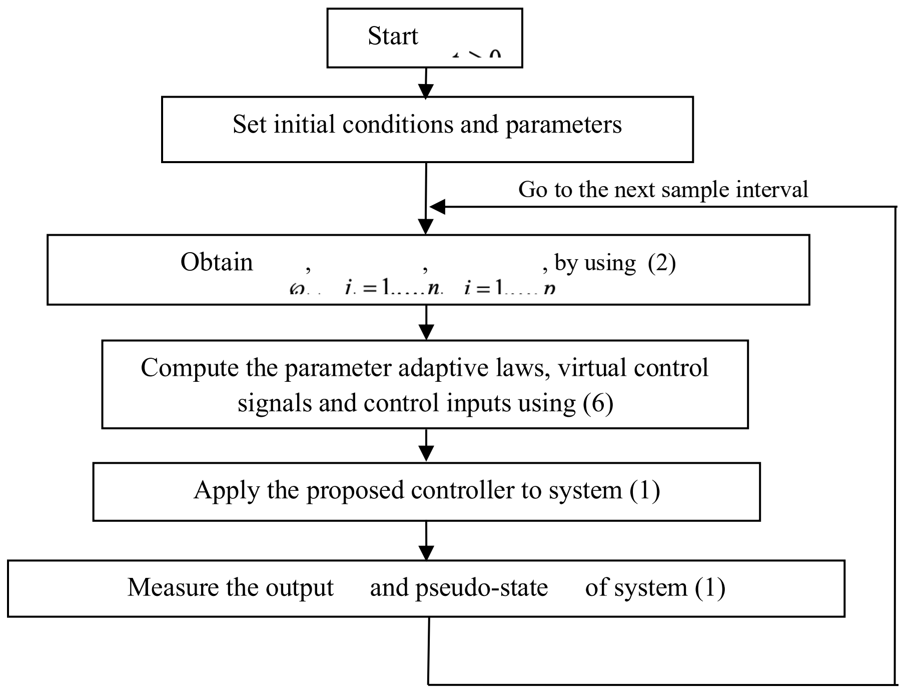

Figure 1. Flow chart of the proposed neural adaptive control algorithm

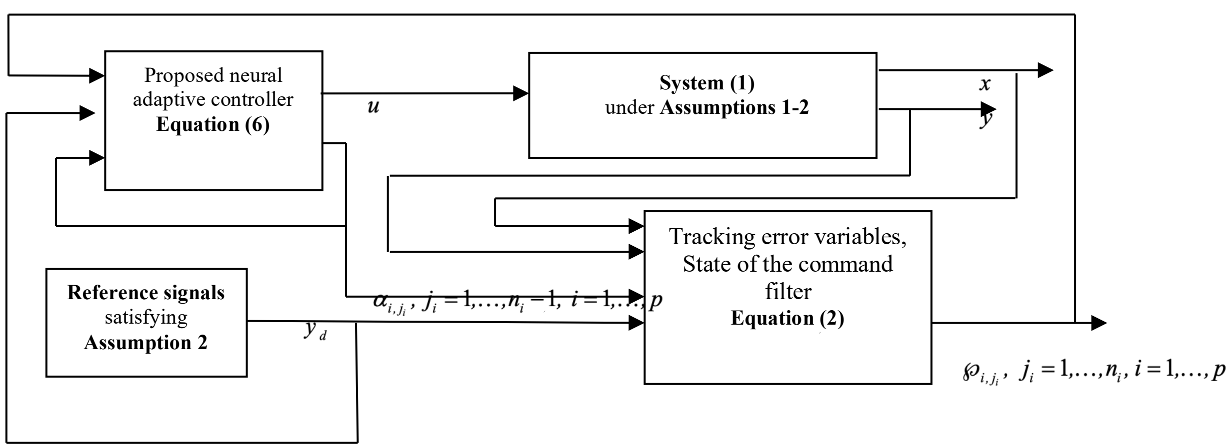

Figure 2. Overall block diagram of system (1) using the proposed neural adaptive controller

Remark 24. Theoretically speaking, an inordinately large number of nodes (neurons) in the hidden layers can minimize the neural approximation errors but may exacerbate the design complexity of the control system [3, 21, 23-28]. Generally, the number of nodes in the input layer is close to that of hidden layer neurons in the neural networks [21, 28]. Convergence of both the parameter estimation errors and tracking control errors can be achieved simultaneously in finite time by selecting sufficiently large design parameters (b_{i,j_i} and χ_{l,i,j_i}, $l = 1,\ldots,5$, $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$) and a sufficiently small function σ. Nevertheless, this might provoke huge amplitudes of the control input signals. It should be emphasized is that the tradeoffs between the control effort and the convergence performance must be taken into account for real systems and applications [1, 21, 23-28].

3.6 Comparison to existing control scheme studies

The main benefits of this paper are as follows:

- To the best of the authors' knowledge, there has been no research to date of neural adaptive control issues for the considered category of non-square incommensurate fractional-order plants with non-affine characteristics, unknown control directions, asymmetric time-varying pseudo-State constraints, disturbances, and input nonlinearities.
- Unlike [1, 4, 7, 9-11, 16, 19, 32, 41- 43, 51, 52, 55, 58, 64], many hypotheses and conditions correlated to the functions, dynamics, control directions, input nonlinearities, number of outputs and inputs, constraints, and estimation errors of controlled plants are not indispensable in this study.
- New corollaries and lemmas based on Caputo definitions are proposed to extend approaches and methods concerning the controllability and stability analysis of integer-order plants as opposed to fractional-order ones.
- In contrast to control schemes [17, 41], the proposed control method handles a wide variety of issues existing in practical engineering plants and applications (issues resulting from non-square structures, non-affine characteristics, unknown control directions, asymmetric time-varying pseudo-State constraints, dynamical disturbances, input nonlinearities, algebraic loops, steady-state errors and inhomogeneous (or incommensurate) fractional-orders).
- Neither singularity issues nor the chattering phenomenon emerge in the plant's response because the parameter adaptive laws, virtual control signals and control inputs are continuous over the sets $\Omega_{z_{i,j_i}} = \left\{ z_{i,j_i} \in \sim \middle| -\kappa_{3,i,j_i} < z_{i,j_i} < \kappa_{4,i,j_i} \right\}$, $j_i = 1,\ldots,n_i$, $i = 1,\ldots,p$.

- Unlike [33], a new adaptive dynamic surface control method based on fractional order filters and Caputo fractional derivative definitions is proposed in this study to solve the ‘explosion of complexity’ issue and to ensure the semi-global practical finite-time stability of the system signals regardless of initial pseudo-states bounded values and without violating the constraints, simultaneously.

Remark 25. The limitations of our research are that:

- The proposed control scheme cannot handle the encountered difficulties in various industrial systems posed by delay and unmeasured sates. Therefore, it could not be implemented directly to the nonlinear systems with time delays and unmeasured pseudo-states.
- The proposed controller design is somewhat complicated in the sense that many variables are updated during the control command calculation. In fact, the control objective of this paper is attained with relativey high computational burden. This issue will become less important as computer and electronic technologies evolve.

4. Simulation Results and analyses

In this part, two examples are given to illustrate the validity, feasibility and effectiveness of the proposed control scheme. Similar to [21, 23-28], the simulations are carried out using the Matlab routine with the time step size 10^{-4} and by applying an improved version of the Adams Bashforth-Moulton algorithm based on a predictor-corrector scheme.

Example 1: (An academic system)

Consider the following fractional-order plant

$$\begin{cases} D^{\delta_{i,1}} x_{i,1} = f_{i,1}\left(x,\ x_{i,2},\ d\ \right), \\ D^{\delta_{i,2}} x_{i,2} = f_{i,2}\left(x,\ \nu_i,\ d\right), \\ \nu_i = \eta_{0,i}\left(u_i\right), \\ y_i = x_{i,1}\text{, for } i = 1,2 \end{cases} \tag{16}$$

where

- the fractional orders of system (16) are $\delta_{i,1} = 0.45$, and $\delta_{i,2} = 0.55$, for $i = 1,2$.
- $y = \left[y_1, y_2\right]^T \in \sim^{2}$, $x = \left[x_1^T, x_2^T\right]^T \in \sim^{4}$, and $u = \left[u_1^T, u_2^T\right]^T \in \sim^{6}$ denote the system output, pseudo-state and input, respectively, with $x_i = \left[x_{i,1}, x_{i,2}\right]^T \in \sim^{2}$, $u_i = \left[u_{i,1}, u_{i,2}, u_{i,3}\right]^T \in \sim^{3}$, $x_{i,2} \in \sim$ and $y_i \in \sim$, for $i = 1,2$.
- $d = \left[d_{1,1}, d_{1,2}, d_{2,1}, d_{2,2}\right]^T \in \sim^{4}$ represents an external disturbance with $d_{i,j_i} = \dfrac{\sin\left(1+t\right)}{4+i+j_i}$, for $j_i = 1,2$, $i = 1,2$.
- The pseudo-states need to satisfy $-\kappa_{1,i,j_i} < x_{i,j_i} < \kappa_{2,i,j_i}$, with $\kappa_{1,i,j_i} = 13 + \dfrac{j_i}{1+i} + \dfrac{1}{j_i}\sin\left(\dfrac{t}{7}i\right)$ and $\kappa_{2,i,j_i} = 14 + \dfrac{2j}{3+i} + \dfrac{1}{j_i}\cos\left(\dfrac{t}{4}i\right)$, $j_i = 1,2$, $i = 1,2$ and $t \in \left[0,\ 40\right]$.
- The initial conditions of the system pseudo-states are given as $x_1\left(0\right) = \left[-4,\ 8.26\right]^T$ and $x_2\left(0\right) = \left[9.8,\ -9\right]^T$.
- $\eta_{0,i}\left(.\right) \in \sim^{3}$ is a vector of continuous functions, for $i = 1,2$.

The outputs of actuator nonlinearities and system functions are mathematically described respectively as follows

$$
\begin{cases}
v_{i,k_i} = \begin{cases} -\dfrac{15}{k_i}, \ \text{if}\ u_{i,k_i} < -\dfrac{15}{k_i} \\ u_{i,k_i}, \quad \text{if} -\dfrac{15}{k_i} \le u_{i,k_i} \le \dfrac{20}{k}, \ \text{for}\ k_i = 1,\ldots,3 \\ \dfrac{20}{k_i}, \ \text{if}\ u_{i,k_i} > \dfrac{20}{k_i} \end{cases} \\
v_{i,4} = \begin{cases} \left(1+\exp\left(-\left|\sum_{k_i=1}^{3} v_{i,k_i}\right| i\right)\right)\left(1.2u_{i,4}-2.4\right), \ \text{if}\ u_{i,4} \ge 2 \\ 0, \qquad \text{if}\ -1 \le u_{i,4} \le 2 \\ \left(3+\exp\left(-\left|\sum_{k_i=1}^{3} v_{i,k_i}\right| 3i\right)\right)\left(0.8u_{i,4}+0.8\right), \quad \text{if}\ u_{i,4} \le -1 \end{cases} \\
u_i = \left[u_{i,1}, \ldots\, u_{i,3}, u_{i,4}\right]^T \in \sim^{4} \\
v_i = \left[v_{i,1}, \ldots\, v_{i,3},\ v_{i,4}\right]^T \in \sim^{4}, \ \text{for}\ i = 1,2 \\
v = \left[v_1^T, v_2^T\right]^T \in \sim^{8}
\end{cases} \tag{17}
$$

$$
\begin{cases}
f_{i,1}\left(x,\ x_{i,2},\ d\ \right) = \left[1 + \dfrac{\sin\left(1+\|x\|\right)}{5i+1}\right] x_{i,2} + d_{i,1}, \\
f_{i,2}\left(x, v_i,\ d\right) = -\dfrac{\|x_i\|^2 i}{2} + \exp\left(-\dfrac{\|x\| + \|v_i\|}{4}\right) + \sum_{k_i=1}^{4}\left[3i + \dfrac{\cos\left(1+2\|x_i\|\right)}{7k_i+3i}\right] v_{i,k_i} + d_{i,4}, \ \text{for}\ i = 1,2
\end{cases} \tag{18}
$$

The reference signal is selected as $y_d = \begin{bmatrix} y_{d1} \\ y_{d2} \end{bmatrix} = \begin{bmatrix} 5\cos\left(\dfrac{t}{3}\right) \\ 7\sin\left(\dfrac{t}{5}\right) \end{bmatrix}$.

From [8, 19, 25, 27, 35, 43, 50, 56, 69, 70], the outputs of actuator nonlinearities can be rewritten as

$$\nu_i = \Delta_{i,1} \odot \eta_i\left(u_i\right) + \Delta_{i,2}$$
(19)

with $\Delta_{i,1} = \left[\Delta_{i,1,1}, \Delta_{i,1,2}, \Delta_{i,1,3}, \Delta_{i,1,4}\right]^T \in \sim^4$, $\Delta_{i,2} = \left[\Delta_{i,2,1}, \Delta_{i,2,2}, \Delta_{i,2,3}, \Delta_{i,2,4}\right]^T \in \sim^4$

$\eta_i\left(u_i\right) = \left[\eta_{i,1}\left(u_i\right), \eta_{i,2}\left(u_i\right), \eta_{i,3}\left(u_i\right), \eta_{i,4}\left(u_i\right)\right]^T \in \sim^4$, $\Delta_{i,1,k_i} = \left(\dfrac{5}{2k_i}\right) + \left(\dfrac{35}{2k_i}\right)\operatorname{sgn}\left(u_{i,k_i}\right)$,

$\eta_{i,k_i}\left(u_i\right) = erf\left(u_{i,k_i}\right)$,

and $\Delta_{i,2,k_i}$ being a bounded function, for $k_i = 1,\ldots,3$, $i = 1,2$.

$$\Delta_{i,1,4} = \begin{cases} \left(3 + \exp\left(-\left|\sum_{k_i=1}^{3} \nu_{i,k_i}\right| 3i\right)\right) 0.8, & \text{if } u_{i,4} \le 0 \\ \left(1 + \exp\left(-\left|\sum_{k_i=1}^{3} \nu_{i,k_i}\right| i\right)\right) 1.2, & \text{if } u_{i,4} \ge 0 \end{cases}, \quad \eta_{i,4}\left(u_i\right) = u_{i,4},$$

and $\Delta_{i,2,4}$ being a bounded function, for $i = 1,2$.

The gauss error function $erf(.)$ is defined as $erf\left(u_{i,k_i}\right) = \dfrac{2}{\sqrt{\pi}} \int_0^{u_{i,k_i}} \exp\left(-t^2\right) dt$.

It easy to see in this Example that Assumptions 1–2 are satisfied.

The key goal of this simulation is to render system output y to follow reference signal y_d without violating any of the pseudo-state constraints, by virtue of the proposed controller defined in Section 3.

Subsection 3.5 presents the proposed controller design algorithm.

Similar to [21, 23-28, 32, 34], to approximate the unknown nonlinear functions, we utilize RBF neural networks within the simulation. Especially, the RBF neural network $w_{i,j_i}^{*T} \xi_{i,j_i}\left(\wp_{i,j_i}\right)$ uses $10 + i + j_i$ nodes in its hidden layer (i.e., $q_{i,j_i} = 10 + i + j_i$) whose centers are evenly spaced in the area of $\overbrace{[-16,\ 16]\times\cdots\times[-16,\ 16]}^{c_{i,j_i}}\subset\sim^{c_{i,j_i}}$ and the widths are equal to 32, for $j_i=1,2$, $i=1,2$. In accordance with subsection 3.2, $c_{i,j_i}=6+4j_i$ is the input layer size of the neural network, for $j_i=1,2,\ i=1,2$.

For the controller, intermediate control and adaptive laws, the appropriate function and the design parameters are taken as follows, $\sigma=\frac{1}{2}\exp(-3t)$, $\chi_{l,i,j_i}=100i+j_i+l$ and $b_{i,j_i}=5i+10j_i$, for $l=1,\ldots,5$, $j_i=1,2$, $i=1,2$.

The simulation is carried out with the initial conditions $\hat{\theta}_{i,j_i}(0)=0$, $\hat{w}_{i,j_i}(0)=0_{(10+i+j_i)\times 1}$, and $\zeta_{i,j_i}(0)=1.05$, for $j_i=1,2$, $i=1,2$.

Figures 3-18 display the Simulation results of Example 1 obtained by applying the proposed adaptive control scheme. The curves in Figures 3-4 show that system output y effectively tracks reference signal trajectory y_d with very short errors in a very short time. Figures 3-6 clearly show that the full pseudo-state constraints are not overstepped. The trajectories of actuator nonlinearity output v and control input u are depicted in Figures 7-14. Figure 15-18 explain the trajectory behaviors of the tracking errors z_{i,j_i}, $j_i=1,2$, $i=1,2$. These Figures show that both the tracking errors and outputs stay strictly within the constrained region.

Figures 3-18 clearly show that the transient responses are reasonable and the tracking errors converge to zero in finite time without transgressing the constraints. What is more, these results show that the proposed controller is capable of following a reference signal and possesses good robustness in the presence of actuator nonlinearities, disturbances and uncertain dynamics. This agrees with Theorem 1 and satisfactorily attests the efficiency of the proposed control scheme.

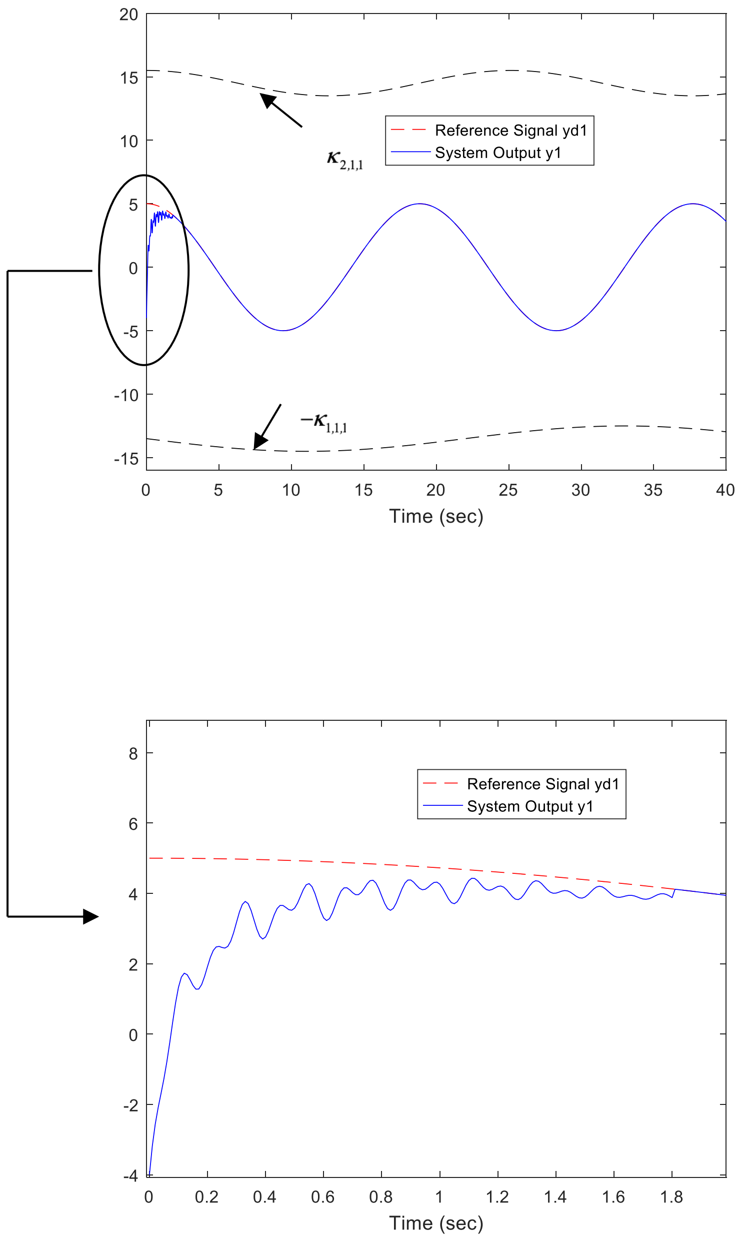

Figure 3. Reference signal y_{d1} and System output y_1.

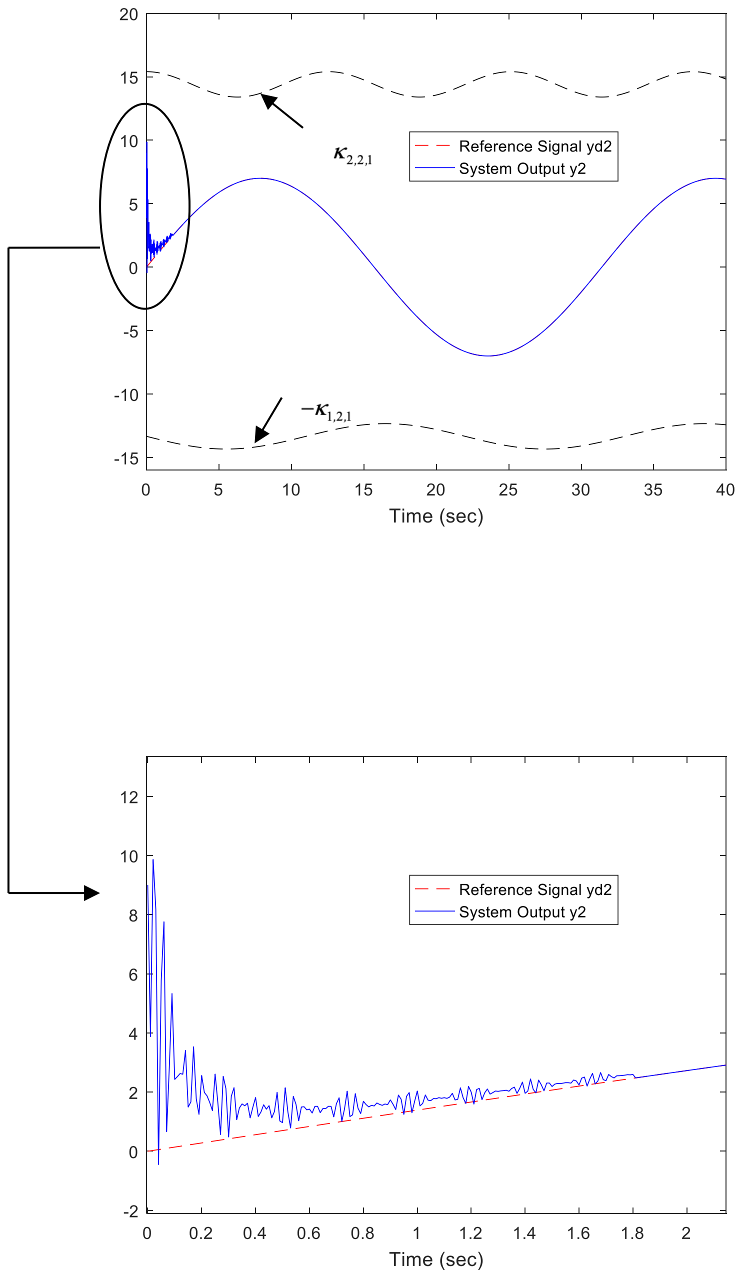

Figure 4. Reference signal y_{d2} and System output y_2.

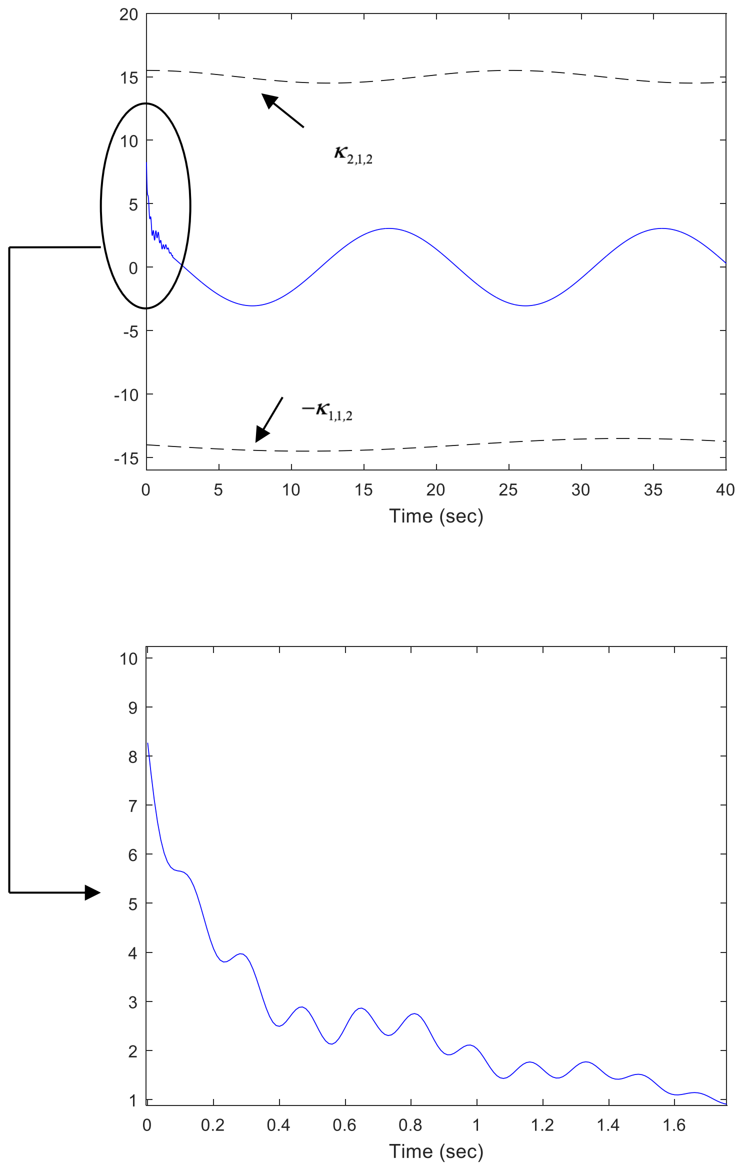

Figure 5. Trajectory of pseudo-state $x_{1,2}$

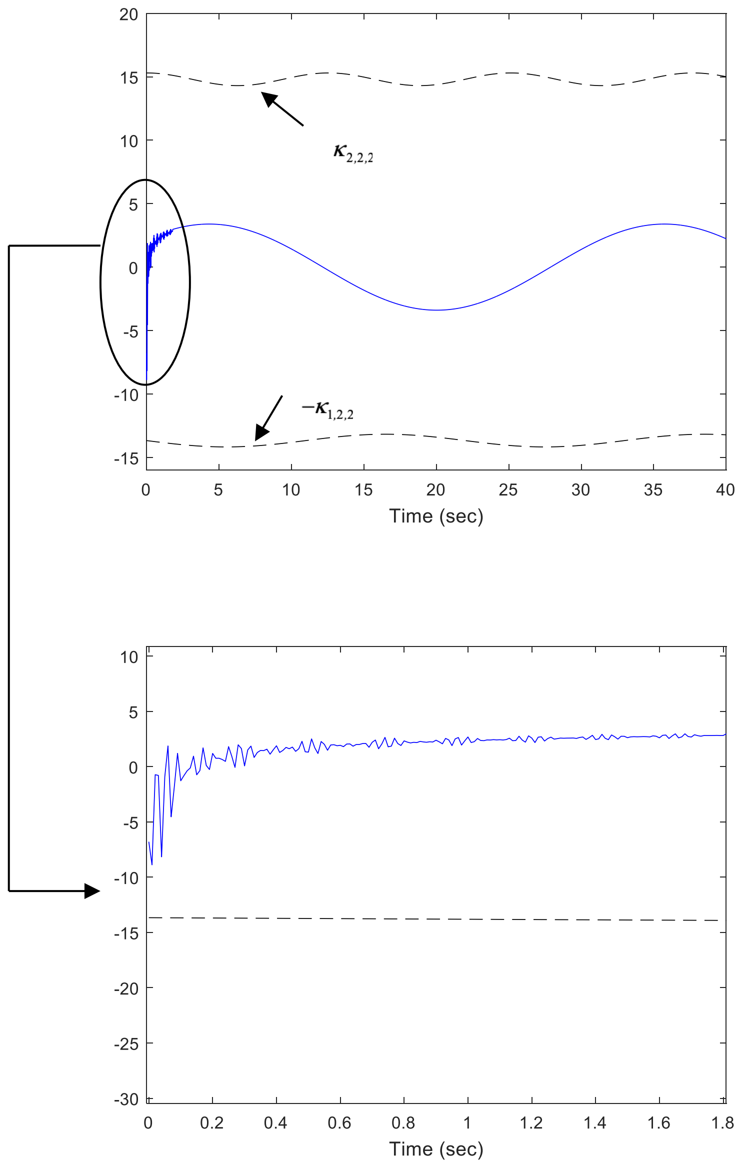

Figure 6. Trajectory of pseudo-state $x_{2,2}$

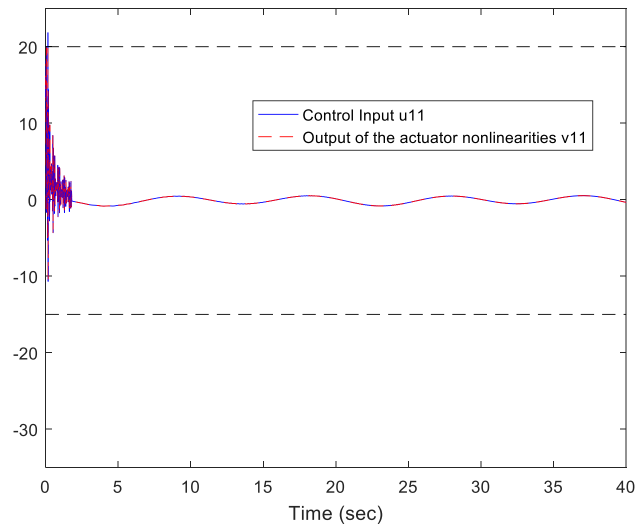

Figure. 7. Output of actuator nonlinearities $v_{1,1}$ and Control input $u_{1,1}$.

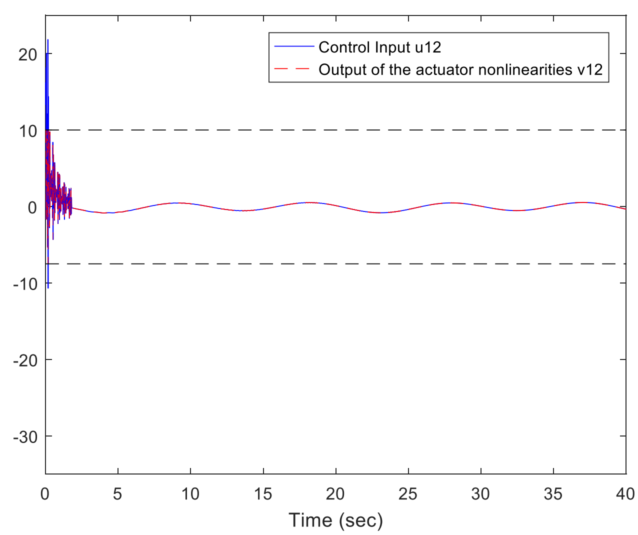

Figure. 8. Output of actuator nonlinearities $v_{1,2}$ and Control input $u_{1,2}$.

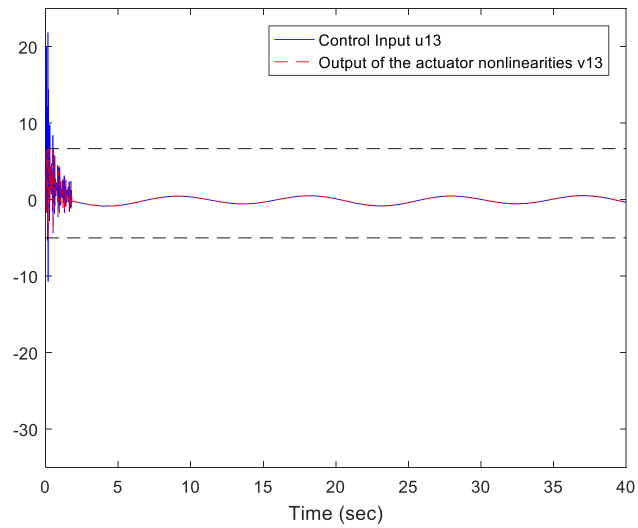

Figure. 9. Output of actuator nonlinearities $v_{1,3}$ and Control input $u_{1,3}$.

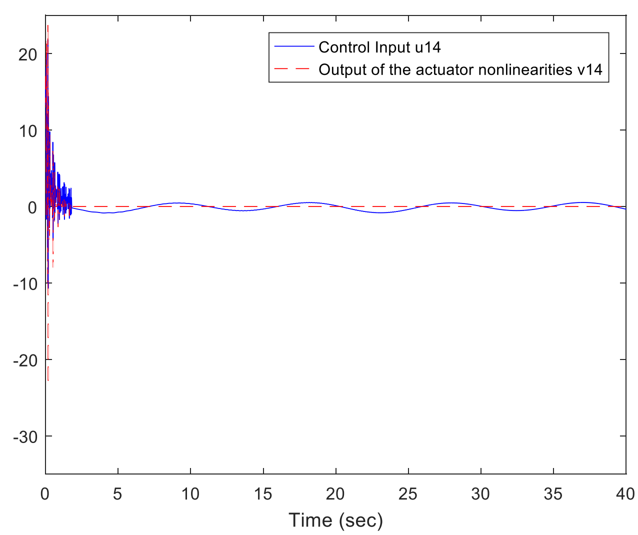

Figure. 10. Output of actuator nonlinearities $v_{1,4}$ and Control input $u_{1,4}$.

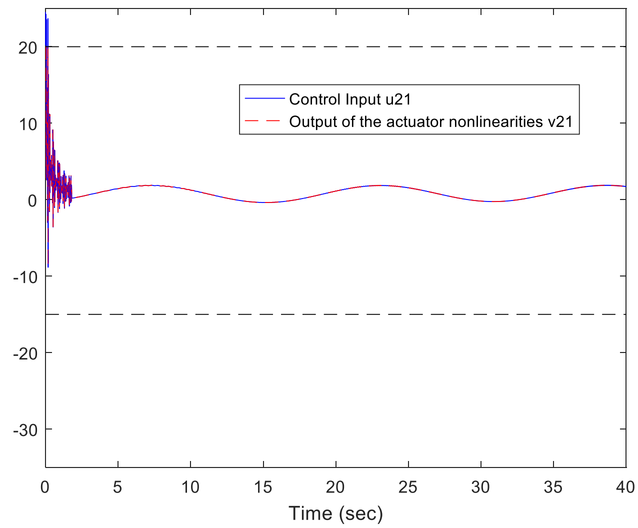

Figure. 11. Output of actuator nonlinearities $v_{2,1}$ and Control input $u_{2,1}$.

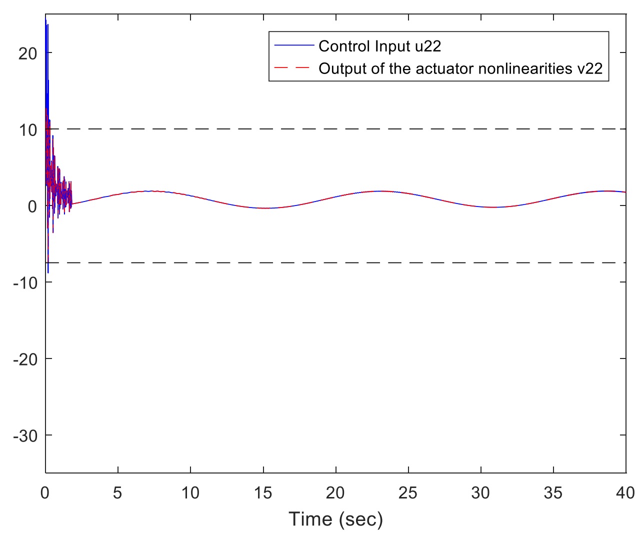

Figure. 12. Output of actuator nonlinearities $v_{2,2}$ and Control input $u_{2,2}$.

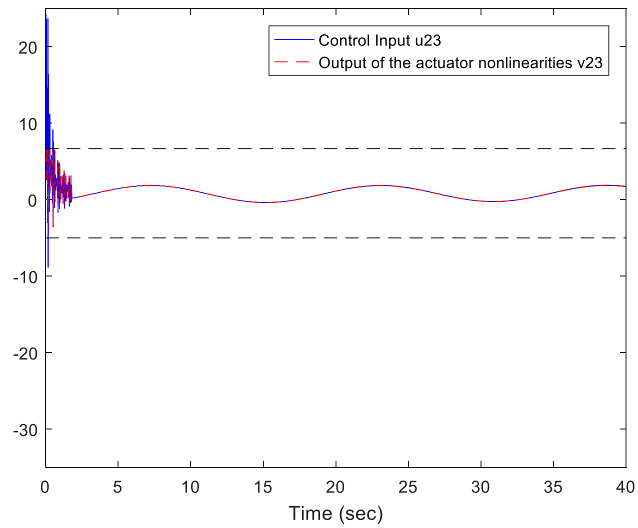

Figure. 13. Output of actuator nonlinearities $v_{2,3}$ and Control input $u_{2,3}$.

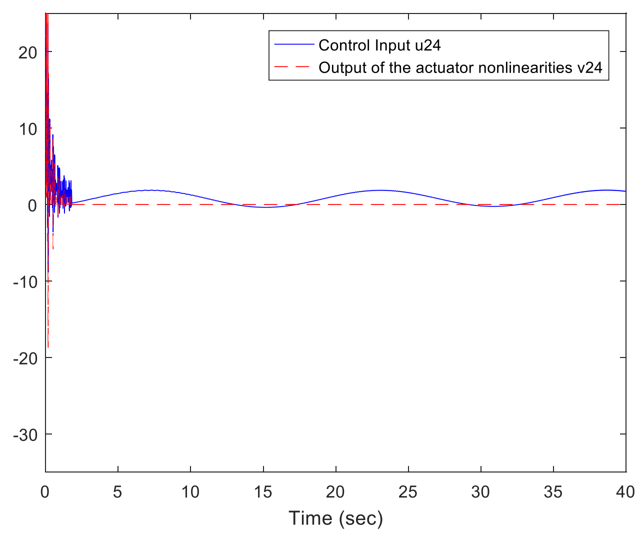

Figure. 14. Output of actuator nonlinearities $v_{2,4}$ and Control input $u_{2,4}$.

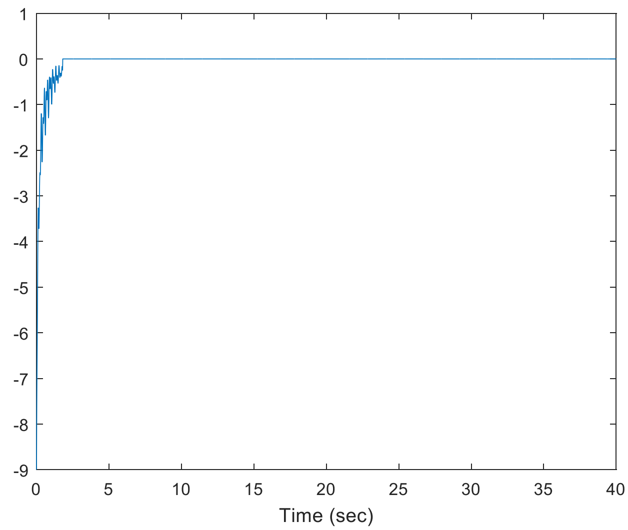

Figure. 15. Tracking error $z_{1,1}$

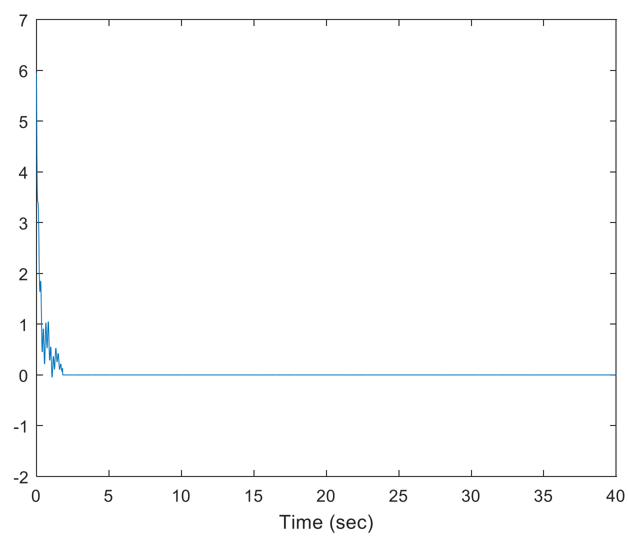

Figure. 16. Tracking error $z_{1,2}$

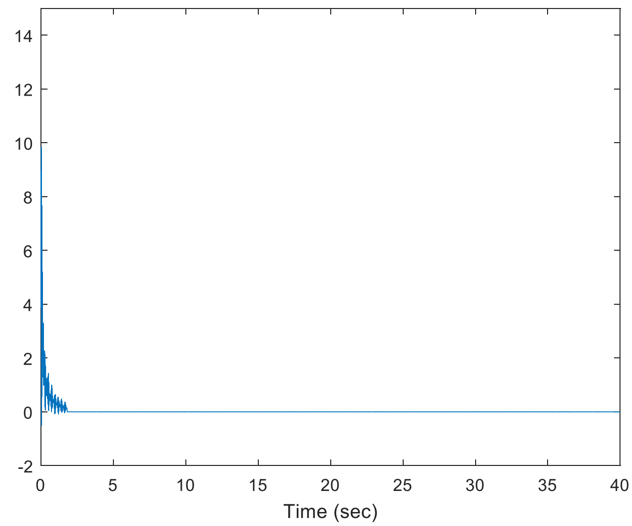

Figure. 17. Tracking error $z_{2,1}$

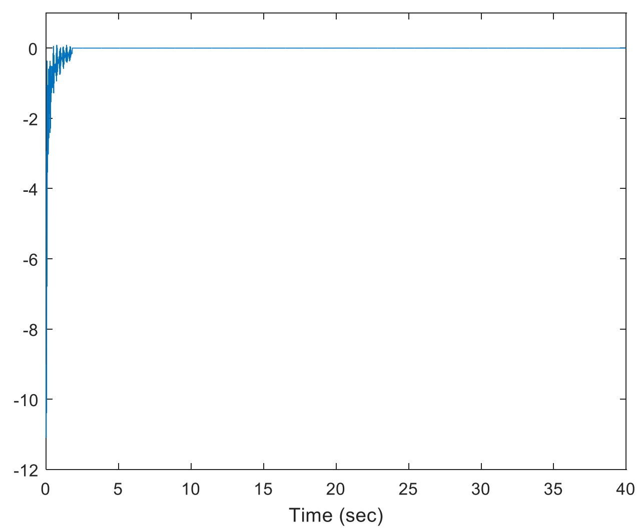

Figure. 18. Tracking error $z_{2,2}$

Example 2: (A Practical system: an electromechanical system [1, 47, 28])

Considering an electromechanical system, the dynamics of the system can be expressed as follows

$$\begin{cases} D^{\delta_{1,j}} x_{1,j} = f_{1,j}\left(x,\ x_{1,j+1},\ d\ \right), j = 1,\ldots,5 \\ D^{\delta_{1,6}} x_{1,6} = f_{1,6}\left(x,\ v_1,\ d\ \right), \\ v_1 = \eta_0\left(u_1\right) \\ y = x_{1,1} \end{cases} \tag{20}$$

where

- $\delta_{1,1} = 0.45$, $\delta_{1,2} = 0.55$, $\delta_{1,3} = 0.35$, $\delta_{1,4} = 0.65$, $\delta_{1,5} = 0.41$and $\delta_{1,6} = 0.59$ are the system fractional orders.
- $y \in \sim$, $x = x_1^T \in \sim^6$, and $u_1 \in \sim$ stand for the system output, pseudo-state and input, respectively, with $x_1 = \left[x_{1,1},\ldots,x_{1,6}\right]^T \in \sim^6$.
- The pseudo-states are limited in the sets $-\kappa_{1,1,j} < x_{1,j} < \kappa_{2,1,j}$, $j = 1,\ldots,6$, with $\kappa_{1,1,j} = 8 + \sin\left(\frac{t}{3j}\right)$,and $\kappa_{2,1,j} = 9 + \cos\left(\frac{t}{4j}\right)$, and $t \in [0, 100]$.
- $d = \left[d_{1,1},\ldots, d_{1,6}\right]^T \in \sim^6$ represents an external disturbance with $d_{1,j} = \frac{\cos(t)\sin(1+t)}{5+j}$ for $j = 1,\ldots,6$.
- The initial conditions of the system pseudo-states are fixed at $x_1(0) = [-5,\ 7.5,\ 8,\ -5,\ -5, -5.26\ ]^T$.
- $x_{1,1}$ is the angular motor position (and hence the position of the load).
- $x_{1,5}$ is the motor armature current.
- u_1 is the input control voltage.
- $\eta_0(.) \in \sim$ is a continuous function.

Table 1 specifies the system parameters.

Table 1. System parameters.

Description	Unit	Value
Rotor inertia	$kg.m^2$	1.625×10^{-3}
Link mass	kg	0.506
load mass	kg	0.434
Link length	m	0.305
Load radius	m	0.023
Gravity coefficient	m/s^2	9.8
Coefficient of viscous friction at the joint	N.m.s/ rad	16.25×10^{-3}
Coefficient characterizing the electromechanical conversion of armature current to torque	N.m/ A	0.9
Armature inductance	H	0.025
Armature resistance	Ω	5
back-emf coefficient	N.m/ A	0.9

The outputs of input nonlinearities and system functions are represented as follows

$$v_1 = \begin{cases} -5, \text{ if } u_1 < -5 \\ u_1, \quad \text{if } -5 \le u_1 \le 4, \\ 4, \text{ if } u_1 > 4 \end{cases}$$

(21)

$$\begin{cases} f_{1,1}\left(x,\ x_{1,2},\ d\ \right) = x_{1,2} + d_{1,1}, \\ f_{1,2}\left(x,\ x_{1,3},\ d\ \right) = x_{1,3} + d_{1,2}, \\ f_{1,3}\left(x,\ x_{1,4},\ d\ \right) = x_{1,4} + d_{1,3}, \\ f_{1,4}\left(x,\ x_{1,5},\ d\ \right) = 15.5764 x_{1,5} - 35.5391 \sin\left(x_{1,1}\right) - 0.28 x_{1,3} + 0.28 \cos\left(x_{1,3}\right) \sin\left(x_{1,5}\right) + d_{1,4} \\ f_{1,5}\left(x,\ x_{1,6},\ d\ \right) = x_{1,6} + d_{1,5} \\ f_{1,6}\left(x,\ v_1,\ d\ \right) = 40 v_1 - 36 x_{1,3} - 200 x_{1,5} \end{cases} \tag{22}$$

The reference signal is specified as $y_d = 6\sin\left(\frac{t}{8}\right)$.

From [8, 19, 25, 27, 35, 43, 50, 56, 69, 70], the outputs of actuator nonlinearities can be rewritten as

$$v_1 = \Delta_{1,1}\eta_1\left(u_1\right) + \Delta_{1,2} \tag{23}$$

with, $\Delta_{1,1} = -\left(\frac{1}{2}\right) + \left(\frac{9}{2}\right)\text{sgn}\left(u_1\right)$, $\eta_1\left(u_1\right) = erf\left(u_1\right)$ and $\Delta_{1,2}$ being a bounded function.

It is easy to verify from this Example that Assumptions 1–2 are satisfied.

Our purpose here is designing an appropriate neural adaptive controller in order for system output y to track the reference output y_d without transgressing any of the pseudo-state constraints, using the proposed method described in Section 3

Note that the proposed controller design algorithm is defined in Subsection 3.5.

Similar to [3, 15, 19, 21, 23-28, 32, 34, 47, 49, 51, 55, 58, 59], to model the unknown dynamics, we employ RBF neural networks in the following simulation. In particular, the RBF neural network $w_{1,j}^{*T}\xi_{1,j}\left(\wp_{1,j}\right)$utilizes $11+j$ nodes in its hidden layer (i.e., $q_{1,j}=11+j$) with centers chosen in the area of $\overbrace{[-10,\ 10]\times\cdots\times[-10,\ 10]}^{c_{1,j}}\subset\mathbb{R}^{c_{1,j}}$ and widths being equal to 20, for $j=1,\ldots,6$. In reference to subsection 3.2, $c_{1,j}=8+4j$ denotes the input layer size of the neural network, for $j=1,\ldots,6$.

The appropriate function and design parameters for the controller, intermediate control and adaptive laws are selected as, $\sigma=\frac{1}{2}\exp(-3t)$, $\chi_{l,1,j}=50+j+3l$ and $b_{i,j}=5+10j$, $l=1,\ldots,5$, $j=1,\ldots,6$.

The simulation is carried out with the initial conditions $\hat{\theta}_{1,j}(0)=0$, $\hat{w}_{1,j}(0)=0_{(11+j)\times 1}$, and $\zeta_{1,j}(0)=1.06$, for $j=1,\ldots,6$.

The Simulation results for Example 2 obtained using the proposed adaptive control scheme are shown in Figures 19-29. Figure 19 reveals that system output y effectively follows reference signal trajectory y_d with very short errors in a very short time. Figures 19-24 show that the full pseudo-state constraints are not transgressed. The trajectories of actuator nonlinearity output v and control input u are depicted in Figure 25. Figures 26-31 elucidate the trajectory behaviors of the tracking errors $z_{1,j}$, $j=1,\ldots,6$. From these Figures, it is observed that the outputs as well as the tracking errors reside strictly within the constrained region.

Figures 19-31 clearly show that favorable control performances with fast convergences of the tracking errors are attained, which means that the transient responses are reasonable and the tracking errors tend to zero in finite time without violating the constraints. Furthermore, these results show that the proposed controller is able to follow a reference signal and has good robustness in the presence of actuator nonlinearities, disturbances and uncertain dynamics. These concur with Theorem 1 and satisfactorily confirm the superiority and effectiveness of our approach.

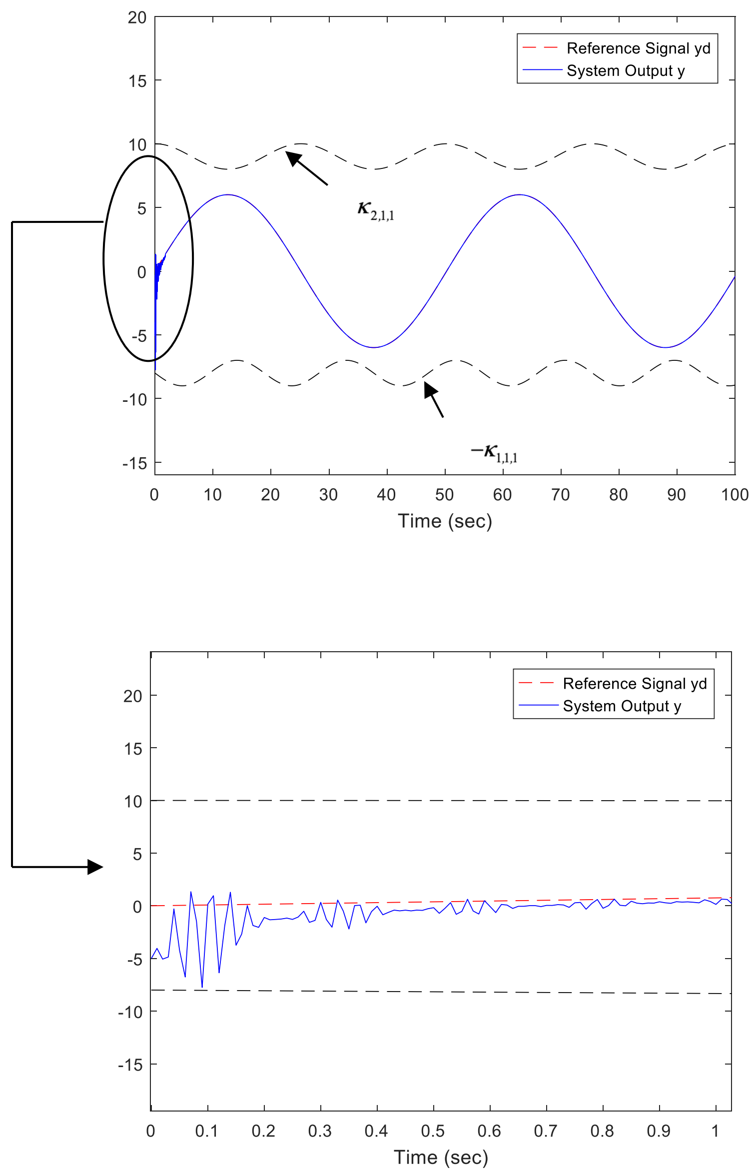

Figure 19. Reference signal y_d and System output y.

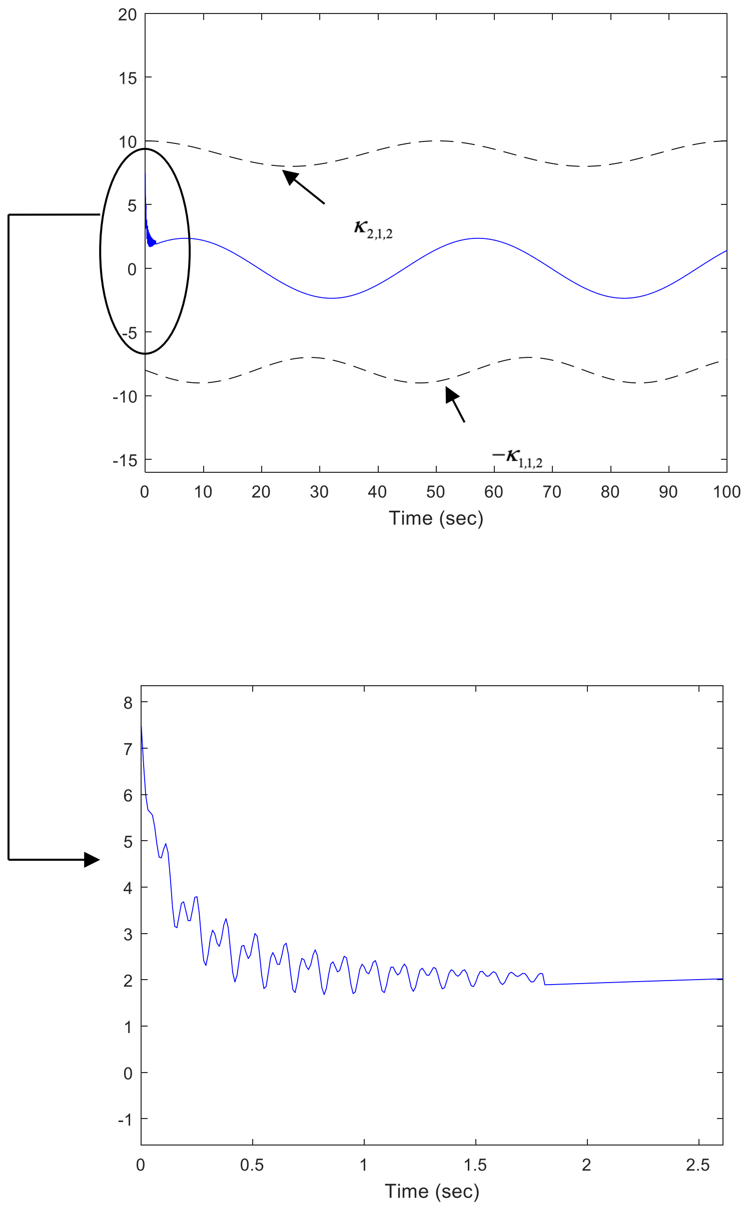

Figure 20. Trajectory of pseudo-state $x_{1,2}$

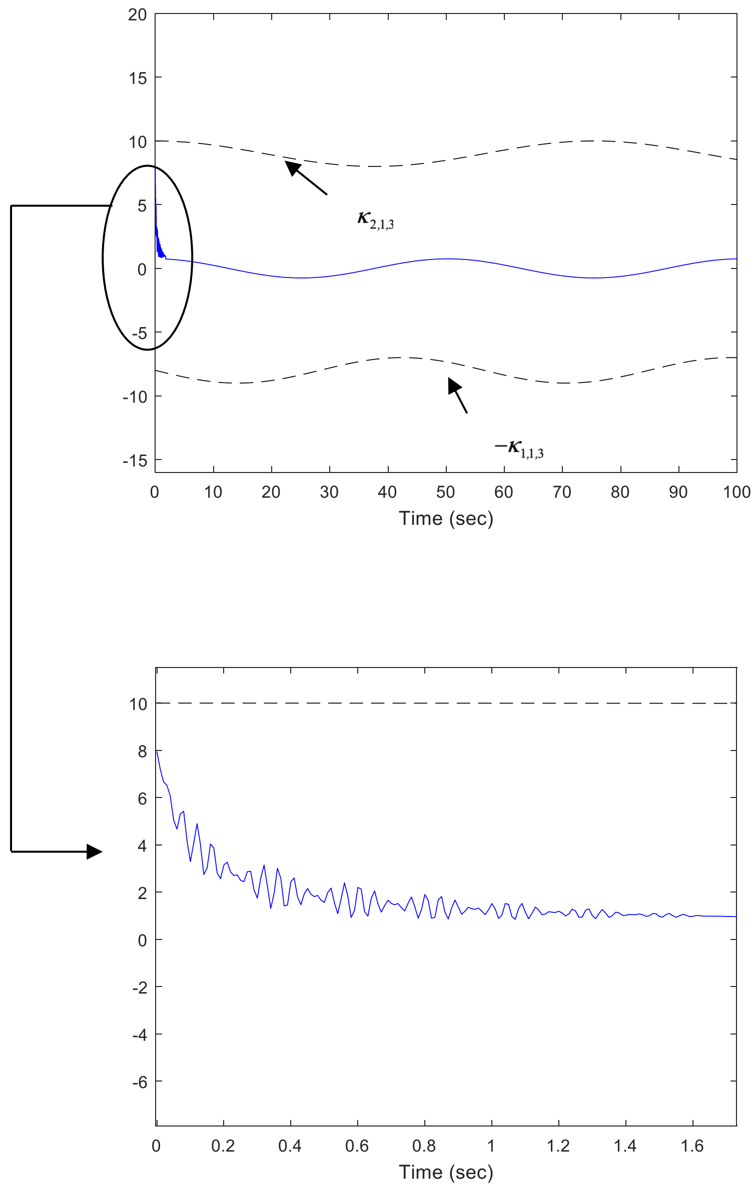

Figure 21. Trajectory of pseudo-state $x_{1,3}$

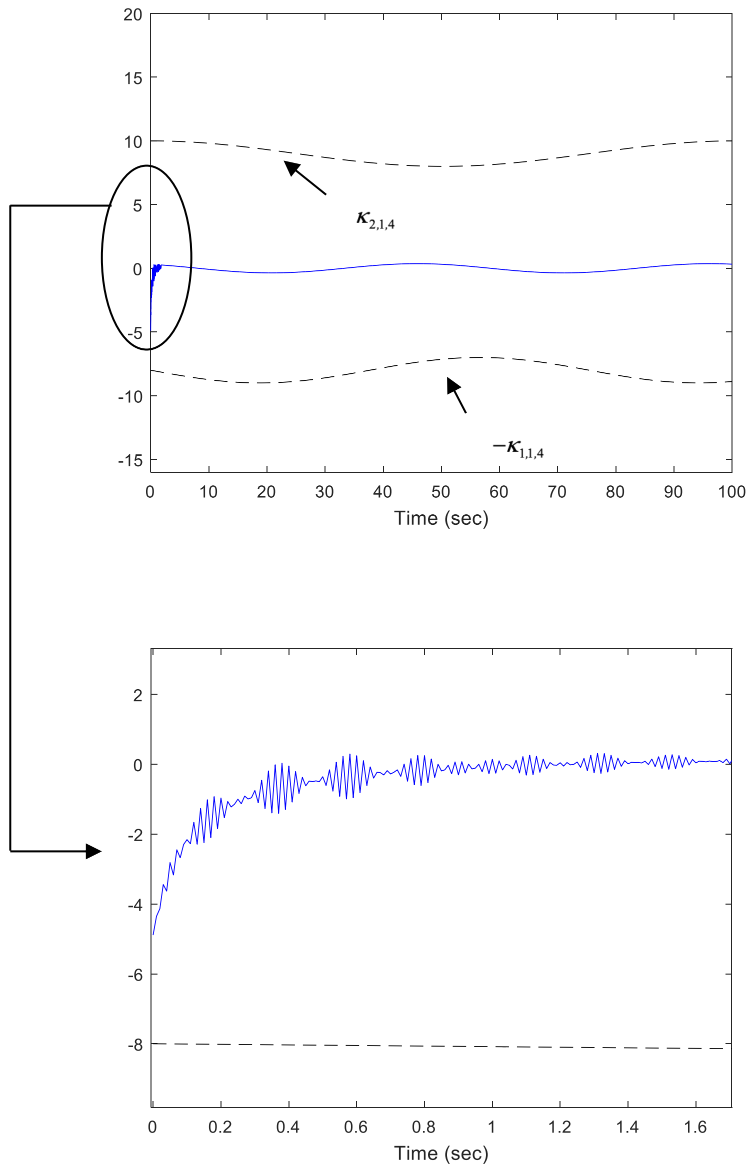

Figure 22. Trajectory of pseudo-state $x_{1,4}$

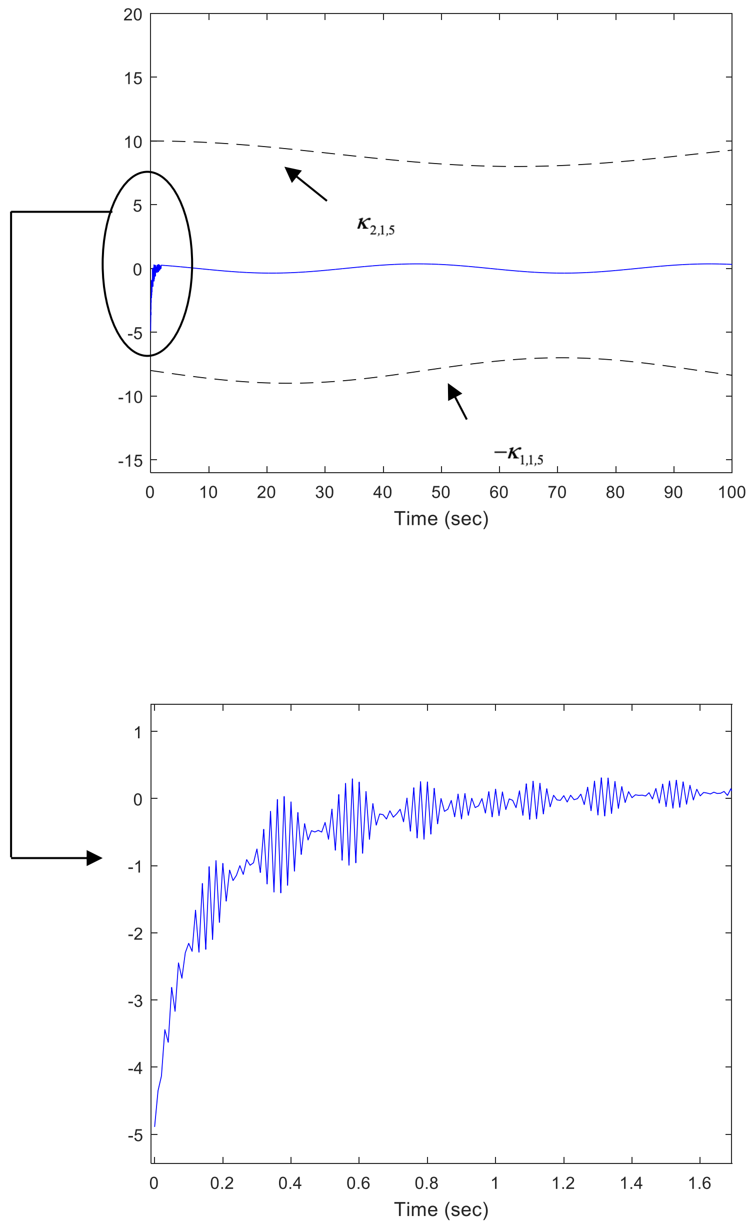

Figure 23. Trajectory of pseudo-state $x_{1,5}$

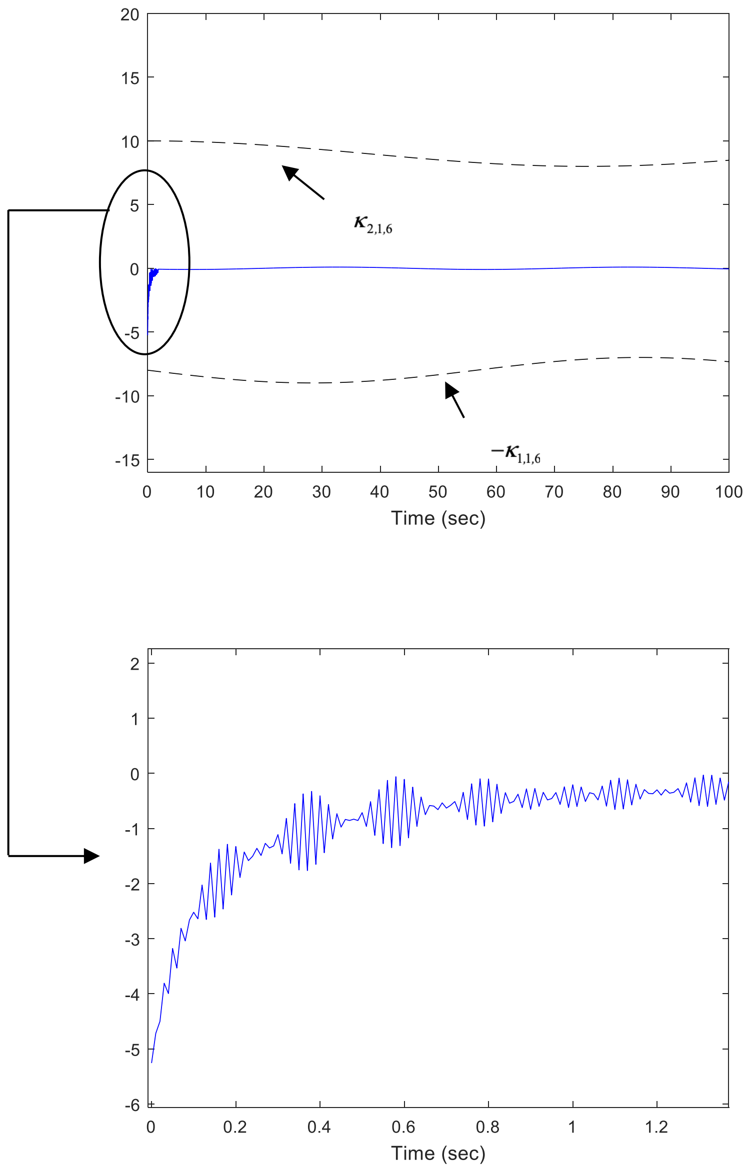

Figure 24. Trajectory of pseudo-state $x_{1,6}$

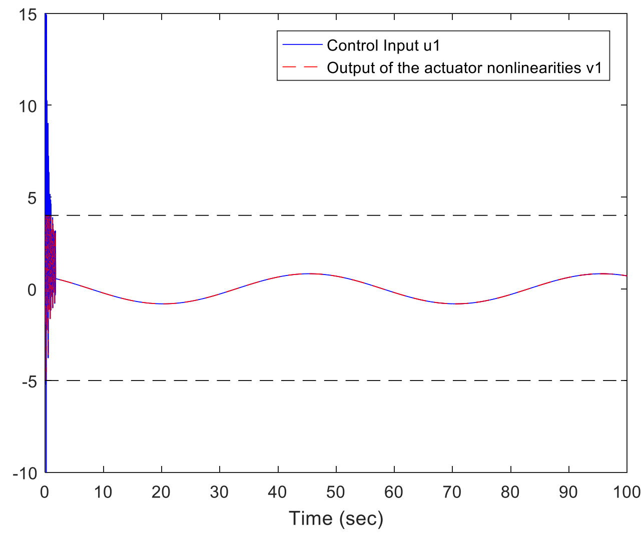

Figure. 25. Output of actuator nonlinearities v_1 and Control input u_1.

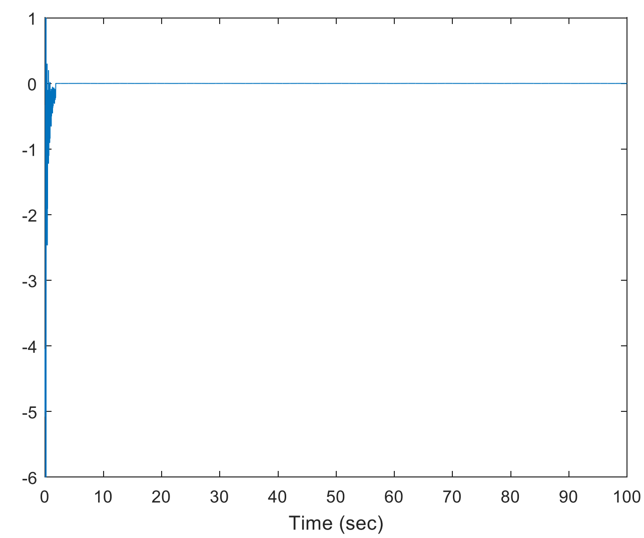

Figure. 26. Tracking error $z_{1,1}$

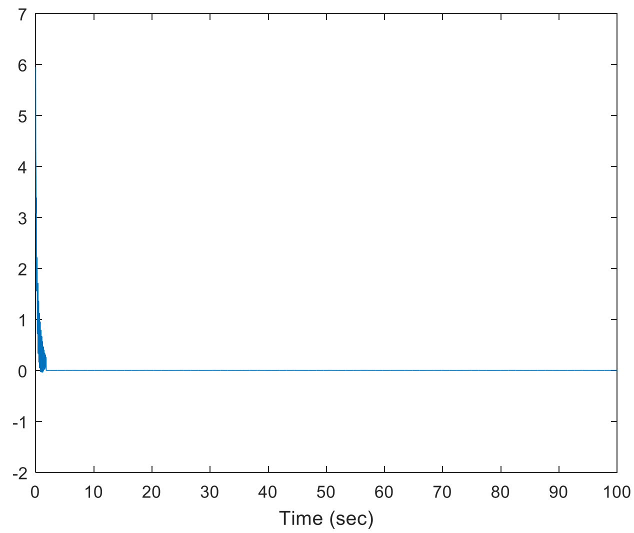

Figure. 27. Tracking error $z_{1,2}$

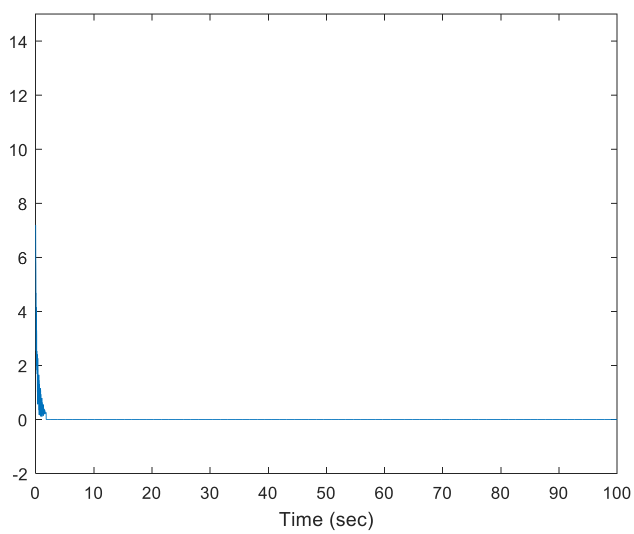

Figure. 28. Tracking error $z_{1,3}$

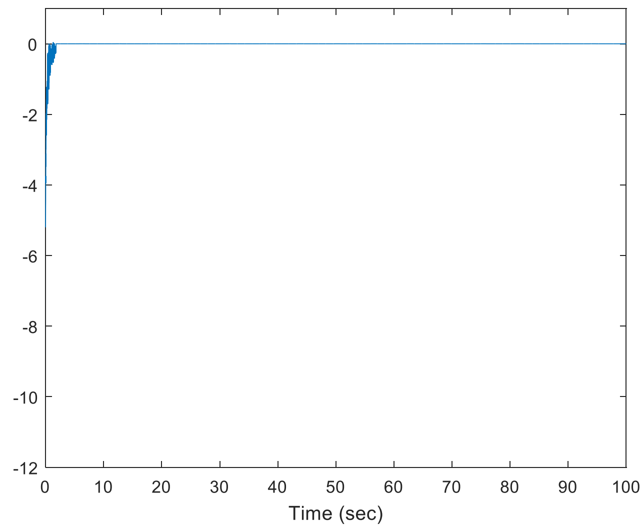

Figure. 29. Tracking error $z_{1,4}$

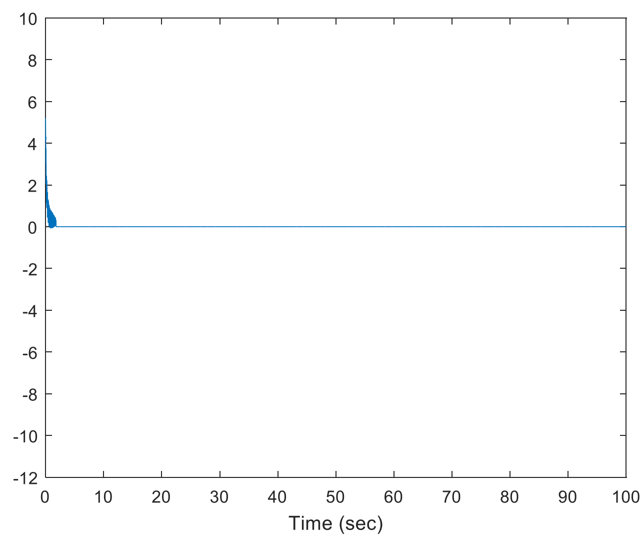

Figure. 30. Tracking error $z_{1,5}$

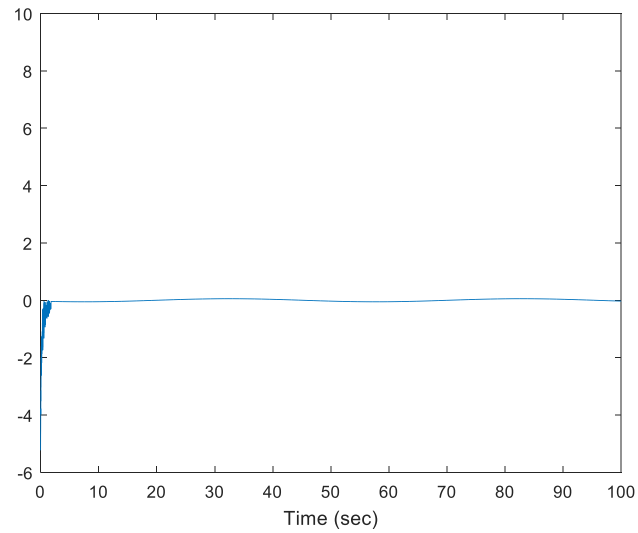

Figure. 31. Tracking error $z_{1,6}$

Remark 26. The Figures show that the tracking errors, system outputs and control inputs present transient behaviors (overshoots and oscillations) before meeting the control target. This can logically be due to:

–the Nussbaum function behavior (performing as a high gain controller)

–the absence of a *priori information* on the system variables and parameters,

–and the effects of actuator nonlinearities in the transient response.

Consequently, throughout the initial adaptation and estimation phases, the controller exhibits oscillations and vibrations, which disappear over time. These types of vibrations and oscillations naturally occur when employing adaptive controllers [1-13, 15-29, 32-39, 41-44, 46-48, 61, 62].

5. Conclusion

In this paper, a new neural adaptive control scheme has been developed for a category of factional-order non-square systems in the presence of Asymmetric pseudo-State Constraints, actuator Nonlinearities and uncertain nonlinear dynamics. In the control design process, the mean-value theorem, neural networks, Asymmetric time-varying barrier Lyapunov functions and Nussbaum functions have been used to conquer the problems caused by input Nonlinearities, uncertain nonlinear dynamics, pseudo-State Constraints and unknown control directions, respectively. In addition, auxiliary variables and a new dynamic surface control method grounded on Caputo fractional derivative definitions and fractional order filters have been incorporated to cope with the issues of disturbance compensation and explosion of complexity. The proposed adaptive control scheme is able to ensure that all signals in the closed-loop system are semi-global practical finite-time stability and the tracking errors asymptotically converge to zero without any constraint violations. Finally, some comparison and simulation results have been carried out to illustrate the effectiveness of the proposed control scheme. In the future, the event-triggered adaptive finite-time control issues will be studied for fractional order nonlinear systems based on this article.

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Appendix 1. Proof of Lemma 3.

Note from [21, 28] that:

- $\sum_{i=1}^{n} \left.\frac{\partial f\left(x_1(t),\ldots,x_n(t)\right)}{\partial x_i}\right|_{t=0} D^{\delta} x_i(t) \le D^{\delta} f\left(x_1(t),\ldots,x_n(t)\right) \le \sum_{i=1}^{n} \frac{\partial f\left(x_1(t),\ldots,x_n(t)\right)}{\partial x_i} D^{\delta} x_i(t)$ and $\operatorname{sgn}\left(\operatorname{tr}\left(H_0\right)\right)=1$, if $f(.)$ is convex with respect to the variables $x_1,\ldots,x_n$.
- $\sum_{i=1}^{n} \frac{\partial f\left(x_1(t),\ldots,x_n(t)\right)}{\partial x_i} D^{\delta} x_i(t) \le D^{\delta} f\left(x_1(t),\ldots,x_n(t)\right) \le \sum_{i=1}^{n} \left.\frac{\partial f\left(x_1(t),\ldots,x_n(t)\right)}{\partial x_i}\right|_{t=0} D^{\delta} x_i(t)$ and $\operatorname{sgn}\left(\operatorname{tr}\left(H_0\right)\right)=-1$, if $f(.)$ is concave with respect to the variables $x_1,\ldots,x_n$.

where $H_0 = \begin{pmatrix} \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_1^{\,2}} & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_2 \partial x_1} & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_2^{\,2}} & \cdots & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_n \partial x_1} & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f(x_1,\ldots,x_n)}{\partial x_n^{\,2}} \end{pmatrix}$ is the Hessian matrix of the function $f(.)$.

Based on the above results, it is deduced that

$$\begin{aligned} \operatorname{sgn}(\operatorname{tr}(H_0)) \sum_{i=1}^{n} \left. \frac{\partial f(x_1(t),\ldots,x_n(t))}{\partial x_i} \right|_{t=0} D^{\delta} x_i(t) &\le \operatorname{sgn}(\operatorname{tr}(H_0)) D^{\delta} f(x_1(t),\ldots,x_n(t)) \\ &\le \operatorname{sgn}(\operatorname{tr}(H_0)) \sum_{i=1}^{n} \frac{\partial f(x_1(t),\ldots,x_n(t))}{\partial x_i} D^{\delta} x_i(t) \end{aligned} \tag{A1.1}$$

if $f(.)$ is convex or concave with respect to the variables $x_1,\ldots,x_n$.

□

Appendix 2. Proof of Lemma 7

We have

$$\begin{cases} V(t) \ge 0 \\ D^{\delta} V(t) \le -\ell_1 V(t) + \ell_2 \end{cases} \tag{A2.1}$$

Since $\frac{\partial^2}{\partial t^2}\left[-\frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)} \right) \right] \right] \ge 0$, the function $-\frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)} \right) \right]$ is convex with respect to the variable t.

Using Definition 2, Properties 1-2, and Lemmas 3, 11, the following relations hold

$$\begin{cases} D^{\delta}V(0)=0 \\ D^{\delta}\left[-\frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\right]\right] \le -\ell_2 \exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\frac{D^{\delta}t^{\delta}}{\Gamma(1+\delta)} \\ \frac{D^{\delta}t^{\delta}}{\Gamma(1+\delta)}=1 \end{cases} \tag{A2.2}$$

From (A2.1) - (A2.2), it can be verified that

$$D^{\delta}\left[V(t)-V(0)-\frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\right]\right] \le -\ell_1 V(t)+\ell_2-\ell_2 \exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right) \tag{A2.3}$$

In addition, one has

$$\begin{cases} V(t)-V(0)-\frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\right]=0, \text{ if } t=0 \\ D^{\delta}\left[V(t)-V(0)-\frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\right]\right]\le 0, \text{ if } V(t)\ge \frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\right] \\ V(t)\le V(0)+\frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\right], \text{ if } V(t)\le \frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\right] \end{cases} \tag{A2.4}$$

Then, based on the last inequalities (A2.3) - (A2.4) and Property 4, it can be written that

$$V(t)\le V(0)+\frac{\ell_2}{\ell_1}\left[1-\exp\left(-\frac{\ell_1 t^{\delta}}{\Gamma(1+\delta)}\right)\right], \quad \forall t\ge 0 \tag{A2.5}$$

□

Appendix 3. Proof of Lemma 8

We have

$$\begin{cases} V(x) \geq 0 \\ D^{\delta} V(x) \leq -\ell_0 V(x) - \ell_3 V^2(x) - \ell_1 \dfrac{V(x)}{V(x)+\sigma} + \ell_2 \end{cases} \tag{A3.1}$$

Let $\Omega_1 = \{x | V(x) > \ell_4\}$ and $\Omega_2 = \{x | V(x) \leq \ell_4\}$ be two sets, where $\ell_4 = \sqrt{\left(\frac{\ell_0}{2\ell_3}\right)^2 + \frac{\ell_2}{\ell_3}} - \frac{\ell_0}{2\ell_3}$ is a strictly positive constant.

There are two cases to be considered.

In the first case, suppose $x \in \Omega_1$ and therefore (A3.1) can be expressed as

$$\begin{aligned} \dot{V}_1(t) = D^{\delta} V(x) &\leq -\ell_1 \frac{V(x)}{V(x)+\sigma} \\ &\leq -\ell_1 + \frac{\ell_1 \sigma}{\ell_4} \end{aligned} \tag{A3.2}$$

with $V_1(t) = D^{\delta-1} V(x)$, $\forall t \geq 0$.

Afterwards, integrate the above inequality (A3.2) and we get

$$0 \leq V_1(t) \leq V_1(0) - \ell_1 t + \frac{\ell_1}{\ell_4} \int_0^t \sigma(\tau) d\tau \tag{A3.3}$$

Since $\int_0^t \sigma(\tau) d\tau$ is bounded, there is a strictly positive constant ℓ_3 such that $\int_0^t \sigma(\tau) d\tau \leq \ell_5$.

Hence, when $t \geq \frac{\ell_5}{\ell_4} + \frac{V_1(0)}{\ell_1}$, we have $x \in \Omega_2$.

In the second case, suppose $x \in \Omega_2$

Consequently, based on case 1, the trajectory of the system $x(t)$ does not exceed the set Ω_2 .

It can be deduced that the solution for $D^{\delta}x = f(x)$ is bounded in finite time.

□

Appendix 4. Proof of Lemma 9

We have

$$D^{\delta}\alpha^{c}(t) = b\left[\alpha(t) - \alpha^{c}(t)\right] \tag{A4.1}$$

Let $\tilde{\alpha}(t) = \alpha^{c}(t) - \alpha(t)$ and select the Lyapunov candidate $V = \dfrac{\tilde{\alpha}^{2}(t)}{2}$

From Lemma 3 and (A4.1), one has

$$\begin{aligned} D^{\delta}V &\le \tilde{\alpha}(t)D^{\delta}\tilde{\alpha}(t) \\ &\le -b\tilde{\alpha}^{2}(t) - \tilde{\alpha}(t)D^{\delta}\alpha(t) \\ &\le -\left(2b - \frac{1}{2\ell^{2}}\right)V + \ell^{2}M_{1}^{2} \end{aligned} \tag{A4.2}$$

where ℓ and M_1 are strictly positive constants such that $\ell > \dfrac{1}{2\sqrt{b}}$ and $\left|D^{\delta}\alpha(t)\right| \le M_1$.

Moreover, it follows from the above inequality (A4.2) and Lemma 7 that

$$\left|\tilde{\alpha}(t)\right| \le \sqrt{\frac{4\ell^{4}M_{1}^{2}}{4b\ell^{2}-1}\left[1 - \exp\left(-\frac{\left(4b\ell^{2}-1\right)t^{\delta}}{2\ell^{2}\Gamma(1+\delta)}\right)\right]} \tag{A4.3}$$

Taking the fractional time derivative of order δ of both side of $D^{\delta}\alpha^{c}(t) = b\left[\alpha(t) - \alpha^{c}(t)\right]$, we have $D^{2\delta}\alpha^{c}(t) = b\left[D^{\delta}\alpha(t) - D^{\delta}\alpha^{c}(t)\right]$.

It can be deduced that $D^{2\delta}\alpha^c(t)$ is bounded by employing the boundedness of $D^{\delta}\alpha(t)$ and $D^{\delta}\alpha^c(t)$.

Let $D^{\delta}\tilde{\alpha}(t) = D^{\delta}\alpha^c(t) - D^{\delta}\alpha(t)$ and choose the Lyapunov candidate $V_{\delta} = \frac{\left[D^{\delta}\tilde{\alpha}(t)\right]^2}{2}$, one has

$$\begin{aligned} D^{\delta}V_{\delta} &\le D^{\delta}\tilde{\alpha}(t) D^{2\delta}\tilde{\alpha}(t) \\ &\le -b\left[D^{\delta}\tilde{\alpha}(t)\right]^2 - \left[D^{\delta}\tilde{\alpha}(t)\right]\left[D^{2\delta}\alpha(t)\right] \\ &\le -\left(2b - \frac{1}{2\ell^2}\right)V_{\delta} + \ell^2 M^2 \end{aligned} \tag{A4.4}$$

where M is a strictly positive constant such that $\left|D^{2\delta}\alpha(t)\right| \le M$.

In addition, it can be obtained from the above inequalities (A4.4) and Lemma 7 that

$$\left|D^{\delta}\alpha^c(t) - D^{\delta}\alpha(t)\right| \le \sqrt{\left(D^{\delta}\alpha(0)\right)^2 + \frac{4\ell^4 M^2}{4b\ell^2 - 1}\left[1 - \exp\left(-\frac{\left(4b\ell^2 - 1\right)t^{\delta}}{2\ell^2\Gamma(1+\delta)}\right)\right]} \tag{A4.5}$$

□

Appendix 5. Proof of Lemma 10

From Definitions 1-2, we get

$$\begin{cases} D^{\delta}\left[\mu_1(t)\mu_2(t)\right] = \dfrac{1}{\Gamma(1-\delta)}\displaystyle\int_0^t \dfrac{\mu_1'(\tau)\mu_2(\tau)+\mu_2'(\tau)\mu_1(\tau)}{(t-\tau)^{\delta}}d\tau \\ D^{\delta}\left[\mu_1(t)\right] = \dfrac{1}{\Gamma(1-\delta)}\displaystyle\int_0^t \dfrac{\mu_1'(\tau)}{(t-\tau)^{\delta}}d\tau \\ D^{\delta}\left[\mu_2(t)\right] = \dfrac{1}{\Gamma(1-\delta)}\displaystyle\int_0^t \dfrac{\mu_2'(\tau)}{(t-\tau)^{\delta}}d\tau \\ D^{-\delta}\mu_1(t) = \dfrac{1}{\Gamma(\delta)}\displaystyle\int_0^t \dfrac{\mu_1(\tau)}{(t-\tau)^{1-\delta}}d\tau \\ D^{-\delta}1 = \dfrac{1}{\Gamma(\delta)}\displaystyle\int_0^t \dfrac{1}{(t-\tau)^{1-\delta}}d\tau = \dfrac{t^{\delta}}{\Gamma(\delta+1)} \end{cases} \tag{A5.1}$$

Since $\mu_1(t)$ and $\mu_2(t)$ are monotonic functions, the following inequalities are therefore satisfied for any time instants $0 \le \tau \le t$

$$\begin{cases} \operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\dfrac{\partial\mu_2(t)}{\partial t}\right)\left[\mu_2(0)-\mu_2(\tau)\right]\mu_1'(\tau) \le 0 \\ \operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\dfrac{\partial\mu_2(t)}{\partial t}\right)\left[\mu_1(0)-\mu_1(\tau)\right]\mu_2'(\tau) \le 0 \\ \operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\dfrac{\partial\mu_2(t)}{\partial t}\right)\left[\mu_2(\tau)-\mu_2(t)\right]\mu_1'(\tau) \le 0 \\ \operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\dfrac{\partial\mu_2(t)}{\partial t}\right)\left[\mu_1(\tau)-\mu_1(t)\right]\mu_2'(\tau) \le 0 \\ \left[\mu_1(0)-\mu_1(\tau)\right]\operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\right) \le 0 \\ \left[\mu_1(\tau)-\mu_2(t)\right]\operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\right) \le 0 \end{cases} \tag{A5.2}$$

Using the above inequalities (A5.1) - (A5.2) and analysis, we can conclude that

$$\begin{cases} \left[\mu_1(0)D^{\delta}\left[\mu_2(t)\right]+\mu_2(0)D^{\delta}\left[\mu_1(t)\right]\right]\operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\dfrac{\partial\mu_2(t)}{\partial t}\right) \\ \qquad \le \operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\dfrac{\partial\mu_2(t)}{\partial t}\right)D^{\delta}\left[\mu_1(t)\mu_2(t)\right] \\ \qquad \le \left[\mu_1(t)D^{\delta}\left[\mu_2(t)\right]+\mu_2(t)D^{\delta}\left[\mu_1(t)\right]\right]\operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\dfrac{\partial\mu_2(t)}{\partial t}\right) \\ \operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\right)\mu_1(0)\dfrac{t^{\delta}}{\Gamma(\delta+1)} \le \operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\right)D^{-\delta}\left[\mu_1(t)\right] \\ \qquad \le \operatorname{sgn}\left(\dfrac{\partial\mu_1(t)}{\partial t}\right)\mu_1(t)\dfrac{t^{\delta}}{\Gamma(\delta+1)} \end{cases} \tag{A5.3}$$

□

Appendix 6. Proof of Lemma 12

We have

$$\begin{cases} 0 \le V_1 \le V_2 \\ \sigma \ge 0 \end{cases} \tag{A6.1}$$

From (A6.1), one has

$$\begin{cases} -V_2 \le -V_1 \\ \dfrac{V_1}{V_1+\sigma}-\dfrac{V_2}{V_2+\sigma}=\dfrac{\sigma(V_1-V_2)}{(V_2+\sigma)(V_1+\sigma)} \\ \qquad \le 0 \\ V_1^2-V_2^2=(V_1-V_2)(V_1+V_2). \\ \qquad \le 0 \end{cases} \tag{A6.2}$$

Therefore, from (A6.2), we get the following inequality

$$\begin{cases} -\dfrac{V_2}{V_2+\sigma} \le -\dfrac{V_1}{V_1+\sigma} \\ -{V_2}^2 \le -V_1^2 \end{cases} \tag{A6.3}$$

□

Appendix 7. Proof of Corollary 1

According to [21], the following inequalities are satisfied

$$:\begin{cases} D^{\delta}\log\left(\dfrac{1}{\mu_2(t)}\right) = -D^{\delta}\log\left(\mu_2(t)\right) \le -\dfrac{1}{\mu_2(t)}D^{\delta}\mu_2(t) \\ D^{\delta}\left(\dfrac{\mu_1^2(t)}{\mu_2(t)}\right) \le 2\dfrac{\mu_1(t)}{\mu_2(t)}D^{\delta}\mu_1(t) - \left(\dfrac{\mu_1(t)}{\mu_2(t)}\right)^2 D^{\delta}\mu_2(t) \\ 2n{\mu_1}^{2n-1}(0)D^{\delta}\mu_1(t) \le D^{\delta}{\mu_1}^{2n}(t) \le 2n{\mu_1}^{2n-1}(t)D^{\delta}\mu_1(t) \\ D^{\delta}\left(\dfrac{\mu_1(t)}{\mu_3(t)}\right)^2 \le 2\left(\dfrac{\mu_1(t)}{{\mu_3}^2(t)}\right)D^{\delta}\mu_1(t) - \left(\dfrac{{\mu_1}^2(t)}{{\mu_3}^4(t)}\right)D^{\delta}{\mu_3}^2(t) \\ -D^{\delta}{\mu_3}^2(t) \le -2\mu_3(0)D^{\delta}\mu_3(t) \end{cases} \tag{A7.1}$$

Therefore, from (A7.1), we get the following inequality

$$\begin{aligned} D^{\delta}\left(\dfrac{\mu_1(t)}{\mu_3(t)}\right)^{2n} &\le n\left(\dfrac{{\mu_1}^{2n-2}(t)}{{\mu_3}^{2n-2}(t)}\right)D^{\delta}\left(\dfrac{\mu_1(t)}{\mu_3(t)}\right)^2 \\ &\le 2n\left(\dfrac{{\mu_1}^{2n-1}(t)}{{\mu_3}^{2n}(t)}\right)D^{\delta}\mu_1(t) - n\left(\dfrac{{\mu_1}^{2n}(t)}{{\mu_3}^{2n+2}(t)}\right)D^{\delta}{\mu_3}^2(t) \\ &\le 2n\left(\dfrac{{\mu_1}^{2n-1}(t)}{{\mu_3}^{2n}(t)}\right)D^{\delta}\mu_1(t) - 2n\mu_3(0)\left(\dfrac{{\mu_1}^{2n}(t)}{{\mu_3}^{2n+2}(t)}\right)D^{\delta}\mu_3(t) \end{aligned} \tag{A7.2}$$

□

Appendix 8. Proof of Corollary 2

In accordance with Lemma 3 and Corollary 1, one consequently obtain

$$
\begin{aligned}
\frac{1}{2n} D^{\delta} \log\left(\frac{\mu_3^{2n}(t)}{\mu_3^{2n}(t) - \mu_1^{2n}(t)} \right) &= \frac{1}{2n} D^{\delta} \log\left(\frac{1}{1 - \left(\frac{\mu_1(t)}{\mu_3(t)} \right)^{2n}} \right) \\
&\le \frac{1}{2n} \left(\frac{1}{1 - \left(\frac{\mu_1(t)}{\mu_3(t)} \right)^{2n}} \right) D^{\delta} \left[\left(\frac{\mu_1(t)}{\mu_3(t)} \right)^{2n} \right] \\
&\le \frac{\mu_1^{2n-1}(t)}{\mu_3^{2n}(t) - \mu_1^{2n}(t)} \left(D^{\delta} \mu_1(t) - \mu_3(0) \left(\frac{\mu_1(t)}{\mu_3^{2}(t)} \right) D^{\delta} \mu_3(t) \right)
\end{aligned}
\tag{A8.1}
$$

□

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